DOE/ET/53088-28 IFSR #28

# ENERGY CONSERVATION AND RELATED CONSTRAINTS IN DRIFT WAVE TURBULENCE

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June 1981

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# Energy Conservation and Related Constraints
# In Drift Wave Turbulence

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Abstract

The problem of energy conservation for the renormalization of the drift wave instability in a sheared magnetic field is considered. It has been suggested previously that there is a connection between a certain constraint on the nonlinear term in the drift kinetic equation and energy conservation. Arguments are presented to dissolve this connection; and in turn, energy conservation is formulated in the physically meaningful statistically averaged sense. Finally, energy conservation is proven for the system of nonlinear equations, renormalized by the Normal Stochastic Approximation, describing the drift wave instability.

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## I. Introduction

There has been much interest in the energy conservation and related constraint properties for the renormalization of the nonlinear drift wave instability[1,2,3]. In particular, it has been suggested that energy conservation arguments lead to a constraint on the nonlinear term in the drift kinetic equation

$$-\frac{C}{B}\vec{\nabla}\phi \times \frac{\vec{B}}{B}\cdot\vec{\nabla} F_{N_e} \quad ;$$

such that its $\phi$ moment vanishes

$$-ne\int d^3x\int du\ \phi\left[-\frac{C}{B}\vec{\nabla}\phi \times \frac{\vec{B}}{B}\cdot\vec{\nabla} F_{N_e}\right] = 0.$$

Here, $F_{N_e}$ is the Klimontovich function for electrons. This constraint has been called "energy conservation", since it can be rewritten as $\vec{J}_\perp\cdot\vec{E}_\perp=0$. Actually, this terminology is misleading, as will be shown in the following. Furthermore, a proper definition and proof of energy conservation for the Normal Stochastic Approximation[4,5] previously thought[3] to violate energy conservation, will be given. This proof will be carried out in detail for the case of electrostatic drift wave turbulence in a sheared magnetic field. The model considered is a slab model where the shear and the electron density gradient are in the inhomogeneous x direction, while the system is taken to be homogeneous in the y and z coordinates. Our analysis is also restricted to the case of stationary turbulence.

Any theory starting from the drift kinetic equation is approximate at the outset, such that it does not properly conserve the perpendicular energy. In this approximation the perpendicular velocity $(\vec{v}_\perp)$ is dominantly $\tilde{\vec{E}} \times \vec{B}$ in nature. Thus $\vec{v}_\perp \cdot \tilde{\vec{E}} = 0$, so that the power into the perpendicular velocity is zero. On the other hand, $\tilde{\vec{E}}$ can change dynamically as the waves grow or damp enabling the perpendicular energy to change. This inconsistency is an artifact of the drift kinetic equation approximation, while in an exact theory the change in $\vec{v}_\perp$ is accounted for by the polarization drift. Consequently, consistent with this drift kinetic equation approximation, the perpendicular energy is to be taken as small compared to the parallel energy. In fact, the perpendicular energy $(mv_\perp^2/2)$ and thus the $\phi$ moment constraint, being the perpendicular interaction energy, should not be included in an energy conservation argument; rather, particle kinetic energy must be limited to the parallel energy $(mv_\parallel^2/2)$. Whether the $\phi$ moment constraint should be conserved under renormalization is still an open question. However, to refer to this constraint as "energy conservation" greatly overstates its importance.

Physically observable energies are statistically averaged quantities. Therefore, the most fundamental and applicable form of energy conservation in this context is that the change in particle energy in a statistically averaged sense must be equal to negative the change in fluctuation energy. In particular, energy conservation will be determined by a parallel energy moment of the statistically averaged one point equation. The equation which propagates the average distribution has been derived by Swartz and Molvig[6] using the Normal Stochastic Approximation (NSA)[5] renormalization scheme. This is essentially the approximation used by Hirshman and Molvig[4] in the somewhat simpler problem of nonlinear stability of the universal mode. The NSA does not satisfy the $\phi$ moment constraint for the fluctuation dynamics; however, for the average distribution it conserves energy in the statistically averaged sense. The result of this paper will be the usual statement of energy conservation, that the change in particle energy density is equal to negative the change in wave energy density plus negative the divergence of the power flux.

## II. Energy Conservation Proof

Our starting point for the analysis of the drift wave problem is the Drift Kinetic Klimontovich Equation (DKKE) for electrons

$$\left[\frac{\partial}{\partial t} + u\,\frac{\vec{B}}{B}\cdot\frac{\partial}{\partial\vec{x}} - \frac{C}{B}\left(\vec{\nabla}\phi\times\frac{\vec{B}}{B}\right)\cdot\frac{\partial}{\partial\vec{x}} + \frac{e}{m_e}\,\vec{\nabla}\phi\cdot\frac{\vec{B}}{B}\,\frac{\partial}{\partial u}\right]F_{N_e} = 0 \quad . \qquad (1)$$

In order to proceed with an energy conservation proof in a statistically averaged sense, a closed system of equations for the one point distribution function and for the two point correlation function must be derived. We consider here the system which results from the application of the NSA to the statistical hierarchy.[6] This approximation includes the effect of stochastic orbit scattering on the correlations.

