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Nonlinear Fluid Equations for a Low Collisionality Toroidal Plasma

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Abstract

Nonlinear fluid equations are rigorously derived to describe resonant resistive perturbations localized near a rational surface in a torus, including neoclassical dissipative effects such as rotation damping and bootstrap current production which arise when the collisional mean free path exceeds the equilibrium scale length. A systematically ordered analysis with two parallel scales is used, for a low aspect ratio, starting directly from the drift-kinetic equation. The radial mode width is formally taken of order the poloidal gyroradius, and frequencies are taken to be sufficiently rapid that the perturbed poloidal and toroidal ion flows do not necessarily relax to satisfy the neoclassical equilibrium relation. This is realistic for many modes, and leads to some departures from standard neoclassical results. Neoclassical effects for electrons are relatively smaller than for ions so to include both consistently next order ion corrections are also included. For simplicity temperature perturbations are neglected. The new dissipative terms are shown to satisfy a field theoretic generalization of the Onsager symmetries, and the equations satisfy an H theorem.

I Introduction

Fluid equations are widely used to study linear and nonlinear resistive plasma behavior involving magnetic reconnection in toroidal confinement geometries.[1,2,3,4,5] However, these fluid equations are derived in the limit where the collision time is shorter than the time scale of the fluid motion, and when the collisional mean free path is short compared to fluid length scales parallel to the magnetic field.[6,7] The first requirement is satisfied for many phenomenon of interest in toroidal confinement devices. The second requirement is also often satisfied near the resistive layer in one sense, since the reconnection occurs near a mode rational surface where the resonant parallel length scales are very long. However, the equilibrium magnetic field strength varies along a field line on the scale of the machine size, which is often much smaller than a mean free path. Investigations of linear instabilities by Callen and Shaing[8,9] and by Connor and Chen[10] have shown that this leads to additional dissipation which is closely related to that arising in neoclassical transport theory.[11] They found that bootstrap current production lead to new, interchange-like modes, and the flow rotation damping increased the effective inertia of the plasma. However, these pioneering investigations were significantly incomplete. Several crucial assumptions about equilibrium neoclassical theory do not apply to typical instabilities. For example, a key ingredient of standard neoclassical theory[11] is that the radial length scale is much larger than a poloidal gyroradius, whereas resistive modes usually have reconnection regions which are thinner than a poloidal gyroradius. Here, a set of nonlinear fluid equations is derived with a systematic ordering appropriate for present operating regimes of interest to examine the effect of neoclassical dissipation on resistive modes near the resonant mode rational surface.

The neoclassical effects which arise in an axisymmetric torus are rotation damping, bootstrap current production, and the Ware pinch. Depending on characteristics of the perturbation, rotation damping can either act non-dissipatively to increase the effective inertia, or to produce strong dissipation.[8] Rotation damping in a tokamak is also closely related to the non-intrinsically ambipolar transport which arises in a stellarator, which has been shown by Kotschenreuther and Aydemir to inhibit formation of magnetic islands and stochasticity.[12,13]

In the plateau regime, the bootstrap current is crudely the same magnitude as the Pfirsch-Schlüter current. Perturbations in the plasma pressure produce perturbations in the bootstrap current. This has been found to be a destabilizing interchange-like effect, both in linear calculations by Callen and Shaing[8,9] and by Connor and Chen,[10] and in nonlinear analysis by Carrera et al.,[14] and Qu et al.,[15] where bootstrap effects caused otherwise stable magnetic islands to grow in the Rutherford regime.

To obtain general non-linear equations which can accurately describe such effects we implement a two-space scale analysis for localized perturbations. The calculation is similar in spirit to previous MHD analysis by Glasser et al.,[13] and by Kotschenreuther et al.,[17] but with kinetic effects. It may be considered a nonlinear complement to previous linear analysis in ballooning representation.[9,10] We require that there is a scale separation between the typical parallel gradient scale of the fluid perturbation and the parallel distance over which the equilibrium magnetic field varies – as occurs in a radially thin a resistive layer where magnetic field lines reconnect. All quantities are split into slowly and rapidly varying parts. Equations are derived for the slowly varying fluid quantities, in which there appear the average of the beat together of rapidly varying quantities. The rapidly varying quantities are computed by kinetic theory. Note that we do not need to assume that quantities like the fluid density or electrostatic potential are constant on a flux surface, or even that flux surfaces exist, but only that there is a parallel scale separation.

This calculation differs from previous work[18] in several respects. Previously, equilibrium neoclassical results were used for resistive instabilities which can have have much different length and time scales than those assumed in equilibrium theory. The ordering of physical parameters used here is realistic for resistive modes. Previous results have been derived by assuming that the mode width greatly exceeds the poloidal ion gyroradius. This is almost never satisfied for realistic parameters. Here we explicitly take the radial width of order the poloidal gyroradius. In addition, we compute next order corrections for the ion contributions, so that somewhat thinner layers may still be described accurately. This has surprisingly little effect on the neoclassical dissipation.

Significant modifications from equilibrium neoclassical results do arise, however. Standard neoclassical theory assumes that flux surface average quantities change very slowly in time, so that the flows are well equilibrated. However, resistive instabilities often have frequencies faster than the *parallel* ion flow equilibration rate, which is slower than the poloidal damping rate by roughly $B_p^2/(B_p^2+B_t^2)$, with B_p and B_t the poloidal and toroidal magnetic field strengths. Unlike equilibrium calculations,[11] we do not assume the ion flows have equilibrated – which leads to modifications of the equilibrium neoclassical results.

Also, the detailed ordering used here allows us to assess the importance of possible nonlinear modifications of these. For example, previous calculations have used axisymmetric neoclassical results to describe perturbations which nonlinearly modify the plasma in a non-axisymmetric way. We adopt the common ordering that the pressure gradients are nonlinearly modified from the equilibrium gradient by order one. We find that nonlinear corrections to the neoclassical dissipation are small.

In addition, we order other physical parameters to include other physical effects of interest consistently with the two scale analysis. We choose a maximal ordering which fully includes parallel compressibility, and includes sufficient parallel electron dynamics to describe a semicollisional resistive layer. This has more than formal significance. A derivation of the neoclassical dissipative processes for localized perturbations requires that the perturbations satisfy certain consistencies in the frequency, radial mode width, collision frequency and plasma β. The importance of the physical effects mentioned above depends on similar parameters, so these effects might be inconsistent with the neoclassical derivation. The demonstration and implementation of a consistent maximal ordering to include these effects resolves these issues, and demonstrates that the neoclassical dissipative effects are robust and apply to a wide variety of resistive phenomenon. This is a major feature which distinguishes this work from previous results.

In general, we find that the final nonlinear equations are similar to those recently obtained by Callen and Shaing by applying the equilibrium neoclassical theory concept of a parallel "viscous" force.[16] However, we find an additional large cross field viscosity for poloidal flows, which can be important for perturbations much thinner than a poloidal gyroradius.

The neoclassical terms in these fluid equations are shown to satisfy Onsager symmetries. We use a field theoretic generalization of the concept of Onsager symmetries which is due to P.J. Morrison.[19] A generalization beyond equilibrium neoclassical theory is needed because of the presence of flow and derivative operators in these dissipative terms. The equations also satisfy an H theorem for the physical free energy. They also conserve toroidal angular momentum for the axisymmetric case. Thus, the derived equations are physically consistent and provide a rigorous basis for examining magnetic reconnection in toroidal geometries.

The outline of this paper is as follows. In Section II we present and explain the results. We give a hueristic argument for the size of the poloidal rotation damping. Also, the nonlinear fluid equations are presented and the neoclassical dissipative terms are described. In Section III the ordering scheme is given, and the lowest order equations are derived. In Section IV the next order results are derived, which include the bootstrap current term.

The generalized Onsager symmetries and H-theorem are shown in Section V. Concluding remarks are in Section VI.

II Discussion of Physical Mechanisms and Results

A Hueristic estimate of rotation damping rate

The poloidal flow damping rate in a torus is surprisingly rapid. Despite the fact that it is due to ion-ion collisions, the damping rate in the plateau or banana regime can exceed the ion collision frequency ν_{ii} by up to roughly $\epsilon^{-3/2}$, and thus by between one and two orders of magnitude. It is instructive to physically interpret the kinetic mechanism for this.

Consider a toroidal plasma with poloidal $\mathbf{E} \times \mathbf{B}$ motion from an electrostatic field E_r perpendicular to the flux surfaces. If there is a radial plasma current j_r produced by the radial electric field, then the plasma kinetic energy due to rotation energy due to joule heating at a rate locally given by $E_r j_r$.

Here consider an axisymmetric plasma for simplicity. Collisionless single particle motion is constrained by the conservation of canonical toroidal angular momentum, which implies that guiding center particle drifts do not cause a secular radial particle drift across flux surfaces. Thus there is no flux surface averaged radial current, and no rotation damping. Rotation damping in an axisymmetric torus results from collisional modifications of a particle's motion. Ion-ion self-collisions only conserve the canonical angular momentum when summed over all particles – and the resulting summed conservation law allows a radial current (if there is a toroidal acceleration at the same time, which is found in the detailed calculation in Sec. III.)

Collisions operate on a distribution function which deviates from a Maxwellian in velocity space. Consider a distribution function which is initially a Maxwellian f_0 constant in space. Suppose we induce a radial electrostatic potential Φ in this plasma, with Φ nearly constant along field lines. We now examine how deviations from a Maxwellian arise and how collisional modifications of particle motion lead to a radial current.

For low collisionality, as in the plateau or banana regime, a useful first step is to consider the collisionless motion. We suppose that E_r is sufficiently small to consider its effects perturbatively. The lowest order collisionless motion consists of streaming along a field line with velocity $v_{||}$ and perpendicular motion from the equilibrium curvature and ∇B drifts. These drifts lead to no secular drift off flux surfaces, but the particle has periodic, oscillatory displacements Δr from a flux surface. The displacement Δr is of the order of the radial drift velocity v_{dr}^0 multiplied by the time to poloidally transit the torus by parallel motion, $1/\omega_t$, where is the ω_t the transit frequency $\approx v_{||}/qR$, with q the safety factor and R the major radius. During these displacements the kinetic energy remains constant, so the distribution remains Maxwellian. When E_r is included, however, the kinetic energy oscillates because of the periodic displacements in the radially varying Φ, so

$$\Delta\mathcal{E} \approx eE_r\Delta r \approx eE_r v_{dr}^0/\omega_t. \tag{1}$$

Equation (1) implies that $v_\perp^2$ and $v_{||}^2$ are changed; thus, we expect the drift velocity to be changed, and by an amount of roughly $\Delta v_{dr} \approx (\Delta\mathcal{E}/\mathcal{E})v_{dr}^0$. For collisionless single particle motion in an axisymmetric plasma, toroidal angular momentum conservation again implies that no secular radial drift results from the perturbed drift, but only a perturbed oscillation. However the energy oscillation in Eq. (1) forces the distribution away from a Maxwellian. When collisions are included, the perturbation in v_{dr} does lead to a net drift from the flux surface, and thus a radial current. In the plateau regime, trajectories are perturbed by order one for particles with small parallel velocity, $v_{||}^c$, for which the transit frequency is of order the collision frequency for *small* pitch angle scattering, $\nu_{ii}(v_i/v_{||}^c)^2$. This group of particles, which constitutes a fraction $v_{||}/v_i$ of all particles, has a perturbed radial drift of the same order as in the collisionless case, leading to a radial current

$$j_r \approx en_i\Delta v_{dr}(v_{||}^c/v_i) \approx e^2 n_i E_r v_{dr}^2 (qR/v_{||}^c)(v_{||}^c/v_i)T_i. \tag{2}$$

In a torus, $v_{dr} \approx v_i\rho_i/R$.

Dividing $E_r j_r$ by the kinetic energy density, we obtain the poloidal rotation damping rate in the plateau regime, $\nu \approx qv_i/R$.

Since the estimate of ν above neglects trapped particles, it is not valid in the banana regime. Similar arguments for that regime reveal that $\nu \approx \nu_{ii}\epsilon^{-3/2}$ which matches smoothly to the plateau result at the collisionality boundary. The ϵ in this expression is the local aspect ratio, which can easily be less than 1/10 at, say, the $q = 1$ surface. Thus the true poloidal damping rate can exceed ν_{ii} by one to two orders of magnitude. Part of the reason for this is because the dissipation occurs in a narrow region of velocity space, so the effective collision frequency is larger than ν_{ii}.

The analytical calculation of Sec. III reveals that a radial current is also driven if there is a net parallel fluid velocity. In an low aspect ratio axisymmetric torus, this cancels the current from E_r if $v_{\|} = (B/B_p)(cE_r/B)$. In this case the component of this parallel velocity in the poloidal direction cancels that from the $\mathbf{E} \times \mathbf{B}$ velocity, so that there is only purely toroidal rotation in the symmetry direction. The perpendicular velocity equilibrates to this state at the rate ν; however the parallel velocity equilibrates at a rate slower by a roughly factor $(B_p/B)^2$.

In a non-axisymmetric torus in the plateau regime, both E_r and $v_{\|}$ damp to zero at rates which are crudely comparable to the ones above. When ion diamagnetic effects are included, the radial electric field decays to a value which confines the ions, $E_r \approx (dn_iT_i/dr)/en_j$. This is due to the well known fact that neoclassical transport is not intrinsically ambipolar in non-symmetric configurations, so the radial current drives E_r to a particular value. In a lower collisionality regime where super banana effects arise, the poloidal damping rate is expected to be orders of magnitude larger than this.

Note that in the argument above, we neglected the radial current arising from collisionless motion and Δv_{dr}, invoking axisymmetry. This strictly applies if $\partial\Phi/\partial\zeta = 0$, but not to the kind of helical Φ which arises in a resistive mode. Note that if Φ varies only slowly along $\mathbf{B}$, as is true in a radially thin resistive layer, a particle with predominantly parallel motion perceives little difference from the case with $\partial\Phi/\partial\zeta = 0$, so one might expect the resulting current to be small. A rigorous justification of the neglect of this current for resistive modes requires a precisely ordered calculation, given in Sec. III.

B Discussion of results

The calculation demonstrates that near the resistive layer the perturbed quantities can be described by the following fluid variables (to requisite order in the expansion): the pressure p, the parallel velocity u, the electrostatic potential Φ for the perpendicular $\mathbf{E}\times\mathbf{B}$ velocity, and a helical flux function ψ for the magnetic field. (In general, variables to describe temperature perturbations are needed, but these are neglected for simplicity). The equations of evolution for these quantities are similar to fluid equations in a slab geometry, but with additional large dissipative terms from the average effects of the toroidicity.

