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Momentum and Thermal Transport in Neutral-Beam-Heated Tokamaks

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Abstract

We have explored the relation between momentum and thermal transport in neutral-beam-heated tokamaks with subsonic toroidal rotation velocity. A theory of diffusive momentum transport driven by ion-temperature-gradient-driven turbulence[1] (η_i-turbulence) is presented. In addition, the level of η_i-turbulence is enhanced by radially sheared toroidal rotation. The resulting ion shear viscosity is:

$$\chi_\varphi = 1.3\left[\frac{1+\eta_i}{\tau} + \left(\frac{L_n}{2c_s}\frac{dV_0}{dr}\right)^2\right]^2 \frac{\rho_s^2 c_s}{L_s},$$

The associated ion thermal diffusivity, χ_i, is identical to χ_φ. Thus, a scenario based on velocity-shear-enhanced η_i-turbulence is consistent with the experimentally observed relationship between thermal and momentum confinement.

I. Introduction

For tokamaks to attain ignition, auxiliary heating is probably necessary. The most common means of auxiliary heating is by the tangential injection of an energetic beam of neutrals (NBI heating). Unfortunately, NBI heating generally results in the degradation of energy confinement time (τ_E) which decreases with increasing power so that the heating becomes less efficient the more it is applied.[2] Conventional wisdom explains the heat loss in terms of enhanced electron thermal conduction, but experimental results from D-III indicate that *ion* losses are of comparable importance.[3] Neoclassical predictions of ion conductivity (χ_i) are too low by about an order of magnitude. A good understanding of the ion loss mechanism is at present still developing.

Experimental clues to the nature of the ion conductivity are sparse, since direct ion temperature profile measurements have become possible only recently, with charge exchange spectroscopy. One commonly observed feature is that the confinement times of ion temperature (τ_i) and of the toroidal rotation rate (τ_φ) tend to behave alike, with similar scalings. This suggests that momentum transport arises from the same cause as the anomalous χ_i, and so a unified description of the two processes is desirable. Further incentive for the study of momentum confinement comes from its inherent ability to isolate ion from electron dynamics less ambiguously than thermal studies, since momentum is carried almost exclusively by the ions.

Attempts to explain momentum loss rates by classical or neoclassical mechanisms have all come up short for various reasons. Classical predictions ($\tau_\varphi^{-1} \sim \nu_{i,i}\rho_i^2/a^2$) are far too slow to agree with observed dissipation rates. The gyroviscous theory of Stacey and Sigmar[4,5] assumes a plasma rotation aligned with the magnetic field, and then argues that the subsequent deviation from solid body motion is damped at a rate governed by the classical *gyro*viscosity ($\tau_\varphi^{-1} \sim v_{th_i}^2/\Omega_i R^2$), which is more consistent with experimental observations. However, it has been noted by Connor et al.[6] that the strong *parallel* damping provided by the gyroviscosity leaves the plasma with a predominantly *toroidal* (rigid rotator) flow. Connor et al. then demonstrate that, excluding up-down asymmetries, etc., collisional damping of this flow is classical, again too slow to agree with experiment. Perhaps more contrary to neoclassical theories (and all nonlocal theories) are recent experiments on TFTR which demonstrate the *diffusive* nature of momentum transport.[7] A different approach to describing momentum transport is desirable.

The present work examines the possibility that ion-temperature-gradient-driven turbulence (hereafter "η_i-turbulence") plays a substantial role in determining both momentum and thermal transport in NBI plasmas. This mode is destabilized in plasmas with steep ion temperature profiles and relatively flat density profiles, such that $\eta_i \equiv d \ln T_i / d \ln n_0 > \eta_{ic} \simeq 1.5$. This theory has several distinct advantages. First, we should expect the value of η_i to be greatly affected by the NBI process, since the beam applies heat directly to the ions in a localized region of the plasma. Second, since the mode is essentially a parallel ion sound wave, with fluctuations in ion pressure (heat) and ion parallel velocity (momentum), it offers a good chance to explain a causal connection between the transport of these two quantities. Third, being a localized *micro*instability, the nature of the resulting turbulent momentum transport is inherently diffusive in nature. Fourth, its dependence on the temperature *profile* offers an immediate explanation for the observed decrease of momentum transport in TFTR when the heated region is changed from the plasma core to the edge.[7]

All of these suggest that η_i-turbulence is a good candidate for the cause of the anomalous transport in NBI plasmas. However, current theories of the η_i-instability do not include the effects of a radially sheared toroidal rotation, $dV_{i0}/dr \neq 0$, as introduced by the neutral beam. We find that it has two important effects. First, the sheared velocity field naturally facilitates prediction of the turbulent viscosity required to explain anomalous momentum transport. Second, it acts as an additional free energy source that enhances the existing ion-temperature-gradient turbulence level. Hence, one of our goals here is to improve the current theory of η_i-turbulence by incorporating these two aspects of toroidal shear flow.

We shall assume the following about present-day NBI regimes. First, the incoming beam of neutrals is rapidly thermalized so that a one-fluid description of the ions is adequate. Second, the value of η_i should be sufficiently above η_{ic} that a fluid theory applies.[9] Third, we assume that the rotation rate of the plasmas is below the sound speed $(V_0/c_s < 1)$, so that a shock wave is not excited. Finally, we assume that the transport of momentum is governed by shear viscosity, and that particle diffusivity plays a much smaller role.

In this paper, we examine the effects of a parallel velocity shear on η_i-turbulence, generalizing the results of Ref. 1. The principal results are the following:

1. The turbulent shear viscosity, calculated from $\chi_\varphi \equiv -\langle \tilde{v}_\parallel \tilde{v}_r \rangle \left(dV_\parallel / dr \right)^{-1}$, is given by:
$$\chi_\varphi = 3.3 \left[\frac{1+\eta_i}{\tau} + \left(\frac{L_n}{2c_s} \frac{dV_{i0}}{dr} \right)^2 \right]^2 \frac{\langle k_y \rho_s \rangle}{L_s} \rho_s^2 c_s,$$
for $\eta_i > \eta_{ic} \simeq 1$. Here $\langle k_y \rangle$ is the rms spectrum-averaged poloidal wavenumber, and $\langle k_y \rho_s \rangle \simeq 0.4$.
2. The ion thermal conductivity, χ_i, is found to equal the value of χ_φ given above, which is suggestive of experimental observation. This agrees with the basic scaling of Lee and Diamond[1] $\left(\chi_i \simeq \left[(1+\eta_i)/\tau \right]^2 \langle k_y \rangle / L_s \right)$. The enhancement factor $\left(\frac{L_n}{2c_s} \frac{dV_{i0}}{dr} \right)^2$ represents the role of the shear flow as an additional free energy source, and is related to the hydrodynamic Richardson number.[10]
3. For dissipative trapped electron dynamics, a simple estimate shows that the electron heat conductivity due to η_i-turbulence is enhanced by the velocity shear:
$$\chi_e \simeq 10 \epsilon^{3/2} \left[\frac{1+\eta_i}{\tau} + \left(\frac{L_n}{2c_s} \frac{dV_{i0}}{dr} \right)^2 \right]^3 \frac{\rho_s^2 c_s^2}{\nu_e L_s^2},$$
where ϵ is the inverse aspect ratio.
4. We demonstrate that the calculation of saturated turbulent diffusivity as an *eigenvalue* of the renormalized equations, as opposed to the more standard mixing-length method, takes far better account of the structure of the eigenmodes. Specifically, it is the *only* available method for accurately determining the saturation levels in the presence of multiple free energy sources, as here. Also, this technique allows resolution of purely numerical factors, like the coefficient of 3.3 above. Furthermore, it allows a prediction of the nonlinear frequency shift, which comes from the imaginary component of the diffusion eigenvalue.

The current theory is relevant in the regime $\eta_i > \eta_{ic} \simeq 1$, $V_0/c_s < 1$, and $\frac{dV_0}{dr} < \frac{\tau c_s}{L_n}\left(\frac{1+\eta_i}{\tau}\right)^{3/2}$.

The remainder of the paper is organized as follows. In Sec. II, the basic model is reviewed, and modifications due to the sheared flow are discussed. In Sec. III, the modified linear theory is presented. In Sec. IV, the one-point renormalization is performed, with subsequent solution for saturated turbulence levels. Section V contains calculations of transport coefficients. Section VI contains the summary and comparisons with various experiments.

