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Theory of Trapped-Particle-Induced Resistive Fluid Turbulence

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ABSTRACT

A theory of anomalous electron heat transport, evolving from trapped-particle-induced resistive interchange modes, is proposed. These latter are a new branch of the resistive interchange-ballooning family of instabilities, destabilized when the pressure carried by the unfavorably-drifting trapped particles is sufficiently large to overcome stabilizing contributions coming from favorable average curvature. Expressions for the turbulent heat diffusivity and anomalous electron thermal conductivity at saturation are derived for two regimes of trapped particle energy: *i)* a moderately-energetic regime, which is "fluid-like" in the sense that the unstable mode grows faster than the time that it takes for particles in this energy range to precess once around the torus, and *ii)* a highly-energetic regime, where the trapped species has sufficiently high energy as to be able to *resonantly* interact with the mode. Unlike previous theories of anomalous transport, the estimates of diffusion and transport obtained here are self-consistent, since the trapped particles do not "see" the magnetic flutter due to their rapid bounce motion. The theory is valid for moderate electron-temperature, high ion-temperature (auxiliary-heated) plasmas, and as such, is relevant for present and future-generation experimental fusion devices.

I INTRODUCTION

It has become increasingly apparent over the years that some form of auxiliary heating will be required in order to realize the temperatures necessary for thermonuclear fusion breakeven. An undesirable consequence of this design has been an almost universally-observed degradation in confinement with increasing input power.[1] Recent years have seen a substantial amount of theoretical effort invested in pursuit of possible mechanisms which can explain this experimentally-observed anomaly in the transport of heat. Since auxiliary heating acts to raise the plasma beta content (beta is the ratio of kinetic to magnetic pressure, $\beta = 8\pi p/B^2$), a natural source of free energy to consider has been that associated with the relaxation of the pressure gradient. Thus, resistive pressure-gradient-driven turbulence, which evolves naturally from the robust linear resistive interchange-ballooning family of instabilities, has emerged as a prime candidate to explain the experimental findings. Plasma configurations with inherently unfavorable curvature, are prone to resistive interchange instabilities. Turbulence evolving from these modes has thus been proposed as a cause of anomalous thermal transport in both stellarators[2,3] and reversed-field pinches (RFP).[4,5] In tokamaks, on the other hand, where resistive interchange modes are dormant because of favorable average curvature, the plasma utilizes the free energy source associated with the pressure gradient by localizing the mode to the outer region of unfavorable curvature, and exploiting a weak aneurism to "balloon" outward. Turbulence evolving from one branch of these resistive ballooning modes has been advanced[6] as an explanation for the degradation in energy confinement associated with auxiliary heating experiments (such as those on the ISX-B tokamak[6]) where the poloidal beta is sufficiently large, and the electron temperature sufficiently low, so that the modes can grow on a time scale faster than the time that it takes for acoustic waves to propagate ballooning information along a connection length from the unfavorable to the favorable curvature region, or for diamagnetic effects to become operative, i.e., $\gamma > c_s/qR$, ω_*, where γ is the linear mode growth rate, c_s is the acoustic speed, qR is the connection length, and ω_* is the diamagnetic drift frequency. While relatively successful in accounting for experimental observations in the ISX-B tokamak, the resistive ballooning turbulence theory has not fared as well in accounting for experimental observations in other devices such as PDX[7] or Doublet III,[8] even at high poloidal beta. The principal reason for the absence of agreement in these devices is that the electron temperature, other than at the very edge, is too high to satisfy the validity criterion of the theory. In particular, compressional stabilization through acoustic wave propagation enters and significantly alters the stability picture.[9] A less unrealistic criterion for most tokamaks is a situation where the mode grows slower than sound propagation along a connection length but faster than the communication of ballooning information by acoustic waves along a parallel wavelength of the mode, i.e., $c_s/qR > \gamma > k_{\|}c_s$. Under such conditions, it has been shown[9–11] that the threshold for the onset of ballooning modes becomes so stringent [i.e., $\Delta' \sim {S_M}^{1/3} \gg 1$, where Δ' is the well-known measure of resistive tearing (ballooning) stability,[12] and S_M is the magnetic Reynolds number] that, except for cases of proximity to marginal stability, it is thought unlikely that they would pose as a serious candidate for poor confinement in tokamaks.

Now, all the afore-mentioned theories have used a resistive magnetohydrodynamic description to model the plasma. Such a formulation does not take account of the different behavior of individual particles, an issue which becomes manifest only at the kinetic level. In particular, implicit in these theories has been the notion that *all* particles sample both regions of adverse and favorable curvature, so that what they "feel" is, in some sense, an average effect, which in the case of tokamaks is favorable. In fact, however, a significant number of trapped particles are restricted in their excursions to only the unfavorable curvature region. It is not difficult to see then, that if these were to carry sufficient pressure, they would be able to exploit the aneurism and drive an instability. Naturally, such trapped-particle-driven modes would be of interest only insofar as the particles are able to execute single-particle bounce motion many times more rapidly than the time it takes for them to be collisionally scattered or detrapped. Thus, unlike previous theories which concerned themselves with the collisional magnetohydrodynamic (MHD) approximation, recent theories have sought to explore the plateau[13] and banana[14,11] collisionality regimes, and have found that resistive interchange-ballooning modes can be excited even with favorable average curvature, as in tokamaks. It is the purpose of this paper to consider the implication of these trapped-particle-induced resistive interchange modes for turbulence and anomalous transport in auxiliary-heated plasmas. The analysis presented here represents the first attempt at calculating transport coefficients with the influence of energetic trapped particles incorporated.

As an aid to the reader, we give here a general discussion of the analysis. To begin with, since the only participants in the dynamics that enjoy a privileged role are the trapped particles, we adopt a simplified plasma model which handles the latter kinetically, while treating the rest of the plasma as a resistive magnetohydrodynamic fluid. Such a representation has sufficient detail to render the qualitative picture outlined in the previous paragraph, yet remains sufficiently simple as to be analytically tractable without an excess of algebraic baggage. It then turns out that the relevant temporal ordering can be described in terms of two parameters: a characteristic resistive interchange time-scale, given by $\omega_R \sim {S_M}^{-1/3}\omega_A$ (where S_M is the magnetic Reynolds number, and ω_A is the Alfvén frequency), and the precessional (banana-center) magnetic drift frequency of the trapped particles (ω_{dh}). Since the latter is proportional to the energy of the the trapped species (i.e., $\omega_{dh} \propto E$), we have found it convenient to discriminate between two (energy-) classes of trapped particles. One class of these, we categorize as moderately-energetic, in the sense that the unstable linear mode manages to grow faster than the time it takes for the trapped species to precess once around the torus (i.e., $|\omega| \gg \bar{\omega}_{dh}$). We refer to this regime as "fluid-like." Another class of trapped particles, which we single out as highly-energetic, are those with sufficient energy as to be able to resonantly interact with the mode (i.e., $|\omega| \simeq \bar{\omega}_{dh}$). This latter type of resonant wave-particle interaction, in fact, was the original motivation behind the work of Ref. 11, which sought to explain the high-frequency ($\omega \sim 100$ kHz), low-amplitude, precursor oscillations to the so-called "fishbone" bursts,[15] as due to energetic trapped particle-induced resistive interchange modes. The bulk of the trapped particles in a plasma fall into the non-resonant "fluid-like" regime. On the other hand, energetic trapped particles created during auxiliary heating, as well as those in the high-energy tail of the bulk distribution, fall into the "resonant" regime.

As discussed in Sec. II, the "fluid-like" regime engenders a purely-growing trapped-particle-modified resistive interchange mode which is destabilized when the pressure due to the unfavorably-drifting (i.e., $\omega_{*h}\bar{\omega}_{dh} > 0$, where ω_{*h} is the diamagnetic drift frequency) trapped particles, overcomes the stabilizing influence of favorable average curvature. Using standard methods of renormalized strong turbulence theory, we treat the nonlinear evolution of a "test" mode in the presence of a background spectrum of multiple-helicity turbulence. The nonlinear dynamics we envision can be described as a cascade from dominant, long-wavelength trapped-particle-driven modes, to a short-wavelength viscous energy sink, with the transfer process itself dynamically mediated by turbulent $\mathbf{E} \times \mathbf{B}$ advection. The pressure perturbations are so localized that their nonlinear response is principally diffusive in character. An expression for the trapped-particle turbulent radial diffusion coefficient is obtained which is found to be larger than the simple linear mixing-length prediction. We interpret this discrepancy as arising because of nonlinear radial broadening of the highly-localized linear eigenmode due to the turbulent viscous dissipation of its high-k_r components. An estimate of the anomalous electron thermal conductivity (χ_e) is then derived by methods similar to the resistive ballooning turbulence theory,[6] which gives roughly the same parameter scalings as in the theory of Carreras and Diamond. For characteristic TFTR parameters, we compute a χ_e value on the order of 10^4 cm^2/sec, which is not inconsistent with typical experimental measurements. The χ_e radial profile, which is found to be increasing with minor radius, is also in accord with experimental trends.