Statistically averaging the DKKE Eq. (1) yields an exact equation for the averaged one point distribution function

$$\left[\frac{\partial}{\partial t} + u\,\frac{\vec{B}}{B}\cdot\frac{\partial}{\partial\vec{x}}\right]f_e = \left\langle\left[\frac{C}{B}\left(\vec{\nabla}\phi\times\frac{\vec{B}}{B}\right)\cdot\frac{\partial}{\partial\vec{x}} - \frac{e}{m_e}\vec{\nabla}\phi\cdot\frac{\vec{B}}{B}\,\frac{\partial}{\partial u}\right]F_{N_e}\right\rangle \equiv C\,(f_e) \quad . \qquad (2)$$

Here, $f_e \equiv \langle F_{N_e}\rangle$ is the average distribution function. The collision operator in Eq. (2) has been computed within the NSA by Swartz and Molvig[6] and is

$$C(f_e) = \frac{\partial}{\partial u}\left\{\frac{-e^2}{m_e T_e}\int\frac{d\omega}{2\pi}\int\frac{d\vec{k}}{(2\pi)^2}\int dx' k_{\|}(x)\,[\omega-\omega_e^*(x')]\,R_e H^{+}_{\vec{k}\omega}(x,u,x')\,f_e(u)\,S_{\vec{k}\omega}(x,x')\right\}$$
$$+\frac{\partial}{\partial x}\left\{\frac{eL_n}{T_e}\int\frac{d\omega}{2\pi}\int\frac{d\vec{k}}{(2\pi)^2}\int dx'\omega_e^*(x)\,R_e H^{+}_{\vec{k}\omega}(x,u,x')\,f_e(u)\left[-i2\mathrm{Im}E^{-1*}_{\vec{k}\omega}(x,x')+\frac{e}{T_e}[\omega-\omega_e^*(x')]S_{\vec{k}\omega}(x,x')\right]\right\}$$
$$+\frac{\partial}{\partial u}\left\{i\,\frac{2e}{m_e}\int\frac{d\omega}{2\pi}\int\frac{d\vec{k}}{(2\pi)^2}\int dx' k_{\|}(x)\,R_e H^{+}_{\vec{k}\omega}(x,u,x')\,f_e(u)\,E^{-1*}_{\vec{k}\omega}(x,x')\right\}\quad, \tag{3}$$

where

$$\omega_e^*(x) = \frac{cT_e}{eB}\,\frac{\hat{x}\cdot\vec{k}\times\frac{\vec{B}}{B}}{L_n} \quad ,$$

is the electron drift frequency, $L_n$ is the density scale length, $L_s$ is the shear scale length,

$$k_{\|}(x) = k_{\|}'x = \frac{K_y}{L_s}\,x \quad ,$$

is the parallel wave number,

$$S(\vec{x},\vec{x}') = S(x,x',y-y',z-z',t-t') = \langle\phi(\vec{x})\,\phi(\vec{x}')\rangle \quad ,$$

is the potential correlation function,

$$h_{\vec{k}\omega}(x,u) = \int_{-\infty}^{\infty} dx' H^{+}_{\vec{k}\omega}(x,u,x') = \int_0^{\infty} d\tau \exp\{i[\omega-k_{\|}(x)u]\tau - \tfrac{1}{3}D(k_{\|}'u)^2\tau^3\}\, ,$$

is the resonance function, and $E^{-1}_{\vec{k}\omega}(x,x')$ is the Green's function of the drift wave eigenmode equation.

~~Energy conservation follows from the parallel energy~~ moment of the average Eq. (2)

$$\int d\vec{x}\, \frac{\partial}{\partial t}\, \mathcal{E}_p = \int d\vec{x} \int du\, \tfrac{1}{2}\, m_e u^2 n\, C(f_e)\,, \qquad (4)$$

where

$$\mathcal{E}_p = \int du\, \tfrac{1}{2}\, m_e u^2 n f_e\,,$$

is the parallel particle energy density. Using Eq. (3) for the collision operator, the second term is annihilated when integrated over x, and the third term is higher order in the discreteness parameter, Eq. (4) becomes

$$\int d\vec{x}\, \frac{\partial}{\partial t}\, \mathcal{E}_p = \int d\vec{x} \int du\, \tfrac{1}{2}\, m_e n u^2$$

$$\times\, \frac{\partial}{\partial u}\left[\frac{-e^2}{m_e T_e}\int \frac{d\omega}{2\pi}\int \frac{d\vec{k}}{(2\pi)^2}\int dx' k_{\parallel}(x)\,[\omega-\omega_e^*(x')]\, R_e H^{\pm}_{\vec{k}\omega}(x,u,x')\, f_e(u)\, S_{\vec{k}\omega}(x,x')\right] .$$

Certain relevant features of the following functions are pictorially presented in Fig. 1: the two point resonance function, $R_e H^{\pm}_{\vec{k}\omega}(x,u,x')$, has width $|x-x'| \lesssim x_c$ for $|x+x'|/2 \lesssim x_c$; the spectrum, $S_{\vec{k}\omega}(x,x')$, has width $|x-x'| \lesssim x_T$ for $|x+x'|/2 \lesssim x_T$; and $\phi(x) \sim e^{-i\mu x^2/2}$ is the approximate eigenmode structure.

Since $x_T \approx (L_s/L_n)^{1/2}\rho_i$, $x_c = (2D\tau_c)^{1/2}$, and $\tau_c = [2D(k_{\parallel}' v_e)^2]^{-1/3}$, typically $x_c/x_T \ll 1$. In this limit, after integration by parts with respect to u, the energy equation becomes

$$\int d\vec{x}\, \frac{\partial}{\partial t}\, \mathcal{E}_p$$

$$\cong \int d\vec{x} \int \frac{d\omega}{2\pi}\int \frac{d\vec{k}}{(2\pi)^2}\int du\, \frac{ne^2}{T_e}[(k_{\parallel}(x)u-\omega)+\omega]\,[\omega-\omega_e^*(x)]\, R_e h_{\vec{k}\omega}(x,u)\, f_e(u)\, S_{\vec{k}\omega}(x,x) \quad .$$