II. Basic Model

To describe the nonlinear ion dynamics of a beam-heated tokamak, we shall adopt a simple one-fluid ion model.[1] In this model, we assume fluid ions and adiabatic electrons, and thereby the phase velocity regime $v_{th_i} \lesssim \left(\omega/k_{\|}\right) < v_{th_e}$. Also, we consider a sheared slab configuration, with all inhomogeneities in the radial $(\hat{x})$-direction, which necessitates the macroscopic gradient ordering of $\eta_i > 1 \gg L_n/L_s$ for consistency of a fluid treatment.

In sheared slab geometry, the magnetic field is given by $\vec{B} = B_0\left(\hat{z} + \left(x/L_s\right)\hat{y}\right)$, and so the parallel wavenumber is given by $k_{\|} = \left(x - x_s\right) k_y/L_s$ in the neighborhood of a mode rational surface x_s, where $\vec{k} \cdot \vec{B} = 0$. Since the background plasma is inhomogeneous in the x-direction only, perturbations have the form $\tilde{f}(x) \exp\left[-i\omega t + ik_y y + ik_z z\right]$.

Here, the primary modification to previous such models is the inclusion of a radially-dependent toroidal ion velocity, $V_{\varphi 0}(x)$ (alternately referred to as rotation velocity, toroidal momentum, and shear flow). We assume that the rotation velocity is subsonic $\left(V_{\varphi 0}/c_s < 1\right)$, apparently consistent with regimes of current experimental interest,[11] but make no assumption yet as to the degree of velocity shear, $\left(L_V\right)^{-1} = d\ln V_{\varphi 0}/dr$, except that it be consistent with the fluid model, as verified *a posteriori*.

In the sheared slab model of a tokamak, the toroidal direction is given by $\hat{\varphi} = \cos\alpha\hat{z} + \sin\alpha\hat{y}$, where $\alpha = \tan^{-1}\left(\epsilon_0/q_0\right)$ (i.e., the angle between $\hat{\varphi}$ and $\hat{b} = \vec{B}/|B|$ at x_s), and ϵ_0 and q_0 are the safety factor and the ratio of minor to major radius, each evaluated at the rational surface. Since $\epsilon_0/q_0 \ll 1$ then $\hat{\varphi}$, $\hat{z}$, and $\hat{b}$ are approximately parallel. The slight deviation between $\hat{b}$ and $\hat{\varphi}$ is crucial to the gyroviscous theory, and ultimately leads to the problems discussed in Ref. 6. However, in the present case the distinction is much less important, since the primary mechanism of viscosity is temperature gradient driven fluctuations, and is insensitive to this small difference. The difference between the toroidal and parallel components of the velocity is small, since $V_{\|0} = V_{\varphi 0} B_\varphi \Big/ \sqrt{B_\varphi^2 + B_\theta^2} = V_{\varphi 0} \Big/ \left(1 + \left(\frac{\epsilon_0}{q_0}\right)^2\right)^{1/2}$.

In this fluid model, the ion dynamics are described by the ion density, $n_i = \langle n_0 \rangle + \tilde{n}_i(\vec{x}, t)$, the ion parallel velocity, $v_{\|i} = \langle V_{\|0} \rangle + \tilde{v}_{\|i}(\vec{x}, t)$, and the ion pressure, $P_i = \langle P_{i0} \rangle + \tilde{p}_i(\vec{x}, t)$, where $\langle \quad \rangle$ denotes an ensemble average. These quantities evolve according to the ion continuity equation, the parallel momentum equation, and the equation of adiabatic pressure evolution

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{v}_{\perp i}) + \nabla_\| \left(n_i v_{\|i}\right) = 0 \tag{1}$$

$$m_i n_i \left(\frac{\partial v_{\|i}}{\partial t} + \left(\vec{v}_E + \vec{v}_{\|i}\right) \cdot \nabla v_{\|i}\right) = -e n_i \nabla_\| \Phi - \nabla_\| P_i + \mu_\| \nabla_\|^2 v_{\|i} \tag{2}$$

$$\frac{\partial P_i}{\partial t} + \left(\vec{v}_E + \vec{v}_{\|i}\right) \cdot \nabla P_i + \Gamma P_i \nabla_\| v_{\|i} = 0, \tag{3}$$

where Φ is the electrostatic potential, $\mu_\|$ is the parallel viscosity (due to either Landau damping or collisional viscosity) required for saturation of the turbulence, Γ is the ratio of specific heats, and the perpendicular ion dynamics are due to $\vec{E} \times \vec{B}$, diamagnetic and, in next order, polarization drifts, where respectively,

$$\vec{v}_E = \frac{c}{B} \hat{b} \times \nabla \Phi$$

$$\vec{v}_{Di} = \frac{c}{eBn_i} \hat{b} \times \nabla P_i$$

$$\vec{v}_p = -\frac{c^2 m_i}{eB^2} \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right) \nabla \Phi$$

Electron dynamics are assumed adiabatic, and the equations are closed with the quasineutrality condition

$$\tilde{n}_i = \tilde{n}_e = \frac{e\tilde{\Phi}}{T_e},$$

where $\Phi = \langle\Phi(x)\rangle + \tilde{\Phi}(\vec{x},t)$. The background radial electric field, generally present in beam-heated tokamaks,[12] is the by-product of a flow which deviates from the direction of $\vec{B}$, and obeys the radial force balance equation. In a slab model, it plays no direct role in the evolution of the instabilities, and is thus ignored.

To simplify this set of equations, we eliminate $\tilde{n}_i$ and $\vec{v}_\perp$ from Eqs. (1)–(3), and make the assumption that the radial width of the fluctuations is much less than the scale length of any of the macroscopic gradients. To simplify notation, we rescale time and distance to units of Ω_i^{-1} and ρ_s $(= c_s/\Omega_i)$ respectively, and undimensionalize the remaining fields as $\tilde{\phi} = e\tilde{\Phi}/T_e$, $\tilde{v}_\parallel = \tilde{v}_{\parallel i}/c_s$, and $\tilde{p} = \left[\tilde{p}_i/\langle P_{i0}\rangle\right](T_i/T_e)$. This yields the following three equations in $\tilde{\phi}$, $\tilde{v}_\parallel$, and $\tilde{p}$

$$\left(\frac{\partial}{\partial t} + \vec{V}_0\cdot\nabla\right)\left(1-\nabla_\perp^2\right)\tilde{\phi} + v_D\left[1+\left(\frac{1+\eta_i}{\tau}\right)\nabla_\perp^2\right]\nabla_y\tilde{\phi}$$

$$-\hat{b}\times\nabla\tilde{\phi}\cdot\nabla\left(\nabla_\perp^2\tilde{\phi}\right) + \nabla_\parallel\tilde{v}_\parallel = 0 \tag{4}$$

$$\left(\frac{\partial}{\partial t} + \vec{V}_0\cdot\nabla\right)\tilde{v}_\parallel - \frac{V_0}{L_V}\nabla_y\tilde{\phi} + \hat{b}\times\nabla\tilde{\phi}\cdot\nabla\tilde{v}_\parallel = -\nabla_\parallel\tilde{\phi} - \nabla_\parallel\tilde{p} + \mu\nabla_\parallel^2\tilde{v}_\parallel \tag{5}$$

$$\left(\frac{\partial}{\partial t} + \vec{V}_0\cdot\nabla\right)\tilde{p} + v_D\left(\frac{1+\eta_i}{\tau}\right)\nabla_y\tilde{\phi} + \hat{b}\times\nabla\tilde{\phi}\cdot\nabla\tilde{p} + \Upsilon\nabla_\parallel\tilde{v}_\parallel = 0, \tag{6}$$

where

$$\eta_i = \frac{d\ln T_i}{d\ln n} \qquad v_D = -\frac{cT_e}{eB}\frac{d\left(\ln n_0\right)}{dx} \qquad L_V = \left(\frac{d\ln V_0}{dx}\right)^{-1}$$

$$V_0 = \langle V_{\varphi 0}\rangle/c_s \qquad \mu = \frac{\mu_\parallel\omega_{ci}}{c_s^2} \qquad \Upsilon = \frac{\Gamma}{\tau} \qquad \tau = \frac{T_e}{T_i},$$

and we have retained only the $\vec{E}\times\vec{B}$ nonlinearities, since others are of relative order $k_\parallel/k_y(\ll 1)$.

