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AREA AS A DEVIL'S STAIRCASE IN TWIST MAPS

Q. Chen
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

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Abstract In area-preserving maps, the area under an invariant set as a function of frequency is a devil's staircase. We show this staircase is the derivative of the average action of the invariant set with respect to frequency. This implies that resonances fill the phase space completely when there are no invariant curves.

The study of transport in low degree-of-freedom Hamiltonian systems is of considerable importance to various fields of physics[1] (eg. plasma confinement, particle accelerators, celestial mechanics, etc). Typically, when the system is near the integrable case, there are invariant tori almost everywhere, and they act as absolute barriers to transport; therefore, no global transport is observed. However, as the perturbation increases, some of the tori are broken and become invariant cantor sets (or "cantori") which don't restrict the global transport any more. The region over which transport can occur is limited by the remaining tori; as these are successively destroyed, this can become the entire phase space.[2] We call this situation "supercritical."

Since the phase space is usually a complex mixture of regular and irregular regions, we need to divide it into different "quasi-local" states to account for the regular behavior (or stickiness). Recently, it was shown that such a decomposition is possible: the states are called "resonances."[3] They are the regions bounded by partial separatrices, and they never overlap. Transport takes place by transition between resonances whose turnstiles overlap. Combined with the flux theory,[4] this model gives us a detailed description of global transport as successive jumps between different resonances. It also successfully predicts various statistical quantities.

In order that resonances take over the whole phase space in the supercritical case, the area under an invariant as a function of its frequency should be a complete devil's staircase (see the following for details). Indeed, numerical calculation of the area function by MacKay et al. indicates this.[3] We show here that the area staircase is complete in the supercritical case. In fact it is the same devil's staircase discovered by Aubry[5,6] in the study of the commensurate-incommensurate transition in the classical Frenkel-Kontorava model. Additionally, our results give a formula for the area under an invariant circle. This is important in the study of adiabatic invariance.[7]

We will consider an area-preserving twist map T on the torus:

$$T(x,p) = (x',p') \tag{1}$$

Sometimes it is convenient to lift this map to the plane R^2, then it can be written in the action representation in the form:

$$p'dx' - pdx = dF(x,x') \tag{2a}$$

or

$$\begin{aligned} p &= -\, \partial F(x,x')/\partial x \\ p' &= \partial F(x,x')/\partial x' \end{aligned} \tag{2b}$$

We assume the generating function $F(x,x')$ has the form:

$$F(x,x') = K(x-x') - V(x) \tag{3}$$

where the potential, $V(x)$, is a periodic function with period 1 and the kinetic enengy, $K(x)$, a convex function. The twist condition, $\partial^2 F(x,x')/\partial x'\partial x < 0$, is assumed to be satisfied everywhere.

An example of the above model is the much-studied standard map[2] where

$$V(x) = -\frac{k\cos(2\pi x)}{(2\pi)^2},$$

$$K(x) = {}^1/_2\, x^2 \tag{4a}$$

or:

$$\begin{aligned} p' &= p - k/2\pi \sin(2\pi x) \\ x' &= x + p' \end{aligned} \tag{4b}$$

An orbit $\{ x_i \}$ of the map T is therefore a stationary state of the action:

$$W(\{x_i\}) = \sum_i F(x_i, x_{i+1}) \tag{5}$$

Equation (5) defines the classical Frenkel-Kontorava model as a one dimensional harmonic chain in a periodic external potential.

Among various orbits of the map T, those which minimize the action (5) are the most important ones. Generally, they correspond to the hyperbolic Poincare-Birkhoff orbits. The map preserves the angular order of these orbits,[5] so they have frequencies defined as the average rotation per iteration. We will say an periodic orbit is of type (m,n) if it has period n and frequency m/n.

The way to construct the (m,n) resonance is shown in figure 1.[3] We illustrate the process for a (1,3) resonance of the standard map. Starting from the minimizing (1,3) orbit, we form the upper partial separatrix by following the right-going unstable manifold of some point M_2, till this unstable manifold crosses the stable manifold of its right neighboring point at some point $M^+{}_3$. In fact there are an infinity of such points and they form the homoclinic orbits from minimizing (1,3) orbit to itself. A homoclinic orbit on the right-going stable and unstable manifolds will be denoted as $(1,3)_+$ orbit. There are also an infinity of such orbits, we will be interested in the one with the least action, the minimizing homoclinic orbit. From the point $M^+{}_3$, we follow the stable manifold of the minimizing (1,3) orbit till it reaches M_0. This defines the upper partial separatrix in the largest gap of the (1,3) resonance, the upper partial separatrix in the other gaps are formed by taking two preimages of this one. The lower partial separatrix is formed similarly. The region bounded by the upper and lower partial separatrices is the (1,3) resonance.