Turning next to the case of the high-energy trapped particles, linear perturbative analysis reveals that these can now *resonantly* destabilize the resistive interchange mode via wave-particle interaction. Not only is this mode different from the usual resistive MHD interchange mode in its destabilization mechanism, but also in that it has a real component to its frequency comparable to the magnetic drift frequency of the rapidly-precessing trapped particles. As in the "fluid-like" case, there is a hot-particle beta-threshold associated with this instability which is significantly lower than the core plasma beta-threshold for the onset of resistive ballooning modes. In analyzing the turbulence that evolves from these modes, we find the intuitively plausible result that only the hot particles undergo *resonant* turbulent diffusion, whereas the bulk plasma electrostatic turbulence remains non-resonant. Interestingly, hot-particle diffusion dominates that of the bulk. The electrostatic instability itself, couples to magnetic flutter, and begets anomalous electron thermal transport along perturbed field lines. Depending upon the relative ordering among the hot ion magnetic precessional drift frequency, the bulk electron parallel transit frequency, and the wave-particle decorrelation rate, two transport scenarios are identified. In the event that the latter two are the faster temporal scales in the turbulence, then there is no qualitative difference between this situation and the "fluid-like" case, and the analysis goes through as before. In the event, however, that the precessional drift dominates, there will be a reduction in the transport level due to a gradual detuning of the wave-particle resonance, much as in the case of runaways.[16] Thus, high-frequency, oscillatory, non-resonant motion replaces the usual picture of irreversible, parallel diffusion of electrons along perturbed field lines. This is a situation of weak turbulence where the anomalous thermal conductivity varies quadratically with the fluctuation level in contrast to the usual situation in strong turbulence where the scaling is linear. For the illustrative case of TFTR, all three temporal scales are probably of comparable importance.

Finally, we remark that an interesting aspect of the present theory is that since the trapped particles do not "see" the magnetic flutter due to their rapid bounce motion, the estimates of diffusion and transport obtained are self-consistent. This is to be contrasted with previous theories of resistive ballooning turbulence,[6] which did not self-consistently account for the back-reaction of the induced magnetic flutter on the evolution of the pressure fluctuations, and hence suffered from the criticism that the level of turbulence would wane if proper account were taken of thermal conduction along *perturbed* field lines.[17] This does not present itself as an issue in the present theory.

The paper is organized as follows. In Sec. II, we highlight the main features of the linear theory. The formal nonlinear theory is developed in Sec. III. We retain the $\mathbf{E} \times \mathbf{B}$ advective nonlinearity in the evolution equations for the vorticity, the core pressure, and the hot particle pressure, and renormalize these via iterative substitution techniques *à la* the Direct Interaction Approximation (DIA).[18] Providing separate treatments for the "fluid-like" and "resonant" regimes, respectively, we obtain expressions for the turbulent diffusion and transport coefficients, and compare their efficacy for anomalous transport. We conclude the paper in Sec. IV with a summary of our results.

II HIGHLIGHTS OF THE LINEAR THEORY

A comprehensive linear theory of these modes has been given elsewhere.[11] We restrict ourselves here to providing information pertinent to the construction of our nonlinear model. The problem is theoretically formulated by separating the plasma into two components: a warm core (denoted in the text by a subscript 'c') which is treated using a resistive MHD prescription, and a trapped-particle component (denoted by a subscript 'h') which is dealt with using drift-kinetic formalism. The ordering is carried out using two smallness parameters: the inverse aspect ratio, ϵ, and the inverse magnetic Reynolds number, $\lambda = (n^2/S_M)^{1/3}$. Here, n is the toroidal mode number, $S_M = \tau_R/\tau_A$ is the magnetic Reynolds number, $\tau_R = 4\pi a^2/\eta_{\|}c^2$ is the resistive diffusion time, and $\tau_A = \omega_A^{-1} = qR(1+2q^2)^{1/2}/v_A$ is the Alfvén time. Focusing on the ideally-stable β_c operating regime, the β-ordering is given by $\beta_{pc} \sim O(1)$, and $\beta_{ph} \sim O(\epsilon)$. Defining a characteristic inverse resistive interchange time scale by $\omega_R = \lambda\omega_A$, a frequency ordering given by $|\omega| \sim \bar{\omega}_{dh} \sim \varpi_R = \epsilon^{4/3}\omega_R$ is adopted. Here, $\omega_{dh} = -i\mathbf{v}_d \cdot \nabla$ is the magnetic precessional drift frequency, $\mathbf{v}_d \simeq E\,\hat{\mathbf{e}} \times \boldsymbol{\kappa}/\Omega_h$ is the magnetic drift velocity, E is the energy per unit mass, $\boldsymbol{\kappa} = \hat{\mathbf{e}}_{\|} \cdot \nabla\hat{\mathbf{e}}_{\|}$ is the curvature, $\Omega_h = e_h B/m_h c$ is the cyclotron frequency, and an overhead bar denotes bounce-averaging. The frequency ordering $\bar{\omega}_{dh}/\omega_R \sim \epsilon^{4/3} < 1$ is consistent with large present-generation tokamaks such as JET or TFTR, due to their large geometric dimensions and high magnetic field strengths. Following a procedure first proposed by Glasser, Greene, and Johnson,[12] an averaging strategy involving an expansion of field quantities in powers of λ and ϵ is embraced, which eventually redeems a set of coupled eigenmode equations for the leading-order mean, i.e., DC, quantities as solubility conditions on the higher-order fluctuating, or AC terms. The results are thus valid for $c_s/qR > \gamma > k_{\|}c_s$. In the highly-simplified incompressible limit, the final eigenmode equation is reduced to

$$\Gamma\Phi_{xx} + \Big(\frac{T_\Omega + \tilde{D}_{Rc}}{\Gamma} - x^2\Big)\,\Phi = 0, \qquad (1)$$

where

$$D_{Rc} = \frac{1}{q'^2\Psi'^4}\Big\langle\frac{B^2}{|\nabla V|^2}\Big\rangle\Big[8\pi P_c'\langle\kappa_V\rangle - q'\Psi'^2\frac{\langle\sigma B^2\rangle}{\langle B^2\rangle} + (\Big\langle\frac{\sigma^2 B^2}{|\nabla V|^2}\Big\rangle - \frac{\langle\sigma B^2/|\nabla V|^2\rangle^2}{\langle B^2/|\nabla V|^2\rangle})\Big] \sim O(\epsilon^2{\beta_{pc}}^2)$$

is the well-known measure of resistive stability associated with average curvature ($D_{Rc} < 0 \Leftrightarrow$ favorable average curvature), $\tilde{D}_{Rc} = D_{Rc}/\epsilon^2$,

$$T_\Omega = \frac{2^{7/2}\pi^2}{q'^2\Psi'^4}\Big\langle\frac{B^2}{|\nabla V|^2}\Big\rangle m_h \int_{tr} d(\alpha B)\, \bar{\kappa}_W^2 \tilde{\tau}_b \times \int dE\, E^{3/2}\frac{\omega}{\hat{\omega}_{dh}}\frac{\bar{\omega}_{dh}\partial_E + \hat{\omega}_{*h}}{\omega - \bar{\omega}_{dh}} F_{0h} \sim O(\frac{\beta_{ph}}{\epsilon})$$