The shear flow V_0 has two effects on the η_i equations. First, it introduces a toroidal Doppler shift in all time derivatives, which we eliminate by performing a Galilean transformation in the $\hat{\varphi}$ direction to the co-moving frame. More importantly, the radial $\vec{E}\times\vec{B}$ convection of ion momentum, represented by the second term in Eq. (5), determines radial momentum transport. The fact that $\vec{E}\times\vec{B}$ motion also determines ion thermal transport, represented by the second term in Eq. (6), is the underlying reason for the eventual result that $\chi_i = \chi_\varphi$.

Finally, we note that the inclusion of toroidal ion momentum does not modify the nonlinear structure of Ref. 1, and so the energetics, renormalization, etc., are all quite similar. However, the fact that the shear flow provides an additional free energy source underlies our result that the inclusion of dV_0/dx effects *enhances* transport.

III. Linear Theory

The linear theory of the η_i-instability has been addressed by many authors, and we do not repeat all the basic details here. However, no one has considered the effects of a sheared toroidal ion flow on the η_i-mode, so we find it necessary to modify the basic linear theory to include this effect. Also, we shall consider the possibility that the sheared velocity field, acting as the *dominant* free energy source, might destabilize the plasma Kelvin-Helmholtz mode, as described in Ref. 13.

Linearizing Eqs. (4)–(6), Fourier transforming in the y- and z-directions, neglecting Υ (which gives corrections of order $\left(k_\parallel/k_y\right)^4$), and taking $k_\parallel = k_y x/L_s$, we obtain the following eigenmode equation

$$\frac{d^2\tilde{\phi}_{\vec{k}}}{dx^2} + Q(x,\Omega)\tilde{\phi}_{\vec{k}} = 0, \tag{7}$$

where the "potential" function is given by

$$Q(x,\Omega) = \left\{-k_y^2 + \frac{1-\Omega}{\Omega+K} - \frac{J^{1/2}}{\Omega(\Omega+K)}sx + \frac{s^2x^2}{\Omega^2}\right\}, \tag{8}$$

and we have used the notation

$$\Omega = \frac{\omega}{k_y v_D}, \qquad s = \frac{L_n}{L_s} \ll 1, \qquad K = \frac{1+\eta_i}{\tau}, \qquad \text{and} \qquad J = \left(\frac{V_0 L_n}{L_V}\right)^2 .$$

As will become apparent, K and J serve to parameterize the free energy content of the ion temperature gradient and the ion shear flow, respectively. The parameter J is analogous to the Richardson number, used to describe shear flows in classical fluid dynamics, here inverted for convenience. The difference is that the buoyancy terms due to the gravitational effect on the density gradient $\left(g/L_n\right)$ are replaced by drift frequency terms $\left(k_y^2 v_D^2\right)$, and adjusted to fit into the present scheme of parameter de-dimensionalization.

Equation (7) is a simple Weber's equation, and the lowest mode is given by

$$\tilde{\phi}_{\vec{k}}(x) = \phi_0 \exp\left[-\frac{is}{\Omega}(x - x_0)^2\right], \tag{9}$$

where

$$x_0 = \frac{J^{1/2}}{2s}\frac{\Omega}{\Omega + K}, \tag{10}$$

with the dispersion relation:

$$\left(1 + k_y^2\right)\Omega^2 + \left[K k_y^2 + is - 1\right]\Omega + isK = -\frac{J}{4}\frac{\Omega}{\Omega + K}. \tag{11}$$

The left-hand side of Eq. (11) is the standard dispersion relation for the slab η_i-mode,[8,14] and the right-hand side represents the modification due to shear flow. This equation describes three modes: the usual Pearlstein-Berk electron drift mode, the shear-flow modified η_i-mode, and also the Kelvin-Helmholtz instability. The drift mode is stable everywhere for adiabatic electrons; however, the last two of these are potentially more important, and hence are the focus of the rest of this section.

We first consider the η_i-instability. For the regime where the present fluid theory is applicable (discussed below) it suffices to solve for Ω by iteration, assuming that the right-hand side of Eq. (11) is small. Neglecting the drift-wave root, a first order iteration gives the unstable η_i-root as

$$\Omega_{\eta_i} \cong \frac{isK}{1 - J/4K} \simeq is\left(K + \frac{J}{4}\right) \tag{12}$$

in the low-k_y regime (i.e., $k_y^2 \ll 1/K$).

From this simple analysis, we see that the shear flow has two effects on the η_i-mode. First, it enhances the growth rate at low k_y, with leading correction of order $J \sim \left(V_0/L_V\right)^2$. This enhancement is without regard to the sign of either V_0 or L_V. Second, we see from Eqs. (9)–(10) that the shear flow shifts the center of the mode away from the $x = 0$ rational surface. While this latter effect is not too important for regimes of current interest, it underlies a third effect that is not described by our simple fluid equations.

This third effect, which is apparent in the kinetic analogue of Eq. (7), is a cross term combining effects of shear flow and magnetic shear damping, and varies as $J^{1/2}x^3$ (see

Appendix A). This effect is not represented in the fluid mode equation, Eq. (7), although it sets an upper limit on J for the model to be valid. We find that the best way to describe this limit involves a gyrokinetic analysis. Since this analysis is not germane to our central purpose, it is outlined in Appendix A. The upshot of this analysis is that in order for the cross-term to be unimportant, we must require

$$J^{1/2} \ll \left| \frac{\tau\Omega(\Omega + K)}{3sx_{\max}} \right|.$$

Using $\Omega \simeq isK$ and $x_{\max} \simeq x_T \simeq K^{1/2}$ (the WKB turning point), we find that $J^{1/2} \ll \frac{\tau}{3}K^{3/2}$. Beyond this limit, the simple quadratic well structure embodied by the fluid approximation is no longer valid, due to the disappearance of one of the WKB turning points. While the mode may still be *locally* unstable, the *eigenfunction* characteristics are drastically altered and require a more detailed description than that given here. Shooting code comparisons of Eqs. (7) and (A4) verify this result. Comparison of the above limit with the measurements of Isler et al.,[11] reveals that this restriction does not exclude present-day parameter regimes, where typically $J < 1$ and $K \sim 3 - 5$.

We next consider the question of whether or not the Kelvin-Helmholtz instability is described by our equations. A similar situation has been addressed previously by Catto, Rosenbluth, and Liu,[13] who found unstable modes, which in various limits ($\eta_i \rightarrow 0$, $\tau \gg 1$, $L_n \rightarrow \infty \ldots$) seem to agree with the solution of Eq. (11). However, their study only addresses the limits $L_s \rightarrow \infty$ and then $L_n \rightarrow \infty$ individually, so that in both cases the potential is approximated as a simple quadratic in x. The more realistic case of a shear damped mode with moderate Richardson number is never addressed. Since a consistent treatment of this situation involves solution of a Schrödinger-like equation with a complicated cubic potential (coming from the same effect that limits the validity of the η_i mode above), analytical results are difficult to obtain; however, it is possible to examine the situation numerically using a shooting code with the full kinetic potential, Eq. (A4). From this analysis, we find no regime where the unstable Kelvin-Helmholtz modes predicted by the fluid theory persist in the more detailed kinetic analysis. The reason appears to be that if the Richardson number is above the threshold of instability predicted by the fluid theory ($J > 1$), the subsequent shift of the mode center is so large that the potential is drastically altered by terms of order x^3 and higher. However, we must stress that the above only implies that the pure Kelvin-Helmholtz instability is not well described by the fluid equations and the particular geometry of the present simplified model. It is possible that toroidal effects, FLR effects, and so forth are present in a more realistic situation to give a strong Kelvin-Helmholtz instability.

IV. Nonlinear Theory

Approximation of the one-point nonlinear η_i equations has been performed in Ref. 1, using DIA renormalization of the nonlinearities. Then, an augmented mixing-length scheme was used to estimate the saturated turbulence levels. In the following, we adopt a similar approach, but differ in two significant ways. First, the vorticity nonlinearity in the continuity equation is renormalized so as to include qualitatively the effects of the nonlinearly driven potential fluctuations, which are generally neglected. Second and more importantly, we improve upon the arguments used in Ref. 1 by following a method whereby the renormalized diffusivity is treated as an eigenvalue necessary for turbulent saturation.[15]

The following calculations are also valid in the zero-flow limit ($J \to 0$) and hence supersede the one-point results of Ref. 1.