Since $(k_{\|}(x)u-\omega)\,\mathrm{Re}\,h_{\vec{k}\omega}(x,u)$ is odd in $k_{\|}(x)u-\omega$, and vanishes on integration over x, the energy equation is finally

$$\int d\vec{x}\,\frac{\partial}{\partial t}\,\mathcal{E}_p = \int d\vec{x}\int\frac{d\omega}{2\pi}\int\frac{d\vec{k}}{(2\pi)^2}\;2\omega\;\frac{\mathrm{Im}\,\alpha_e}{k^2\lambda D_e^2}\,\mathcal{E}_{\vec{k}\omega}(x)\quad, \tag{5}$$

where

$$\mathcal{E}_{\vec{k}\omega}(x) = \frac{\langle \vec{E}^2(x)\rangle_{\vec{k}\omega}}{8\pi} = \frac{k^2\langle\phi^2(x)\rangle_{\vec{k}\omega}}{8\pi} = \frac{k^2 S_{\vec{k}\omega}(x)}{8\pi}\quad,$$

and

$$\alpha_e = \int du\; i\,[\omega-\omega_e^*(x)]\,\mathrm{Re}\,h_{\vec{k}\omega}(x,u)\,f_e(u)\quad.$$

In order to convert the energy Eq. (5) into a familiar energy conservation relation, the response of the plasma to a small external current source is examined. The following is a many wave statistical generalization of the well known result from one wave small signal theory. The external current is $\vec{J}$, and the self consistent field is $\vec{E}$. Using the electrostatic approximation, $\vec{E}=-\vec{\nabla}\phi$, and the continuity equation, $\partial/\partial t\rho+\vec{\nabla}\cdot\vec{J}=0$; the equation for the current interaction is

$$-\int d\vec{x}\,\langle\vec{J}\cdot\vec{E}\rangle = \int d\vec{x}\,\langle\vec{J}\cdot\vec{\nabla}\phi\rangle = -\int d\vec{x}\,\langle(\vec{\nabla}\cdot\vec{J})\,\phi\rangle = \int d\vec{x}\,\langle\frac{\partial\rho}{\partial t}\,\phi\rangle\quad. \tag{6}$$

Rewriting $\rho$ and $\phi$ in terms of their inverse Fourier transforms in t, y, z, while leaving the inhomogeneous x dependence explicit, Eq. (6) becomes

$$-\int d\vec{x} \langle \vec{J}\cdot\vec{E} \rangle = \int d\vec{x} \int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^2} \int \frac{d\omega'}{2\pi} \int \frac{d\vec{k}'}{(2\pi)^2}$$

$$\times \left[ -i\omega \langle \rho_{\vec{k}\omega}(x)\, \phi_{\vec{k}'\omega'}(x) \rangle \exp\{i[(\vec{k}+\vec{k}')\cdot\vec{x} - (\omega+\omega')t]\} \right] \quad . \tag{7}$$

Poisson's equation in eigenmode form is

$$k^2 \left\{ \frac{-1}{k^2 \lambda_{D_e}^2} \left[ \frac{d^2}{dx^2} - (\Lambda - \mu^2 x^2 + \alpha_e) \right] \right\} \phi_{\vec{k}\omega}(x) = 4\pi \rho_{\vec{k}\omega}(x) \quad , \tag{8}$$

where $\Lambda \approx 1 - \omega_e^*/\omega$, and $\mu \approx L_n \omega_e^*/L_s \omega$. Using a W.K.B. expansion for $\phi_{\vec{k}\omega}(x)$,

$$\phi_{\vec{k}\omega}(x) = [\phi_{\vec{k}\omega}^{(0)} + \phi_{\vec{k}\omega}^{(1)} + \ldots] e^{i\int k_x dx} = \phi_{\vec{k}\omega}\, e^{i\int k_x dx} \quad ,$$

with notation such that

$$\phi_{\vec{k}\omega}^{(0)}(x) = \phi_{\vec{k}\omega}^{(0)}\, e^{i\int k_x dx} \quad , \quad \ldots \quad .$$

The ordering of terms is due to slow spatial and time variation; where $\alpha_e$, $\gamma$, source $\rho_{\vec{k}\omega}$, and d/dx of $k_x$ or $\phi_{\vec{k}\omega}$ are all first order. The slow time scale being represented by $\gamma$, $\omega = \omega_r + i\gamma$, where

$$\frac{\partial}{\partial t} \mathcal{E}_{\vec{k}\omega}(x) = 2\gamma\, \mathcal{E}_{\vec{k}\omega}(x) \quad .$$

The necessary derivatives of $\phi_{\vec{k}\omega}(x)$ are

$$\frac{d}{dx}\,\phi_{\vec{k}\omega}(x) = \left(ik_x\phi_{\vec{k}\omega} + \frac{d\phi_{\vec{k}\omega}}{dx}\right) e^{i\int k_x dx} \quad ,$$

and

$$\frac{d^2}{dx^2}\,\phi_{\vec{k}\omega}(x) = \left(-k_x^2\phi_{\vec{k}\omega} + i2k_x\frac{d\phi_{\vec{k}\omega}}{dx} + i\,\frac{dk_x}{dx}\,\phi_{\vec{k}\omega} + \frac{d^2\phi_{\vec{k}\omega}}{dx^2}\right) e^{i\int k_x dx} \quad .$$

Collecting zeroeth order terms in the eigenmode Eq. (8), it follows that

$$\bar{\epsilon}\Big|_{\omega=\omega_r} = 0 \tag{9}$$

where

$$\bar{\epsilon} = \frac{k_x^2 + \Lambda - \mu^2 x^2}{k^2\lambda_{D_e}^2} \quad , \quad \bar{\epsilon} = \frac{k_x^2}{k^2\lambda_{D_e}^2} + \epsilon \quad ,$$

and

$$\epsilon = \frac{\Lambda - \mu^2 x^2}{k^2\lambda_{D_e}^2} \quad .$$

Collecting first order terms and using Eq. (9), it is found that

$$\frac{k^2}{4\pi}\left[\frac{-i}{k^2\lambda_{D_e}^2}\left(2k_x\frac{d\phi_{\vec{k}\omega}^{(0)}}{dx} + \frac{dk_x}{dx}\phi_{\vec{k}\omega}^{(0)}\right) + i\gamma\frac{\partial\epsilon}{\partial\omega}\phi_{\vec{k}\omega}^{(0)} + \frac{\alpha_e}{k^2\lambda_{D_e}^2}\phi_{\vec{k}\omega}^{(0)}\right] e^{i\int k_x dx} = \rho_{\vec{k}\omega}(x) \quad . \tag{10}$$