The area under the upper or lower partial separatrix of an (m,n) resonance gives the upper or lower area of this resonance; the difference of the two is its area. By integrating eq(2) it is found that these two areas are given by:[3]

$$A^+(m/n) = \sum_{i=-\infty}^{\infty} \left\{ \sum_{t=1}^{n} \left[F(x^+_{in+t}, x^+_{in+t+1}) - F(x_t, x_{t+1}) \right] \right\}$$

$$A^-(m/n) = \sum_{i=-\infty}^{\infty} \left\{ \sum_{t=1}^{n} \left[F(x_t, x_{t+1}) - F(x^-_{in+t}, x^-_{in+t+1}) \right] \right\} \qquad (6)$$

where $\{ x^+_t \}$ is the $(m,n)_+$ minimizing orbit and $\{ x^-_t \}$ the $(m,n)_-$ minimizing orbit.

Obviously the area under an invariant circle is well defined, it turns out we can also define the area under a hyperbolic cantorus of irrational frequency ν, the formula is:[3]

$$A(\nu) = -\sum_{t=-\infty}^{\infty} t \left[F(x^r_t, x^r_{t+1}) - F(x^l_t, x^l_{t+1}) \right] \qquad (7)$$

where $\{ x^l_t \}$ is the orbit of the left endpoints of a gap in the cantorus and $\{ x^r_t \}$ the orbit of the right endpoints.

So the area as a function of frequency has a jump at each rational number and is monotonically increasing; therefore, it is a devil's staircase.

The average action for all minimizing configurations as a function of the frequency ν is defined as:

$$e(\nu) = \lim_{N-N' \to \infty} \left\{ \sum_{n=N'}^{N-1} F(x_n, x_{n+1}) \right\} / (N-N') \qquad (8)$$

It was shown that this average action exists and is a convex function of the frequency ν; therefore, it has monotonically increasing left and right derivatives $e'^{-}(\nu)$ and $e'^{+}(\nu)$ which are equal almost everywhere.[6]

Now we show that the derivative of the average action with respect to frequency gives us the area devil's staircase:

$$
\begin{aligned}
A(\nu) &= de(\nu)/d\nu && \nu \text{ irrational} \\
A^{+}(\nu) &= de(\nu)^{+}/d\nu && \\
A^{-}(\nu) &= de(\nu)^{-}/d\nu && \Big\} \ \nu \text{ rational}
\end{aligned}
\tag{9}
$$

According to Aubry,[6] the derivative of the average action with respect to frequency is a complete devil's staircase when the lower bound of the Lyapunov exponents of all the minimizing orbits is positive; this implies all invariant circles are destroyed and all cantori are hyperbolic, therefore, resonances fill the entire phase space.

Let us calculate $e'^{+}(\nu)$ and $e'^{-}(\nu)$ for a rational frequency m/n. To approach the minimizing (m,n) configuration from above and below, we use the minimizing $(mk+m_1, nk+n_1)$ and $(mk+m_2, nk+n_2)$ configuration, respectively, and let the integer k go to infinity.[3] The integers m_1, n_1, m_2, n_2 are uniquely determined by:

$$
m_1 n - n_1 m = 1, \quad m_2 n - n_2 m = -1, \quad n_1, n_2 < n \tag{10}
$$

As k goes to infinity, these two configurations limit to the upper and lower minimizing orbits homoclinic to the (m,n) orbit, respectively. Let $\{ x^{k+}_t \}$ denote the minimizing $(mk+m_1, nk+n_1)$ orbit, then the derivative of $e(\nu)$ from above is given by:

$$e'^{+}(\nu) = \lim_{k\to\infty} \frac{1}{(mk+m_1)/(nk+n_1) - m/n} \times$$

$$\left[\frac{\sum_t F(x^{k+}{}_t, x^{k+}{}_{t+1})}{nk+n_1} - \frac{\sum_t F(x_t, x_{t+1})}{n} \right]$$

$$= \lim_{k\to\infty} \left[n \sum_t F(x^{k+}{}_t, x^{k+}{}_{t+1}) - (nk+n_1) \sum_t F(x_t, x_{t+1}) \right]$$

$$= \sum_{i=-\infty}^{\infty} \left\{ \sum_{t=1}^{n} \left[F(x^{+}{}_{in+t}, x^{+}{}_{in+t+1}) - F(x_t, x_{t+1}) \right] \right\}$$

The sums in the first two equalities are taken over the period of the orbits. Similar result holds for $e'^{-}(\nu)$. Since these equations are identical to eq(6), this implies eq(9) for rational frequency.