A. Renormalization

Fourier transforming Eqs. (4)–(6) in y and z yields

$$\frac{\partial}{\partial t}\left(1-\nabla_\perp^2\right)\tilde{\phi}_{\vec{k}} + i\omega_{*e}\left(1+K\nabla_\perp^2\right)\tilde{\phi}_{\vec{k}} + ik_\parallel \tilde{v}_{\parallel\vec{k}} = -N_{\vec{k}}\left(\tilde{\phi},\nabla_\perp^2\tilde{\phi}\right) \tag{13}$$

$$\frac{\partial}{\partial t}\tilde{v}_{\parallel\vec{k}} - \frac{V_0}{L_V} ik_y\tilde{\phi}_{\vec{k}} + ik_\parallel\tilde{\phi}_{\vec{k}} + ik_\parallel\tilde{p}_{\vec{k}} + \mu k_\parallel^2 \tilde{v}_{\parallel\vec{k}} = N_{\vec{k}}\left(\tilde{\phi},\tilde{v}_\parallel\right) \tag{14}$$

$$\frac{\partial}{\partial t}\tilde{p}_{\vec{k}} + iK\omega_{*e}\tilde{\phi}_{\vec{k}} + ik_\parallel\Upsilon\tilde{v}_{\vec{k}} = N_{\vec{k}}\left(\tilde{\phi},\tilde{p}\right), \tag{15}$$

where the symmetrized nonlinear convolutions have the form

$$N_{\vec{k}}\left(\tilde{\phi},\tilde{f}\right) \equiv \left\{\frac{\partial}{\partial x}\left[\sum_{\vec{k}'}\left(-ik_y'\right)\tilde{\phi}_{-\vec{k}'}\tilde{f}_{\vec{k}''}\right] - ik_y\sum_{\vec{k}'}\frac{\partial\tilde{\phi}_{-\vec{k}'}}{\partial x'}\tilde{f}_{\vec{k}''}\right\} - \left\{\tilde{f}\leftrightarrow\tilde{\phi}\right\}, \tag{16}$$

where $\vec{k}$, $\vec{k}'$, and $\vec{k}''$ denote the "test," "background," and "driven" modes, respectively, such that $\vec{k}+\vec{k}' = \vec{k}''$. Using the standard weak coupling closure approximation to renormalize the nonlinearities, we iteratively substitute the nonlinearly driven fields $\tilde{\phi}^{(2)}_{\vec{k}''}$, $\left(\nabla^2_\perp \tilde{\phi}_{\vec{k}''}\right)^{(2)}$, $\tilde{v}^{(2)}_{\parallel\vec{k}''}$, and $\tilde{p}^{(2)}_{\vec{k}''}$ for the corresponding modal $\left(\vec{k}''\right)$ fluctuations. The superscript (2) denotes the "driven" fluctuation resulting from the direct beating of test and background modes. Hence,

$$\Delta\omega_{\vec{k}''}\left(1-\nabla^2_\perp\right)\tilde{\phi}^{(2)}_{\vec{k}''} + i\omega''_{*e}\left(1+K\nabla^2_\perp\right)\tilde{\phi}^{(2)}_{\vec{k}''} + ik''_\parallel \tilde{v}^{(2)}_{\parallel\vec{k}''} = -S\left(\nabla^2_\perp\tilde{\phi}\right) \tag{17}$$

$$\Delta\omega_{\vec{k}''}\tilde{v}^{(2)}_{\parallel\vec{k}''} - \frac{V_0}{L_V} ik''_y\tilde{\phi}^{(2)}_{\vec{k}''} + ik''_\parallel\tilde{\phi}^{(2)}_{\vec{k}''} + ik''_\parallel\tilde{v}^{(2)}_{\parallel\vec{k}''} = S(\tilde{v}) \tag{18}$$

$$\Delta\omega_{\vec{k}''}\tilde{p}^{(2)}_{\vec{k}''} + iK\omega''_{*e}\tilde{\phi}^{(2)}_{\vec{k}''} + ik''_\parallel\Upsilon\tilde{v}^{(2)}_{\vec{k}''} = S(\tilde{p}), \tag{19}$$

where the nonlinear sources are given by

$$S(\tilde{f}) = \left\{ik'_y\tilde{\phi}_{\vec{k}'}\frac{\partial}{\partial x}\tilde{f}_{\vec{k}} - ik_y\frac{\partial\tilde{\phi}_{\vec{k}}}{\partial x'}\tilde{f}_{\vec{k}} + ik_y\tilde{\phi}_{\vec{k}}\frac{\partial}{\partial x'}\tilde{f}_{\vec{k}'} - ik'_y\frac{\partial\tilde{\phi}_{\vec{k}}}{\partial x}\tilde{f}_{\vec{k}'}\right\}, \tag{20}$$

which will yield phase coherent terms when substituted into the nonlinearities. Here, $\Delta\omega_{\vec{k}''}$ may be regarded as the rate of decorrelation for three wave resonance.

At this point, we depart slightly from the previous treatments of renormalization.[15] The standard procedure is to neglect the driven potential, $\tilde{\phi}^{(2)}_{\vec{k}''}$, completely, based on its smoothness relative to the other driven fields, and the fact that its direct inclusion renders the equations intractable. While this is probably adequate for the $\tilde{v}_\parallel$ and $\tilde{p}$ equations, the convective quantity of the continuity equation, $\nabla^2_\perp\tilde{\phi}$, has a simple and direct (linear) relation to the field that convects it, $\tilde{\phi}$. Therefore, it is not clear that the convection and the subsequent back-reaction of the convecting velocity are independent effects, as in the other equations.

Due to the mathematical difficulty of explicitly including the driven potential, $\tilde{\phi}^{(2)}_{\vec{k}''}$, it is better to express it in terms of $\left(\nabla^2_\perp\tilde{\phi}_{\vec{k}''}\right)^{(2)}$, which may be done via an "integration by parts" in the vorticity nonlinearity (i.e., Eq. (16) with $\tilde{f}\rightarrow\nabla^2_\perp\tilde{\phi}$) with respect to x. This latter operation is performed by noting that near the mode rational surface of $\vec{k}$, $k'_\parallel \simeq k''_\parallel$, and hence

$$\frac{\partial}{\partial x'} \simeq \frac{k'_y}{k''_y}\frac{\partial}{\partial x''},$$

which allows us to rewrite the vorticity nonlinearity as

$$N_{\vec{k}}\left(\tilde{\phi}, \nabla_{\perp}^{2} \tilde{\phi}\right) = \left[\frac{\partial}{\partial x} \sum_{\vec{k}'} (-ik_y) \frac{k_y^2 + 2k_y k_y'}{k''^2_y} \tilde{\phi}_{-\vec{k}'} \nabla_{\perp}^{2} \tilde{\phi}_{\vec{k}''} \right. \\ \left. -ik_y \sum_{\vec{k}'} \left(\frac{k_y^2 + 2k_y k_y'}{k''^2_y}\right) \frac{\partial \tilde{\phi}_{-\vec{k}'}}{\partial x'} \nabla_{\perp}^{2} \tilde{\phi}_{\vec{k}''}\right]. \tag{21}$$

However, since the η_i-mode has, to lowest order, incompressible mass flow ($\nabla \cdot (n\vec{v}) \simeq 0$), this amended renormalization of the vorticity nonlinearity will have only secondary importance relative to the final results.