For easy comparison with past results, here we write the equations with the same normalizations for the fluid quantities as in previous reduced MHD fluid equations.[20] The variable τ is the time normalized by v_A/R, where v_A is the Alfvén speed $(B^2/4\pi n_i m_i)^{1/2}$. Spacial coordinates x, y, giving the radial and poloidal directions perpendicular to the magnetic field are normalized to the minor radius r_0 of the resonant surface. The main magnetic field is in the z-direction.The pressure is normalized to $\epsilon B_0^2/8\pi$ with $\epsilon = r_0/R$. All velocities are normalized to ϵv_A. The normalized perpendicular $\mathbf{E}\times\mathbf{B}$ velocity is $\hat{z}\times\nabla\phi$, and the parallel velocity is u. Finite larmor radius terms enter proportional to a parameter λ. This parameter is defined so that the perpendicular diamagnetic velocity $(cT_e/neB_0)(dn/dr)$ divided by ϵv_A is equal to $\lambda(dp/dx)$. We also define the following notation for any quantities Q_1 and Q_2

$$[Q_1, Q_2] \equiv (\partial Q_1/\partial x)(\partial Q_2/\partial y) - (\partial Q_2/\partial x)(\partial Q_1/\partial y). \tag{3}$$

Gradients along the magnetic field are given by

$$\nabla_{\|} Q = \partial Q/\partial z - [\psi, Q]. \tag{4}$$

Also define

$$\nabla_\perp^2 Q \equiv \partial^2 Q/\partial x^2 + \partial^2 Q/\partial^2 y. \tag{5}$$

The stream function for the ions, giving the $\mathbf{E}\times\mathbf{B}$ and ion diamagnetic velocities, is

$$F = \varphi + (T_i/T_e)p \tag{6}$$

and the normalized current is

$$J = \nabla_\perp^2 \psi. \tag{7}$$

For simplicity we give here only the results for an axisymmetric torus. The fluid equations are

$$\partial\nabla_\perp^2 F/\partial\tau + [F, \nabla_\perp^2 F] - \nabla_\parallel J = \partial\left[K_i^d + (\alpha/\lambda)K_e + K_e^d\right]/\partial x + (\partial^3/\partial^3 x)K_i^v \quad (8)$$

$$\partial\psi/\partial\tau + \nabla_\parallel\varphi - \lambda\nabla_\parallel p = K_e \quad (9)$$

$$\begin{aligned}\partial u/\partial\tau + [\varphi, u] + \frac{(1+T_i/T_e)}{2}\nabla_\parallel p - \mu_{\perp c}\nabla_\perp^2 u + \Theta\left[K_i^d + (\alpha/\lambda)K_e + K_e^d\right] \\ +\mu_\parallel^c(1 - k_{i2}^2/2k_{i1})\nabla_\parallel^2 u + \nu_\perp^{nc}\nabla_\perp^2 u \\ -\Theta(\partial^2/\partial x^2)K_i^v + 2(k_{i2}/k_{i1})(q_0\beta_i^{1/2}\lambda)^{-1}\nabla_\parallel K_i^v = 0\end{aligned} \quad (10)$$

$$\partial p/\partial\tau + [\varphi, p] + \beta_e(\nabla_\parallel v + 2\lambda\nabla_\parallel J) = 2\beta\left(\alpha\partial K_e/\partial x\lambda\partial K_e^d/\partial x\right) \quad (11)$$

where

$$K_i^d = -\nu[\partial F/\partial x + \Theta u], \quad (12)$$

$$K_e = \eta\left\{J + \alpha(\partial/\partial x)(1 + T_i/T_e)p - (\alpha/\lambda)[\partial F/\partial x + \Theta u]\right\}, \quad (13)$$

$$K_i^v = (q_0\beta_i\lambda/4)(v_i/R\nu_i)[k_{i1}(q_0/2)\lambda\beta_i^{1/2}(\partial F/\partial x + \Theta u) + k_{i2}\nabla_\parallel u] \quad (14)$$

$$K_e^d = \sqrt{\frac{m_e T_e}{m_i T_i}}\nu\left[\lambda\left(1 + \frac{T_i}{T_e}\right)\frac{dp}{dx} - \frac{dF}{dx} - \theta u\right]. \quad (15)$$

β_e is $8\pi n_e T_e/B^2$, for a low aspect ratio tokamak with nearly circular flux surfaces $\Theta = B_p/B$, $\nu = (\pi q_0\beta_i)^{1/2}$, $\alpha = k_{e1}\epsilon v_e/\nu_e R$, ν_e and ν_i are the electron and ion Coulomb collision frequencies, given by Eq. (100) and (147). The quantities k_{e1}, k_{i2} and k_{i1} are numerical constants of order one which arise from inversions of the collision operators, and are defined by Eqs. (138),(150) and (154). The quantities $\mu_\perp^c, \mu_\perp^{nc}$ are classical and neoclassical viscosities. In these units, $\mu_\perp^c = k'(R\nu_i/v_i)\lambda\beta_i^{3/2}$, $\mu_\perp^{nc} = k_{i3}q_0^2(R\nu_i/v_i)\lambda\beta_i^{3/2}$ and $\mu_\parallel = \beta_i^{1/2}(v_i/R\nu_i)$, where k_{i3} is defined in Eq. (153), and k' depends on Z_{eff} and can be found in Ref. 16.

The neoclassical dissipative terms arising in the plateau regime are on the right with K_i and K_e arising from ions and electrons respectively.

The largest terms are from K_i^d. The other ion terms are higher order corrections. The electron terms are smaller than K_i^d by roughly $\sqrt{m_e/m_i}$, but they nonetheless are a qualitatively significant modification to Ohm's law. The coefficients ν, α, and $(v_i/\nu_i R)$ determine the strength of the dissipation.

Flow rotation damping arises from K_i^d. Note that in Eq. (5), the first term in K_i gives vorticity damping at a rate ν. In Eq. (7), the second term gives a parallel velocity damping at a rate $\Theta^2\nu$. In both equations the damping tends to equilibrate the flow to make $u_{||} = \Theta^{-1}\partial F/\partial x$. This corresponds to the condition found in equilibrium neoclassical theory, where the poloidal flow is zero (recall we have an isothermal model), and where the toroidal flow velocity is larger than the equilibrium perpendicular flow in a cylinder by $1/\Theta$.

The flow damping rate is much larger than the mode frequency, so the right side of Eq. (5) greatly exceeds the inertial term on the left. For moderate β, this is much larger than the frequency of resistive instabilities, which go as fractional powers of η. In fact, in the expansion used in the derivation, the inertial term is so high order that it may be neglected entirely. It is included here to only to make contact with previous results, and to preserve the usual structure in the equations. It is also convenient to include it in nonlinear numerical calculations in order to easily make use of already existing reduced MHD initial value codes.

Note that toroidal momentum is conserved by these damping terms. The following argument holds to requisite order in ϵ. The perpendicular $\mathbf{E} \times \mathbf{B}$ velocity and diamagnetic flow both have small components into the toroidal direction due to radial gradients of Φ and n, and the poloidal component of $\mathbf{B}$. In the present normalization this toroidal velocity is $-\Theta\partial F/\partial x$. The toroidal component of the parallel velocity is u. The toroidal angular momentum density is then proportional to $u - \Theta\partial F/\partial x$. The space integral of this quantity is conserved (assuming homogeneous boundary conditions).

The K_e terms give the bootstrap current and ware pinch. To see this, first suppose that the flows have equilibrated so that the last term in K_e vanishes. Then the additional current driving term in Ohm's law proportional to $\partial p/\partial x$ gives bootstrap effects. Using Ohms law, we can solve for K_e in terms of $E_{||}$, and when substituted into Eq. (8). The resulting term can be interpreted as the divergence of a flow in the radial direction driven by $E_{||}$, which is the Ware pinch. In the plateau regime the coefficient α is proportional to $\epsilon v_e/R\nu_e$.

Note that in equilibrium neoclassical theory the bootstrap current is not only driven by the electron pressure gradient; ion pressure gradients drive an equally large current (for $T_i = T_e$). The latter effect arises via friction with the toroidally varying parallel ion velocity. If the flow has not equilibrated the bootstrap current is modified in the manner given by the last term in K_e.

K_e^d gives neoclassical density diffusion, which is modified from the equilibrium result by nonzero poloidal flow.

Higher order ion corrections give various neoclassical viscous effects. The term $\mu_\perp^{nc}$ is a cross field viscosity of roughly the same size as the classical viscosity $\mu_\perp^c$, and has been obtained previously in equilibrium theory. The terms involving K_i^v give a novel cross field viscosity for poloidal flows $\partial F/\partial x + \Theta u$, which is coupled to a parallel viscosity. The cross field poloidal flow viscosity greatly exceeds the classical value, and is as important as the flow damping terms from K_i^d for flows with radial scale lengths significantly less than a poloidal gyroradius, as arise in many modes of interest.

For equilibrated ion flows, the usual Onsager symmetries hold between the bootstrap and Ware pinch terms. Many new cross terms arise with non-equilibrated flows, from both K_i and K_e. The simplest way to demonstrate the symmetries in this case is to use a recent field theoretic generalization of the concept of Onsager symmetries due to P.J. Morrison. This demonstration is performed in Sec. IV. The full equations (5)-(8) are also shown to satisfy and H theorem.

III Derivation of Nonlinear Fluid Equations

A Initial equations of motion

We describe the plasma with four fluid equations – the electron continuity equation, the parallel equation of motion for electrons and ions to give the parallel velocities and currents, and the vorticity equation, also called the shear-Alfvén law, to describe the evolution of the perpendicular velocity. We also need Ampere's law to obtain the perturbed magnetic field from the current. These are closed by solving the drift kinetic equation for the pressure tensor. This closure will be carried out for low frequency perturbations localized near a resonant surface in a low aspect ratio torus. The aspect ratio expansion also simplifies the non-dissipative parts of the equations as in previous reduced fluid equations. For simplicity we neglect temperature perturbations in both species.

We begin with the exact equation of motion

$$m_s n_s (d/dt)\mathbf{v}_s = q_s n_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}/c) - \nabla \cdot \Pi + F_s \tag{16}$$

where m_s is the species particle mass, n_s the particle density, q_s the charge, $\mathbf{v}_s$ the fluid velocity, $d/dt = \partial/\partial t + \mathbf{v}_s \cdot \nabla, \mathbf{E}$ the electric field, F_s the friction force from Coulomb collisions, and Π the pressure tensor. We consider perturbation frequencies such that the d/dt is much smaller than the gyrofrequency, so that the perpendicular velocity $\mathbf{v}_{s\perp}$ is

$$\mathbf{v}_{s\perp} = (c/B^2)(\mathbf{E} - \nabla \cdot \Pi/n_s q_s) \times \mathbf{B}. \tag{17}$$

This velocity will be used in the electron continuity equation,

$$\partial n/\partial t = -\nabla_{s\perp} - \nabla \cdot n\mathbf{v}_{s\|} = -\nabla \cdot n\mathbf{v}_{s\perp} + \nabla \cdot (j_{\|}/e - n\mathbf{v}_{i\|}). \tag{18}$$

where we have used $en(\mathbf{v}_e - \mathbf{v}_i) = -j$, the plasma current. The parallel component of j will be given by the Ohm's law, found from the parallel electron equation of motion. In this way the parallel ion velocity enters the electron density equation, and gives the often important effects of parallel compressibility on pressure evolution (in our isothermal approximation pressure is equivalent to density). Of course the parallel ion velocity is found from the parallel component of Eq. (12) for ions.

The evolution of v_s will be found from the shear Alfvén law (i.e., vorticity equation). Summing Eq. (12) over species and using quasineutrality

$$\sum m_s n_s (d/dt)\mathbf{v}_s = j \times \mathbf{B}/c + \nabla \cdot \pi. \tag{19}$$

This can be solved for the perpendicular current $j_\perp$, and then using quasineutrality $\nabla \cdot j = \nabla \cdot j_\perp + \nabla \cdot j_\| = 0$ we obtain the exact shear-Alfvén law

$$\nabla \cdot \left[(c/B^2)\mathbf{B} \times \left(\sum m_s n_s (d/dt)\mathbf{v}_s\right)\right] + \nabla \cdot j_\| - \nabla \cdot \left[(c/B^2)\mathbf{B} \times \nabla \cdot \pi\right] = 0 \tag{20}$$

where $j_\|$ is the parallel current. The first term is the usual inertial term familiar from MHD, the second term gives line bending effects, and for scalar pressure the last term gives interchange effects. Rotation damping arises from the non-scalar part π driven by kinetic effects in Eq. (16). We will see that the rotation damping effects can exceed the MHD interchange effects. This is superficially surprising, since the non-scalar part of π

is smaller than the scalar pressure. However for perturbations localized near a rational surface, the resonant part of the divergence of the scalar pressure term vanishes to lowest order, whereas the the non-scalar part does not – so the latter has a relatively large effect on the resonant average current.

B Mathematical preliminaries

For simplicity we initially consider the case of a resonant perturbation with a single dominant resonant helicity. This is easily generalized to multi-helicity.

We first define slowly and rapidly varying coordinates along the magnetic field. We consider perturbations on an equilibrium magnetic field B_e with flux coordinates

$$B_e = \nabla\chi \times \nabla\left[\theta - \zeta/q(\chi)\right]. \tag{21}$$

It is convenient to define a flux variable $r(\chi)$ to approximately correspond to minor radius. For this, note that χ is the toroidal flux over π, which is roughly a typical value for the toroidal field, B_0, times r^2. (We precisely define B_0 momentarily). Thus a suitable definition of r is

$$2r(\chi)dr(\chi)/d\chi \equiv 1/B_0. \tag{22}$$

The resonant perturbation has poloidal and toroidal mode number m and n. We define the resonant angle

$$\alpha = \theta - n\zeta/m. \tag{23}$$

Our angle coordinates will be α and ζ. In terms of these, the parallel gradient with the equilibrium field of any quantity Q is

$$\nabla_{\|}^{0} Q = \left[(m - n/q)\partial Q/\partial\alpha + \partial Q/\partial\zeta\right](\mathbf{B}_e \cdot \nabla\theta)/B. \tag{24}$$

Resonant, slowly varying quantities are functions of α and χ alone. Denote such quantities by $\bar{Q}$. In the single resonant helicity case for resonant quantities $\partial/\partial\alpha$ is $O(1)$. Near the rational surface $m - n/q$ is small, so $\nabla_{\|}^{0}\bar{Q}$ is small. Specifically, if we are radially localized to within Δr of the rational surface, we have

$$\nabla_{\|}^{0}\bar{Q} \approx \left[(B_e \cdot \nabla\theta)/B\right]\Delta r n(dq/dr)/q^2\bar{Q} \equiv \bar{k}_{\|}\bar{Q} \tag{25}$$

where $\bar{k}_{\|}$ is the parallel wavenumber for average quantities.