In order to prove eq(9) for irrational frequency, we use the following lemma:[8]

Lemma: If R is a rotation of the circle [0,1) through an irrational angle, and f is a Riemann integrable function, then the time average of the function f is equal to its spatial average, i.e.

$$\lim_{N\to\infty} \left[\sum_{i=0}^{N-1} f(R^i(x)) \right] / N = \int_0^1 f(\theta)d\theta \qquad (11)$$

For an irrational frequency ν, the minimizing configuration is either an invariant circle or a cantorus. In either case there exists a monotonically increasing function f such that:

$$x_n = f(n\nu+\alpha) \qquad (12)$$

Periodicity implies that $f(\theta+1) = f(\theta) + 1$. Besides its explicit dependence on the frequency ν in its argument, the functional form of f also depends on the frequency ν. For a subcritical invariant circle, f is analytic, and for a hyperbolic cantorus, f can be written as the sum of step functions:

$$f(x) = \sum_i f_i H(x-x_i) \tag{13}$$

where $H(x) = 1,\ x > 1;\ H(x) = 0,\ x < 0$ is the Heaviside function.

It follows from the above lemma that for an irrational frequency ν:

$$e(\nu) = \int_0^1 F(x(\theta),x(\theta+\nu))d\theta \tag{14}$$

Indeed, Percival used this average action to formulate the variational principle for invariant tori and cantori.[9]

For a subcritical circle, $x(\theta)$ is analytic, so we can interchange the order of differentiation and integration; therefore,

$$de(\nu)/d\nu = \int_0^1 \frac{\partial F(x(\theta),x(\theta+\nu))}{\partial x(\theta+\nu)} \frac{dx(\theta+\nu)}{d\theta} d\theta \ +$$

$$\int_0^1 \frac{\partial F(x(\theta),x(\theta+\nu))}{\partial x(\theta+\nu)} \frac{\partial x(\theta+\nu)}{\partial \nu} d\theta + \int_0^1 \frac{\partial F(x(\theta),x(\theta+\nu))}{\partial x(\theta)} \frac{\partial x(\theta)}{\partial \nu} d\theta$$

The partial derivative $\partial x(\theta)/\partial\nu$ is due to the dependence of the orbit configuration $x(\theta)$ on the frequency ν. Using eq(2b), this becomes:

$$= \int_0^1 p(\theta+\nu)dx(\theta+\nu) + \int_0^1 p(\theta+\nu)\frac{\partial x(\theta+\nu)}{\partial \nu}d\theta - \int_0^1 p(\theta)\frac{\partial x(\theta)}{\partial \nu}d\theta$$

$$= \int_0^1 p(\theta)dx(\theta) = A(\nu)$$

In fact, the condition that $x(\theta)$ be analytic can be relaxed. As long as $x(\theta)$ is transitive, the above equation holds so that it is applicable to the critical invariant circle.

For a hyperbolic cantorus, since $x(\theta)$ is a sum of step functions, the integrand $F(x(\theta),x(\theta+\nu))$ in (14) is also a sum of step functions:

$$F(x(\theta),x(\theta+\nu)) = \sum_t F_t H(\theta-\theta_t) \qquad (15)$$

F_t is nothing but the discontinuity of the generating function at the gap in the cantorus. Let $\{ x^l_t \}$ be the orbit of the left endpoints of a gap in the cantorus, $\{ x^r_t \}$ the orbit of the right endpoints, thus:

$$F_t \equiv F(x^r_t,\, x^r_{t+1}) - F(x^l_t, x^l_{t+1})$$
$$\theta_t \equiv \nu t+\alpha \bmod 1 \qquad (16)$$

The function on the right hand side of eq(15) is periodically extended to the whole real line.

Taking the derivative of the average action $e(\nu)$ yields:

$$de(\nu)/d\nu = \int_0^1 \sum_t \left\{ \frac{\partial F(x^r_t, x^r_{t+1})}{\partial x^r_t} \frac{\partial x^r_t}{\partial \nu} + \frac{\partial F(x^r_t, x^r_{t+1})}{\partial x^r_{t+1}} \frac{\partial x^r_{t+1}}{\partial \nu} \right.$$

$$\left. - \frac{\partial F(x^l_t, x^l_{t+1})}{\partial x^l_t} \frac{\partial x^l_t}{\partial \nu} - \frac{\partial F(x^l_t, x^l_{t+1})}{\partial x^l_{t+1}} \frac{\partial x^l_{t+1}}{\partial \nu} \right\} H(\theta-\theta_t)\, d\theta$$