is a measure of the energetic trapped-particle pressure, $\Gamma = \gamma/\varpi_R$ is the normalized frequency, $x = (V - V_s)/x_R$ is the radial variable signifying the distance away from a mode rational surface, V is the volume inside the flux surface and plays the role of the global radial coordinate, V_s designates the mode rational surface, $x_R = \lambda\,(nq')^{-1}$ is a characteristic resistive scale length, $\sigma = 4\pi J_\|/cB$ is the normalized equilibrium parallel current, $\kappa_W = \kappa_V - q'\theta\kappa_S$ with κ_V and κ_S the normal and geodesic curvature, respectively, $\hat{\omega}_{*h} = (nm_h c/e_h\Psi')\,\partial\ln F_{0h}/\partial V$ is the diamagnetic drift frequency per unit energy, $\bar{\omega}_{dh} = \hat{\omega}_{dh}E = -(nm_h c/e_h\Psi')\,\kappa_W E$ is the bounce-averaged magnetic precessional drift frequency, q is the safety factor, Ψ is the poloidal magnetic flux, F_{0h} is the trapped-particle distribution function, $\tilde{\tau}_b = \oint d\theta/(1-\alpha B)^{1/2}$ is the normalized bounce time, $\alpha = \mu/E$ is the pitch angle variable, $\langle\cdots\rangle$ denotes a flux-surface average, $\overline{(\cdots)} = \oint d\theta/v_\|$ denotes a bounce-average, and finally, primes and x-subscripts denote differentiation with respect to V and x, respectively. The first term in Eq. (1) represents finite inertial effects associated with the divergence of the polarization current. The second is the trapped-particle pressure drive, and is obtained as a second velocity moment of the trapped-particle distribution function which, in turn, is obtained from solving the drift kinetic equation. The third term comes from the interaction of the core pressure with curvature and is the usual measure of flux-surface-averaged curvature. Finally, the last term represents the influence of field line bending, and comes from the divergence of the parallel current. The linear eigenmode width, which is obtained from balancing the field-line-bending term with the finite inertia term, is simply

$$(\Delta x)_l \sim \Gamma^{1/4}, \tag{2}$$

and the dispersion relation for the lowest (most unstable) eigenstate is given by

$$-\Gamma^{3/2} + \tilde{D}_{Rc} + T_\Omega = 0. \tag{3}$$

We distinguish three interesting limits of Eq. (3):

i) **MHD regime:** In the absence of the trapped-particle component, the usual resistive interchange mode driven by adverse curvature is recovered, i.e,

$$\gamma_{MHD} = \left(\frac{n^2}{S_M}\right)^{1/3} \omega_A \, {D_{Rc}}^{2/3}. \tag{4}$$

This is the dominant unstable mode in experimental machines with unfavorable curvature, such as RFP's and stellarators. As is well-known, tokamaks benefit from favorable average curvature, i.e., $D_{Rc} < 0$, and hence the mode is suppressed in these devices.

ii) **Fluid-like regime:** In the limit where the trapped species is only moderately energetic so that the mode grows faster than the time it takes for the trapped-particle banana-center to precess once around the torus, i.e., $|\omega| \gg \bar{\omega}_{dh}$, a trapped-particle-modified resistive interchange mode is obtained, i.e.,

$$\gamma_F = \left(\frac{n^2}{S_M}\right)^{1/3} \omega_A \left(\epsilon^2 \frac{\hat{\omega}_{*h}}{\hat{\omega}_{dh}} \frac{\beta_{ph}}{\hat{s}^2} + D_{Rc}\right)^{2/3}, \tag{5}$$

where the subscript 'F' denotes "fluid-like." Thus, when the trapped particles undergo unfavorable drift such that $\hat{\omega}_{*h}/\hat{\omega}_{dh} > 0$ (which is the prevalent situation), a purely-growing resistive interchange mode will be promoted even if the system has favorable average curvature. The threshold for instability when the system has favorable average curvature is given by

$$\beta_{ph,cr} = \hat{s}^2 \left(\hat{\omega}_{dh}/\hat{\omega}_{*h}\right) |\tilde{D}_{Rc}|,$$

or $(\epsilon\beta_{ph})_{cr} \sim (\epsilon\beta_{pc})^2 \sim O(\epsilon^2)$, which is an order of magnitude lower than the core plasma beta-threshold for the onset of resistive ballooning modes. For typical TFTR parameters, this threshold is typically always exceeded by core trapped particles. We refer to this regime as the "fluid-like" regime. We remark that these modes, though originating from different physical considerations, are qualitatively similar to the resistive ballooning mode of Connor and Chen,[14] which is derived rigorously from kinetic theory. We are reassured then that the simple model of the plasma employed here is capable of successfully reproducing the essential features of a rigorous, fully kinetic theory.

iii) **Resonant Regime:** Finally consider the case for resonant destabilization. Recalling that $\bar{\omega}_{dh} \propto E$, it is clear that the criterion $|\omega| \sim \bar{\omega}_{dh}$ can easily be satisfied for energetic trapped particles created during auxiliary heating. By the same token however, it should be pointed out that even though the bulk of the core trapped particles fall into the "fluid-like" range, the resonance condition can still be satisfied by the high-energy tail of the distribution. Thus, there is broader interest in examining resonant destabilization than just for auxiliary-heated plasmas. Detailed stability analysis of this instance using Nyquist diagrammatic techniques reveals that an instability will be excited when the hot trapped-particle beta exceeds a threshold given by

$$(\epsilon\beta_{ph})_{cr} \sim \max\Big(|D_{Rc}|, (\bar{\omega}_{dh}/\omega_R)^{3/2}\Big) \sim O(\epsilon^2).$$

This threshold condition was found to be magnitude-wise insensitive to the choice of the trapped particle distribution function. It is important to note that the mode is different from the conventional resistive interchange mode not only in its destabilization mechanism, but also in that it has a real component to its frequency comparable to the magnetic drift frequency of the rapidly-precessing trapped particles, and a growth rate given near marginal stability by

$$\gamma_R \simeq \bar{\omega}_{dm} \Big(\frac{\beta_{ph}}{\beta_{ph,cr}} - 1\Big), \qquad (6)$$

where the subscript 'R' in γ_R denotes "resonant."

III RENORMALIZED NONLINEAR THEORY

Having identified the linear instability, it becomes relevant to explore the level of turbulence that can be expected to evolve from it. We begin, as before, with the set of three equations describing the evolution of the vorticity, the core pressure, and the hot particle distribution function, while retaining the $\mathbf{E} \times \mathbf{B}$ advective nonlinearity in all three equations. Since the mode is electrostatic, this is the only nonlinearity of relevance. This coupled set of equations is then analyzed by renormalizing the nonlinearities through iterative substitution techniques via a coherent approximation to the DIA[19] (to be discussed below), whereby the nonlinear evolution of a "test" trapped-particle-induced mode is tracked through a background spectrum of multiple-helicity turbulence. In this way, the primitive nonlinearities are replaced by phase-independent nonlinear operators which simulate their roles.

As representative, we outline here the derivation of the turbulent viscosity operator, which approximates the vorticity nonlinearity. As in Ref. 11, we adopt (modified Hamada[12]) flux surface coordinates (V, θ, ν), where V is the volume inside a flux surface and is a global radial variable, θ is the poloidal angle, and $\nu = \zeta - q\theta$ is a toroidal-like angle (ζ is the true toroidal angle). In this representation, the magnetic field is given by $B = \Psi' \nabla\nu \times \nabla V$. The nonlinearity can then be written as

$$
\begin{aligned}
N_U &= \rho c^3 \frac{|\nabla V|^2}{B^4} \mathbf{B} \cdot \nabla\delta\phi \times \nabla\delta U \\
&= -\rho c^3 \frac{|\nabla V|^2}{B^4} \mathbf{B} \cdot \nabla V \times \nabla\nu \sum_{n+n'=n''} \left(in''\delta U_{n''} \frac{\partial \delta\phi^*_{n'}}{\partial x} + in'\delta\phi^*_{n'} \frac{\partial \delta U_{n''}}{\partial x} \right) \\
&= i\frac{\rho c^3}{2\Psi'} \frac{|\nabla V|^2}{B^2} \sum_{n+n'=n''} \Big[n' \frac{\partial}{\partial x} (\delta\phi^*_{n'} \delta U_{n''}) + n \, (\frac{\partial \delta\phi^*_{n'}}{\partial x} \delta U_{n''} - \delta\phi_{n''} \frac{\partial \delta U^*_{n'}}{\partial x}) \Big].
\end{aligned}
\tag{7}
$$

Here, $\delta U = \partial^2 \delta\phi / \partial x^2$ is the vorticity. For a continuum of localized modes, we can approximate the sum as an integral, i.e., $\sum_{n'} \to \int dn' \simeq \int dx |m'q'/q^2|$. Moreover, from the sum rule $\mathbf{k} + \mathbf{k}' = \mathbf{k}''$ [$\mathbf{k} = (m, n)$], and an expansion about the mode rational surface, we have $dx''/dx' = n'/n''$, so that we can rewrite

$$\sum_{n+n'=n''} (\delta\phi^*_{n'}\delta U_{n''} - \delta\phi_{n''}\delta U^*_{n'}) \to \sum_{n+n'=n''} \delta\phi^*_{n'} \frac{n^2+2nn'}{n''^2}\,\delta U_{n''}.$$