Rapidly varying quantities $\tilde{Q}$ are functions of ζ, χ and α. For rapidly varying, nonresonant quantities $\partial/\partial\zeta$ is of order one. Thus, $\nabla_{\|}^{0}\tilde{Q}$ is of order $B_0 \cdot \nabla\theta/B$.

Define the Jacobian

$$J^{-1} \equiv \nabla\zeta \cdot B_e = \nabla\zeta \cdot \nabla\chi \times \nabla\theta. \qquad (26)$$

For the two scale analysis define an averaging operator, and the appropriate choice is

$$\begin{aligned} \bar{Q} &\equiv \oint Q d\zeta \Big/ \oint d\zeta \\ \tilde{Q} &\equiv Q - \bar{Q}. \end{aligned} \qquad (27)$$

We also need an operator to annihilate the largest, nonresonant part of $\nabla_{\|}$ proportional to $\partial/\partial\zeta$. First, let us define

$$R(r,\theta,\zeta) \equiv B_e/\mathbf{B}_e \cdot \nabla\zeta. \qquad (28)$$

We note that R is roughly equal to the major radius. The nonresonant term is annihilated by taking the average

$$\langle Q \rangle \equiv \oint QRd\zeta \Big/ \oint Rd\zeta. \qquad (29)$$

Note that exactly on the rational surface $Rd\zeta$ is equal to the differential path length element along B_e. We now define

$$B_0 \equiv \langle B_0 \rangle \qquad (30)$$

$$R_0 \equiv \langle R \rangle . \qquad (31)$$

The resonant magnetic field structure for the localized resonant quantities resembles that in a slab. We make several further definitions to transparently make contact with previous slab results.

First, in typical slab calculations, $B \cdot \nabla Q = B_{0z}\partial Q/\partial z + \mathbf{z}_s \times \nabla\psi_s \cdot \nabla Q$, with constant B_{0z} field in the direction of the unit vector $\mathbf{z}_s$, and ψ_s is the z_s component of the vector potential. In a slab equilibrium ψ_s is $B_{0z}x^2/Ls$, with x the distance from the "rational surface" and L_s the shear length.

We thus define ψ_0 so that

$$B_e \cdot \nabla Q = \frac{B_e}{R}\left[\left(\frac{\partial Q}{\partial\zeta}\right) + \frac{\partial\psi_0}{\partial r}\left(\frac{\partial Q}{\partial\alpha}\right)_{\zeta}\right] \qquad (32)$$

with

$$d\psi_0/dr \equiv [1 - n/q(r)m]\,(rB_0/R_0). \tag{33}$$

Near the rational surface, $\psi_0 \approx B_0(r-r_s)^2/2L_{se}$, with $L_{se} = R_0 q^2\,[r(dq/dr)]^{-1}$.

Now consider the perturbed magnetic field $\delta\mathbf{B}$. This can be described in terms of a vector potential $\mathbf{A}$

$$\delta\mathbf{A} = R_0 \delta\psi \nabla\zeta + r_0 \delta\Psi \nabla\alpha \tag{34}$$

where we have chosen a guage where $\nabla\zeta \times \nabla\alpha \cdot \mathbf{A} = 0$, and where r_0 is $r(\psi)$ evaluated at the resonant surface; thus $r_0\nabla\theta$ is order unity. We will find that the magnetic field from the last term can be neglected, so that the form of $(B_e + \delta B)\cdot\nabla Q$ is directly analogous to the slab case. Finally, it is convenient to define a bilinear operater

$$[Q_1, Q_2] \equiv (m/r_0)\,\{(\partial Q_1/\partial r)(\partial Q_2/\partial\alpha) - (\partial Q_2/\partial\alpha)(\partial Q_1/\partial r)\} \tag{35}$$

which is directly analogous to $\mathbf{z}_s \cdot \nabla Q_1 \times \nabla Q_2 = \partial Q_1/\partial x)(\partial Q_2/\partial y) - (\partial Q_2/\partial x)(\partial Q_1/\partial y)$ in the slab case. We then have

$$\begin{aligned}(B_e + \delta B)\cdot\nabla Q &= (B_e/R)(\partial Q/\partial\zeta) + (R_0 B/RB_0)[\psi_0 + \delta\psi, Q] \\ &+ \text{ higher order terms in } \psi\end{aligned} \tag{36}$$

and

$$\begin{aligned}\left\langle (B_e + \delta B)\cdot\nabla(\bar{Q} + \tilde{Q})/B_e \right\rangle &= \left[\bar{\psi}_0 + \delta\bar{\psi}, \bar{Q}\right] + [\delta\tilde{\psi}, \tilde{Q}] \\ &+ \text{ higher order terms in } \psi.\end{aligned} \tag{37}$$

Also define

$$\tilde{\nabla}^0_{||} Q \equiv (B_e/R)\partial Q/\partial\zeta. \tag{38}$$

To generalize to multihelicity, one would define a parallel coordinate z so that $\left\langle (B_e + \delta B)\cdot\nabla(\bar{Q} + \tilde{Q})/B_e \right\rangle = B_0 \dfrac{\partial}{\partial z}\bar{Q} + \left[\bar{\psi}_0 + \delta\psi, \bar{Q}\right] + \left[\delta\tilde{\psi}, \tilde{Q}\right]$.

C Ordering for physical quantities

There are two small parameters which are specific to the problem. The first is $\Delta r/r$, the ratio of the layer width to minor radius. The smallness of this parameter implies the existence of two parallel space scales, and makes possible the rigorous determination of the pressure tensor and closure of the fluid equations. The second parameter is the smallness of the variation of the magnetic field strength along a field line, and this results from the aspect ratio ϵ expansion. Its smallness greatly simplifies the kinetic equations on the short parallel scale.

To obtain a systematic expansion we must order $\Delta r/r$ with respect to ϵ. We must also order the perturbation amplitudes in ϵ. This is important because nonlinearly the equilibrium is changed by the perturbation, and the possible importance of this needs to be assessed. In addition, we must order other parameters to include physical effects of interest consistently with the two scale analysis. This has more than formal significance. A derivation of the neoclassical dissipative processes requires that the perturbations satisfy a certain consistency in the frequency, radial mode width, collision frequency and plasma β. The importance of other physical effects depends on similar parameters – for example to what extent parallel compressional effects are important, or whether kinetic effects such as semi-collisionality can be important in the resistive layer. The demonstration and implementation of a consistent maximal ordering to include these effects resolves these issues, demonstrates that the neoclassical dissipative effects are potentially important for a wide variety of resistive phenomenon, and provides a rigorous set of equations for their description.

The expansion described presently makes strong use of small ϵ and plateau collisionality. We mention that in the banana collisionality regime, a calculation to rigorously derive fluid equations for the resistive layer without assuming parallel flow equilibration is significantly more difficult. We nonetheless believe that the physical processes and thus results would be similar.

For the magnetic field geometry, we take

$$r/R \approx \nabla B/B \approx \epsilon \tag{39}$$

$$\mathbf{B} \cdot \boldsymbol{\nabla} B/B^2 \approx \epsilon^2 \tag{40}$$

$$\bar{k}_{\|} \equiv [(B_0 \cdot \nabla\theta)/B]\,\Delta r(dq/dr)/q^2 \approx \Delta r/rR. \tag{41}$$

We also adopt a flux coordinate system so that for some constant k

$$\mathbf{B}_e = k\boldsymbol{\nabla}\zeta[1 + O(\epsilon)]. \tag{42}$$

For axisymmetric systems, this is satisfied by taking ζ to be the symmetry coordinate and

$B_p/B \approx \epsilon$. Low aspect ratio stellarator orderings typically take $B_p/B \approx \epsilon^{1/2}$, but in this case equation Eq. (42) can still be satisfied by choosing coordinates of the type discussed by Boozer.

We also wish to include common nonlinear effects of interest. As is typical in nonlinear orderings, we suppose that the nonlinearity modifies the equilibrium gradient dp_0/dr by order one, so

$$dp/dr \approx dp_0/dr \approx p_0/r. \tag{43}$$

To include the fact that the equilibrium parallel magnetic field gradient is nonlinearly modified,

$$\delta\bar{B}_r(d/dr) \approx B_0 \cdot \nabla \approx B_0 \Delta r/rR, \tag{44}$$

so that islands with a width of Δr can be considered. With $\delta\mathbf{B} \approx \nabla\delta\bar{\psi} \times R^0\nabla\zeta$, this implies

$$\delta\bar{\psi}/\Delta r \approx B_0 \Delta r/rR. \tag{45}$$

To keep the convective derivative nonlinearity, we wish to have that the nonlinear convective frequency is of order the mode frequency,

$$c(d\bar{\phi}/dr)/B \approx \bar{v}_\theta/r \approx \omega. \tag{46}$$

We note that with $\omega \approx \omega_*$ this is equivalent to

$$(d/dr)e\bar{\phi}/T \approx 1/r. \tag{47}$$

The orderings of physical parameters we wish to satisfy are as follows. To include diamagnetic effects we take

$$\omega \approx \omega_* \approx v_i \rho_i/r^2, \tag{48}$$

with ω_* the diamagnetic frequency and ρ_i the ion gyroradius. For typical parameters, and low mode numbers ω_* is of order the *parallel* flow equilibration rate $\nu_{||}$, and for moderate m modes ω_* easily exceeds $\nu_{||}$. Thus for maximum generality we adopt an ordering which does not predjudice the issue of whether the parallel flows have equilibrated or not

$$\omega \approx \nu_{||} \approx \epsilon^2 v_i/R. \tag{49}$$

Since the frequency is of order the equilibration rate, we expect the parallel flow to be of order the equilibrium value,

$$v_{\|i} \approx cE_r/B_p \approx c\bar{\Phi}/\epsilon\Delta r B. \tag{50}$$

The *poloidal* rotation damping rate greatly exceeds the mode frequency for low and moderate m modes. Thus the line bending term in Eq. (16) is not balanced by the inertial term, but by the kinetically driven part Π in the last term. For a strongly magnetized plasma, the tensor Π has the usual form

$$\Pi_{ij} = p_\perp \delta_{ij} + (p_\| - p_\perp)\tilde{b}_i \tilde{b}_j \tag{51}$$

where $\tilde{b} = \mathbf{B}/B$. To estimate the kinetic part of Π we use the drift kinetic equation for the guiding center distribution function $f(\mathbf{x}, \mathcal{E}, \mu)$

$$\partial f/\partial t + (\mathbf{v}_E + \mathbf{v}_d + v_\|\tilde{n}) \cdot \boldsymbol{\nabla} f + q_s(\mathbf{v}_d + \mathbf{v}_\|) \cdot \mathbf{E}\partial f/\partial \mathcal{E} + C(f, f) = 0 \tag{52}$$

where $\mathbf{v}_E = \mathbf{E} \times \mathbf{B}/B^2$, $\mathbf{v}_d$ is the curvature plus ∇B drift, $\mathbf{v}_d = (m_s c/q_s B)\tilde{n} \times (\mathbf{v}_\perp^2 \boldsymbol{\nabla} B/2B + \mathbf{v}_\|^2 \tilde{n} \cdot \nabla \tilde{n})$, and C is the bilinear collision operator. The discussion in Sec. 2 shows that rotation damping arises from the energy variation caused by E_r, and so is driven by the term $\mathbf{v}_d \cdot \mathbf{E}$. This driving term is balanced by $v_\| \nabla_\| f$, which allows us to estimate the size of the perturbed $\tilde{f}$, using $\nabla_\| \approx \epsilon/r$. This $\tilde{f}$ is peaked near $v_\| \approx 0$, so it produces a non-scalar perturbed pressure with $p_\perp > p_\|$. The kinetic perpendicular pressure $p_\perp^k$ has size

$$p_\perp^k \approx n_i T_i (q_i v_d \cdot E/T_i)/(v_i \epsilon/r). \tag{53}$$

The resulting contribution from the last term in Eq. (16) to leads us to balance the line bending term $\boldsymbol{\nabla} \cdot \bar{j}_\|$ thus

$$\Delta r \bar{j}_\| / rR \approx (c/BR) p_\perp^k / \Delta r \tag{54}$$

To describe resistive layers, we take

$$\bar{E}_\| \approx \omega \delta \bar{\psi}/c \approx \nabla_\| \bar{\phi} \approx \eta j \approx (4\pi/c)\eta \delta \bar{\psi}/\Delta r^2 \tag{55}$$

with

$$\eta = \nu_e m_e / n e^2 \tag{56}$$

the resistivity. Consistently with the plateau regime, in Eq. (49) we must take ν_e to satisfy

$$v_e/R < \nu_e < \epsilon^{-3/2} v_e/R, \tag{57}$$

and a similar relation for the ions.

Previous analysis of resistive layers for typical tokamak parameters have found that they are often in a certain collisionality regime where electron kinetic effects arise, termed semi-collisional effects. To include the possibility for such effects we order

$$\omega \nu_e \approx \bar{k}_{||}^2 v_e^2. \tag{58}$$

To include full parallel compressibility effects we take

$$\omega \approx k_{||} v_{||i}, \tag{59}$$

We also assume $T_e \approx v_i$, so

$$v_e \approx \sqrt{(m_i/m_e)} v_i. \tag{60}$$

Remarkably, all of the physical orderings given by Eqs. (39)-(60) are satisfied by a simple single parameter ordering of all quantities in terms of ϵ. We thus have the following maximal ordering to consistently derive equations capable of describing a wide variety of resistive phenomenon in toroidal plasmas. Most of the parameters in this ordering also roughly correspond to actual values in many current experimental regimes of interest

$$\begin{aligned}
&\Delta r/r \approx \epsilon^2 \\
&\rho_i/r \approx \epsilon^3 \\
&\omega \approx \epsilon^2 v_i/R \\
&\beta \approx \epsilon^2 \\
&\nabla B_0/B_0 \approx \epsilon/r \\
&\nu_e R/v_e \approx \epsilon \text{ and } \nu_i R/v_i \approx \epsilon \\
&\sqrt{(m_e/m_i)} \approx \epsilon \\
&\bar{v}_\theta \approx c\bar{\phi}/\Delta r B \approx v_i \epsilon^3 \\
&v_{||i} \approx v_i \epsilon^2 \\
&\delta\bar{\psi} \approx B_0 r \epsilon^5
\end{aligned} \tag{61}$$

We now apply these orderings to the fundamental equations.