Now that $N_{\vec{k}}\left(\tilde{\phi}, \nabla_{\perp}^{2} \tilde{\phi}\right)$ can be expressed in terms of $\left(\nabla_{\perp}^{2} \tilde{\phi}_{\vec{k}''}\right)^{(2)}$ alone, we can neglect $\tilde{\phi}^{(2)}_{\vec{k}''}$ in the remaining two nonlinearities, as usual. Furthermore, we neglect the terms in Eqs. (17)–(19) which vary as k_y''. Hence

$$\left(\nabla_{\perp}^{2} \tilde{\phi}_{\vec{k}''}\right)^{(2)} \simeq \frac{S\left(\nabla_{\perp}^{2} \tilde{\phi}\right)}{\Delta\omega_{\vec{k}''}} \tag{22}$$

$$\tilde{v}^{(2)}_{\parallel \vec{k}''} \simeq \frac{S\left(\tilde{v}_{\parallel}\right)}{\Delta\omega_{\vec{k}''}} \tag{23}$$

$$\tilde{p}^{(2)}_{\vec{k}''} \simeq \frac{S(\tilde{p})}{\Delta\omega_{\vec{k}''}}. \tag{24}$$

Substituting these for the $\tilde{f}_{\vec{k}''}$ in the nonlinearities yields

$$N_{\vec{k}}\left(\tilde{\phi}, \nabla_{\perp}^{2} \tilde{\phi}\right) = \frac{\partial}{\partial x} \mu^{xx}_{\vec{k}} \frac{\partial}{\partial x} \nabla_{\perp}^{2} \tilde{\phi}_{\vec{k}} - k_y^2 \mu^{yy}_{\vec{k}} \nabla_{\perp}^{2} \tilde{\phi}_{\vec{k}} + \frac{\partial}{\partial x} \beta^{xx}_{\vec{k}} \frac{\partial}{\partial x} \tilde{\phi}_{\vec{k}} - k_y^2 \beta^{yy}_{\vec{k}} \tilde{\phi}_{\vec{k}} \tag{25}$$

$$N_{\vec{k}}\left(\tilde{\phi}, \tilde{v}_{\parallel}\right) = \frac{\partial}{\partial x} D^{xx}_{\vec{k}} \frac{\partial}{\partial x} \tilde{v}_{\parallel \vec{k}} - k_y^2 D^{yy}_{\vec{k}} \tilde{v}_{\parallel \vec{k}} \tag{26}$$

$$N_{\vec{k}}(\tilde{\phi}, \tilde{p}) = \frac{\partial}{\partial x} D^{xx}_{\vec{k}} \frac{\partial}{\partial x} \tilde{p}_{\vec{k}} - k_y^2 D^{yy}_{\vec{k}} \tilde{p}_{\vec{k}}, \tag{27}$$

where the various diffusion coefficients are given by

$$\mu^{xx}_{\vec{k}} \equiv \sum_{\vec{k}'} \frac{k_y^2}{\left(k_y''\right)^2} \frac{k'^2_y \left|\tilde{\phi}_{\vec{k}'}\right|^2}{\Delta\omega_{\vec{k}''}}$$

$$\mu_{\vec{k}}^{yy} \equiv \sum_{\vec{k}'} \frac{k_y^2}{\left(k_y''\right)^2} \frac{\left|\partial\tilde{\phi}_{\vec{k}'}/\partial x'\right|^2}{\Delta\omega_{\vec{k}''}}$$

$$\beta_{\vec{k}}^{xx} \equiv \sum_{\vec{k}'} \frac{k_y^2}{\left(k_y''\right)^2} \frac{k'^2_y\left|\nabla'_\perp\tilde{\phi}_{\vec{k}'}\right|^2}{\Delta\omega_{\vec{k}''}}$$

$$\beta_{\vec{k}}^{yy} = \sum_{\vec{k}'} \frac{k_y^2}{\left(k_y''\right)^2} \frac{\left|\nabla'_\perp\partial\tilde{\phi}_{\vec{k}'}/\partial x'\right|^2}{\Delta\omega_{\vec{k}''}}$$

$$D_{\vec{k}}^{xx} = \sum_{\vec{k}'} \frac{k'^2_y\left|\tilde{\phi}_{\vec{k}'}\right|^2}{\Delta\omega_{\vec{k}''}}$$

$$D_{\vec{k}}^{yy} = \sum_{\vec{k}'} \frac{\left|\partial\tilde{\phi}_{\vec{k}'}/\partial x'\right|^2}{\Delta\omega_{\vec{k}''}}.$$

We propose the following physical interpretations for the above renormalized nonlinearities. First, $D_{\vec{k}}^{xx}$ and $D_{\vec{k}}^{yy}$, which appear in both the momentum and pressure equations, act as non-Markovian turbulent diffusivities that scatter the $\tilde{v}$ and $\tilde{p}$ fluctuations radially, away from the rational surface. This is directly reflects the property that the unrenormalized nonlinearities couple incoming fluctuation energy from the low-k_y modes (bound and growing) to high k_y (which couple to radially outgoing waves), so that fluctuation energy is transported away from the mode rational surfaces in a diffusive manner (Appendix B).

In a similar way, $\mu_{\vec{k}}^{xx}$ and $\mu_{\vec{k}}^{yy}$ serve as nonlinear eddy diffusivities acting on the vorticity, $\nabla_\perp^2\tilde{\phi}$, while $\beta_{\vec{k}}^{xx}$ and $\beta_{\vec{k}}^{yy}$ act as a turbulent back-reaction to $\mu_{\vec{k}}^{xx}$ and $\mu_{\vec{k}}^{yy}$, maintaining the property that $\tilde{\phi}$ ($\sim \tilde{n}$) not be convected by the $\vec{E}\times\vec{B}$ motion. The fact that μ and β vanish as $k_y \to 0$ may be readily demonstrated from the unrenormalized equation. This does not pose a problem for determining the low-k_y mode saturation level however, since energy cascading may proceed by linear coupling of $\tilde{\phi}$ to $\tilde{v}_\parallel$, which can then cascade via the mode coupling represented by $D_{\vec{k}}$.

Finally, it is useful to estimate the relative magnitude of the various diffusivities in the $k_y^2 \ll \left\langle k_y^2 \right\rangle_{\text{rms}}$ limit as

$$\mu_{\vec{k}}^{xx} \simeq \frac{k_y^2}{\left\langle k_y^2 \right\rangle_{\text{rms}}} D_{\vec{k}}^{xx} \tag{28}$$

$$\beta_{\vec{k}}^{xx} \simeq \frac{k_y^2}{\left\langle k_y^2 \right\rangle_{\text{rms}}} \frac{1}{(\Delta x)_{\text{rms}}^2} D_{\vec{k}}^{xx} \tag{29}$$

with similar relations for $\mu_{\vec{k}}^{yy}$ and $\beta_{\vec{k}}^{yy}$. Here,

$$\left\langle k_y^2 \right\rangle_{\text{rms}} \equiv \frac{\sum_{\vec{k}} k_y^2 \left| \tilde{\phi}_{\vec{k}} \right|^2}{\sum_{\vec{k}} \left| \tilde{\phi}_{\vec{k}} \right|^2} \qquad \text{and} \qquad \frac{1}{(\Delta x)_{\text{rms}}^2} \equiv \frac{\sum_{\vec{k}} \left(\partial \tilde{\phi}_{\vec{k}} / \partial x \right)^2}{\sum_{\vec{k}} \left| \tilde{\phi}_{\vec{k}} \right|^2}.$$

Thus, while μ and β are small relative to D, and have little influence on thermal and momentum transport, we retain them because of their physical significance for the model used here. The rms quantities must remain as free parameters, since their evaluation requires a two-point, spectrum theory. However, for the purpose of estimation, we can use the result from Ref. 1 that $\langle k_y \rho_s \rangle_{\text{rms}} \simeq 0.4$.

B. Solution at Saturated Turbulence

The renormalized equations may now be regarded as analogous to the linear equations, and the renormalized nonlinearities play the role of $\vec{k}$-dependent free parameters that account for transport of energy to and from various parts of $\vec{k}$-space. A one-point "closure" calculation may now be completed by considering only the lowest k_y part of the spectrum, which is almost purely growing, and asking how large the renormalization quantities D, μ, and β must be in order to couple energy to smaller scales as fast as it is fed in by the instability mechanism.