The essence of the coherent approximation to the DIA[19] is that statistically, the spectrum is sufficiently close to Gaussian that in order to effect closure, we need only extract the lowest-order piece of the nonlinearity that is phase-coherent with the fluctuations at n. In other words, for $\delta U_{n''}$ we iteratively substitute the field that is nonlinearly driven by the direct beat interaction of a test mode n with a mode n' in the bath of background fluctuations. This nonlinearly-induced beat term, for the case of vorticity fluctuations, is given by

$$\delta U^{(2)}_{n''} = -\frac{c}{2\Psi'}\,g^U_{n''}\left[n\,(\frac{\partial\delta\phi_{n'}}{\partial x}\,\delta U_n - \delta\phi_n\,\frac{\partial\delta U_{n'}}{\partial x}) + n'\,(\frac{\partial\delta\phi_n}{\partial x}\,\delta U_{n'} - \delta\phi_{n'}\,\frac{\partial\delta U_n}{\partial x})\right], \tag{8}$$

where the vorticity propagator

$$g^U_{n''} = (\gamma'' + \Delta\omega_U{}'')^{-1}, \tag{9}$$

is an approximation to the "response function" of the DIA theory,[18] and physically represents a dynamical memory of three-mode interactions. The term $\Delta\omega''$ appearing in the propagator, is a nonlinear (i.e., amplitude-dependent) decorrelation time which acts to limit the extent of this dynamical memory. Without it, the theory would allow phase correlation between Fourier modes to persist indefinitely in the advection field of all the other modes, which is clearly unphysical. Iterating Eq. (8) in Eq. (7), and casting all quantities in non-dimensional form, we finally get

$$N_U = \Lambda^{xx}_{n,\Omega} U_{xx} - n^2\Lambda^{\nu\nu}_{n,\Omega} U + C^{xx}_{n,\Omega}\Phi_{xx} - n^2 C^{\nu\nu}_{n,\Omega}\Phi, \tag{10}$$

where

$$\Lambda^{xx}_{n,\Omega} = \frac{1}{4}\sum_{n',\Omega'} G^U_{n''}(\frac{nn'}{n''})^2 \langle\langle \Phi^2 \rangle\rangle_{n'},$$

$$\Lambda^{\nu\nu}_{n,\Omega} = \frac{1}{4}\sum_{n',\Omega'} G^U_{n''}\Big(\frac{n'\lambda}{n''\lambda'}\Big)^2 \langle\langle \Phi^2_{xx} \rangle\rangle_{n'}$$

$$C^{xx}_{n,\Omega} = \frac{1}{4}\sum_{n',\Omega'} G^U_{n''}\Big(\frac{n'^2\lambda}{n''\lambda'}\Big)^2 \langle\langle \Phi^2_{x} \rangle\rangle_{n'}, \tag{11}$$

$$C^{\nu\nu}_{n,\Omega} = \frac{1}{4}\sum_{n',\Omega'} G^U_{n''}\Big(\frac{n'\lambda^2}{n''\lambda'^2}\Big)^2 \langle\langle \Phi^2_{xx} \rangle\rangle_{n'}.$$

In these equations, the potential fluctuation and propagator have been nondimensionalized to $\Phi = (c/\varpi_R x_R \Psi')\,\delta\phi$ and $G^U_{n''} = \varpi_R\, g^U_{n''}$, respectively, the x-subscripts denote differentiation with respect to x, $\Omega = i\Gamma$, and the notation $\langle\langle \cdots \rangle\rangle_{n'}$ represents a spectral average. In deriving the above results, we have exploited symmetry considerations to eliminate terms odd in n' (e.g., $\sum_{n'} n'\Phi^2_{n'} = 0$), and have used the assumption of quasi-normality to remove odd spectral moments (e.g., $\sum_{n'} n'^2 \Phi_{n'} \partial\Phi_{n'}/\partial x = 0$). The first two terms in Eq. (11) are respectively, the radial and poloidal turbulent viscosities, and the second set of terms are their energy-conserving counterparts. We draw the reader's attention to the non-Markovian [i.e., (n, Ω) dependence] character of the turbulent collision operators, which is not only necessary for energy conservation, but also indicative of the temporal and spatial non-locality of the renormalized turbulence theory.

In keeping with our objective to prevent algebraic pollution from obfuscating the physics, we make some simplifying assumptions. To begin with, we argue that since short-wavelength fluctuations can be expected to be dissipated by the turbulence, we may reasonably confine our attention to the evolution of long-wavelength modes in a bath of short-wavelength fluctuations. We thus focus the scope of our investigation to answering the question: What level of turbulence would be required to saturate the growth of a long-wavelength mode, i.e., $n < n_{rms}$, where n_{rms} is the root mean square toroidal mode number, obtained as an average over the spectrum ($n_{rms} = \langle\langle n^2 \rangle\rangle^{1/2}$)? With this assumption, one can easily show that the radial turbulent viscosities are a more efficient vehicle for nonlinear interaction than their poloidal counterparts. A second assumption, one that has implicitly been made in deriving Eq. (10), is to ignore the contributions from the driven potential fluctuations, i.e., $\delta\phi^{(2)}_{n''}$, relative to those from the driven vorticity fluctuations, i.e., $\delta U^{(2)}_{n''}$. The reasoning here follows from an explicit calculation of the driven potential fluctuations, which requires the inversion of differential eigenmode operators. This, in turn, leads to spatial convolution contributions to the renormalized response equations which, due to the radially localized nature of the eigenmodes, are smooth relative to the more singular driven vorticity contributions. Hence, their neglect is justified.

Following these same arguments and analysis through for the other nonlinearities, we finally arrive at the following set of coupled, renormalized equations for the evolution of the vorticity and core pressure fluctuations [$\Pi = \delta p_c/(inx_R P_c')$ is the non-dimensionalized pressure fluctuation]

$$\Gamma\Phi_{xx} - \Lambda^{xx}_{n,\Omega}\Phi_{xxxx} + \Big[\frac{T^{nl}_{\Omega}(x)}{\Gamma} - x^2\Big]\Phi = -\tilde{D}_{Rc}\Pi, \tag{12}$$

$$\Big(\Gamma - \Delta^{xx}_{n,\Omega}\frac{\partial^2}{\partial x^2}\Big)\Pi = \Phi, \tag{13}$$

where the nonlinear trapped-particle pressure response has been incorporated into the vorticity equation as

$$T^{nl}_{\Omega}(x) = \frac{2^{7/2}\pi^2}{q'^2\Psi'^4}\Big\langle\frac{B^2}{|\nabla V|^2}\Big\rangle m_h \int_{tr} d(\alpha B)\,\bar{\kappa}^2_W\tilde{\tau}_b \times \int dE\, E^{3/2}\frac{\omega}{\bar{\omega}_{dh}}\,\frac{\bar{\omega}_{dh}\partial_E + \hat{\omega}_{*h} + i\varpi_R\mathcal{B}^{xx}_{n,\Omega}\partial^2_x}{\omega - \bar{\omega}_{dh} + i\varpi_R\mathcal{D}^{xx}_{n,\Omega}\partial^2_x}F_{0h}.$$