D Applications of orderings to fluid and field equations

We now define appropriate averages of the fluid equations. For the shear-Alfvén law and continuity equation, we take $\langle(1/B)\cdots\rangle$,

$$\Big\langle(1/B)\nabla\cdot\Big[(c/B^2)\mathbf{B}\times((\Sigma m_s n_s(d/dt)\mathbf{u}_s\Big]\Big\rangle+\Big\langle\nabla_{||}(j_{||}/B)\Big\rangle$$
$$=\Sigma\Big\langle(1/B)\nabla\cdot\Big[(c/B^2)B\times\nabla\cdot\Pi_s\Big]\Big\rangle \tag{62}$$

$$\partial\langle n/B\rangle/\partial t=\Big\langle(\nabla_{||}[(j_{||}-nu_{||i})/B]\Big\rangle-\langle\nabla\cdot nu_{\perp}\rangle\,, \tag{63}$$

where we have used $\nabla\cdot j_{||}=B\cdot\nabla(j_{||}/B)$, and similarly for $v_{||}$. Consider Eq. (55). The first term on the left, the inertial term, is relatively small by $O(\epsilon^2)$. In addition, it can be shown that the off diagonal parts of π are relatively small by ϵ^2 so

$$\Big\langle\nabla_{||}(j_{||}/B)\Big\rangle=\sigma\Big\langle c\mathbf{B}\times\nabla(p_{s||}+p_{s\perp})\cdot\boldsymbol{\nabla}(1/B^2)/B\Big\rangle[1+O(\epsilon^2)] \tag{64}$$

where we have also used the fact that when $\beta\approx\epsilon^2$,

$$(\mathbf{b}\cdot\boldsymbol{\nabla}\mathbf{b})_{\perp}=2(\boldsymbol{\nabla}_{\perp}B)/B[1+O(\epsilon^2)]. \tag{65}$$

Equations (46) and (40) together imply that

$$dp_{\perp}^{k}/dr\approx(p_0/r)(\rho_i/\Delta r)\approx\epsilon p_0/r. \tag{66}$$

This would seem to imply that the scalar pressure interchange term dominates the rotation damping term in Eq. (57) by order ϵ. MHD interchange effects have been considered in related contexts by Kotschenreuther et al., and by Cary and Kotschenreuther, and in fact the average of this term vanishes to lowest order. We briefly consider the magnitude of this term in Appendix A, and find that it is smaller than its apparent size by at least $\Delta r/r\approx\epsilon^2$. Thus, the rotation damping term dominates.

The parallel dynamics can be found by taking the parallel velocity moment of the drift kinetic equation,

$$n_i m_i(\partial/\partial t+v_E\cdot\nabla)u_{||i}=-\nabla_{||}p_{||}+q_i n_i E_{||}+R_{ie}+(\mathbf{B}\cdot\nabla B/B^2)(p_{||}-p_{\perp})+w$$
$$w=-2(\mathbf{v}_E\cdot\nabla B/B)n_i m_i v_{||}+m_i\int d\mathbf{v}v_{||}\mathbf{v}_d\cdot\nabla f \tag{67}$$

where R_{ie} is the collisional force on the ions. The convenient average for this equation is $\langle (B_0^2/B^2)\cdots\rangle$,

$$m_i\left[\partial\left\langle n_i u_{\|i} B_0^2/B^2\right\rangle/\partial t+\left\langle n_i v_E\cdot\nabla u_{\|i} B_0^2/B^2\right\rangle\right][1+O(\epsilon^2)]$$
$$\left\langle\nabla_{\|}\bar{p}_{\|} B_0^2/B^2\right\rangle+q_i n_i\left\langle E_{\|} B_0^2/B^2\right\rangle+\left\langle R_{ie} B_0^2/B^2\right\rangle+\left\langle\omega B_0^2/B^2\right\rangle$$
$$+\left\langle B_0^2(\tilde{\nabla}_{\|} B/B^3)(\tilde{p}_\perp-\tilde{p}_{\|})\right\rangle=0. \tag{68}$$

A similar equation can be written for the electrons; however, the inertial term and the w term are relatively smaller by ϵ^2 because of the small electron mass. Adding the electron result to Eq. (61) cancels the electric field and friction terms, so that

$$m_i\left[\partial\left\langle n_i u_{\|i} B_0^2/B^2\right\rangle/\partial t+\left\langle n_i v_E\cdot\nabla u_{\|i} B_0^2/B^2\right\rangle\right][1+O(\epsilon^2)]$$
$$=\left\langle\nabla_{\|}\bar{p}_{\|i}+\bar{p}_{\|e} B_0^2/B^2\right\rangle+\langle w\rangle+\Sigma\left\langle B_0^2(\tilde{\nabla}_{\|} B/B^3)(\tilde{p}_{\perp s}+\tilde{p}_{\|s})\right\rangle. \tag{69}$$

The last term drives the parallel flow toward the equilibrium neoclassical value.

We next simplify the form of the perturbed magnetic and electric field using the size of the currents and Ampere's Law. With Eq. (29), Ampere's Law becomes

$$4\pi j/c=\left[\nabla r\times R_0\nabla\zeta)d^2\psi/dr^2+(\nabla r\times r_0\nabla\theta)d^2\Psi/d^2r\right]\times[1+O(\epsilon^2)]. \tag{70}$$

The perpendicular and parallel currents can be found from Eqs. (15), (16) and (57)-(59). In particular, we find

$$\tilde{j}_{\|}\approx\epsilon\bar{j}_{\|}\approx\epsilon^2 cB/r. \tag{71}$$

With Eqs. (63)-(64) we can estimate the sizes of the $\bar{\psi}$, $\tilde{\psi}$, $\bar{\Psi}$, and $\tilde{\Psi}$. We find

$$\delta\mathbf{B}\cdot\nabla\equiv\delta\bar{\mathbf{B}}\cdot\nabla+\delta\tilde{\mathbf{B}}\cdot\nabla \tag{72}$$

with

$$\delta\bar{\mathbf{B}}\equiv R_0\nabla\zeta\times\nabla\bar{\psi}[1+O(\epsilon^2)],\quad \delta\tilde{B}\equiv R_0\nabla\zeta\times\nabla\tilde{\psi}[1+O(\epsilon^2)] \tag{73}$$

and to lowest order

$$\delta\tilde{\mathbf{B}}\approx\epsilon\delta\bar{\mathbf{B}}. \tag{74}$$

We now use Eq. (18) and Eqs. (65)-(67) to simplify the averaged fluid equations. Also note that $B/B_e=1+O(\epsilon^4)$. We make use of some results of Sec. IV.E, where it is shown that $\tilde{n}\approx\epsilon\bar{n}$ and $\tilde{u}_{\|i}\approx\epsilon\bar{u}_{\|i}$. Thus

$$\partial\bar{n}/\partial t[1+O(\epsilon^2)]=\left[\bar{\psi}_0+\delta\bar{\psi},\bar{j}_{\|}/e-n_0\bar{u}_{\|i}\right]/B_0[1+O(\epsilon^2)]+B_0\langle\nabla\cdot nu_\perp/B\rangle \tag{75}$$

$$\left[\bar{\psi}_0 + \delta\bar{\psi}, \bar{j}_{||}\right]/B_0[1+O(\epsilon^2)] = B_0\Sigma\left\langle c\mathbf{B}\times\nabla(p_{s||}+p_{s\perp})\cdot\nabla(1/B^2)/B\right\rangle[1+O(\epsilon^2)] \quad (76)$$

$$m_i n_i \quad \left\langle(\partial/\partial t + v_E\cdot\bar{\nabla})\bar{u}_{||i}\right\rangle[1+O(\epsilon^2)] = [\bar{\psi}_0 + \delta\bar{\psi}, (\bar{p}_{||i}+\bar{p}_{||e})]/B_0$$
$$\times \quad [1+O(\epsilon^2)] + \langle w\rangle + \Sigma\left\langle B_0(\nabla^0_{||}B/B^3)(\tilde{p}_{\perp s}+\tilde{p}_{||s}\right\rangle. \quad (77)$$

The size of the the electromagnetic part of the perpendicular electric field can be estimated using Faraday's law, using estimates for ψ and Ψ. The result shows that $E_\perp$ is electrostatic to lowest order

$$E_\perp = \nabla_\perp\Phi[1+O(\epsilon^2)]. \quad (78)$$

With this, we simplify the $v\cdot\nabla$ terms in the fluid equations. From Eq. (13), (34), (58) and (71), we find

$$\nabla\cdot nu_\perp = c\left\{[B\times\nabla B/B^3]\cdot[\nabla(p_{e\perp})/q_i + n\nabla\Phi]\right\} + \mathbf{v}_E\cdot\nabla n. \quad (79)$$

With this, we find that Eq. (68) becomes

$$\begin{aligned}\partial\bar{n}/\partial t[1+O(\epsilon^2)] &= \left[\bar{\psi}_0+\delta\bar{\psi},\bar{j}_{||}/e - n_0\bar{u}_{||i}\right]/B_0[1+O(\epsilon^2)] \\ &+ c[\bar{\phi},\bar{n}]/B_0[1+O(\epsilon^2)] + cB_0\Big\langle[B\times\nabla B/B^4] \\ &\cdot \nabla r\partial/\partial r\left[(\tilde{p}_{e||}+\tilde{p}_{e\perp})/q_i + n_0\tilde{\bar{\Phi}}\right]\Big\rangle[1+O(\epsilon^2)]. \quad (80)\end{aligned}$$

Similarly, we find that

$$m_i n_{i0}(\partial\bar{u}_{||i}/\partial t + [\bar{\Phi},\bar{u}_{||i}]/B_0)[1+O[\epsilon^2)] = \left[\bar{\psi}_0+\delta\bar{\psi},(\bar{p}_{||i}+\bar{p}_{||e})\right]$$
$$\times[1+O(\epsilon^2)] + < w > + \sigma\left\langle B_0^2(\nabla^0_{||}B/B^3)(\tilde{p}_{\perp s}+\tilde{p}_{||s})\right\rangle. \quad (81)$$

Note that with the ordering in the previous section, *all explicit terms in Eq. (69), (73) and (74) are of the same order.*

We now turn to a kinetic derivation of $\tilde{p}_{s\perp}+\tilde{p}_{s||}$ in terms of the average fluid variables in order to close these fluid equations.

E Application of orderings to the drift kinetic equation

The drift kinetic distribution function $f(x, v_\perp, v_\|)$ is split into three parts, $f = f_M + \bar{f} + \tilde{f}$. The distribution f_M is the background distribution, and taken to be constant in space and time. It is a Maxwellian with the local density and temperature at the rational surface. All spatial and time dependence is contained in $\bar{f}$ and $\tilde{f}$. They include both the the equilibrium plasma gradients and the nonlinearly modified gradients. For example, in an *equilibrium* with local density n_0 and gradient dn/dr_0 at the rational surface radius r_s, $f_M = n_0 \exp(-m_s v^2/T_s)$, and $\bar{f} = (r - r_s)(dn/dr_0)\exp(-m_s v^2/T_s)$. As is typical in nonlinear orderings, we assume the nonlinear perturbation alters the radial gradient by order one. Thus the average distribution $\bar{f}$ has a radial gradient scale length of order the minor radius r. We consider perturbations localized to within Δr of the resonant rational surface. Given the radial gradient scale of $\bar{f}$, we have

$$\bar{f} \approx (\Delta r/r) f_M \approx \epsilon^2 f_M. \tag{82}$$

Equation (46) and the arguments in the paragraph preceding it imply

$$\tilde{f} \approx \epsilon^3 f_M. \tag{83}$$

In the plateau regime it is convenient to write the drift kinetic equation in the the variables $v_\|$ and $v_\perp$ instead of μ and ϵ; we then have

$$\begin{aligned} \partial f/\partial t + v_\| \nabla_\| f + C(f,f) \;&+\; (\mathbf{v}_d + \mathbf{v}_E)\cdot\nabla f - (q_s/m)E_\| \partial f/\partial v_\| \\ &+\; (c/2B^3)\mathbf{B}\times\nabla\Phi\cdot\nabla B(v_\perp \partial/\partial v_\perp + 2v_\| \partial/\partial v_\|)f \\ &+\; v_\perp^2 v_\| (\nabla_\| B/B)(\partial/\partial v_\perp^2 - \partial/\partial v_\|^2)f = 0 \end{aligned} \tag{84}$$

where for simplicity we have used Eq. (58) to neglect terms which are higher order. The sixth term gives the heating effect of $v_d \cdot E$. Recall from Sec. II that this leads to rotation damping. The last term gives the "mirror force" which arises since for single particle motion $v_\perp$ and $v_\|$ change as the magnitude of B changes.

F Lowest order kinetic ion results

Before proceeding with the expansion it is convenient to regroup some terms in Eq. (77). We multiply the last two terms by B/B and $(B/B)^2$, taking $B/B = \bar{B}/B + \tilde{B}/B$, so that

$$\partial f/\partial t \;+\; v_\| \nabla_\| f + C(f,f) + (\mathbf{v}_d + \mathbf{v}_E)\cdot\nabla f - (q_s/m)E_\| \partial f/\partial v_\|$$

$$+ \quad (c/2B^3)(\bar{B}/B + \tilde{B}/B)\mathbf{B} \times \nabla\Phi \cdot \nabla B(v_\perp \partial/\partial v_\perp + 2v_\| \partial/\partial v_\|)f$$

$$+ \quad v_\perp^2 v_\| (\nabla_\| B/B)(\bar{B}/B + \tilde{B}/B)^2(\partial/\partial v_\perp^2 - \partial/\partial v_\|^2)f = 0. \qquad (85)$$

Recall that

$$\tilde{B}/B \approx \epsilon$$

$$\tilde{n} \cdot \nabla B/B \approx \epsilon^2$$

and for ions

$$v_{di} d/dr \approx \epsilon^3 v_i/r. \qquad (86)$$

We now expand Eq. (77) in powers of ϵ.

We define the size of f_M as order zero, and define frequencies $\approx v_i/r$ of order zero. Equations of order ϵ^0 through ϵ^3 are satisfied by taking f_M to be a Maxwellian independent of space and time. In next order the perturbed collision operator arises,

$$c(f) \equiv C(f_M, f) + C(f, f_M). \qquad (87)$$

Distribution functions $\tilde{f}$ will occur which are sharply peaked in velocity space, with width $\Delta v_\| \approx v_\|^c \approx v_i \epsilon^{-1/3}$. The operator c has a diffusive character, it is larger than its nominal ordering in this region by $(v_i/\Delta v)^2 \approx \epsilon^{-2/3}$ i.e., $c \approx \epsilon^{1/3} v_i/R$, and also $v_\|$ is smaller, $v_\| \approx \epsilon^{1/3} v_i$. We could formally perform a boundary layer analysis in $v_\|$, dividing velocity space into separate regions. However, for simplicity of presentation we only write one equation to describe all of velocity space. At a given order in ϵ this leads to the inclusion of c in equations for which it seems one higher order in ϵ, and the reader needs to recall that c is larger than its nominal order in the boundary layer.