$$- \sum_t F_t \frac{d(\nu t+\alpha)}{d\nu}$$

$$= \int_0^1 \sum_t \left(p^r_{t+1} \frac{\partial x^r_{t+1}}{\partial \nu} - p^l_{t+1} \frac{\partial x^l_{t+1}}{\partial \nu} \right) H(\theta-\theta_{t+1}+\nu)\, d\theta$$

$$- \int_0^1 \sum_t \left(p^r_t \frac{\partial x^r_t}{\partial \nu} - p^l_t \frac{\partial x^l_t}{\partial \nu} \right) H(\theta-\theta_t)\, d\theta - \sum_t tF_t$$

where in the last equality, the integrand of the first term is the shift in the θ variable by an angle ν of that of the second term. Again these two terms cancel due to the periodicity of the integrand. Thus:

$$de(\nu)/d\nu = - \sum_t t \left[F(x^r_t, x^r_{t+1}) - F(x^l_t, x^l_{t+1}) \right]$$

This is exactly the same formula as Eq(7).

Equation (9) gives a convenient formula for computing the area under an invariant circle. We use periodic orbits whose frequencies are the successive rational approximants of the irrational frequency of the invariant circle to obtain the derivative of the average action. The results for the golden mean invariant circle at two parameter values for the standard map are shown in table 1. $A(\ell)$ is the area calculated using rational approximants at level ℓ, the convergence rate $\delta(\ell)$ is defined as:

$$\delta(\ell) = (A(\ell+1) - A(\ell))/(A(\ell) - A(\ell-1)) \qquad (17)$$

Notice this method converges considerably faster than the linear interpolation method (see table 2). In the latter case, we compute the area under a periodic orbit by connecting neighboring points with straight lines. The convergence rate is so slow that it virtually impossible to find the area without the cost of extremely long approximating periodic orbits.

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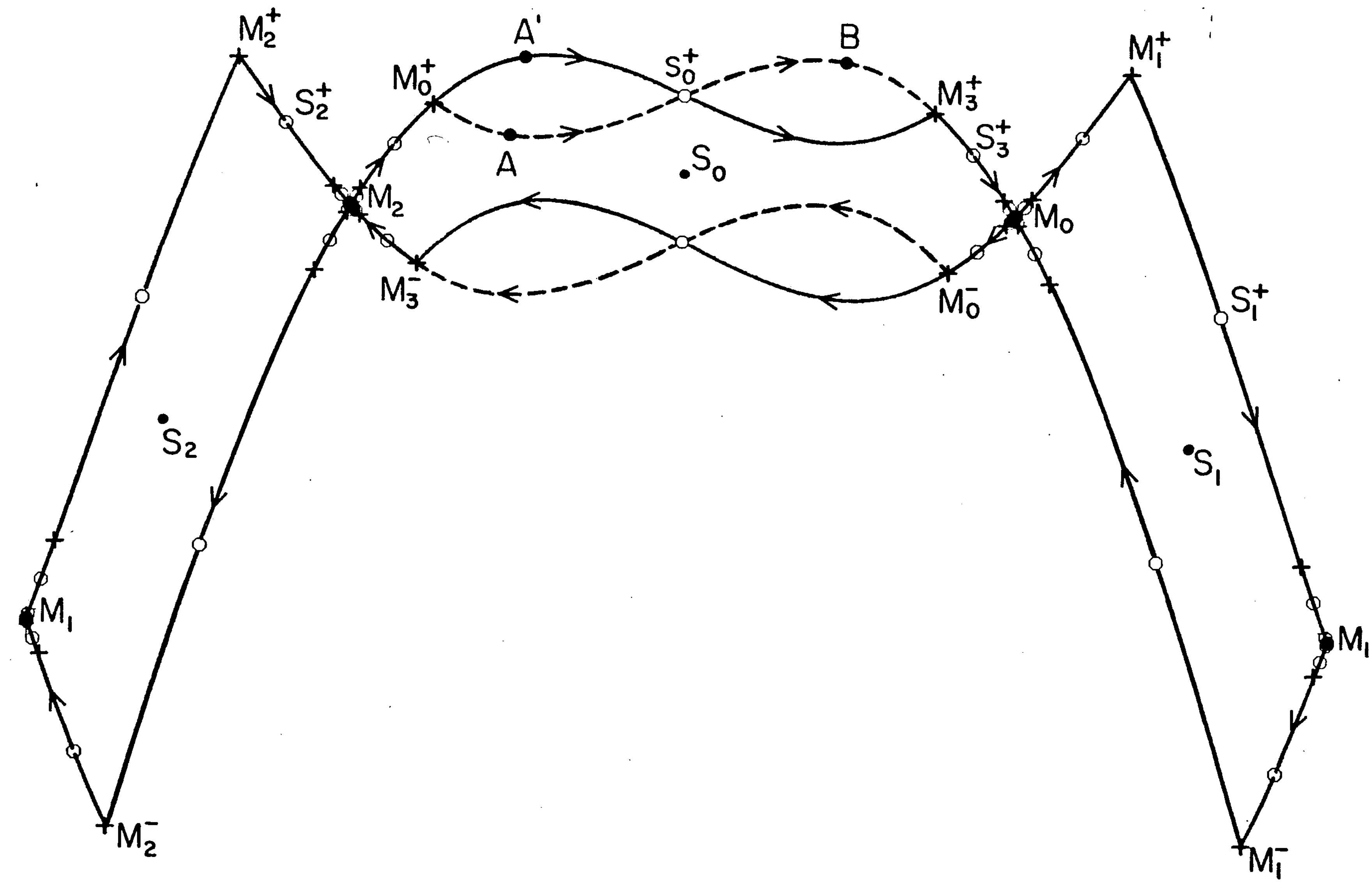