The terms

$$\Delta^{xx}_{n,\Omega} = \frac{1}{4}\sum_{n',\Omega'} G^P_{n''}n'^2\langle\langle\Phi^2\rangle\rangle_{n'}, \tag{14a}$$

$$\mathcal{D}^{xx}_{n,\Omega} = \frac{1}{4}\sum_{n',\Omega'} G^H_{n''}n'^2\langle\langle\Phi^2\rangle\rangle_{n'}, \tag{14b}$$

$$\mathcal{B}^{xx}_{n,\Omega} = \frac{1}{4}\sum_{n',\Omega'} G^H_{n''}n'^2\Big\langle\Big\langle\frac{\bar{\omega}_{dh}'\partial_E + \hat{\omega}_{*h}'}{\omega' - \bar{\omega}_{dh}'}\Phi^2\Big\rangle\Big\rangle_{n'}, \tag{14c}$$

are respectively, the turbulent core pressure diffusivity, the turbulent trapped-particle pressure diffusivity, and finally, a term which takes account of the renormalization of the background trapped-particle distribution function. In deriving Eqs. (12) and (13), we have, as before, invoked the highly-localized nature of the eigenmodes to keep only the dominant (radial) differential operators. The propagators in Eq. (14) are given by

$$G^P_{n''} = (\Gamma'' + \Delta\Omega''_P)^{-1},$$
$$G^H_{n''} = (\Gamma'' - \bar{\Gamma}_{dh}{}'' + \Delta\Omega''_H)^{-1},$$

where $\bar{\Gamma}_{dh}{}'' = i\bar{\omega}_{dh}{}''/\varpi_R$, and the $\Delta\Omega''$ are the (nonlinear scrambling) decorrelation frequencies, as discussed earlier. We have adopted the convention of writing all the diffusion coefficients in non-dimensional form. For convenience, we note here that these are related to the dimensional coefficients $D = \sum_{n'} g_{n''} \langle\langle v_r^2\rangle\rangle_{n'}$ through $D = (x_R^2\varpi_R/|\nabla V|^2)\,\mathcal{D}$. Useful progress can be made by transforming to Fourier (k-) space, where we can combine Eqs. (12) and (13) into a single renormalized eigenmode equation:

$$\tilde{\Phi}_{kk} + \Big[\frac{T^{nl}_{\Omega}(k)}{\Gamma} + \frac{\tilde{D}_{Rc}}{\Gamma + \Delta^{xx}_{n,\Omega}k^2} - \Gamma k^2 - \Lambda^{xx}_{n,\Omega}k^4\Big]\,\tilde{\Phi} = 0, \tag{15}$$

where

$$T^{nl}_{\Omega}(k) = \frac{2^{7/2}\pi^2}{q'^2\Psi'^4}\Big\langle\frac{B^2}{|\nabla V|^2}\Big\rangle m_h \int_{tr} d(\alpha B)\,\bar{\kappa}_W^2\bar{\tau}_b$$
$$\times \int dE\,E^{3/2}\frac{\omega}{\bar{\omega}_{dh}}\,\frac{\bar{\omega}_{dh}\partial_E + \hat{\omega}_{*h} - i\varpi_R\mathcal{B}^{xx}_{n,\Omega}k^2}{\omega - \bar{\omega}_{dh} - i\varpi_R\mathcal{D}^{xx}_{n,\Omega}k^2}\,F_{0h},$$

and $\tilde{\Phi}(k) = \int_{-\infty}^{+\infty} dx\,\exp(ikx)\Phi(x)$. Following the discussion of Sec. II, we find it convenient, at this juncture, to continue our analysis by distinguishing between two classes of trapped particles: moderately-energetic and highly-energetic trapped particles.

III.A Fluid-like Regime ($|\omega| \gg \bar{\omega}_{dh}$)

First, consider the more tractable "fluid-like" limit, i.e., $|\omega| \gg \bar{\omega}_{dh}$. In that case,

$$\mathcal{B}^{xx}_{n,\Omega} \simeq \frac{1}{4}\sum_{n',\Omega'}(\Gamma'' + \Delta\Omega''_H)^{-1}\,n'^2\,\frac{\hat{\omega}_{*h}{}'}{\omega'}\langle\langle\Phi^2\rangle\rangle_{n'} \propto \sum_{n'} n'^3\Phi^2_{n'} \equiv 0,$$

and the problem simplifies to the following form

$$\tilde{\Phi}_{kk} + \Big[\frac{(\hat{\omega}_{*h}/\hat{\omega}_{dh})\,\beta_{ph}/\hat{s}^2}{\Gamma + \mathcal{D}^{xx}_{n,\Omega}k^2} + \frac{\tilde{D}_{Rc}}{\Gamma + \Delta^{xx}_{n,\Omega}k^2} - \Gamma k^2 - \Lambda^{xx}_{n,\Omega}k^4\Big]\,\tilde{\Phi} = 0, \tag{16}$$

where

$$\frac{\beta_{ph}}{\hat{s}^2} = \frac{2^{7/2}\pi^2}{q'^2\Psi'^4}\Big\langle\frac{B^2}{|\nabla V|^2}\Big\rangle m_h \int_{tr} d(\alpha B)\,\bar{\kappa}^2_W \tilde{\tau}_b \int dE\,E^{3/2}\,F_{0h},$$

β_{ph} is the trapped-particle poloidal beta, and $\hat{s} = V_s q_s'/q_s$ is the shear parameter with $q_s = q(V_s)$ evaluated at the mode rational surface. At saturation (i.e., $\Gamma = 0$), Eq. (16) can be solved in terms of integer-order Bessel functions. Approximating $\mathcal{D}^{xx}_{n,\Omega} \sim \Delta^{xx}_{n,\Omega}$, the eigenvalue condition is given by

$$\mathcal{D}^{xx}_{n,\Omega} \simeq 4\Big(\frac{\hat{\omega}_{*h}}{\hat{\omega}_{dh}}\frac{\beta_{ph}}{\hat{s}^2} - |\tilde{D}_{Rc}|\Big), \tag{17}$$

where as usual, we have specialized to the case of favorable average curvature. The mode-width at saturation, determined now by the balance of field-line-bending and viscous diffusion, is

$$(\Delta x)_{nl} \sim (\Lambda^{xx}_{n,\Omega})^{1/6}. \tag{18}$$

Note, in particular, that the nonlinear mode width is amplitude-dependent. The dynamics can heuristically be described as one of cascade from dominant long-wavelength trapped-particle-driven modes, to a short-wavelength viscous energy sink, with the transfer process itself dynamically regulated by turbulent $\mathbf{E}\times\mathbf{B}$ advection. To prevent any confusion, it serves to remark here that although we have written the $\mathbf{E}\times\mathbf{B}$ nonlinearity as a turbulent diffusion operator, it should not be construed to be dissipative in the *strict* sense of the word. This is clear physically, since particles and waves cannot exchange electrostatic energy through purely $\mathbf{E}\times\mathbf{B}$ perpendicular motion. In other words, since the electrostatic field can do no work on the advective current ($\mathbf{J}_\perp\cdot\mathbf{E}_\perp = 0$), there can be no heating. On the other hand, we may think of the $\mathbf{E}\times\mathbf{B}$ nonlinearity as dissipative in the weaker sense that it allows a mode to lose energy by coupling to the other modes in the system.

The predicted diffusion coefficient derived above is about four times the linear mixing-length result, i.e., $\mathcal{D}^{xx}_{n,\Omega} \sim 4\gamma_l(\Delta x)^2_l$, where $(\Delta x)_l$, and γ_l are the linear mode width and growth rate, given by Eqs. (2) and (5), respectively. The source of this discrepancy can be sought in the nonlinear broadening of the eigenmodes beyond their linear values.[3] Previous numerical simulations of resistive pressure-gradient-driven turbulence[3] have found that while the linear mixing-length result correctly predicts the parameter scalings, it falls short, magnitude-wise, of the numerically-obtained fluctuation amplitude. It is, therefore, at the least reassuring to find that in spite of the approximations that have gone into the theory, the final answer is compatible with simulation results, and predicts an enhancement over and above the linear mixing-length estimate. The conventional wisdom of the linear mixing-length estimate as an *upper* bound on the level of turbulence, based on using the *linear* mode width and the fastest-growing *linear* growth rate, is thus seriously called into question.