With these points in mind, at order ϵ^4 we obtain the important equation for the ions

$$v_{\|}\tilde{\nabla}^0_{\|}\tilde{f}_{i3} + c(\tilde{f}_{i3} + c(\langle f_i\rangle_2) + (q_i/T)\nabla_{\|}\tilde{\bar{\Phi}}_3 v_{\|} f_M$$
$$-(cT\bar{B}/q_iB^4)\mathbf{B}\times\nabla r\cdot\mathbf{\nabla}B(\partial/\partial r)(q_i\bar{\Phi}_2 f_M/T_i + \langle f_i\rangle_2)[(v_\perp^2+2v_{\|}^2)/v_i^2]$$
$$+v_{\|}v_\perp^2\left(\bar{B}^2\nabla_{\|}B/B^3\right)\left(\partial/\partial v_\perp^2 - \partial/\partial v_{\|}^2\right)\langle f_i\rangle_2 = 0. \tag{88}$$

Taking the average of this, we obtain

$$c(\langle f_i\rangle) = 0. \tag{89}$$

This implies

$$\langle f_i\rangle_2 = n(r,\alpha)/n_0 f_M + 2u_{\|}(r,\alpha)(v_{\|}/v_i^2)f_M \tag{90}$$

where we have neglected temperature perturbations. A result such as Eq. (83) is necessary so that *closed* fluid equations can be *rigorously* derived.

It is convenient to consider the part of $\tilde{f}_{i3}$ driven by $\partial u_{\|}/\partial r$ separately from the rest, so define

$$\tilde{f}_{i3} = f_{i3}^r + f_{i3}^v - (q_i\tilde{\Phi}/T_i)f_M \tag{91}$$

where

$$v_{\|}\tilde{\nabla}^0_{\|}f_{i3}^r + c(f_{i3}^r) = (cT/q_i\bar{B})[\bar{B}^2(\mathbf{B}\times\mathbf{\nabla}r\cdot\mathbf{\nabla}B)/B^4]$$
$$\times(\partial/\partial r)[q^i\Phi/T_i + n/n_0][(v_\perp^2+2v_{\|}^2)/v_i^2]f_M - u_{\|}(\bar{B}^2\mathbf{\nabla}_{\|}B/B^3)(v_\perp^2/v_i^2)f_M \tag{92}$$

$$v_{\|}\tilde{\nabla}^0_{\|}f_{i3}^v + c(f_{i3}^v) = 2(cT/q_i\bar{B})([\bar{B}^2(\mathbf{B}\times\mathbf{\nabla}r\cdot\mathbf{\nabla}B)/B^4)$$
$$[(\partial/\partial r)u_{\|}](v_{\|}/v_i^2)[(v_\perp^2+2v_{\|}^2)/v_i^2]f_M. \tag{93}$$

Furthermore, it is convenient to define

$$f_{i3}^r \equiv h^r + g^r, \tag{94}$$

with

$$g^r \equiv -u_{\|}(v_{\|}/v_i^2)\bar{B}^2(\tilde{1}/B^2) \tag{95}$$

and

$$v_{\|}\tilde{\nabla}^0_{\|}h^r + c(h^r) = \Big\{(cT/q_i\bar{B})[\bar{B}^2(\mathbf{B}\times\mathbf{\nabla}r\cdot\mathbf{\nabla}B)/B^4]$$
$$\times(\partial/\partial r)[q_i\Phi/T_i + n/n_0] - u_{\|}(\bar{B}^2\nabla_{\|}B/B^3)\Big\}\times[(v_\perp^2+2v_{\|}^2)/v_i^2]f_M. \tag{96}$$

Now let us consider the electrons.

G Lowest order electron kinetic results

The kinetic development for the electrons is similar to that for the ions. The operator $v_{||}\nabla_{||}$ is relatively larger for electrons than for ions by $(m_i/m_e)^{1/2}$. Thus the part of $\tilde{f}_e$ that is driven by electron toroidicity is smaller than $\tilde{f}_i$ by order ϵ. Electron neoclassical effects arising from $\tilde{p}_e$, such as bootstrap current production, enter by keeping terms of one additional higher order in ϵ than is necessary to obtain the lowest order ion neoclassical effects. In this section we derive only the lowest order effects.

We again take $f_e = f_{eM} + \bar{f}_e + \tilde{f}_e$, with f_{eM} a Maxwellian independent of space and time, with the equilibrium density and temperature at the rational surface. We define f_{eM} to be of order one, $\bar{f}$ to be $\approx \epsilon^2$, the lowest order $\tilde{f}$ to be $\approx \epsilon^3$, and frequencies $\approx v_i/r$ to be of order one. The collision operator $C_{ee}(f_e, f_e) + C_{ei}(f_e, f_i)$ is of order $\epsilon v_e/R \approx \epsilon v_i/r \approx \epsilon$.

Orders ϵ^0, ϵ^1, and ϵ^2 are trivial. The collision operator can, to requisite accuracy, be taken as

$$C_{ee}(F_{eM} + \delta f_e, f_{eM} + \delta f_e) + C_{ei}(f_{eM} + \delta f_e, f_{iM}) = c_e(\delta f_e) + \mathbf{R}^k_{ei}, \tag{97}$$

where

$$c_e(f) \equiv C_{ee}(f, f_{eM}) + C_{ee}(f_{eM}, f) + L(f), \tag{98}$$

with

$$L(f) = \nu_{ei}(v)[(\partial/\partial\sigma)(1-\sigma^2)(\partial/\partial\sigma)f]/2, \tag{99}$$

where $\sigma = v_{||}/v$, and $\nu(v)$ is the electron ion collision frequency, and

$$\mathbf{R}^k_{ei} = 2\nu(v)v_{||}u_{||i}f_M/v_e^2. \tag{100}$$

At order ϵ^3 we obtain

$$\begin{aligned} c_e(\langle f\rangle_2) \;&+\; v_{||}\widetilde{\nabla}^0_{||}\tilde{f}_3 + c_e(\tilde{f}_3) - (e/T_e)\widetilde{\nabla}^0_{||}\tilde{\Phi}_3 f_{eM} \\ &+\; v_{||}v_\perp^2(\bar{B}^2\nabla_{||}B/B^3)(\partial/\partial v_\perp^2 - \partial/\partial v_{||}^2)\langle f_2\rangle = 0. \end{aligned} \tag{101}$$

The average of Eq. (94) implies

$$\langle f\rangle_2 = n(r,\alpha)f_M \tag{102}$$

where again we have neglected temperature perturbations. Note that unlike the ion case, $v_{||}f_M$ is not a solution of $c_e = 0$. Since $\langle f\rangle_2$ is a function of v^2, the last term in Eq. (94) vanishes, and we have

$$\tilde{f}_3 = -(e\tilde{\Phi}/T)f_{eM} \tag{103}$$

At order ϵ^2, we have

$$
\begin{aligned}
&c_e(\langle f\rangle_3) + (e/T_e)\left\langle E_{\|}\right\rangle f_{eM} + v\|\nabla_{\|}\langle f\rangle_2 + 2\nu(v)v_{\|}\left\langle u_{\|i}\right\rangle f_M/v_e^2 \\
&+v_{\|}v_{\perp}(\bar{B}^2\nabla_{\|}B/B^3)(\partial/\partial v_{\perp}^2 - \partial/\partial v_{\|}^2)\tilde{f}_3 \\
&= -v_{\|}\widetilde{\nabla}_{\|}^0\tilde{f}_4 - c_e(\tilde{f}_4) + (e/T_e)\widetilde{\nabla}_{\|}^0\tilde{\Phi}_4 f_{eM} - 2\nu(v)v_{\|}\tilde{u}_{\|i}f_M/v_e^2 \\
&-v_{\|}v_{\perp}^2(\bar{B}^2\nabla_{\|}B/B^3)(\partial/\partial v_{\perp}^2 - \partial/\partial v_{\|}^2)\langle f\rangle_3 \\
&+(cT_e\bar{B}/eB^4)(B\times\nabla r\cdot\nabla B)(\partial/\partial r)(-e\langle\Phi\rangle_2/T_e + \langle n\rangle_2)f^M \\
&\times\left[(\partial v_{\|}^2 + v_{\perp}^2)/v_e^2\right].
\end{aligned} \tag{104}
$$

Since $\tilde{f}_3$ is a function of v^2, the last term on the left vanishes. The terms from friction with the ions can be eliminated by defining

$$
\begin{aligned}
\langle f\rangle_3 &= \langle f\rangle_3' + 2v_{\|}\left\langle u_{i\|}\right\rangle f_M/v_e^2 \\
\tilde{f}_4 &= \tilde{f}_4' + 2v_{\|}\tilde{u}_{i\|}f_M/v_e^2.
\end{aligned} \tag{105}
$$

When operated on by the collision operator the second terms in the expressions above cancel the ion friction terms. The $c(\tilde{f}_4)$ term has been included at this order for reasons mentioned previously, and to make the cancellation above readily apparent we have also included the $\tilde{\mathbf{R}}_{ie}^k$ term at this order.

Define the Spitzer function $F_s(v_{\|}, v_{\perp})$ as the solution to

$$
c_e(F_s) = v_{\|}f_M, \tag{106}
$$

and define the collision frequency by

$$
\nu_e^{-1} \equiv (2/v_e^2)\int dv v_{\|}F_s. \tag{107}
$$

Upon averaging Eq. (97), we find

$$
\langle f\rangle_3' = (-eE_{\|}/T_e - \nabla_{\|}\langle n\rangle)F_s \equiv \mathbf{F}_{\|}F_s, \tag{108}
$$

so that the lowest order current is found from the usual Ohm's Law

$$
E_{\|} + T_e\bar{\nabla}\langle n\rangle/e = \eta\langle j\rangle. \tag{109}
$$

The nonresonant part of Eq. (97) gives

$$v_{\|}\widetilde{\nabla}_{\|}^{0}\tilde{f}_4 + c(\tilde{f}_4) = (e/T_e)v_{\|}\widetilde{\nabla}_{\|}\tilde{\Phi}_4 f_{eM} + \Big[v_{\perp}^2 u_{i\|}(\bar{B}^2\widetilde{\nabla}_{\|}^{0}B/B^3)$$
$$+2v_{\|}^2(\widetilde{\nabla}_{\|}^{0}u_{i\|})\Big] f_M/v_e^2 + v_{\|}v_{\perp}^2(\bar{B}^2\nabla_{\|}B/B^3)(\partial/\partial v_{\perp}^2 - \partial/\partial v_{\|}^2)\left\langle f\right\rangle_3'$$
$$-(cT_e\bar{B}/eB^4)(B\times\nabla r\cdot\nabla B)(\partial/\partial r)(-e\left\langle\Phi\right\rangle_2/T_e + \left\langle n\right\rangle_2)f_M$$
$$\times\left[(v_{\|}^2+v_{\perp}^2)/v_i^2\right]. \tag{110}$$

H Lowest order evaluation of fluid equations

We begin by noting that $\bar{n}$, $\bar{j}$, $\bar{p}$, and $\bar{u}_{i\|}$ can be replaced by their average values $\langle n\rangle$, $\langle j\rangle$, $\langle p\rangle$, $\langle\Phi\rangle$ and $\left\langle u_{i\|}\right\rangle$ respectively with a relative error of $O(\epsilon^2)$, because $\tilde{n}$, $\tilde{j}$, $\tilde{p}$, $\tilde{\Phi}$ and $\tilde{u}_{i\|}$ are relatively small by $O(\epsilon)$. For example,

$$\bar{n} = \langle n/R\rangle / \langle 1/R\rangle = \left\langle(\langle n\rangle+\tilde{n})/(\langle R\rangle+\tilde{R})\right\rangle / \left\langle 1/(\langle R\rangle+\tilde{R})\right\rangle$$
$$= \left\langle(\langle n\rangle+\tilde{n})\left[1-\tilde{R}/\langle R\rangle+(\tilde{R}/\langle R\rangle)^2+\cdots\right]\right\rangle / \left\langle\left[1-\tilde{R}/\langle R\rangle+\cdots\right]\right\rangle$$
$$= \langle n\rangle - \left\langle\tilde{n}\tilde{R}\right\rangle/\langle R\rangle + O\left[\tilde{n}(\tilde{R}/\langle R\rangle)^2\right] = \langle n\rangle\left[1+O(\epsilon^2)+\text{ higher order terms}\right] \tag{111}$$

and similarly for the other quantities.