Substituting $\rho_{\vec{k}\omega}$ from Eq. (10) into Eq. (7), while keeping lowest order terms, results in

$$-\int d\vec{x}\langle\vec{J}\cdot\vec{E}\rangle = \int d\vec{x}\int\frac{d\omega}{2\pi}\int\frac{d\vec{k}}{(2\pi)^2}\int\frac{d\omega'}{2\pi}\int\frac{d\vec{k}'}{(2\pi)^2}\,(-i\omega)$$

$$\times\left\langle -i\,\frac{2}{k^2\lambda_{D_e}^2}\,\frac{k^2}{8\pi}\left[2k_x\,\frac{d\phi^{(0)}_{\vec{k}\omega}}{dx}\,e^{i\int k_x dx}\,\phi_{\vec{k}'\omega'}(x) + \frac{dk_x}{dx}\,\phi^{(0)}_{\vec{k}\omega}(x)\phi_{\vec{k}'\omega'}(x)\right]\right.$$

$$\left. + 2\,\frac{k^2}{8\pi}\left(i\gamma\frac{\partial\epsilon}{\partial\omega} + \frac{\alpha_e}{k^2\lambda_{D_e}^2}\right)\phi^{(0)}_{\vec{k}\omega}(x)\;\phi_{\vec{k}'\omega'}(x)\right\rangle\exp\{i[(\vec{k}+\vec{k}')\cdot\vec{x}-(\omega+\omega')t]\}\;. \tag{11}$$

Spatial homogeneity in Y-Z and stationarity in t implies that

$$\frac{k^2}{8\pi}\left\langle\phi_{\vec{k}\omega}(x)\,\phi_{\vec{k}'\omega'}(x)\right\rangle = \mathcal{E}_{\vec{k}\omega}(x)\,(2\pi)^3\,\delta(\omega+\omega')\,\delta(\vec{k}+\vec{k}')\quad. \tag{12}$$

Using Eq. (12) and the consistent approximation that

$$\phi_{\vec{k}\omega}(x) \approx \phi^{(0)}_{\vec{k}\omega}(x) = \phi^{(0)}_{\vec{k}\omega}e^{i\int k_x dx}\quad,$$

it follows from Eq. (11) that

$$-\int d\vec{x}\,\langle\vec{J}\cdot\vec{E}\rangle = \int d\vec{x}\int\frac{d\omega}{2\pi}\int\frac{d\vec{k}}{(2\pi)^2}\left[2\gamma\omega\,\frac{\partial\epsilon}{\partial\omega}\,\mathcal{E}_{\vec{k}\omega}(x) + 2\omega\left(\frac{\mathrm{Im}\,\alpha_e}{k^2\lambda_{D_e}^2}\right)\mathcal{E}_{\vec{k}\omega}(x)\right]$$

$$+\int d\vec{x}\int\frac{d\omega}{2\pi}\int\frac{d\vec{k}}{(2\pi)^2}\int\frac{d\omega'}{2\pi}\int\frac{d\vec{k}'}{(2\pi)^2}\left(\frac{-2\omega}{k^2\lambda_{D_e}^2}\right)\frac{k^2}{8\pi}\left\langle 2k_x\,\frac{d\phi_{\vec{k}\omega}(x)}{dx}\,\phi_{\vec{k}'\omega'}(x)\right.$$

$$\left.-2ik_x^2\,\phi_{\vec{k}\omega}(x)\,\phi_{\vec{k}'\omega'}(x) + \frac{dk_x}{dx}\,\phi_{\vec{k}\omega}(x)\phi_{\vec{k}'\omega'}(x)\right\rangle\exp\{\,i[(\vec{k}+\vec{k}')\cdot\vec{x}-(\omega+\omega')t]\,\}\;. \tag{13}$$

From the parity of $\epsilon_{\vec{k}\omega}$ and $\mathcal{E}_{\vec{k}\omega}$, $\epsilon_{\vec{k}\omega}=\epsilon_{-\vec{k}\,-\omega}$ and $\mathcal{E}_{\vec{k}\omega}=\mathcal{E}_{-\vec{k}\,-\omega}$, it is clear that $\omega\epsilon\mathcal{E}_{\vec{k}\omega}(x)$ is odd in $\omega$-k so that

$$\int d\omega \int d\vec{k}\ \omega\epsilon\mathcal{E}_{\vec{k}\omega}(x) = 0 \quad . \tag{14}$$

Equation (14) can be rewritten using Eq. (9),

$$\epsilon = -\frac{k_x^{\ 2}}{k^2\lambda_{D_e}^{\ 2}} \quad ,$$

resulting in

$$\int d\omega \int d\vec{k}\ \frac{\omega k_x^{\ 2}}{k^2}\mathcal{E}_{\vec{k}\omega}(x) = 0 \quad . \tag{15}$$