We are now adequately equipped to make an estimate of the anomalous electron thermal conductivity. We assume that magnetic fluctuations are induced by these electrostatic modes via a coupling through the parallel Ohm's law. The perturbed radial magnetic field (δB_V) is related to the parallel vector potential by

$$\frac{\delta B_V}{B} = \frac{\mathbf{B} \times \nabla \delta A_{\|}}{B^2}.$$

The flux-surface-averaged Ohm's law, in non-dimensionalized form, is given by

$$A_{xx} = \frac{\epsilon}{\Psi'} (\mathbf{B} \cdot \nabla)\, \Phi = -i\epsilon^{4/3} x\, \Phi,$$

where $A = (nq'/\lambda^2 \epsilon^{4/3} \Psi'^2)\, \delta\mathbf{A} \cdot \mathbf{B}$ is the non-dimensionalized parallel vector potential. Thus, magnitude-wise,

$$|A| \simeq \epsilon^{4/3} (\Delta x)^3 |\Phi|. \tag{19}$$

To determine the spectrum of the potential fluctuations with any degree of confidence would require a two-point correlation theory,[20] which is beyond the scope of the present investigation. Nonetheless, we can obtain a rough estimate of $|\Phi|$ from Eqs. (14b) and (17):

$$|\Phi| \sim \frac{4\Gamma\, \Delta x}{n_{rms}}. \tag{20}$$

Putting Eqs. (19) and (20) together, we arrive at an estimate for the magnetic fluctuation level:

$$\left|\frac{\delta B_V}{B}\right| \simeq 4\lambda^2 \epsilon^{11/3} \frac{\Gamma(\Delta x)^4}{n_{rms} q_s \hat{s}}. \tag{21}$$

An expression for the (collisionless) anomalous electron thermal conductivity can be straightforwardly derived from the drift kinetic equation for the electrons:

$$\chi_e \simeq \sum \frac{i v_\parallel^2 \left\langle\left\langle \left(\delta B_V/B\right)^2 \right\rangle\right\rangle}{\omega_r' - k_\parallel' v_\parallel + i/\tau_c}, \tag{22}$$

where τ_c, the wave-particle decorrelation time, is recursively defined in terms of the thermal conductivity, and we may roughly approximate this relationship by $\tau_c \sim x^2/\chi_e$. We postulate a strong turbulence situation where the nonlinear scrambling time (τ_c) is the shortest time scale in the problem. This assumption must then be verified aposteriori for consistency. The anomalous electron thermal conductivity is then given by

$$\chi_e \simeq \sum v_\parallel^2 \left\langle\left\langle \left(\frac{\delta B_V}{B}\right)^2 \right\rangle\right\rangle \tau_c,$$

and after substituting for τ_c and some rearrangement,

$$\chi_e \simeq \left[\sum v_\parallel^2 \left\langle\left\langle \left(\frac{\delta B_V}{B}\right)^2 \right\rangle\right\rangle (\Delta x)^2\right]^{1/2} \simeq C_1 v_{te} \left|\frac{\delta B_V}{B}\right| \Delta x, \tag{23}$$

where C_1 is a measure of the spectral sum. Using Eqs. (2), (5), and (21), we finally arrive at the following estimate for the anomalous electron thermal conductivity:

$$\chi_e \sim C_0 \left(q_s \hat{s}\right)^{-2} \frac{a v_t}{S_M} \epsilon \left(\epsilon^2 \frac{\hat{\omega}_{*h}}{\hat{\omega}_{dh}} \frac{\beta_{ph}}{\hat{s}^2} - |D_{Rc}|\right)^{3/2}. \tag{24}$$

In Eq. (24), C_0 is a numerical coefficient that depends on the magnitude of the spectral sums and other numbers that enter into the analysis. As mentioned before, C_0 must be obtained from a two-point analysis and, base on previous experience with such problems,[20] the spectral coefficients can be anticipated to be quite large. In order of magnitude, we estimate it here to be on the order of 10. For typical TFTR parameters, $T_e \simeq 1\ keV$,

$S_M \sim 10^7$; $\beta_p \simeq 1.85$, $a = 80\, cm$, $\epsilon = .33$, and $|D_{Rc}| \simeq .2$, we obtain a value on the order of $10^4\ cm^2/sec$. This value agrees reasonably with typical experimental measurements. Moreover, noting that $\chi_e(r) \propto T_e^{-1}(r)$, we see that the radial profile of χ_e increases as one moves outward in radius, again in accordance with experimental observations. On the other hand, it is fair to point out that inclusion of finite Larmor radius (FLR) effects will act to reduce the estimated value of χ_e.[21]

The expression, Eq. (24), was obtained under the assumption of strong turbulence, i.e., $\tau_c^{-1} > k_\parallel v_\parallel$. To check the consistency of this assumption, we note that

$$\tau_c^{-1} \sim \frac{\chi_e}{x^2} \simeq \frac{v_{te}}{x}\left|\frac{\delta B_V}{B}\right| \simeq C_0\left(\frac{n^2}{S_M}\right)^{1/3} \frac{v_{te}}{a}\,\epsilon\left(\epsilon^2\,\frac{\hat{\omega}_{*h}}{\hat{\omega}_{dh}}\,\frac{\beta_{ph}}{\hat{s}^2} - |D_{Rc}|\right)^{7/6} \sim 2\times 10^5\, sec^{-1},$$

while

$$k_\parallel v_\parallel \simeq k_\theta\,\frac{x}{qR}\,v_{te} \simeq \left(\frac{n^2}{S_M}\right)^{1/3} \frac{v_{te}}{q_s \hat{s} R}\left(\frac{\hat{\omega}_{*h}}{\hat{\omega}_{dh}}\,\frac{\beta_{ph}}{\hat{s}^2} - |\tilde{D}_{Rc}|\right)^{1/6} \sim 6\times 10^4\, sec^{-1}.$$

The assumption is therefore vindicated.

III.B Resonant Interaction ($|\omega| \sim \bar{\omega}_{dh}$)

More work is needed to derive an answer for the case of resonant interaction. For concreteness, we assume a model slowing-down distribution function of the form

$$F_{0h} = \frac{P_h(V)}{\sqrt{2}\pi m_h B \tilde{\tau}_{b0} E_m}\, E^{-3/2}\delta(\alpha - \alpha_0), \qquad E \le E_m. \tag{25}$$

Such a distribution function can, for example, model the slowing down of energetic ions injected into a tokamak during perpendicular neutral beam injection. In Eq. (25), E_m is the injection energy, α_0 the injection angle, and $\tilde{\tau}_{b0} = \tilde{\tau}_b(\alpha = \alpha_0)$. Analytic progress is made possible only by making some simplifying assumptions in order to relate the various turbulent diffusion coefficients together. Since we are primarily interested in resonant interactions, we may plausibly assume that the propagator $G^H_{n''}$ is highly localized with respect to the width of the potential spectrum. This assumption allows us to approximate the propagator by

$$G^H_{n''} \simeq \pi\varpi_R \delta(\omega'' - \bar{\omega}_{dh}'') = \pi\varpi_R\delta\Big[(\omega' - \bar{\omega}_{dh}') + (\omega - \bar{\omega}_{dh})\Big], \tag{26}$$

where $\delta(\cdots)$ is the Dirac Delta function, and thus relate $\mathcal{B}^{xx}_{n,\Omega}$ to $\mathcal{D}^{xx}_{n,\Omega}$:

$$\mathcal{B}^{xx}_{n,\Omega} \simeq -\frac{n_{rms}}{n}\frac{\hat{\omega}_{*h}}{\omega - \bar{\omega}_{dh}}\mathcal{D}^{xx}_{n,\Omega}.$$

The energetic trapped-particle pressure term can now be straightforwardly evaluated as

$$T^{nl}_{\Omega}(k) = \frac{\omega}{\bar{\omega}_{dm}}\frac{\hat{\omega}_{*h}}{\hat{\omega}_{dh}}\frac{\beta_{ph}}{\hat{s}^2}\Big[(1+\frac{n_{rms}}{n})\ln\Big(1-\frac{\bar{\omega}_{dm}}{\omega - i\varpi_R\mathcal{D}^{xx}_{n,\Omega}k^2}\Big)-\frac{n_{rms}}{n}\ln\Big(1-\frac{\bar{\omega}_{dm}}{\omega}\Big)\Big],$$

where now

$$\frac{\beta_{ph}}{\hat{s}^2} = \frac{8P_h(V)\,\bar{\kappa}^2_{W0}}{q'^2\Psi'^4}\Big\langle\frac{B^2}{|\nabla V|^2}\Big\rangle,$$

$\bar{\omega}_{dm} = \bar{\omega}_{dh}(E = E_m)$, and $\bar{\kappa}_{W0} = \bar{\kappa}_W(\alpha = \alpha_0)$. As before, we confine our attention to long-wavelength modes ($n < n_{rms}$), and assume, again for simplicity, that $\Lambda^{xx}_{n,\Omega} \simeq (n/n_{rms})^2\,\mathcal{D}^{xx}_{n,\Omega}$, and $\Delta^{xx}_{n,\Omega} \simeq \mathcal{D}^{xx}_{n,\Omega}$. Looking at the steady-state limit where $\omega_r \simeq \bar{\omega}_{dm}/2$, $\gamma = 0$, we finally obtain

$$\tilde{\Phi}_{\xi\xi} + \frac{1}{\mathcal{D}^{xx}_{n,\Omega}}\Big[i\delta^{-1}\frac{\hat{\omega}_{*h}}{\hat{\omega}_{dh}}\frac{\beta_{ph}}{\hat{s}^2}\ln\Big(1-\frac{2}{1-2i\xi^2}\Big) - \frac{2|\tilde{D}_{Rc}|}{-i+2\xi^2} + \frac{i}{2\mathcal{D}^{xx}_{n,\Omega}R^3}\xi^2 - \frac{\delta^2}{R^3\mathcal{D}^{xx}_{n,\Omega}}\xi^4\Big]\tilde{\Phi} = 0, \qquad (27)$$

where $\xi = (\mathcal{D}^{xx}_{n,\Omega}R)^{1/2}\,k$, $\delta = n/n_{rms} < 1$, and $R = \varpi_R/\bar{\omega}_{dm} \sim 1$. Equation (27) is to be interpreted as a complex eigenvalue equation for the turbulent diffusion coefficient $\mathcal{D}^{xx}_{n,\Omega}$. The equation is too complicated to solve analytically and is currently being investigated numerically. We may make further progress, however, by assuming $\mathcal{D}^{xx}_{n,\Omega}$ as given, and proceed to discuss what form of transport can be expected to evolve from this situation.