The standard method for calculating saturation levels employs a "mixing-length" scheme. While this method is useful for obtaining the correct scalings with certain key parameters, it is deficient in several regards. First, the condition for saturation is derived from "asymptotic balance" of certain parameters, while ignoring others. Such a procedure is inherently insensitive to the detailed phase and amplitude structure of the various modes at different $\vec{k}$, which may be important. Specifically, while some basic parameter scalings are determined, others are ignored completely. It is difficult or impossible to devise a more elaborate mixing-length scheme to incorporate the more subtle scalings. For example, in the present case of a system driven by two free energy sources (gradients), it is not clear how to use asymptotic balance of source and sink for an accurate determination of the relative contributions of η_i and the shear flow to the turbulent excitation. Second, the differential operators are approximated asymptotically using the turbulent mixing length, $\Delta_{\vec{k}}$, and the subsequent approximation of a differential equation as an algebraic equation leaves potentially large numerical factors unresolved. Finally, a consistent picture of turbulent saturation, one based on steady-state energetics, is never established. Using the current approach, such a picture is outlined in Appendix B. All of these deficiencies are corrected in the following "diffusion as eigenvalue" calculation. This technique makes no further assumption on the dynamics of the system, and applies everywhere the fluid equations are valid.

Considering only the low-k_y portion of the spectrum (dominantly growing), we set $\partial/\partial t$ to zero in Eqs. (13)–(15). In this regime, we can also neglect $k_y^2 D_{\vec{k}}^{yy}$, $k_y^2 \mu_{\vec{k}}^{yy}$, $k_y^2 \beta_{\vec{k}}^{yy}$, $k_\parallel^2 \mu$, and $\Upsilon k_\parallel$, and let $\nabla_\perp^2 \simeq \partial^2/\partial x^2$. Then, Fourier transforming Eqs. (13)–(15) (with Eqs. (25)–(27) as the $N_{\vec{k}}$) in x and solving for $\tilde{\phi}_{\vec{k}}$ yields

$$\frac{1}{k_x^2}\frac{\partial}{\partial k_x}\frac{1}{k_x^2}\frac{\partial}{\partial k_x}\psi + \frac{D^{xx}L_s}{k_y}\frac{J^{1/2}}{K}\frac{1}{k_x^2}\frac{\partial}{\partial k_x}\left(\frac{\psi}{1+iD^{xx}k_x^2/K\omega_{*e}}\right) + \frac{\left(D^{xx}\right)^2 L_s^2}{k_y^2}\frac{1}{K}\left(\frac{1-\left(K-i\beta^{xx}/\omega_{*e}\right)k_x^2 - i\mu^{xx}k_x^4/\omega_{*e}}{1+iD^{xx}k_x^2/K\omega_{*e}}\right)\psi = 0, \tag{30}$$

where for convenience we have defined

$$\psi \equiv \left(1 - iK\omega_{*e}/D^{xx}k_x^2\right)\tilde{\phi}_{\vec{k}}.$$

We can reduce the number of parameters, and extract the basic mixing-length scalings of diffusion and mode width by defining $N = \left(L_s/K^2k_y\right)D_{\vec{k}}^{xx}$, $M = \left(L_s/K^2k_y\right)\mu_{\vec{k}}^{xx}$, $B = \left(L_s/Kk_y\right)\beta_{\vec{k}}^{xx}$, $u = \sqrt{K}k_x$. This yields

$$\frac{1}{u^2}\frac{\partial}{\partial u}\frac{1}{u^2}\frac{\partial}{\partial u}\psi + N\left(\frac{J}{K}\right)^{1/2}\frac{1}{u^2}\frac{\partial}{\partial u}\left(\frac{\psi}{1-isNu^2}\right) + N^2\frac{1-(1+isB)u^2+isMu^4}{1-isNu^2}\psi = 0. \tag{31}$$

Cast in this form, it is clear that the resulting dispersion relation for N, M, and B will depend only on $s = L_n/L_s$ and J/K, so the mixing-length results somewhat succeed in resolving basic scalings.

Equation (31) may be manipulated into "Schrödinger form" as

$$\frac{\partial^2}{\partial z^2}\varphi + Q(z; N, M, B)\varphi = 0 \tag{32}$$

with

$$Q = \frac{N^2}{9}\left[\frac{1-(1+isB)|z|^{2/3}+isM|z|^{4/3}}{1-isN|z|^{2/3}} - \frac{J}{4K}\frac{1}{\left(1-isN|z|^{2/3}\right)^2}\right], \tag{33}$$

where

$$z \equiv u^3, \qquad \varphi \equiv \exp\left[\frac{NJ}{6K}\left(\frac{z}{1-isN|z|^{2/3}}\right)\right]\psi,$$

and we have noted that small mode width ($\Delta z \lesssim 1$, shown *a posteriori*) implies that

$$\frac{\partial}{\partial z}\left(\frac{\psi}{1-isN|z|^{2/3}}\right) \simeq \frac{1}{1-iNs|z|^{2/3}}\frac{\partial\psi}{\partial z}.$$

This approximation is only applied to the term dependent on J, so this "Schrödinger approximation" is exact in the limit of no shear flow, $J \to 0$.

The dispersion relation for Eq. (33) may be obtained by a WKB approximation, with the $z_T \simeq 1$ turning point (since $\Delta Z \lesssim 1$ implies a potential smooth relative to the mode). Using $s \ll 1$ as an ordering parameter, we find to order s

$$\int_{-z_T}^{z_T} [Q(z; N, M, B)]^{1/2}\, dz = N\left(1 - \frac{J}{4K}\right)^2 \frac{\pi}{8}\left[1 + \frac{is}{4}(N + 5M - 6B)\right] = \frac{\pi}{2}. \tag{34}$$

That is,

$$N^2 is\left(1 + \frac{5M}{N} - \frac{6B}{N}\right) + 4N - \frac{16}{(1-J/4K)^2} = 0. \tag{35}$$

If we assume that M/N and B/N are independent of N, as with Eqs. (28) and (29), then we can solve Eq. (35) as a quadratic equation in N. The root that is dominantly *real* is, to order s,

$$N = \frac{4}{(1-J/4K)^2}\left[1 - is\left(1 + \frac{5M}{N} - \frac{6B}{N}\right)\right]. \tag{36}$$

Restoring the parameter scalings and using the estimates in Eqs. (28) and (29) yields, for $J/4K \ll 1$,

$$D_{\vec{k}}^{xx} = 4\left(K + \frac{J}{4}\right)^2 \frac{k_y}{L_s}\left[1 - is\left(1 - \frac{k_y^2}{\langle k_y^2\rangle_{\rm rms}}\right)\right]. \tag{37}$$

This is the principal result of this nonlinear analysis.

The first thing to notice is that $D_{\vec{k}}^{xx}$ is dominantly *real*, the imaginary component being of order s. In the $J \to 0$ limit, the scaling agrees with the results of Ref. 1 (i.e., $D \sim K^2 k_y/L_s$) and importantly, we find that this result is enhanced by a numerical factor of 4 in this more accurate calculation. The shear flow, represented by J, further enhances the diffusion rate.

The imaginary part of D^{xx} is a nonlinearly induced frequency shift, and does not affect overall transport. It is interesting because it lends itself to a simple physical interpretation, as follows. From inspection of Eq. (37), one can see that this shift comes from two physical processes. First, nonlinear coupling to modes of different frequency produces the portion of $\mathrm{Im}\,(D^{xx})$ that is independent of μ and β, i.e., which remains in the $k_y^2 \ll \langle k_y^2\rangle_{\rm rms}$ limit. Second, there is a part induced by nonzero eddy diffusivity, which contributes *only* to the imaginary part of D^{xx}. This is because transport of vorticity, which represents momentum fluctuation with no net momentum content, will only affect the fluctuation frequency of the momentum, not its overall rate of diffusion. The latter is a good example of a process that may only be resolved by an *eigenvalue* solution for the turbulent diffusivity, in which the details of the *linear* energy exchange processes are accounted for.

Numerical analysis of Eq. (32) (shooting code) qualitatively confirm the WKB solution. Quantitatively, they show a factor of 3.3 in place of 4, which is constant for $s \lesssim 0.2$.

Finally, we should note that in Ref. 1 it was shown that consideration of spectrum broadening reveals that $D_{\vec{k}}^{xx}$ is enhanced by a factor of about $\left(\frac{\pi}{4} ln(R_e)\right)^2$, where R_e is the η_i-turbulence analogue of the Reynolds number. We expect that a spectrum analysis applied to the present case would show a similar enhancement. However, since this requires a two-point theory, we shall not address this issue here.