Using Eq. (15), Eq. (13) becomes

$$-\int d\vec{x} \langle \vec{J}\cdot\vec{E} \rangle = \int d\vec{x} \int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^2} \left[2\gamma\omega\frac{\partial\epsilon}{\partial\omega}\mathcal{E}_{\vec{k}\omega}(x) + 2\omega\frac{\mathrm{Im}\,\alpha_e}{k^2\lambda_{D_e}^{\ 2}}\mathcal{E}_{\vec{k}\omega}(x)\right]$$

$$+\int d\vec{x} \int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^2} \int \frac{d\omega'}{2\pi} \int \frac{d\vec{k}'}{(2\pi)^2} \frac{-2\omega}{k^2\lambda_{D_e}^{\ 2}} \frac{k^2}{8\pi} \left\langle 2k_x \frac{d\phi_{\vec{k}\omega}(x)}{dx}\phi_{\vec{k}'\omega'}(x) + \frac{dk_x}{dx}\phi_{\vec{k}\omega}(x)\phi_{\vec{k}'\omega'}(x)\right\rangle$$

$$\times \exp\{i[(\vec{k}+\vec{k}')\cdot\vec{x}-(\omega+\omega')t]\} \tag{16}$$

The following identity is obtained by interchanging $\omega$ with $\omega'$ and $\vec{k}$ with $\vec{k}'$,

$$\int d\omega \int d\vec{k} \int d\omega' \int d\vec{k}'\, \omega k_x(\omega,\vec{k}) \left\langle \phi_{\vec{k}\omega}(x) \frac{d\phi_{\vec{k}'\omega'}(x)}{dx}\right\rangle \exp\{i[(\vec{k}+\vec{k}')\cdot\vec{x}-(\omega+\omega')t]\}$$

$$= \int d\omega \int d\vec{k} \int d\omega' \int d\vec{k}'\, \omega' k_x(\omega',\vec{k}') \left\langle \frac{d\phi_{\vec{k}\omega}(x)}{dx}\phi_{\vec{k}'\omega'}(x)\right\rangle \exp\{i[(\vec{k}+\vec{k}')\cdot\vec{x}-(\omega+\omega')t]\} \quad . \tag{17}$$

Using spatial homogeneity in y-z and stationarity in t to replace $\omega'$ with $-\omega$ and $\vec{k}'$ with $-\vec{k}$, Eq. (17) becomes

$$\int d\omega \int d\vec{k} \int d\omega' \int d\vec{k}'\, \omega k_x(\omega,\vec{k}) \left\langle \phi_{\vec{k}\omega}(x) \frac{d\phi_{\vec{k}'\omega'}(x)}{dx} \right\rangle \exp\{i[(\vec{k}+\vec{k}')\cdot\vec{x}-(\omega+\omega')t]\}$$
$$= \int d\omega \int d\vec{k} \int d\omega' \int d\vec{k}'\, (-\omega) k_x(-\omega,-\vec{k}) \left\langle \frac{d\phi_{\vec{k}\omega}(x)}{dx} \phi_{\vec{k}'\omega'}(x) \right\rangle \exp\{i[(\vec{k}+\vec{k}')\cdot\vec{x}-(\omega+\omega')t]\} \quad . \tag{18}$$

The reality of $\phi(x)$ implies that $k_x^*(-\omega,-\vec{k})=-k_x(\omega,\vec{k})$, since $k_x$ is real $k_x(-\omega,-\vec{k})=-k_x(\omega,\vec{k})$, so that Eq. (18) becomes

$$\int d\omega \int d\vec{k} \int d\omega' \int d\vec{k}'\, \omega k_x(\omega,\vec{k}) \left\langle \phi_{\vec{k}\omega}(x) \frac{d\phi_{\vec{k}'\omega'}(x)}{dx} \right\rangle \exp\{i[(\vec{k}+\vec{k}')\cdot\vec{x}-(\omega+\omega')t]\}$$
$$= \int d\omega \int d\vec{k} \int d\omega' \int d\vec{k}'\, \omega k_x(\omega,\vec{k}) \left\langle \frac{d\phi_{\vec{k}\omega}(x)}{dx} \phi_{\vec{k}'\omega'}(x) \right\rangle \exp\{i[(\vec{k}+\vec{k}')\cdot\vec{x}-(\omega+\omega')t]\} \quad . \tag{19}$$

Integrating the homogeneity property Eq. (12) by

$$\int \frac{d\omega'}{2\pi} \int \frac{d\vec{k}'}{(2\pi)^2} \exp\{i[(\vec{k}+\vec{k}')\cdot\vec{x}-(\omega+\omega')t]\} \quad ,$$

yields

$$\mathcal{E}_{\vec{k}\omega}(x) = \int \frac{d\omega'}{2\pi} \int \frac{d\vec{k}'}{(2\pi)^2} \frac{k^2}{8\pi} \langle \phi_{\vec{k}\omega}(x)\, \phi_{\vec{k}'\omega'}(x) \rangle \exp\{i[(\vec{k}+\vec{k}')\cdot\vec{x}-(\omega+\omega')t]\}$$

So that

$$-\int d\vec{x} \int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^2} \frac{d}{dx} \left[ \frac{2k_x}{k^2 \lambda_{D_e}^2}\, \omega\, \mathcal{E}_{\vec{k}\omega}(x) \right] = \int d\vec{x} \int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^2} \int \frac{d\omega'}{2\pi} \int \frac{d\vec{k}'}{(2\pi)^2}$$
$$\times \left( \frac{-2\omega}{k^2 \lambda_{D_e}^2} \right) \frac{k^2}{8\pi} \left\langle k_x \frac{d\phi_{\vec{k}\omega}(x)}{dx} \phi_{\vec{k}'\omega'}(x) + k_x \phi_{\vec{k}\omega}(x) \frac{d\phi_{\vec{k}'\omega'}(x)}{dx} + \frac{dk_x}{dx} \phi_{\vec{k}\omega}(x)\, \phi_{\vec{k}'\omega'}(x) \right\rangle$$
$$\times \exp\{i[(\vec{k}+\vec{k}')\cdot\vec{x}-(\omega+\omega')t]\} \quad . \tag{20}$$