The key feature which sets the present mode apart from all other (fluid-like) varieties of the resistive interchange-ballooning family of instabilities is the fact that it has a large reactive component to its frequency, on the order of the precessional drift frequency of the energetic trapped particles. This characteristic has a number of interesting consequences for transport. To begin with, hot-particle electrostatic diffusion is decoupled from bulk-particle transport. This can be seen by noting the difference between Eq. (26), which when substituted into Eq. (14b) shows that the energetic species undergoes resonant diffusion, and Eq. (9), which when substituted into Eq. (14a) indicates that the bulk diffuses non-resonantly. More quantitatively, the quasilinear (Markovian) limit of the turbulent diffusion operators, i.e.,

$$\begin{aligned}\Delta_0^{xx} &\simeq \sum_{n'} n'^2 \left\langle\langle\Phi^2\rangle\right\rangle_{n'} \frac{\varpi_R\,\Delta\omega'}{\bar{\omega}_{dh}'^2},\\ \mathcal{D}_0^{xx} &\simeq \sum_{n'} n'^2 \left\langle\langle\Phi^2\rangle\right\rangle_{n'} \frac{\varpi_R}{\bar{\omega}_{dh}},\end{aligned} \tag{28}$$

implies $|\Delta_0^{xx}|/|\mathcal{D}_0^{xx}| \sim O(\Delta\omega/\bar{\omega}_{dh}) \ll 1$, which in turn, clearly brings out the fact that hot-particle electrostatic diffusion dominates bulk transport.

The second interesting consequence relates to magnetic flutter transport. To estimate the level of magnetic fluctuations, we assume, as stated before, that $\mathcal{D}^{xx}$ is given. Then, we may estimate the level of potential fluctuations

$$|\Phi| \sim n_{rms}^{-1}\left(\frac{\bar{\omega}_{dh}}{\varpi_R}\,\mathcal{D}^{xx}\right)^{1/2},$$

which in turn implies

$$\left|\frac{\delta B_V}{B}\right| \sim \lambda^2\epsilon^{11/3}\,\frac{(\Delta x)^3}{n_{rms}q_s\hat{s}}\left(\frac{\bar{\omega}_{dh}}{\varpi_R}\,\mathcal{D}^{xx}\right)^{1/2}. \tag{29}$$

Comparing equations (21) and (29), it is noteworthy that for $\mathcal{D}^{xx} \sim \gamma_R \times(\Delta x)^2/\varpi_R$, with γ_R given by Eq. (6), the level of magnetic fluctuations in the resonant case is smaller by a factor $\gamma_R/\bar{\omega}_{dh} \ll 1$ than in the fluid-like case. The anomalous electron conductivity is given by Eq. (22), where we now have a large real frequency competing with the electron transit and nonlinear scambling frequencies in the denominator. Depending on the relative ordering among these frequencies, we envision two possible transport scenarios. In the event that $\bar{\omega}_{dh} < k_\parallel v_\parallel + i/\tau_c$, i.e., a strongly turbulent situation where electrons decorrelate rapidly from field lines, there is no qualitative difference between the present regime and the "fluid-like" case. Following through with the analysis as in that instance, we obtain

$$\chi_e^S \simeq C_0\,(q_s\hat{s})^{-2}\,\frac{av_t}{S_M}\,\epsilon^2\left(\frac{\bar{\omega}_{dh}}{\omega_R}\right)^{3/2}\sqrt{\mathcal{D}^{xx}} \sim 10^4\,\sqrt{\mathcal{D}^{xx}}\;cm^2/sec, \tag{30}$$

where the numerical estimate is for TFTR parameters. In the opposite limit where $\bar{\omega}_{dh} > k_\parallel v_\parallel + i/\tau_c$, a high-frequency, oscillatory motion replaces the previous picture of irreversible parallel diffusion along perturbed field lines. This oscillatory motion gradually detunes the wave-particle resonance, and thereby acts to reduce the transport level.[16] More quantitatively,

$$\chi_e^W \sim C_0 \frac{v_{te}^2}{\bar{\omega}_{dh}} \left|\frac{\delta B_V}{B}\right|^2.$$

We note that unlike the previous case of strong turbulence where the scaling of χ_e with fluctuation amplitude was linear, here it is quadratic. This is characteristic of a weak turbulence regime. Substituting for the fluctuation amplitude from Eq. (29), we finally have

$$\chi_e^W \sim C_0 \frac{v_{te}^2}{\bar{\omega}_{dh}} \frac{n_{rms}^{5/3}\epsilon^4}{{S_M}^{4/3} q_s \hat{s}} \left(\frac{\bar{\omega}_{dh}}{\omega_R}\right)^{5/2} \mathcal{D}^{xx} \sim 3 \times 10^3 \mathcal{D}^{xx}. \tag{31}$$

Clearly, these two scenarios are very different from each other. To get a better feeling for which of the two is the better description of the experimental situation, we need to compare the relative magnitudes of the frequencies in question. Using deutrium (denoted by subscript 'D') as the hot particle species, we make some estimates for TFTR parameters:

$$\bar{\omega}_{dh} \simeq \frac{k_\theta \rho_D}{R} v_D \sim 2 \times 10^5\, sec^{-1},$$

$$k_\| v_\| \simeq k_\theta \frac{x}{qR} v_{te} \simeq \left(\frac{n^2}{S_M}\right)^{1/3} (\hat{s} q R)^{-1} \left(\frac{\bar{\omega}_{dh}}{\varpi_R}\right)^{1/4} \sim 6 \times 10^4\, sec^{-1},$$

and

$$\tau_c^{-1} \simeq \frac{\chi_e^S}{x^2} \simeq C_0 \left(\frac{n^2}{S_M}\right)^{1/3} \frac{v_{te}}{a} \frac{\bar{\omega}_{dh}}{\omega_R} \epsilon^2 \sqrt{\mathcal{D}^{xx}} \sim 10^5 \sqrt{\mathcal{D}^{xx}}\, sec^{-1}.$$

The TFTR case thus appears to fall somewhere between the two scenarios we have presented. On the other hand, we have been evaluating the precessional drift frequency at the maximum (i.e., injection) energy. The energetic particles will, of course, be slowing down in energy. The strong turbulence scenario may thus be the more relevant of the two.

Before concluding this section, we comment on two other points. First, that the rapid perpendicular bounce motion of the trapped species makes them oblivious to parallel variation. In other words, trapped particles average over the magnetic flutter along perturbed field lines, and hence, fail to be affected by it. Second, since $\omega_{*c}/\bar{\omega}_{dh} \sim T_c/\epsilon T_h \ll 1$, finite Larmor radius effects can exert no influence on the transport processes described here. This is in marked contrast to the case discussed in the last subsection, as well to all previous theories of resistive pressure-gradient-driven turbulence.