V. Transport

Having obtained the saturation level of turbulent diffusivity at long wavelength, where most of the turbulent transport takes place, we wish to use this knowledge to obtain the levels of turbulent viscosity, χ_φ (i.e., momentum diffusivity) and ion thermal conductivity, χ_i. This we may do by replacing the nonlinear convection, $\hat{b} \times \nabla\tilde{\phi} \cdot \nabla$ in Eqs. (4)–(6), with the nonlinear decorrelation rate, $\Delta\omega_{\vec{k}''}$, taking $\partial/\partial t \to 0$ and solving. Neglecting Υ, (which gives contributions of order s^2), we obtain

$$\tilde{v}_{\|\vec{k}} = \left[\frac{V_0}{L_V v_D}\frac{i\omega_{*e}}{\Delta\omega_{\vec{k}''}} + i\omega_{*e}\left(1 - \frac{iK\omega_{*e}}{\Delta\omega_{\vec{k}''}}\right)\frac{sx}{\Delta\omega_{\vec{k}''}}\right]\tilde{\phi}_{\vec{k}} \tag{38}$$

$$\tilde{p}_{\|\vec{k}} = -\frac{i\omega_{*e}}{\Delta\omega_{\vec{k}''}}K\tilde{\phi}_{\vec{k}}. \tag{39}$$

The rate of radial flux of the perturbed parallel momentum is given by

$$q_\varphi = \left\langle \tilde{v}_r \tilde{v}_\| \right\rangle = \sum_{\vec{k}'} \left\langle -ik'_y \tilde{\phi}_{\vec{k}'} \tilde{v}_{\|-\vec{k}'} \right\rangle. \tag{40}$$

Using Eq. (38) and noting that the terms which vary as x vanish due to symmetry considerations,

$$q_\varphi = -\frac{V_0}{L_V}\sum_{\vec{k}'}\left\langle \frac{k'^2_y \left|\tilde{\phi}_{\vec{k}'}\right|^2}{\Delta\omega_{\vec{k}''}} \right\rangle = -\frac{V_0}{L_V}\left\langle D^{xx}_{\vec{k}} \right\rangle \tag{41}$$

and hence the turbulent viscosity is

$$\chi_\varphi \equiv -\frac{q_\varphi}{(dV_0/dr)} = 4\left(K + \frac{J}{4}\right)^2 \frac{\langle k_y\rangle_{\mathrm{rms}}}{L_s}, \tag{42}$$

where we have neglected $\mathrm{Im}\,(D^{xx})$, which does not contribute to transport.

A similar calculation, using Eq. (39) to calculate the correlation of the radial velocity with the pressure fluctuation, gives the ion thermal flux as

$$q_i = -(1+\eta_i)\sum_{\vec{k}'}\left\langle \frac{k'^2_y \left|\tilde{\phi}_{\vec{k}'}\right|^2}{\Delta\omega_{\vec{k}''}} \right\rangle \tag{43}$$

with resulting ion thermal conductivity

$$\chi_i = 4\left(K + \frac{J}{4}\right)^2 \frac{\langle k_y\rangle_{\mathrm{rms}}}{L_s}. \tag{44}$$

Redimensionalizing Eqs. (42) and (44), using the numerical coefficient of 3.3 from the shooting code analysis, and taking $\langle k_y \rho_s \rangle_{\mathrm{rms}} \simeq 0.4$ from Ref. 1, we find

$$\chi_\varphi = \chi_i = 1.3 \left[\frac{1+\eta_i}{\tau} + \left(\frac{V_\varphi L_n}{2 c_s L_V} \right)^2 \right]^2 \frac{\rho_s^2 c_s}{L_s}. \tag{45}$$

The result that $\chi_i = \chi_\varphi$ is an important property of η_i-turbulence in the presence of a shear flow, and suggests a plausible explanation of experimental observations at PDX,[16] ISX-B,[11] D-III, TFTR,[7] and other beam-heated tokamaks.

Following Ref. 1, we can crudely estimate the electron thermal conductivity (χ_e) associated with the trapped electron response to the turbulent potential fluctuations.[17] In the dissipative trapped electron regime ($\omega_{*e} < \nu_{e_{\mathrm{eff}}}$), we may estimate χ_e as:

$$\chi_e \simeq 15\sqrt{2}\epsilon^{3/2} \frac{\rho_s^2 c_s^2}{\nu_e} \sum_{\vec{k}'} k_y'^2 \left| \frac{e\tilde{\phi}_{\vec{k}}}{T_e} \right|^2 . \tag{46}$$

Crudely estimating, $\sum_{\vec{k}'} k_y'^2 \left| \tilde{\phi}_{\vec{k}'} \right|^2 \simeq \langle \Delta\omega_{\vec{k}} D_{\vec{k}} \rangle_{\mathrm{rms}} \simeq 3.3 \left(K + \frac{J}{4} \right)^3 \frac{\langle k_y^2 \rho_s^2 \rangle_{\mathrm{rms}}}{L_s^2}$. Again using $\langle k_y \rho_s \rangle \simeq 0.4$, we find only. Hence,

$$\chi_e \simeq 10\epsilon^{3/2} \left[\frac{1+\eta_i}{\tau} + \left(\frac{V_\varphi L_n}{2 c_s L_V} \right)^2 \right]^3 \frac{\rho_s^2 c_s^2}{\nu_e L_s^2}, \tag{47}$$

and thus χ_e is also enhanced by the sheared toroidal flow.

VI. Discussion

In this work, the effects of a subsonic, radially sheared toroidal ion flow on η_i-turbulence have been examined, in an effort to assess its role in neutral-beam-heated tokamaks. We have shown that the levels of fluctuation and turbulent transport increase with (an analogue of) the Richardson number, $J \equiv \left(\frac{L_n}{c_s}\frac{dV_{i0}}{dr}\right)^2$. We have demonstrated that there is significant diffusive momentum transport (viscosity) in the presence of η_i-turbulence, and that the momentum diffusivity and ion thermal diffusivity are the same, thereby providing a plausible explanation for the observation that momentum and thermal transport tend to behave similarly.

There is a striking correlation in the experimental literature between the application of stimuli which alter η_i and/or J, and the concomitant observation of a like change in momentum and/or thermal diffusivity. The well known degradation of χ_i with increasing beam power[2] is one example. As a second example, recent experiments on TFTR[7] compare beam center-heating with edge-heating (which reduces η_i). During edge-heating, both χ_φ and the energy diffusivity are reduced by a factor of about 2. As a third example, a substantial decrease of χ_φ following beam turn-off is observed in TFTR,[7] PDX,[16] and ISX-B.[11] This may be explained as the result of $T_i(r)$ and $V_\varphi(r)$ flattening (thereby reducing η_i and J) as the direct ion heating and shear flow excitation are terminated. Finally, the decrease of χ_i in TFTR "supershots"[18] accompanying balanced injection may possibly be connected with the concomitantly observed peaked density profiles, as well as the $V_\varphi \rightarrow 0$ turnoff of the Richardson number enhancement. However, care is required in interpreting supershot results, since the large density of high energy in this regime makes the applicability of our one-fluid ion model questionable.

The results of this paper indicate that shear flow enhanced η_i-turbulence is quite possibly an important factor in beam-heated tokamaks. However, the present model is a crude one, and there are several possible directions for further study. Among these are consideration of the effects of toroidicity, neoclassical damping of ion flows, the effects of an unthermalized beam, and further investigation into the possibility of the shear flow dominating the temperature gradient drive, and hence destabilizing a predominantly Kelvin-Helmholtz rather than η_i mode.