Using Eq. (19) in Eq. (20), it follows that

$$-\int d\vec{x} \int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^2} \frac{d}{dx} \left[ \frac{2k_x}{k^2 \lambda_{D_e}^2} \omega \mathcal{E}_{\vec{k}\omega}(x) \right] = \int d\vec{x} \int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^2} \int \frac{d\omega'}{2\pi} \int \frac{d\vec{k}'}{(2\pi)^2}$$

$$\times \left( \frac{-2\omega}{k^2 \lambda_{D_e}^2} \right) \frac{k^2}{8\pi} \left\langle 2k_x \frac{d\phi_{\vec{k}\omega}(x)}{dx} \phi_{\vec{k}'\omega'}(x) + \frac{dk_x}{dx} \phi_{\vec{k}\omega}(x) \phi_{\vec{k}'\omega'}(x) \right\rangle \exp\{i[(\vec{k}+\vec{k}')\cdot\vec{x} - (\omega+\omega')t]\} . \tag{21}$$

Substituting Eq. (21) into Eq. (16) results in

$$-\int d\vec{x} \langle \vec{J}\cdot\vec{E} \rangle = \int d\vec{x} \int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^2} \left\{ 2\gamma\omega \frac{\partial\epsilon}{\partial\omega} \mathcal{E}_{\vec{k}\omega}(x) - \frac{d}{dx} \left[ \frac{2k_x}{k^2 \lambda_{D_e}^2} \omega \mathcal{E}_{\vec{k}\omega}(x) \right] + 2\omega \left( \frac{\mathrm{Im}\,\alpha_e}{k^2 \lambda_{D_e}^2} \right) \mathcal{E}_{\vec{k}\omega}(x) \right\} . \tag{22}$$

Regarding Eq. (9) as $\bar{\epsilon}[x,k_x(\omega),\omega]=0$, the total differential is

$$0 = d\bar{\epsilon} = dx \left( \frac{\partial\bar{\epsilon}}{\partial x} + \frac{\partial k_x}{\partial x} \frac{\partial\bar{\epsilon}}{\partial k_x} \right) + d\omega \left( \frac{\partial\bar{\epsilon}}{\partial\omega} + \frac{\partial k_x}{\partial\omega} \frac{\partial\bar{\epsilon}}{\partial k_x} \right) . \tag{23}$$

The last term in Eq. (23) results in

$$\frac{\partial\bar{\epsilon}}{\partial k_x} = -\frac{\partial\omega}{\partial k_x} \frac{\partial\bar{\epsilon}}{\partial\omega} ,$$

with the group velocity defined as $v_g = \partial\omega/\partial k_x$, it follows that

$$-\frac{\partial\bar{\epsilon}}{\partial k_x} = v_g \frac{\partial\bar{\epsilon}}{\partial\omega} .$$

Using the definitions in Eq. (9), $\partial\bar{\epsilon}/\partial k_x = 2k_x/k^2\lambda_{D_e}^2$ and

$\partial\bar{\epsilon}/\partial\omega = \partial\epsilon/\partial\omega$, the last expression results in

$$\frac{-2k_x}{k^2\lambda_{D_e}^2} = v_g \frac{\partial\epsilon}{\partial\omega} \quad . \tag{24}$$

Finally, the equation for the external current source is found by substituting Eq. (24) into Eq. (22),

$$-\int d\vec{x} \langle \vec{J}\cdot\vec{E} \rangle = \int d\vec{x} \int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^2} \left\{ \frac{\partial}{\partial t}\left[\omega \frac{\partial\epsilon}{\partial\omega} \mathcal{E}_{\vec{k}\omega}(x)\right] + \frac{d}{dx}\left[v_g \omega \frac{\partial\epsilon}{\partial\omega} \mathcal{E}_{\vec{k}\omega}(x)\right] + 2\omega \frac{\mathrm{Im}\,\alpha_e}{k^2\lambda_{D_e}^2} \mathcal{E}_{\vec{k}\omega}(x) \right\} \quad . \tag{25}$$

The left hand side (LHS) of this equation is the external source; while the first term on the right hand side (RHS) is the wave energy density, the second term the power flux, and the third term the power dissipation. Setting the external source, $\vec{J}$, to zero in Eq. (25) and substituting this result into the energy moment Eq. (5), results in

$$\int d\vec{x} \frac{\partial}{\partial t} \mathcal{E}_p = -\int d\vec{x} \int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^2} \left\{ \frac{\partial}{\partial t}\left[\omega \frac{\partial\epsilon}{\partial\omega} \mathcal{E}_{\vec{k}\omega}(x)\right] + \frac{d}{dx}\left[v_g \omega \frac{\partial\epsilon}{\partial\omega} \mathcal{E}_{\vec{k}\omega}(x)\right] \right\} . \tag{26}$$

The LHS is the particle energy density, the first term on the RHS is the wave energy density, and the second term is the power flux. This is the final energy conservation relation. The rate of change of the particle energy is equal to negative the rate of change of the wave energy plus negative the divergence of the power flux. The physical interpretation of the energy conservation process is that the unstable electrons near the mode rational surface are feeding energy into the waves which are in turn convecting energy out to the ion layer where ion Landau damping occurs and thus the waves are absorbed.

## III. Conclusion

The $\phi$ moment constraint on the nonlinear drift wave term has been shown to be separate from the energy conservation issue. Moreover, the terminology "energy conservation" applied to this constraint would be misleading. The physically relevant formulation of energy conservation in a statistically averaged sense is utilized throughout this paper. In particular, energy conservation is determined by a parallel energy moment of the average equation. The resultant energy conservation equation, being of well known form, proves energy conservation for the NSA renormalization of the drift wave problem.



## Acknowledgments

This work would not have been possible without the prior calculation of the collision operator in the NSA derived by Swartz and Molvig.[5]

This research was supported in part by the U.S. Department of Energy grant number DE-FG05-80ET 53088.