Fig 1

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Figure caption

figure 1: (1,3) resonance for the standard map in symmetry coordinates. M_t are points on the period 3 minimizing orbit, S_t are points on the period 3 minimax orbit. $M^+{}_t$ are points on the minimizing homoclinic orbit.

Table 1

Area under the golden mean KAM curve calculated by the method mentioned in this paper at different parameter values. Orbits are calculated to precision 10^{-12}. $A(\ell)$ is the area calculated from the ℓ(th) rational approximating periodic orbit of the golden mean curve, $\delta(\ell)$ is the convergence rate at the level ℓ defined by eq(17).

$k = 0.9$

Level	M/N	$A(\ell)$	$\delta(\ell)$
4	5/8	0.6104229	—
5	8/13	0.6164991	-0.39350
6	13/21	0.6141081	-0.39701
7	21/34	0.6150573	-0.39712
8	34/55	0.6146804	-0.39303
9	55/89	0.6148285	-0.38682
10	89/144	0.6147712	-0.38299
11	144/233	0.6147932	-0.38205
12	233/377	0.6147848	-0.38165
13	377/610	0.6147880	-0.38197
14	610/987	0.6147868	—

$k = k_c = 0.971635406$

Level	M/N	$A(\ell)$	$\delta(\ell)$
4	5/8	0.6103039	—
5	8/13	0.6157595	-0.37235
6	13/21	0.6137281	-0.37417
7	21/34	0.6144882	-0.37174
8	34/55	0.6142056	-0.37373
9	55/89	0.6143112	-0.37233
10	89/144	0.6142719	-0.37326
11	144/233	0.6142866	-0.37266
12	233/377	0.6142811	-0.37304
13	377/610	0.6142831	-0.37280
14	610/987	0.6142824	-0.37295
15	987/1597	0.6142827	-0.37286
16	1597/2584	0.6142826	—

Table 2

Area under the golden mean KAM curve calculated by linearly interpolating the neighboring points of the periodic orbits. All Orbits are calculated to precision 10^{-12}. $A(\ell)$ is the area calculated from the ℓ(th) rational approximating periodic orbit of the golden mean curve, $\delta(\ell)$ is the convergence rate at the level ℓ defined by eq(17).

$k = 0.9$

Level	M/N	$A(\ell)$	$\delta(\ell)$
4	5/8	0.7081045	___
5	8/13	0.6708096	0.38843
6	13/21	0.6563233	0.96231
7	21/34	0.6423830	0.60258
8	34/55	0.6339828	0.78309
9	55/89	0.6274048	0.67063
10	89/144	0.6229934	0.69026
11	144/233	0.6199483	0.63581
12	233/377	0.6180122	0.63471
13	377/610	0.6167834	0.61881
14	610/987	0.6102303	___

$k = k_c = 0.971635406$

Level	M/N	$A(\ell)$	$\delta(\ell)$
4	5/8	0.7058759	___
5	8/13	0.6688289	0.38648
6	13/21	0.6545110	0.93595
7	21/34	0.6411104	0.58059
8	34/55	0.6333304	0.75352
9	55/89	0.6274674	0.66256
10	89/144	0.6235831	0.71557
11	144/233	0.6208036	0.69140
12	233/377	0.6188818	0.70734
13	377/610	0.6175225	0.70114
14	610/987	0.6165694	0.70601
15	987/1597	0.6158965	0.70456
16	1597/2584	0.6154224	___