Now let us consider the continuity equation, Eq. (x32). We need $\tilde{f}_{e3}$ and $\tilde{\Phi}_3$ to evaluate the last term to the same order as the other terms in Eq. (x32). However from Eq. (96) we see that these contributions cancel in Eq. (x32), so we have to lowest order

$$= \left[\bar{\psi}_0+\delta\bar{\psi}, \left\langle j_{\|}\right\rangle/e - n_0\left\langle u_{i\|}\right\rangle\right] + \left[\langle\Phi\rangle, \langle n\rangle\right]\partial\langle n\rangle/\partial t\left[1+O(\epsilon)\right]. \tag{112}$$

Now let us consider the $\langle w\rangle$ term in Eq. (z32). Note that the first term in w is higher order by ϵ. To evaluate the second term, we use

$$\left\langle(\bar{B}/B)^2\int dv v_{\|}v_d\cdot\nabla f\right\rangle$$
$$= (cT/q_i\bar{B})\times\left\langle(\bar{B}/B)^2\int dv v_{\|}\left[(2v_{\|}^2+v_{\|}^2)/v_i^2\right](\bar{B}B\times\nabla B\cdot\nabla r/B^3)\partial\left(\langle f\rangle+\tilde{f}\right)/\partial r\right\rangle$$
$$= (cT/q_i\bar{B})\left\langle(\bar{B}/\bar{B})\int dv v_{\|}\left[(2v_{\|}^2+v_{\|}^2)/v_i^2\right](\bar{B}^2B\times\nabla B\cdot\nabla r/B^4)\partial\langle f\rangle/\partial r\right\rangle$$
$$\times[1+O(\epsilon)] + (cT/q_i\bar{B})$$
$$\times\left\langle\left(\int dv v_{\|}(\bar{B}^2B\times\nabla B\cdot\nabla r/B^4\right)\left[\left(2v_{\|}^2+v_{\perp}^2\right)/v_i^2\right]\partial\tilde{f}/\partial r\right\rangle[1+O(\epsilon)]. \tag{113}$$

These terms will be cancelled by the contributions from f_{i3}^{v} to $\left\langle \left(B_0^2\nabla_{\|}B/B^3\right)\tilde{p}_{\|}\right\rangle$ and $\left\langle \left(B_0^2\nabla_{\|}B/B^3\right)\tilde{p}_{\perp}\right\rangle$, respectively. To see this we operate on Eq. (x42) and (y42) with $\left\langle (\tilde{B}/B_0)\int dvv_{\|}\right\rangle$, obtaining

$$-\left\langle (B_0^2\nabla_{\|}B/B^3)\tilde{p}_{\|}\right\rangle [1+O(\epsilon)] = \left(cT/q_i\bar{B}\right)\left\langle (\tilde{B}/\bar{B})\int dvv_{\|}\left[(2v_{\|}^2+v_{\perp}^2)/v_i^2\right]\right.$$
$$\left.\left(\bar{B}^2 B\times\nabla B\cdot\nabla r/B^4\right)\partial\langle f\rangle/\partial r\right\rangle \qquad (114)$$

so that the first term in Eq. (106) is cancelled. The second term in Eq. (106) can be evaluated. First define

$$\partial K(\zeta)/\partial\zeta = \left(\bar{B}^2 B\times\nabla B\cdot\nabla r/B^4\right)/R. \qquad (115)$$

This is a valid definition in view of the arguments in Appendix A. Then operate on Eq. (x42)-(y42) with $(\partial/\partial r)\int Rd\zeta\int dv\left[\left(2v_{\|}^2+v_{\perp}^2\right)/v_i^2\right]$ to obtain

$$\int dvv_{\|}(\partial/\partial r)\tilde{f}_{i3}\left[\left(2v_{\|}^2+v_{\perp}^2\right)/v_i^2\right] = (\partial/\partial r)\int Rd\zeta\int dv(v_{\|}^2/v_i^2)c(\tilde{f}_{i3}^{v})$$
$$+(cT/q_i\bar{B})K(\zeta)\times(\partial/\partial r)^2\left[q_i\Phi/T_i+n/n_0\right]\int dv\left[\left(2v_{\|}^2+v_{\perp}^2\right)/v_i^2\right]^2 f_M$$
$$-\bar{B}^2(\tilde{1}/B^2)(\partial u_{\|}/\partial r)\int(v_{\perp}^2/v_i^2)\left[\left(2v_{\|}^2+v_{\perp}^2\right)\right]f_M/2 \qquad (116)$$

where we have used the energy conservation property of c. The first term on the right is one order higher in ϵ than the rest (because $v_{\|}$ is small where c is large). Neglect this term and operate with $\langle(1/R)\partial K/\partial\zeta\rangle$ on Eq. (109). The term in Eq. (109) containing $K(\zeta)$ is annihilated since

$$\langle K(\zeta)(1/R)\partial K/\partial\zeta\rangle = (1/2)\oint d\zeta\,\partial K^2/\partial\zeta = 0. \qquad (117)$$

To evaluate the contribution to $\left\langle \left(B_0^2\nabla_{\|}B/B^3\right)\tilde{p}_{\perp}\right\rangle$ first define

$$\tilde{f}_{i3}^{v} = \tilde{h}^{v}+\tilde{g}^{v} \qquad (118)$$
$$\tilde{g}^{v} = 2(cT/qB_0)K(\zeta)(\partial u_{\|}/\partial r)\left[\left(2v_{\|}^2+v_{\perp}^2\right)/v_i^4\right]f_M \qquad (119)$$
$$v_{\|}\nabla_{\|}h^{v}+c(h^{v}) = 2\left(cT/qB_0v_i^4\right)K(\zeta)(\partial u_{\|}/\partial r)c(v_{\|}^2 f_M). \qquad (120)$$

The function h^v is one order higher in ϵ than g^v and may be neglected to lowest order. The contribution of g^v to $\left\langle \left(B_0^2\nabla_{\|}B/B^3\right)\tilde{p}_{\perp}\right\rangle$ cancels the second term in Eq. (106). Note that the contribution of g^v to the vorticity equation vanishes in view of Eq. (110). The higher order contribution from h^v is considered later.

Thus, $\tilde{f}_{i3}^{v}$ cancels out of the shear-Alfvén law and the parallel equation of motion to this order. The adiabatic contributions of $\tilde{f}_{i3}$ and $\tilde{f}_{e3}$ (i.e., those proportional to $e\Phi/T$) cancel in $\sum_s \left(p_{\|} + p_{\perp}\right)$. The significant contributions from neoclassical effects come from f_{i3}^{T}, Eq. (x42).

To evaluate these, it is convenient to define the following Fourier transforms

$$
\begin{aligned}
&r_0 B_0^2 (B \times \nabla r \cdot \nabla B)/B^4 = \sum G_{nm} e^{i(n\zeta + m\theta)} \\
&r_0 B_0^2 \nabla_{\|} B/B^3 = \sum P_{nm} e^{i(n\zeta + m\theta)}.
\end{aligned}
\tag{121}
$$

Furthermore define the kinetic integrals

$$
\begin{aligned}
\tau_{nm} = \mathrm{Re}\Big\{ &\int dv \left[\left(2v_{\|}^2 + v_{\perp}^2\right)/v_i^2\right] \\
&\times \left[i(n - m/q)/R + c\right]^{-1} \left[\left(2v_{\|}^2 + v_{\perp}^2\right)/v_i^2\right] F_M \Big\}
\end{aligned}
\tag{122}
$$

where $[i(n - m/q)/R + c]^{-1}$ is to be interpreted as an operator inverse. Then to lowest order, the kinetic terms in the shear-Alfvén Law and parallel momentum equation become

$$
\begin{aligned}
&cB_0 \Big\langle B \times \nabla(p_{\|} + p_{\perp}) \cdot \nabla(1/B^2)/B \Big\rangle = (n_i m_i c^2 v_i^2 / B_0^2 r_0^2) \\
&\times (\partial/\partial r) \sum G_{nm} \tau_{nm} \left\{ G_{nm}^{*} \left[(\partial\Phi/\partial r) - (T_i/q_i n_0)\partial n/\partial r\right] + u_{\|} P_{nm} \right\}
\end{aligned}
\tag{123}
$$

$$
\begin{aligned}
&\Big\langle B_0^2 \nabla_{\|} B/B^3 (p_{\|} + p_{\perp}) \Big\rangle = n_i m_i v_i^2 / r_0^2) \\
&\times \sum P_{nm} \tau_{nm} \left\{ G_{nm}^{*} (c/B_0) \left[(\partial\Phi/\partial r) - (T_i/q_i n_0)\partial n/\partial r\right] + u_{\|} P_{nm} \right\}.
\end{aligned}
\tag{124}
$$

To obtain an explicit expression we need to evaluate τ_{nm}. The operator $[i(n - m/q)/R + c]^{-1}$ has a resonant structure near $v_{\|} = 0$, and most of the contribution in the integral comes from this region. It has been shown in the neoclassical literature that to lowest order in the plateau collisionality regime, integrals containing $[i(n - m/q)/R + c]^{-1}$ may be simplified by replacing the collision operator c by an infinitesimal Krook operator $\delta\nu$, so that $[i(n - m/q)/R + \delta\nu]^{-1}$ becomes an algebraic resonance function. Also, to lowest order we may replace R by R_0. Using this, we obtain

$$
\omega_{nm} = 2\sqrt{\pi} R_0/(n - m/q) v_i \left[1 + \text{ higher order terms}\right].
\tag{125}
$$

I Discussion of rotation damping terms

We now discuss the qualitative features of the new neoclassical dissipative terms. Let us write the lowest order shear-Alfvén law and parallel equation of motion. It is instructive to include the usual inertial terms in the shear-Alfvén law (even though they are higher order), and obtain

$$(n_{i0}m_i c/B)(\partial/\partial t)(\partial^2/\partial r^2)\left[c\left\langle\Phi\right\rangle/B + n(cT_i/q_iB_0n_0)\right] = \left[\left\langle\psi\right\rangle + \left\langle\delta\psi\right\rangle, \left\langle j_{\|}\right\rangle\right]/B_0$$
$$-\nu(n_i m_i c/B)(\partial/\partial r)\left\{(\partial/\partial r)\left[(c/B_0)\left\langle\Phi\right\rangle + n(cT_i/q_iBn_0)\right] - \Theta\left\langle u_{\|}\right\rangle\right\} \quad (126)$$

$$m_i n_{i0}\left\{\partial\left\langle u_{\|}\right\rangle/\partial t + (c/B_0)\left[\left\langle\Phi\right\rangle, \left\langle u_{\|}\right\rangle\right]\right\} = (1 + T + T_i/T_e)\left[\left\langle\psi\right\rangle + \left\langle\delta\psi\right\rangle, \left\langle p\right\rangle\right]$$
$$-\nu m_i n_{i0}\Theta\left\{(\partial/\partial r)\left[(c/B_0)\left\langle\Phi\right\rangle + n(cT_i/q_iBn_0)\right] - \Theta'\left\langle u_{\|}\right\rangle\right\} \quad (127)$$

where

$$\nu = \frac{v_i^2}{\sum}\left|G_{nm}\right|^2\tau_{nm} \quad (128)$$
$$\Theta = \sum G_{nm}P^*_{nm}\tau_{nm}/\sum\left|G_{nm}\right|^2\tau_{nm} \quad (129)$$
$$\Theta' = \sum\left|P_{nm}\right|^2\tau_{nm}/\sum G_{nm}P^*_{nm}\tau_{nm}. \quad (130)$$

Consider Eq. (119). The neoclassical term tends to damp the total perpendicular plasma velocity, $(\partial/\partial r)\left[(c/B_0)\left\langle\Phi\right\rangle + n(cT_i/q_iBn_0)\right]$, toward the value $\Theta\left\langle u_{\|}\right\rangle$, at a rate ν. In Eq. (y65), the neoclassical term tends to damp the parallel velocity $\left\langle u_{\|}\right\rangle$ towards the perpendicular velocity divided by Θ' at a rate $\Theta\Theta'\nu$. We will show momentarily that for axisymmetry $\Theta = \Theta' \approx B_p/B$. Then both damping terms drive the flow to a state with $(\partial/\partial r)\left[(c/B_0)\left\langle\Phi\right\rangle + n(cT_i/q_iBn_0\right] - \Theta\left\langle u_{\|}\right\rangle = 0$, whereupon there is no poloidal flow, but only flow in the symmetry direction. For non-axisymmetry, the Cauchy-Schwartz inequality can be used to show $|\Theta'| > |\Theta|$. One can then demonstrate that the combined effect of both terms is to drive both the perpendicular flow $(\partial/\partial r)\left[(c/B_0)\left\langle\Phi\right\rangle + n(cT_i/q_iBn_0)\right]$ and the parallel flow towards zero.

For axisymmetry,

$$B^0 = I'(\psi')\nabla\zeta + \nabla\psi' \times \nabla\zeta \quad (131)$$

and B^0 depends only on θ. One can then show that

$$(B \times \nabla B \cdot \nabla r/B^4)/(B \cdot \nabla B/B^3) = \Theta. \quad (132)$$

Note that the right side of Eq. (x68j) is independent of poloidal angle. For nearly circular, concentric flux surfaces $\Theta = B_p/B$. Thus, $G_{nm} = \Theta P_{nm}$ for all n and m, and so $\Theta' = \Theta$.

IV Next Order Neoclassical Effects – Bootstrap Current and Ware Pinch

A Electron corrections

We now proceed to next order in ϵ for the average electron distribution function. Taking the average of the drift kinetic equation to order ϵ^5

$$\begin{aligned} c_e\left(\langle f\rangle_4\right) &+ \frac{\partial \langle f\rangle_2}{\partial t} + v_{\|}\nabla_{\|}\langle f\rangle_3 - v_{\|}\frac{e}{T}\left\langle E_{\|}\right\rangle_6 f_{eM} \\ &+ v_{\|}v_{\perp}^2\left\langle \frac{\nabla_{\|}B\bar{B}^2}{B^3}\left(\frac{\partial}{\partial v_{\perp}^2} - \frac{\partial}{\partial v_{\|}^2}\right)\tilde{f}_4\right\rangle + \langle v_E\rangle\cdot\nabla\langle f\rangle_2 + \left\langle v_d\cdot\nabla\tilde{f}_3\right\rangle \\ &+ c\left[\bar{B}\left(B\times\nabla\langle\Phi\rangle\cdot\nabla B\right)/2B^4\right]\left[\left(v_{\perp}^2 + 2v_{\|}^2\right)/v_i^2\right] f_{eM}. \end{aligned} \tag{133}$$

We need $\langle f\rangle_4$ only to compute the current. For this purpose we multiply Eq. (126) by the Spitzer function, f_s/f_M, integrate over velocity, and use

$$\int \frac{f_S}{f_M}C(f) = \int \frac{f}{f_M}C(f_s) = \int f v_{\|} dv.$$

Furthermore, f_s can be written as

$$F_s = v_{\|}\mathcal{G}(v) f_M$$

where $\mathcal{G}$ is a function only of $v = \sqrt{v_{\|}^2 + v_{\perp}^2}$. Thus, we obtain

$$\begin{aligned} \int dv\,\langle f\rangle_4 v_{\|} &= \frac{1}{2}\int dv\ \mathcal{G}(v)\left\langle\frac{\bar{B}^2\nabla_{\|}B}{B^3}\tilde{f}_4\right\rangle\left(v_{\perp}^2 - 2v_{\|}^2\right) \\ &+ \int dv v_{\|}\mathcal{G}(v)\left[v_{\|}\nabla_{\|}\langle f\rangle_3 - v_{\|}\frac{e}{T}\left\langle E_{\|}\right\rangle_6 f_M\right] \\ &+ \text{Higher order terms.} \end{aligned} \tag{134}$$

The first term on the right gives the neoclassical correction to Ohm's law. Note that certain terms in Eq. (126) do not contribute to the current by symmetry.