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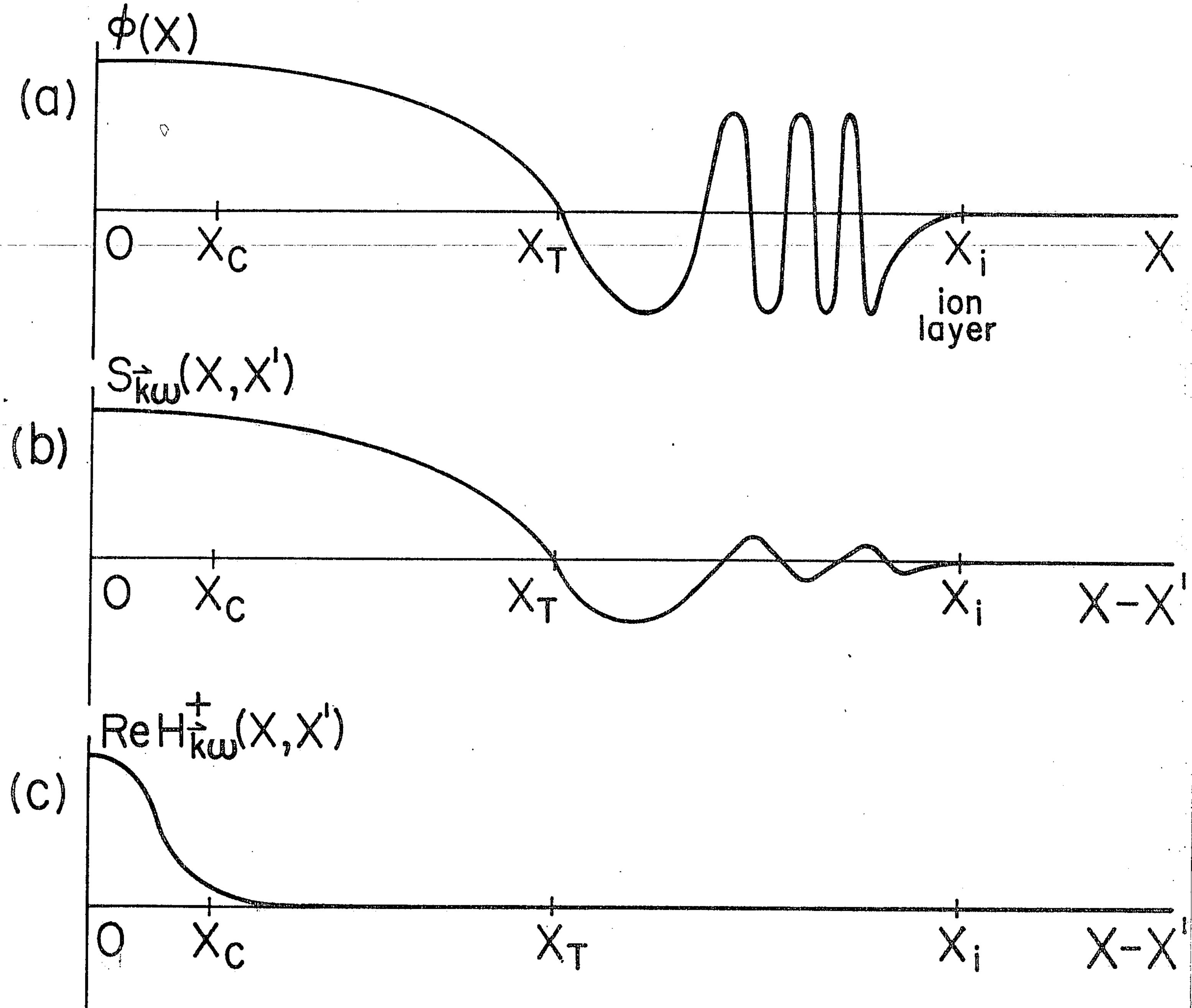
Figures

Figure 1 (a) The eigenmode structure, $\phi(x)$, and

(b) the spectrum, $S_{\vec{k}\omega}(x,x')$, are shown slowly varying up to the turning point, $x_T$; while

(c) the resonance function, $\mathrm{Re}H^{\pm}_{k\omega}(x,x')$, is cut off at the correlation distance, $x_c$.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(c)
$\mathrm{Re}\,H^{+}_{\vec{k}\omega}(X,X')$
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iFSR #28