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Appendix A: Kinetic Limit of the Linear Fluid Equations

Here, we explore the limit of validity of the fluid equations by examining the ion gyrokinetic theory[19] for geometry and gradients similar to the preceding fluid model. A Maxwellian velocity distribution shifted by $V_{\varphi 0}(x)$ in the $\hat{\varphi}$-direction is assumed. This yields the following perturbed phase space ion distribution:

$$\tilde{f}_i\left(\vec{k}; v_\perp, v_\parallel\right) = F_M\left(\vec{v} - V_{\parallel 0}\hat{b}\right)\left\{1 - \frac{J_0^2\left(k_\perp v_\perp/\Omega_i\right)}{\omega - k_\perp V_{\perp 0} - k_\parallel v_\parallel} \times \left[\omega - k_\perp V_{\perp 0} - k_\parallel V_{\parallel 0}\right.\right.$$
$$\left.\left.- \frac{\omega_{*e}}{\tau}\left[1 + \frac{\eta_i}{2}\left(\frac{v_\perp^2 + \left(v_\parallel - V_{\parallel 0}\right)^2}{v_i^2} - 3\right) + \frac{L_n V_{\parallel 0}\left(v_\parallel - V_{\parallel 0}\right)}{L_v v_i^2}\right]\right]\right\} \times \frac{e\tilde{\phi}}{T_i}, \quad (A1)$$

where F_M is the Maxwellian, J_0 is the zeroth Bessel function, and $v_i^2 = T_i/m_i$. Integrating away the $\vec{v}$-dependence and undimensionalizing time and distance to Ω_i^{-1} and ρ_s, we find that

$$\tilde{n}_i = G(\vec{k})\frac{e\tilde{\phi}}{T_i}, \quad (A2)$$

where

$$G(\vec{k}) = -\left[1 + \frac{1}{2\Omega'}Z_p'(\zeta)\Gamma_0\left(\left(\frac{2J}{\tau}\right)^{1/2}\zeta - \frac{\eta_i}{\tau}\zeta^2\right)\right.$$
$$\left.+ \frac{1}{\Omega'}\zeta Z_p(\zeta)\left(\left(\Omega' + \frac{1}{\tau} - \frac{\eta_i}{2\tau}\right)\Gamma_0 + \frac{\eta_i}{\tau^2}k_\perp^2\left(\Gamma_0 - \Gamma_1\right)\right)\right], \quad (A3)$$

where Z_p and Z_p' are the plasma dispersion function and its derivative, $\zeta = \sqrt{\tau/2}\left(\Omega' k_y/L_n k_\parallel\right)$, $\Omega' = \left(\omega - \vec{V}_\varphi \cdot \vec{k}\right)/\omega_{*e}$, and $\Gamma_n = I_n\left(k_\perp^2/\tau\right)\exp\left(-k_\perp^2/\tau\right)$, where I_n is a modified Bessel function. Applying the quasineutrality equation with adiabatic electrons, $\tilde{n}_i = e\tilde{\phi}/T_e$, expanding in k_x^2 to first order, and then taking $k_x^2 = -\partial^2/\partial x^2$ and $k_\parallel = k_y x/L_s$, we obtain the following differential equation in x:

$$\frac{\partial^2 \tilde{\phi}_{\vec{k}}}{\partial x^2} + Q(x;\Omega)\tilde{\phi}_{\vec{k}} = 0, \quad (A4)$$

where

$$Q(x;\Omega) = \frac{1/\tau - G\left(k_y^2\right)}{G'\left(k_y^2\right)}. \quad (A5)$$

This equation reduces to the fluid eigenmode equation, Eq. (7), when expanded to order x^2 and k_y^2 in the limit $|\zeta| \gg 1$ and $k_y^2 \ll 1$. As with the fluid version, the potential is even in x *except* for the terms induced by V_0 through the Richardson number, J. The terms which vary as x shift the fluid potential, but do not alter the quadratic structure; however this is not true of the odd terms which vary as x^3 or higher, which tend to destroy the quadratic structure at large x. Physically, these terms represent the effects of higher shear damping when the velocity shear causes the mode to stray too far from the mode rational surface. Analytically, we can derive a crude but adequate estimate of the regime of fluid validity by requiring that the term cubic in x which is not in the fluid theory, be less than the quadratic term, which *is* in the fluid theory. Upon expanding, we find

$$J^{1/2} \ll \left| \frac{\tau\Omega(\Omega + K)}{3sx_{\max}} \right| . \tag{A6}$$

This is the principal result of this section.

The shooting code analysis of Eq. (A4) may also be used to find the effect of $\frac{dV_\varphi}{dr}$ on the instability threshold, η_{ic}, in the spirit of Ref. 9. Although we have not done a detailed analysis, preliminary studies show that η_{ic} is *lowered* as $\frac{dV_\varphi}{dr}$ is increased, but not by more than about 15% before the limiting effects mentioned in the above paragraphs become important. This result is interesting in light of recent results from transport simulations by Goldston et al.,[20] which indicate that η_i tends to maintain itself at marginal stability, even when strong central ion heating is applied. If this is the case, then in the presence of a shear flow the allowable ion temperature gradient is even lower.

Appendix B: Energy Saturation

Here, we propose a criterion for turbulent saturation based on the ensemble-averaged turbulence energies, and then translate this criterion into a mathematical method for accurately solving for the diffusion in the low-k_y regime of the spectrum, which is responsible for most of the transport.

We may define the following energy-like integrals,[1] which represent the degree of turbulence excited in the various fields:

$$E^W = \frac{1}{2}\int d^3x \left(\left|\tilde{\phi}\right|^2 + \left(\nabla_\perp\tilde{\phi}\right|^2\right) \quad (B1)$$

$$E^K = \frac{1}{2}\int d^3x \left|\tilde{v}_\parallel\right|^2 \quad (B2)$$

$$E^I = \frac{1}{2}\frac{1}{\Upsilon}\int d^3x \left|\tilde{p}\right|^2 . \quad (B3)$$

Evolution equations for these energies may be obtained by integrating Eqs. (4)–(6), and using the conservative property of convective nonlinearities, $\int d^3x\tilde{A}\left(\nabla\tilde{\phi}\times\hat{b}\right)\cdot\nabla\tilde{A} = 0$ for any $\tilde{A}$. This yields

$$\frac{\partial}{\partial t}E^W = -\int d^3x\tilde{\phi}\nabla_\parallel\tilde{v}_\parallel \quad (B4)$$

$$\frac{\partial}{\partial t}E^K = -\int d^3x\left[\tilde{v}_\parallel\nabla_\parallel\tilde{\phi} + \tilde{v}_\parallel\nabla_\parallel p + \frac{V_0}{L_V}\tilde{v}_\parallel\nabla_y\tilde{\phi} + \mu\left|\nabla_\parallel v_\parallel\right|^2\right] \quad (B5)$$

$$\frac{\partial}{\partial t}E^I = -\int d^3x\left[\tilde{p}\nabla_\parallel\tilde{v}_\parallel + \frac{1}{\Gamma}\left(\frac{1+\eta_i}{\tau}\right)v_D\tilde{p}\nabla_y\tilde{\phi}\right] . \quad (B6)$$

Hence, the total energy of the system evolves by

$$\frac{\partial}{\partial t}E = -\int d^3x\left[\frac{1}{\Gamma}\left(\frac{1+\eta_i}{\tau}\right)v_D\left\langle\tilde{p}\nabla_y\tilde{\phi}\right\rangle + \frac{V_0}{L_V}\left\langle\tilde{v}_\parallel\nabla_y\tilde{\phi}\right\rangle + \mu\left|\nabla_\parallel v_\parallel\right|^2\right] . \quad (B7)$$

This energy evolution equations state that turbulence energy so defined enters the spectrum at low k_y through the η_i and dV_0/dr free energy sources, and leaves the spectrum at high k_y through viscous (Landau or collisional) damping. In order for the energy to move from low to high k_y, nonlinear mode coupling must occur, here modelled as triad resonance coupling of $\vec{k}$ and $\vec{k}'$ modes to $\vec{k}''$ over a time of $\left(\Delta\omega_{\vec{k}''}\right)^{-1}$.

In $\vec{k}$-space, this transfer of energy must go in the direction of source (low k_y) to sink (high k_y). Meanwhile, back in configuration space, the transfer to higher k_y translates to energy going out away from the mode rational surface, where turbulent fluctuations can damp through the higher $k_\parallel$.

For the renormalized equations, which replace the nonlinear equations by linear equations with energy sink $D_{\vec{k}}$, saturation occurs when $D_{\vec{k}}$ has become large enough that all the energy fed into the system by instability are carried off to higher k_y, thus turning off the growth of all parts of the spectrum.

Analytically, $D_{\vec{k}}$ may be treated as an eigenvalue of the renormalized equations that regulates energy transfer between various parts of the spectrum. By restricting ourselves to the low-k_y part of the spectrum, where most of the radial transport occurs, all diffusivities except $D_{\vec{k}}^{xx}$ are of small relative importance in Eqs. (25)–(27), and hence we may neglect them.