We now use Eq. (103) for $\tilde{f}_4$, Eq. (82) for $\tilde{u}_{\|i}$. Furthermore, we make use of the resonant nature of $(v_{\|}\nabla_{\|} + c)^{-1}$, which allows us to neglect some terms as higher order in ϵ. We obtain

$$\frac{1}{2}\int dv\mathcal{G}(v)\left\langle\frac{\bar{B}^2\nabla_{\|}B\tilde{f}}{B^3}\left(v_\perp^2-2v_\|^2\right)\right\rangle$$

$$=\frac{\pi}{2}\frac{R_0v_e^2}{r_0^2}\sum_{nm}\frac{1}{|n-m/q|}\left\{u_{\|i}\left|P_{nm}\right|^2\int dv_\perp\frac{\mathcal{G}(v)}{v_e}\left(\frac{v_\perp^2}{v_e^2}\right)^2 f_M\Big|_{v_\|=0}\right.$$

$$+\frac{cT}{e\bar{B}}P_{nm}^*G_{nm}\frac{\partial}{\partial r}\left(-\frac{e\phi}{T}+\frac{n}{n_0}\right)\int dv_\perp\left(\frac{v_\perp^2}{v_e^2}\right)^2\frac{f_M}{v_e}\mathcal{G}(v)\Big|_{v_\|=0}$$

$$\left.-F_\|\left|P_{nm}\right|^2 v_e^2\int dv_\perp\left(\frac{v_\perp^2}{v_e^2}\right)^2\mathcal{G}(v)\;\mathcal{G}(v)f_M\Big|_{v_\|=0}\right\}. \tag{135}$$

The first two terms give the bootstrap current; the last term gives the plateau regime reduction in electrical conductivity due to toroidicity, which is qualitatively similar to that found in the banana regime by Hinton and Oberman.

We also need the neoclassical contribution to the electron continuity equation, Eq. (73).

$$cB_0\left\langle\left(\frac{B\times\nabla B\cdot\nabla r}{B^4}\right)\frac{\partial}{\partial r}\left[\frac{\left(\tilde{p}_{e\|}+\tilde{p}_{e\perp}\right)}{e}+n_0\tilde{\bar{\Phi}}\right]\right\rangle=\pi\frac{v_e^2}{r_0^2}\frac{cn_0m_e}{B_0e}\frac{\partial}{\partial r}\sum_{nm}\frac{R_0}{\left|n-\dfrac{m}{q}\right|}$$

$$\left\{u_{\|i}G_{nm}^*P_{nm}\int dv_\perp\left(\frac{v_\perp^2}{v_e^2}\right)^4 f_M\Big|_{v_\|=0}+\left|G_{nm}\right|^2\frac{cT}{eB_0}\frac{\partial}{\partial r}\left(\frac{n}{n_0}-\frac{e\phi}{T}\right)\int dv_\perp\left(\frac{v_\perp}{v_e}\right)^4 f_M\Big|_{v_\|=0}\right.$$

$$\left.-G_{nm}^*P_{nm}v_e^2F_\|\int dv_\perp G(v)f_M\left(\frac{v_\perp}{v_e}\right)^4\Big|_{v_\|=0}\right\}. \tag{136}$$

Also note that the term needed for the parallel equation of motion, Eq. (74), is

$$m_e \int dv \left\langle \frac{\bar{B}^2 \nabla_{\|} B}{B^3} \tilde{f}_e \right\rangle \left(v_\perp^2 + v_{\|}^2\right) = \frac{\pi R_0 v_e^2}{r_0^2} \sum_{nm} \frac{1}{\left| n - \frac{m}{q} \right|}$$

$$\times \left\{ u_{\|i} \left|P_{nm}\right|^2 \int dv_\perp \frac{v_\perp^4}{v_e^4} f_M \Big|_{v_{\|}=0} \right.$$

$$+ \frac{cT}{eB_0} P_{nm}^* G_{nm} \frac{\partial}{\partial r} \left(-\frac{e\phi}{T_0} + \frac{n}{n_0} \right) \int dv_\perp \frac{v_\perp^4}{v_e^4} f_M \Big|_{v_{\|}=0}$$

$$\left. - F_{\|} \left|P_{nm}\right|^2 v_e^2 \int dv \frac{v_\perp^4}{v_e^4} \mathcal{G}(v) f_M \Big|_{v_{\|}=0} \right\}. \qquad (137)$$

Define

$$\frac{\pi R_0 v_e^2}{r_0^2} \sum_{nm} \frac{P_{nm}^* G_{nm}}{\left| n - \frac{m}{q} \right|} \int dv_\perp \mathcal{G}(v) F_M \Big|_{v_{\|}=0} \left(\frac{v_\perp}{v_e} \right)^4 = k_1 \frac{\epsilon v_e}{R \nu_e} \qquad (138)$$

where K_1 is order one and

$$\frac{\pi R_0 v_e^2}{r_0^2} \sum_{nm} \left|P_{nm}\right|^2 \int dv_\perp \left(\frac{v_\perp}{v_e} \right)^4 \mathcal{G}^2(v) f_M \Big|_{v_{\|}=0} = \frac{k_2 \epsilon^2 v_e}{\nu_e R}. \qquad (139)$$

For axisymmetry, $P_{nm} = \Theta G_{nm}$, we have that the Ohm's law, including $O(\epsilon^4)$ and $O(\epsilon^4)$ terms, is

$$\begin{aligned} \left(\langle j \rangle^4 + \langle j \rangle^5 \right) &= \frac{ne^2}{m_c \nu_e^s} \left[\left\langle E_{\|} \right\rangle_5 + \left\langle E_{\|} \right\rangle_6 + (T/e) \nabla_{\|} \left(\langle n \rangle_2 + \langle n \rangle_3 \right) \right] \\ &+ k_1 n_e \left(\frac{\epsilon v_e}{\nu R} \right) \left[\frac{cT}{eB} \frac{\partial}{\partial r} \left(\frac{\langle n \rangle_2}{n_0} - \frac{e \langle \phi \rangle_2}{T} \right) + \Theta U_{\|i} \right] \\ &- k_2 \left(\frac{n_e^2}{m_e \nu_e} \right) \left(\frac{\epsilon^2 v_e}{\nu R} \right) \left[\left\langle E_{\|} \right\rangle^3 + (T/e) \nabla_{\|} \langle n \rangle^2 \right] \\ &+ \text{ Higher order terms.} \end{aligned} \qquad (140)$$

It is convenient to rewrite this equation in a form which is identical up to order ϵ^5. First, recall that $\langle n \rangle = \langle n \rangle_2 + \langle n \rangle_3 + \cdots, j = \langle j \rangle^4 + \langle j \rangle^5 + \cdots$, etc. Then

$$\langle j \rangle = \frac{1}{\eta_T} \left[\left\langle E_{\|} \right\rangle + (T/e) \nabla_{\|} \langle n \rangle \right]$$

$$+ \quad k_1 ne\left(\frac{v_e}{\nu R}\right)\left[\frac{cT}{eB_0}\frac{\partial}{\partial r}\left(\frac{\langle n\rangle}{n_0}-\frac{e\langle\phi\rangle}{T}\right)+\Theta U_{\|i}\right]$$

$$+ \quad \text{Higher order terms.} \tag{141}$$

Here,

$$\eta_T = \frac{m_e \nu_e^s}{ne^2}\left[1-\frac{k_2\epsilon^2 v_e}{\nu R}\right]^{-1} \tag{142}$$

gives the small $O(\epsilon)$ reduction in electrical conductivity by toroidal effects in the plateau regime.

B Next order ion corrections

To obtain the next order contributions of the ion pressure tensor to Eq. ( ), we must solve the drift kinetic equation for $\tilde{f}_4$ and $\langle f\rangle_3$. At order ϵ^5, we have

$$\begin{aligned}
\frac{\partial\langle f\rangle_2}{\partial t} &+ \langle v_E\rangle\cdot\nabla\langle f_2\rangle + v_\|\nabla_\|\langle f\rangle_2 - \left(\frac{q_i}{m}\right)\nabla_\|\langle\Phi\rangle_2 v_\| f_M \\
&+ c_i\left(\langle f\rangle_3\right) + B_0^2\nabla_\| B/B^3 v_\| v_\perp^2\left(\frac{\partial}{\partial v_\|^2}-\frac{\partial}{\partial v_\perp^2}\right)\tilde{f}_3 \\
&+ \frac{cTB_0}{q}\left(\frac{v_\perp^2+2v_\|^2}{v_i^2}\right)\frac{B\times\nabla B\cdot\nabla\tilde{f}_3}{B^4} = \\
&- v_\|\tilde{\nabla}_\|\tilde{f}_4 - c_i(\tilde{f}_4) + \frac{q_i}{m_i}\nabla_\|\tilde{\Phi}_4 v_\| f_M - \frac{cB\times\nabla\langle\Phi\rangle_3\cdot\nabla B}{B^4}\left(\frac{v_\perp^2+2v_\|^2}{v_i^2}\right)f_M \\
&+ \frac{cTB_0}{q}\left(\frac{v_\perp^2+2v_\|^2}{v_i^2}\right)\frac{B\times\nabla B\cdot\nabla\langle f\rangle_3}{B^4} + \frac{B_0^2\nabla_\| B}{B^3}v_\| v_\perp^2\left(\frac{\partial}{\partial v_\|^2}-\frac{\partial}{\partial v_\perp^2}\right)\langle f\rangle_3 \\
&- \left[cB_0\frac{B\times\nabla\tilde{\Phi}_3\cdot\nabla B}{B^4} - \frac{cB_0\tilde{B}B\times\nabla\langle\Phi\rangle_2\cdot\nabla B}{B^5} + \frac{cTB_0}{q}\frac{B\times\nabla B\cdot\nabla\tilde{f}_3}{B^4}\right. \\
&+ \left.\frac{cTB_0}{q}\frac{\tilde{B}B\times\nabla B\cdot\nabla\langle f\rangle_2}{B^5}\right]\left(\frac{v_\perp^2+2v_\|^2}{v_i^2}\right)f_M \\
&+ \frac{B_0^2\nabla_\| B}{B^3}v_\perp^2 v_\|^2\left(\frac{\partial}{\partial v_\perp^2}-\frac{\partial}{\partial v_\|^2}\right)\left[-\frac{2\tilde{B}}{B}\langle f\rangle_2 - \tilde{f}_3\right]
\end{aligned} \tag{143}$$

To simplify the following calculation of next order ion corrections we will assume both axisymmetry and nearly circular flux surfaces. The average of Eq. (137) gives

$$\frac{\partial\langle f\rangle_2}{\partial t} + \langle v_E\rangle\cdot\nabla\langle f\rangle_2 + v_\|\nabla_\|\langle f\rangle_2 - \left(\frac{q_i}{m_i}\right)\nabla_\|\langle\Phi_2\rangle v_\| f_M$$

$$+ \quad c_i(\langle f\rangle_3) + \left\langle B_0^2(\nabla_\| B/B^3)v_\| v_\perp^2 \left(\frac{\partial}{\partial v_\perp^2} - \frac{\partial}{\partial v_\|^2}\right)\tilde{f}_3\right\rangle$$

$$+ \quad \frac{cTB_0}{q}\left(\frac{v_\perp^2 + 2v_\|^2}{v_i^2}\right)\left\langle\frac{B\times\nabla r\cdot\nabla B}{B_4}\frac{\partial}{\partial r}\left(\frac{q\tilde{\Phi}_3}{T_i}f_M - \tilde{f}_3\right)\right\rangle = 0. \qquad (144)$$

It will be necessary to invert the collision operator c to obtain $\langle f\rangle_3$. The solubility condition of Eq. (139) for $\langle f\rangle_3$ for this is that the lowest three velocity moments, $\int dv$, $\int dvv_\|$, and $\int dvv^2$, of Eq. (138) vanish. This gives the previous ion fluid equations for density, velocity and temperature.

We now subtract the following from Eq. (138): the density equation times f_M, the temperature evolution equation times $\frac{2}{3}\left(v^2/v_i^2 - 3/2\right)$, and the parallel velocity evolution equation times $2v_\| f_M/v_i^2$. We use Eqs. (82)-(89) for $\tilde{f}$; we replace $\left[(n-m/q)v_\| R + c_i\right]^{-1}$ by $\delta(v_\|)\left(R/|n-m/q|\right)$, which is adequately accurate to this order in ϵ. We obtain

$$\begin{aligned}
\langle f\rangle_3 &= \langle n\rangle_3 f_M + 2\left\langle v_\|\right\rangle_3 v_\| F_M/v_i^2 + \langle f\rangle_3^n + \langle f\rangle_3^v, \\
c\left(\langle f\rangle_3^n\right) &= -\frac{2}{3}\nabla_\|\left\langle v_\|\right\rangle_2 \frac{(2v_\|^2 - v_\perp^2)}{v_i^2}f_M \\
&- \frac{cT}{qB_0}\frac{\partial}{\partial r}\sum\frac{R|G_{nm}|^2}{r_0^2|n-m/q|}\left[\frac{cT}{qB_0}\frac{\partial}{\partial r}\left(\frac{q_i\langle\phi\rangle_2}{T_i} - \frac{\langle n\rangle_2}{n_0}\right) - \Theta\left\langle v_\|\right\rangle_2\right] \\
&\times \left\{\frac{v_\perp^2}{v_i^2}\delta(v_\|) - \frac{f_M}{\sqrt{\pi}v_i}\left[1 + \frac{1}{3}\left(\frac{v^2}{v_i^2} - \frac{3}{2}\right)\right]\right\}. \qquad (145)
\end{aligned}$$

The distribution $\langle f\rangle_3$ contributes directly to Eq. (74) through the term $\nabla_\|\left\langle p_\|\right\rangle$. With $\langle n\rangle = \langle n\rangle_2 + \langle n\rangle_3$, etc., we have

$$\nabla_\|\left\langle p_\|\right\rangle = T\nabla_\|\langle n\rangle - n_i m_i\frac{v_i^2}{\nu_i}\nabla_\|^2\left\langle v_\|\right\rangle \qquad (146)$$

$$+n_i m_i\nabla_\|\frac{v_i}{4}\left(\frac{K_2 v_i}{R\nu_i}\right)\sum\frac{|G_{nm}|^2}{\epsilon^2\left|n-\dfrac{m}{q}\right|}\left(\rho_i\frac{\partial}{\partial r}\right)\left[\frac{cT}{qB_0}\frac{\partial}{\partial r}\left(\frac{q\langle\phi\rangle_2}{T_i} - \frac{\langle n_2\rangle}{n_0}\right)\right. \qquad (147)$$

$$\left. -\Theta\left\langle v_\|\right\rangle_2\right] \qquad (148)$$

where

$$\nu_i^{-1} = \frac{2}{q}\int dv\frac{(2v_\|^2 - v_\perp^2)}{v_i^2}C_i^{-1}\left[\frac{2v_\|^2 - v_\perp^2}{v_i^2}f_M\right] \qquad (149)$$

$$k_{i2} = \nu_i\Big/\frac{2}{3}\int dv\left(\frac{2v_\|^2 - v_\perp^2}{v_i^2}\right)C_i^{-1}\left\{\left[\frac{v_\perp^2}{v_i}\delta(v_\|) - \frac{1}{\sqrt{\pi}}\left(\frac{1}{2} + \frac{v^2}{3v_i^2}\right)\right]f_M\right\}. \qquad (150)$$

The second term on the right is the classical parallel viscosity. The last term is a novel neoclassical effect driven by poloidal flow.

$$C(\langle f\rangle_3^v) = \sum \frac{\Theta|_{nm}|^2}{\left|n - \dfrac{m}{q}\right|}\left[\frac{cT}{qB_0}\frac{\partial}{\partial r}\left(\frac{q_i\langle\phi\rangle_2}{T_i} - \frac{\langle n_2\rangle}{n_0}\right) - \Theta\left\langle u_{\|}\right\rangle_2\right] \times \left[v_{\|}v_\perp^2\left(\frac{\partial}{\partial v_\perp^2} - \frac{\partial}{\partial v_{\|}^2}\right)\frac{v_\perp^2}{v_i^2}f_M\delta(v_{\|}) - \frac{4}{\sqrt{\pi}}\frac{v_{\|}}{v_i}f_M\right]. \tag{151}$$

We choose $\langle f\rangle_3^n$ and $\langle f\rangle_3^v$ to have the first three moments, $\int dv$, $\int dvv_{\|}$, $\int dvv^2$, vanish (this is possible because $c_i(f) = 0$ has homogeneous solutions).