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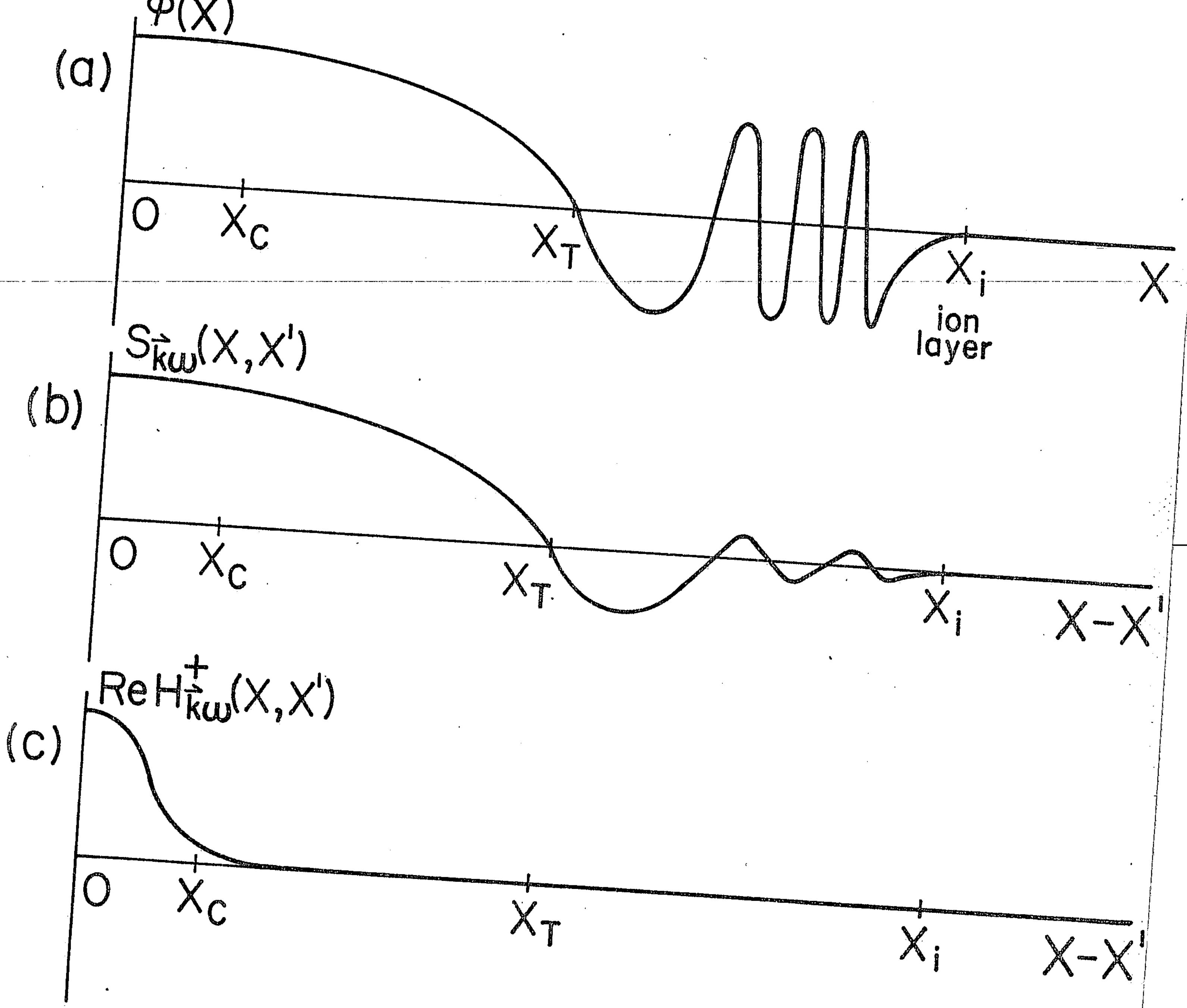

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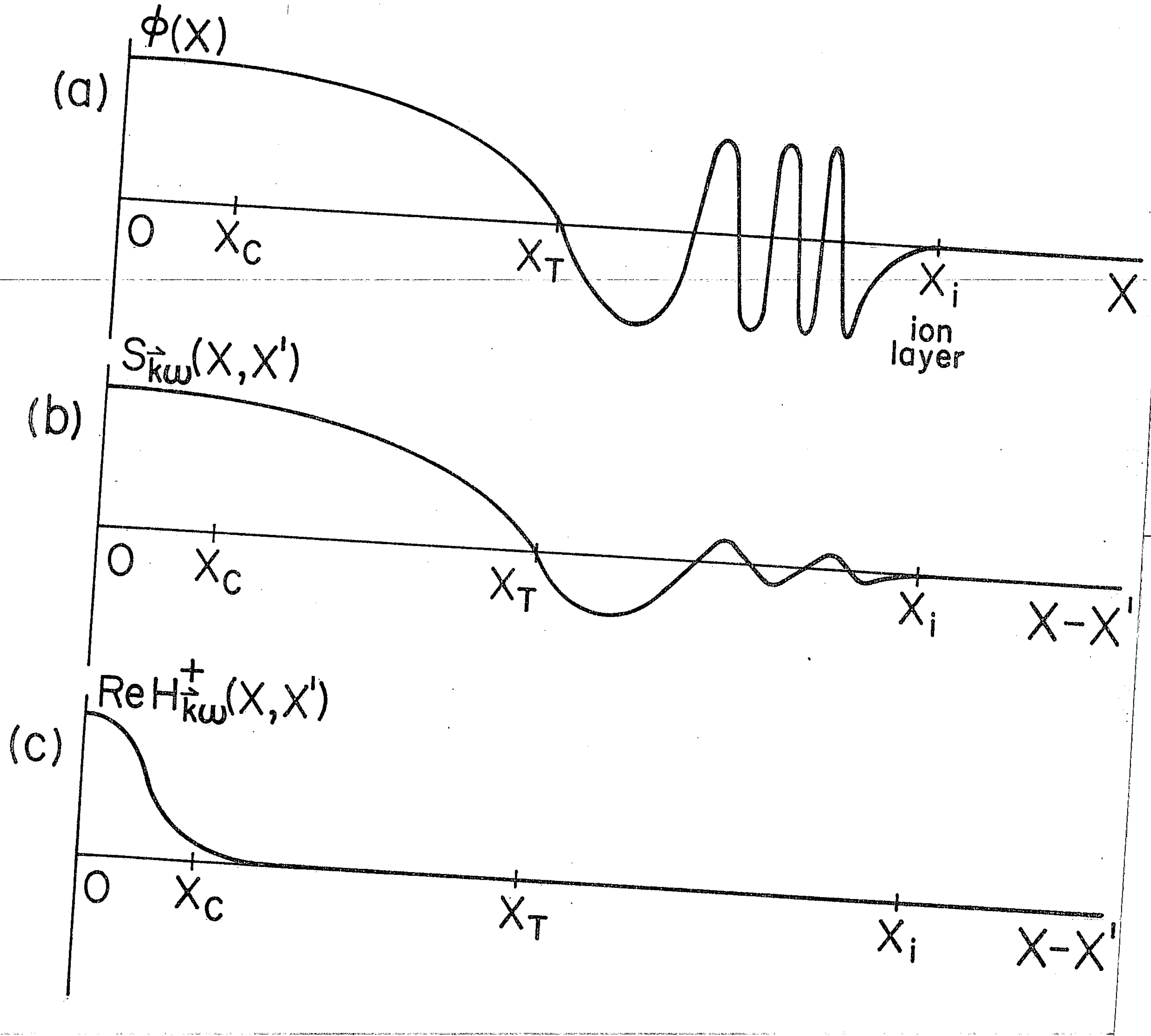
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