IV SUMMARY AND CONCLUSION

We have attempted, in this work, to analyze the transport that is expected to evolve from trapped-particle-induced localized resistive interchange modes in tokamak configurations. The calculation represents the first analytic investigation of transport which incorporates the effect of energetic, trapped particles. Our results are summarized in Table 1. We have found it convenient to discriminate between two regimes of trapped-particle energy: *i)* a moderately-energetic regime, which is "fluid-like" in the sense that the unstable mode grows faster than the time it takes for particles in this energy range to precess once around the torus (i.e., $|\omega| \gg \bar{\omega}_{dh}$), and *ii)* a highly-energetic regime, where the trapped species has sufficiently high energy so as to be able to resonantly interact with the mode (i.e., $|\omega| \sim \bar{\omega}_{dh}$). The "fluid-like" regime engenders a purely-growing mode when the pressure carried by the unfavorably-drifting trapped particles overcomes the stabilizing contribution associated with favorable average curvature. The resonant case distinguishes itself not only by its different destabilization mechanism (i.e., wave-particle interaction), but also by the fact that it has a real component to its frequency on the order of the precessional drift frequency of the trapped species. In both instances, there is an instability threshold associated with the poloidal beta which, magnitude-wise, is significantly lower than that associated with resistive ballooning modes. In the linear theory of these modes, the radial mode-width is obtained from a balance between field-line-bending and finite inertia, and the mode is found to be highly localized. Nonlinearly, however, the mode is broadened due to turbulent viscous dissipation of its high-k_r components, so that the width at saturation is determined by a balance between field-line-bending and viscous diffusion. This fact then leads to an estimate for the turbulent diffusivity that is larger than the linear mixing-length estimate. While the latter correctly reproduces the parameter scaling of the diffusion coefficient, the conventional wisdom that it provides an *upper* bound on the transport level is, at best, misleading and unsubstantiated.

The nonlinear evolution of the mode can be described in the following way. Before reaching saturation, the spectrum gains energy from the trapped-ion pressure gradient source of free energy. The long-wavelength trapped-particle-driven modes nonlinearly couple to the short-wavelength viscous energy sink, with the cascade process dynamically regulated by turbulent $\mathbf{E} \times \mathbf{B}$ advection. Viscous dissipation, and hence the radial extent of the localized mode, both increase. This process continues until energy drain by nonlinear coupling is just sufficient to balance the pressure drive, at which point saturation is achieved. Our entry into the sixth line of Table 1 is the level of electrostatic turbulence that is required to achieve this saturated state. Our results indicate that bulk and trapped-particle electrostatic transport track each other in the fluid-like regime. Of more interest is the resonant regime. Here, the hot-particle transport is resonant and exceeds, in magnitude, the level of bulk-particle electrostatic transport, which remains non-resonant.

The electrostatic instability couples to magnetic flutter and generates anomalous electron transport along perturbed field lines. We have evaluated and entered into column 8 of the table, the poloidal component of magnetic fluctuations. Magnitude-wise, $|\delta B_\theta / B_\theta| \sim 10^{-4}$, which again is consistent with experimental findings. Using our expression for the magnetic fluctuations, we have derived an estimate for the anomalous electron thermal conductivity. In the case of the "fluid-like" regime, we have a situation of strong turbulence where the expression for χ_e is found to exhibit a parameter scaling similar to the resistive ballooning turbulence theory of Carreras and Diamond.[6] For typical TFTR parameters, we compute a χ_e value on the order of 10^4 cm^2/sec, which is not inconsistent with experimental measurements. The resonant regime is more complicated, since the frequency now has a reactive component which competes with the parallel transit and nonlinear decorrelation frequency of the electrons. Depending on the relative ordering between $\bar{\omega}_{dh}$ and $k_\| v_\| + i/\tau_c$, we have identified two possible transport scenarios. In the event that $\bar{\omega}_{dh} < k_\| v_\| + i/\tau_c$, we again envision a strongly turbulent situation, qualitatively similar to the fluid-like regime. If, however, the reverse inequality holds, then there will be a reduction in the transport level *à la* runaway transport theory,[16] where a high-frequency, oscillatory motion replaces the previous picture of irreversible electron diffusion along perturbed field lines. In all the above cases, $\chi_e(r) \propto T_e^{-1}(r)$, so that the radial χ_e profile increases with radius, again in conformity with experimental observations. As for FLR effects, we note that while they act to reduce our χ_e estimates in the "fluid-like" regime, they are ineffectual in the resonant regime since the real part of the frequency in that case is much larger than the core diamagnetic drift frequency. Another interesting point is that since the trapped particles do not "see" the magnetic flutter due to their rapid bounce motion, the estimates of transport obtained here do not suffer from criticisms of non-self-consistency, which has been directed at previous theories of resistive pressure-gradient-driven turbulence. Finally, although the experimantal focus of attention in this paper has been on tokamaks, it may be expected that the level of losses due to these modes for such unfavorable-curvature (i.e., $D_{Rc} > 0$) devices as RFP's or stellarators would be more severe, given that the linear instability would then be threshold-less. In conclusion, we remark that the present theory, in conjunction with the neoclassical[13,14] and collisional resistive MHD[6] theories of resistive interchange-ballooning turbulence, scan a broad spectrum of parameters across the radial profile of the plasma, and when put together, provide a comprehensive theory of resistive pressure-gradient-driven turbulence.

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	Fluid-like Regime ($\lvert\omega\rvert \gg \bar{\omega}_{dh}$)	*Resonant Regime* ($\lvert\omega\rvert \sim \bar{\omega}_{dh}$)	
ω_r	0	$\bar{\omega}_{dh}$	
γ	$\left(\frac{n^2}{S_M}\right)^{1/3} \omega_A \left(\epsilon\beta_{ph} - \lvert D_{Rc}\rvert\right)^{2/3}$	$\bar{\omega}_{dh}\left(\frac{\beta_{ph}}{\beta_{ph,cr}} - 1\right)$	
$(\epsilon\beta_{ph})_{cr}$	$\lvert D_{Rc}\rvert \sim O(\epsilon^2)$	$\max\left(\lvert D_{Rc}\rvert, (\frac{\omega_{dh}}{\omega_R})^{3/2}\right) \sim O(\epsilon^2)$	
x_l	$\frac{a}{q_s\hat{s}}\,(nS_M)^{-1/3}\,\epsilon^{-2}\left(\epsilon\beta_{ph} - \lvert D_{Rc}\rvert\right)^{1/4}$	$\frac{a}{q_s\hat{s}}\,(nS_M)^{-1/3}\left(\frac{\bar{\omega}_{dh}}{\omega_R}\right)^{1/4}$	
x_{nl}	$\frac{a}{q_s\hat{s}}\,(\epsilon/nS_M)^{1/3}\,(\Lambda^{xx})^{1/6}$		
D^{xx}	$\frac{a^2}{\tau_R}\,4\left(\epsilon\beta_{ph} - \lvert D_{Rc}\rvert\right)/(q_s\hat{s})^2$	solution of complex eigenvalue equation [Eq. (27)]	
$\lvert\Delta^{xx}\rvert/\lvert\mathcal{D}^{xx}\rvert$	1	$\frac{\Delta\omega}{\bar{\omega}_{dh}} \ll 1$	
$\lvert\delta B_\theta/B_\theta\rvert$	$4\,(nS_M)^{-1/3}\left(\epsilon\beta_{ph} - \lvert D_{Rc}\rvert\right)^{7/6}$	$(nS_M)^{-1/3}\,\epsilon\,\frac{\omega_{dh}}{\omega_R}\sqrt{\mathcal{D}^{xx}}$	
Transport ordering	$\tau_c^{-1} > k_\parallel v_\parallel \gg \omega_r$	$k_\parallel v_\parallel + i\tau_c^{-1} > \bar{\omega}_{dh}$	$k_\parallel v_\parallel + i\tau_c^{-1} < \bar{\omega}_{dh}$
Turbulence regime	Strong Turbulence	Strong Turbulence	Weak Turbulence
χ_e	$\frac{C_0}{(q_s\hat{s})^2}\,\frac{a v_{te}}{S_M}\,\epsilon\left(\epsilon\beta_{ph} - \lvert D_{Rc}\rvert\right)^{3/2}$	$\frac{C_0}{(q_s\hat{s})^2}\,\frac{a v_{te}}{S_M}\,\epsilon^2\left(\frac{\bar{\omega}_{dh}}{\omega_R}\right)^{3/2}\sqrt{\mathcal{D}^{xx}}$	$C_0\,\frac{v_{te}^2}{\bar{\omega}_{dh}}\,\frac{n^{5/3}\epsilon^4}{S_M{}^{4/3} q_s\hat{s}}\left(\frac{\bar{\omega}_{dh}}{\omega_R}\right)^{5/2}\mathcal{D}^{xx}$
FLR reduction?	Yes	No	
Self-consistent?	Yes	Yes	

TABLE I.