We now examine Eq. (137) for $\tilde{f}_4$. The assumption of nearly circular flux surfaces now allows us to neglect the contributions from the last two terms. To see this, note that the $m \neq 1$ fourier harmonics of both $B \times \nabla B \cdot \nabla r/B^r$ and $B_0^2\nabla_{\|}B/B^3$ are smaller than the $m = 1$ harmonics by $O(\epsilon)$ or more for nearly circular surfaces. Thus, when computing $\left\langle B \times \nabla B \cdot \nabla r/B^4(\tilde{p}_{\perp 4} + \tilde{p}_{\|4})\right\rangle$ and $\left\langle B_0^2\nabla_{\|}B/B^3(\tilde{p}_{\perp 4} + \tilde{p}_{\|4})\right\rangle$, we only need the $m = \pm 1$ piece of $\tilde{f}_4$ to this order. The last two terms in Eq. (137) give a predominantly $m = 2$ contribution to $\tilde{f}_4$, and so they may be neglected.

The contribution from the third to last term in Eq. (137) gives a poloidal rotation damping term with the same effect as Eq. (119)-(123), but with a damping rate which is smaller by $O(\epsilon)$. It may therefore be considered as an $O(\epsilon)$ correction to the coefficient in Eqs. (121)-(123). Such terms do not introduce any qualitatively new physical effects, but only allow the damping rate to be computed with great accuracy. We do not pursue such quantitative corrections here.

However, the contributions to $\tilde{f}$ from the $\langle f\rangle_3^n$ part of $\langle f\rangle_3$ in the fifth and sixth terms on the left produce a novel viscous damping effect on poloidal flows, which is coupled to the standard classical parallel viscosity. Define

$$v_{\|}\nabla_{\|}\tilde{f}_4^n + c_i(\tilde{f}_4^n) = \frac{cT}{q}B_0\left(\frac{v_\perp^2 + 2v_{\|}^2}{v_i^2}\right)\frac{B \times \nabla r \cdot \nabla B}{B^4} \times \frac{\partial}{\partial r}\langle f\rangle_3^n. \tag{152}$$

Then we obtain to requisite order

$$cB_0 \left\langle \frac{B \times \nabla r \cdot \nabla B}{B^4} \int dv \left(v_\perp^2 + v_\|^2\right) \tilde{f}_4^n \right\rangle =$$

$$\frac{n_i m_i c}{4B} \left(\frac{v_i}{R}\right) \left(\frac{v_i}{\nu_i R}\right) \frac{q_0}{2} \rho_i \frac{\partial}{\partial r} \left\{ k_{i1} \frac{q_0 \rho_i}{4} \frac{\partial}{\partial r} \left[\frac{c}{B} \frac{\partial}{\partial r} \left(\phi - \frac{T}{q} \frac{n}{n_0} \right) - \Theta u_\| \right] \right.$$

$$\left. + R \nabla_\| v_\| \right\} \tag{153}$$

where

$$k_{i1} = \nu_i \Big/ \int \left[\frac{v_\perp^2}{v_i} \delta(v_\|) - \frac{1}{\sqrt{\pi}} \left(\frac{1}{2} + \frac{v^2}{v_i^2} \right) \right] C^{-1} \left[\frac{v_\perp^2}{v_i} \delta(v_\|) - \frac{1}{\sqrt{\pi}} \left(\frac{1}{2} + \frac{v^2}{v_i^2} \right) \right]. \tag{154}$$

To obtain the average needed for the parallel equation of motion, note that for an axisymmetric geometry

$$\left\langle \frac{B_0^2 \nabla_\| B}{B^3} \int dv \left(v_\|^2 + v_\perp^2\right) \tilde{f} \right\rangle = \Theta \left\langle \frac{B_0^2 B \times \nabla r \cdot \nabla B}{B^4} \int dv \left(v_\|^2 + v_\perp^2\right) \tilde{f} \right\rangle. \tag{155}$$

There is one final higher order ion effect. Recall that the term $\langle w \rangle$ in Eq. (79) was cancelled by the contribution of $\tilde{f}_{i3}^v$ to $\left\langle B_0^2 \nabla_| B / B(\rho_\perp + \rho_\|) \right\rangle$. This cancellation holds to one additional order for an axisymmetric, nearly circular geometry, except for one term – the first term on the right in Eq. (109). To requisite order we may use Eq. (112) for f_{i3}^v, to obtain

$$\frac{\partial}{\partial r} \int dv \tilde{f} \left(\partial v_\|^2 + v_\perp^2\right) / v_i^2 = -\frac{q^2 \rho_i^2}{4} \int \frac{v_\|^2}{v_i^2} C_i \left(\frac{v_\|^2}{v_i^2} f_M \right) \frac{\partial^2 u_\|}{\partial r^2} = -k_{i3} q^2 \nu_i \rho_i^2. \tag{156}$$

This is a neoclassical cross-field viscosity; it has roughly the same size and scaling as the classical viscosity, and agrees with previous equilibrium results.

V Onsager Symmetries and H Theorem

The equilibrium neoclassical transport coefficients satisfy Onsager symmetry relations among the thermodynamic forces and fluxes. Furthermore, an H theorem holds so that entropy always increases. A definition of Onsager symmetries for the dissipative terms in Eqs. (8)-(11) requires a generalization of neoclassical concepts, because of the presence of flow and derivative operators. Field theoretic concepts have been developed by Morrison, which are applicable to a wide class of systems with dissipation. Using these, we will show that Onsager symmetries and an H theorem apply to the terms in Eqs. (8)-(11).

First we briefly demonstrate the concepts for a system with N quantities as functions of space, $Q_i(x)$. Suppose there is a free energy functional $\mathcal{F}(Q_i, Q_N)$. Then the local thermodynamic forces are $\delta\mathcal{F}/\delta Q_i$, the functional derivative of $\mathcal{F}$. Suppose the dissipative terms in the evolution equation can be written as

$$\dot{Q}_i(x) = \int dx'\delta(x-x') \sum_j M_{ij}(x,x)\delta\mathcal{F}/\delta Q_j \tag{157}$$

for some operator M, which gives the thermodynamic fluxes. If M is symmetric

$$\int A(x) M_{ij} C(x) = \int C(x) M_{ji} A(x) dx \tag{158}$$

then the generalized Onsager symmetries hold. More explicitly, the free energy change due to the part of the dissipative evolution of Q_j is driven by the thermodynamic force from Q_i, $\dot{\mathcal{F}} = \int dx(\delta\mathcal{F}/\delta Q_j) M_{ij} (\delta\mathcal{F}/\delta Q_i)$, is equal to the free energy change from the converse forces and fluxes. If the operator M is negative definite we have an H theorem

$$\dot{\mathcal{F}} = \sum_{ij} \int dx \frac{\delta\mathcal{F}}{\delta Q_i} M_{ij} \frac{\delta\mathcal{F}}{\delta Q_j} < 0. \tag{159}$$

We will demonstrate the symmetries and H theorem by explicitly giving the M_{ij} for each of the dissipative terms in Eqs. (8)-(14).

First, we must define the free energy $\mathcal{F}$. The thermodynamic definition of $\mathcal{F}$ is the total energy plus the temperature times entropy. For a quasineutral Vlasov plasma with Coulomb collision operator, this is

$$\sum_j \int dx dv \left(\frac{1}{2} m v^2 f_0 + T_i j_j |n f_j\right) + \int dx \; B^2/8\pi. \tag{160}$$

Equations (8)-(11) describe the evolution of small local deviations from a Maxwellian, so $f_j = f_{jM} + \delta f$. The first terms in Eq. (157) can be easily evaluated to second order in δf for perturbations of the form in Eq. (84). Including the perpendicular diamagnetic flow and only the poloidal component of B, this becomes in normalized units

$$\mathcal{F} = \frac{1}{z} \int dx \left[|\nabla F|^2 + |\nabla\psi|^2 + u^2 + \frac{(1+k)p^2}{2\beta} \right]. \tag{161}$$

As in previous descriptions of reduced MHD, the field variables are $U = \nabla_\perp^2 F$, ψ, u, p. Note

$$\frac{\delta\mathcal{F}}{\delta U} = -\phi \quad \frac{\delta\mathcal{F}}{\delta\psi} = -J \quad \frac{\delta\mathcal{F}}{\delta v} = v \quad \frac{\delta\mathcal{F}}{\delta p} = \frac{1+k}{z} p. \tag{162}$$

This $\mathcal{F}$ is conserved by the nondissipative terms in Eqs. (8)-(11).

The operators M are most easily written by giving $\sum_{ij} \int \frac{\delta A}{\delta Q_1} M_{ij} \frac{\delta C}{\delta Q_j}$ for each M_{ij} needed to generate each of K_i^α, K_i^v, K_e in Eqs. (8)-(11)

$$K_i^\alpha \quad : \quad \int \frac{\delta A}{\delta Q} M \frac{\delta C}{\delta Q} = -\nu \int dx \left(\frac{d}{dx}\frac{\delta A}{\delta u} - \Theta \frac{\delta A}{\delta u} \right) \left(\frac{d}{dx}\frac{\delta C}{\delta u} - \Theta \frac{\delta C}{\delta u} \right) \tag{163}$$

$$K_e \quad : \quad \int \frac{\delta A}{\delta Q} M \frac{\delta c}{\delta u} = -\eta \int dx \left(-\frac{\delta A}{\delta \psi} + 2\alpha\beta \frac{d}{dx}\frac{\delta A}{\delta p} + \frac{\alpha}{\lambda} \left[\frac{d}{dx}\frac{\delta A}{\delta U} - \Theta \frac{\delta A}{\delta u} \right] \right) \left(-\frac{\delta C}{\delta \psi} + 2\alpha\beta \frac{d}{dx}\frac{\delta C}{\delta p} + \frac{\alpha}{\lambda} \left[\frac{d}{dx}\frac{\delta C}{\delta u} - \Theta \frac{\delta C}{\delta u} \right] \right) \tag{164}$$

$$\begin{aligned} K_i^v \quad : \quad & \int \frac{\delta A}{\delta Q} M \frac{\delta C}{\delta Q} \\ = \; & \left(\frac{4}{q_0 \beta_i \lambda} \right) \left(\frac{v_i}{r \nu_i} \right) \left(\frac{\beta^{1/2}}{2K_{i1}} \right) \int dx \Bigg[k_{1i} \left(\frac{q_0}{z} \beta_i^{1/2} \frac{d}{dx} \left(-\frac{d}{dx}\frac{\delta A}{\delta u} \right) + \Theta \frac{\delta A}{\delta u} \right) \\ + \; & k_{i2} \nabla_\parallel \frac{\delta A}{\delta u} \Bigg] \left(\frac{q_0 \beta_i \lambda}{4} \right) \left[k_{i1} \left(\frac{q_0}{z} \lambda \beta_i^{1/2} \right) \frac{d}{dx} \left(-\frac{d}{dx}\frac{\delta c}{\delta u} + \Theta \frac{\delta C}{\delta u} \right) + k_{i2} \nabla_\parallel \frac{\delta C}{\delta u} \right] \end{aligned} \tag{165}$$

The other dissipative terms in Eq. (10) can be generated from

$$\left(\mu_\perp^c + \mu_\perp^{nc}\right) \nabla_\perp^2 v \colon \int \frac{\delta A}{\delta Q} M \frac{\delta C}{\delta Q} = -\left(\mu_\perp^c + \mu_\perp^{nc}\right) \int dx \left(\nabla_\perp \frac{\delta A}{\delta u} \right) \left(\nabla_\perp \frac{\delta C}{\delta u} \right) \tag{166}$$

$$\mu_\parallel^c \left(1 - k_{i2}^2 / 2K_{i1}\right) : \int \frac{\delta A}{\delta Q} M \frac{\delta C}{\delta Q} = -\mu_\parallel^c \left(1 - \frac{k_{i2}^2}{2k_{1j}} \right) \int dx \left(\nabla_\parallel \frac{\delta A}{\delta u} \right) \left(\nabla_\parallel \frac{\delta C}{\delta u} \right). \tag{167}$$

The quantity $1 - k_{i2}^2/2k_{i1}$ can be shown to be positive using the definiteness of the Coulomb collision operator and the Cauchy-Schwarz inequality.

Direct computation verifies that these operators M generate the dissipative terms. They are clearly symmetric in A and C and negative for $A = C$. Thus the dissipative terms possess Onsager symmetries and imply an H theorem for the free energy F.

VI Conclusions

The rigorously ordered calculation here shows that neoclassical effects present in the resistive layers of magnetic reconnection phenomenon. General nonlinear fluid equations to describe reconnection are given in Eqs. (8)-(14). The neoclassical effects given by the dissipative terms are of the same size, or larger than other effects usually considered for reconnection. Some modifications from equilibrium neoclassical results arise because the parallel flow might not equilibrate in rapidly growing instabilities.

These terms satisfy a recent field theoretic generalization of the Onsager symmetry relations, which allows for the presence of the flow terms and the higher order spatial derivatives. Furthermore the equations satisfy an H theorem with the physical free energy.

Thus, these equations provide a consistent and rigorously motivated tool for describing resistive reconnection in toroidal plasmas in more realistic collisionality regimes.

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