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# LOW BETA RIGID MODE STABILITY CRITERION FOR AN ARBITRARY LARMOR RADIUS PLASMA

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## Abstract

The low beta flute interchange dispersion relation for rigid displacement perturbation of axisymmetric plasma equilibria with arbitrary Larmor radius particles and field line curvature large compared to the plasma radius is derived. The equilibrium particle orbits are characterized by two constants of motion, energy and angular momentum, and a third adiabatic invariant derived from the rapid radial motion. The Vlasov equation is integrated, assuming that the mode frequency, axial "bounce" frequency, and particle drift frequency are small compared to the cyclotron frequency, and it is demonstrated that the plasma response to a rigid perturbation has a universal character independent of Larmor radius. As a result the interchange instability is the same as that predicted from conventional MHD theory. However, a new prediction, more optimistic than earlier work, is found for the low density threshold of systems like Migma, which are disc-shaped, that is, the axial extent $\Delta z$ is less than the radial extent $r_0$. The stability criterion is

$$\frac{\kappa a_h^2}{r} > \frac{8\delta_h}{\left[1 + 2Z \dfrac{r_0}{\Delta z} \dfrac{\Omega_i^2}{\omega_{pi}^2}\right]}$$

where $\omega_{pi}$ is the mean ion plasma frequency, $\Omega_i$ the ion cyclotron frequency, $\delta_h$ the hot particle to total ion particle ratio, $\kappa/r$ is the ratio of the field line curvature and the midplane radius, which in our model is treated as a constant over the entire plasma, $a_h$ is the Larmor radius of the energetic species, $Z$ is approximately given by an interpolation formula $Z = \dfrac{8}{3\pi} + \dfrac{\Delta z}{r_0}$ which goes over to the correct limits if either $\dfrac{\Delta z}{r_0} \gg 1$ or $\dfrac{\Delta z}{r_0} \ll 1$.

For $\frac{\Delta z}{r_0} \ll 1$, the stability criterion is determined by the total particle number. Whereas the older theory $\left(\frac{\Delta z}{r_0} \gg 1\right)$ predicted instability at about the densities achieved in actual Migma experiments, the present theory $\left(\frac{\Delta z}{r_0} \ll 1\right)$ indicates that the experimental results were for plasmas with particle number below the interchange threshold.

# I Introduction

Under a variety of conditions the linear response of hot particles with arbitrary Larmor radius may differ appreciably from the predictions of MHD theory.[1–7] There is a variety of concepts, such as astron,[8,9] Elmo Bumpy Torus,[10] field reversed theta pinch,[11] tokamaks containing hot particles,[12] and Migma,[13,14] that hope to exploit such an effect to achieve favorable containment. In this work we present a theory for arbitrary Larmor radius, for the response of an equilibrium configuration to long wavelength rigid mode perturbations. The assumptions made in the calculation are that there is azimuthal symmetry in the equilibrium, that the magnetic field curvature radius is large compared to the plasma radius and beta is low. Because of the last assumption, we shall neglect the equilibrium induced magnetic fields and consider only electrostatic perturbations. The self-consistent description with equilibrium magnetic fields and electromagnetic perturbation will be reported in a later work, although preliminary results of the fully self-consistent calculation will be discussed in the summary.

The primary result of our calculation is that the response to a rigid perturbation has a universal character[1,15] independent of Larmor radius. As a result the same interchange instability as predicted from conventional MHD theory is predicted. However, if the predicted MHD growth rate is less than the precessional frequency of the particles, stabilization arises. Such stabilization can occur at low densities, if the plasma frequency is appreciably less than the ion cyclotron frequency,[1] or if the ratio of hot particle density to cold particle density is sufficiently small.[4] These stabilization mechanisms have been noted in small Larmor radius theories,[1,4] but we believe that this calculation is the first to rigorously demonstrate that the stability criteria is independent of Larmor radius.

The calculation is applied to the Migma configuration, where the experimentally achieved equilibrium has an axial width, $L_p$, considerably less than the radial extent, $R_p$, which is of order two Larmor radii. Because of the shape, the density threshold for instability is shown to be larger by a factor, $L_p/R_p$, than the case where $L_p \gg R_p$, which is usually studied. The predicted particle number threshold is above the number of particles trapped in the Migma experiment[14] and is therefore compatible with experimental results, where the interchange mode has not been observed.

We expect the universality of the response of rigid modes to apply even in the finite beta fully electromagnetic case. Hence, if we apply known results from small Larmor radius theory, we can indicate the stability criteria for arbitrary beta. This discussion will be given in the summary section.

The structure of the paper is as follows. In Sec. II we discuss the equilibrium. In Sec. III we analyze the dispersion relation that is obtained for a rigid displacement. The algebra needed to obtain the formal response makes use of Hamiltonian perturbation theory. Use of a Hamiltonian formalism in the perturbation theory allows for a systematic analysis of complex configurations. Since we are dealing with large orbits, and will need to find adiabatic invariants in a non-conventional ordering, a Hamiltonian formalism is particularly useful. However, the details are somewhat involved, and as a consequence we have included the algebraic reduction of the problem separately in Sec. IV. In Sec. III the analysis of the electric field coupling for a disc-shaped plasma is presented. In Sec. V we discuss the consequences of the interchange instability to the Migma concept.

## II Equilibrium

In cylindrically symmetric plasma equilibria where the current density $\mathbf{J}_0$ is azimuthal, the equilibrium vector potential $\mathbf{A}_0$ may be represented in terms of the flux variable $\psi(r,z)$:

$$\mathbf{A}_0 = \psi \nabla \theta$$

where $r, \theta, z$ are the usual cylindrical coordinates. The equilibrium magnetic field $\mathbf{B}$ is

$$\mathbf{B} = \nabla \psi \times \nabla \theta. \tag{1}$$

The lines of force lie in surfaces of $\psi =$ constant and $\theta =$ constant.

It is convenient to introduce a function $\chi(r,z)$ so that $\psi, \theta, \chi$ form an orthogonal set of curvilinear coordinates, where

$$\mathbf{B} = \lambda(\psi, \chi) \nabla \chi, \tag{2}$$

so that $\lambda$ is determined by

$$\nabla \times \mathbf{B} = -\frac{\partial \lambda}{\partial \psi} \frac{r B^2}{\lambda} \mathbf{e}_\theta = 4\frac{\pi}{c} J_{0\theta} \mathbf{e}_\theta.$$

Note that

$$\begin{aligned} |\nabla \psi| &= rB \\ |\nabla \theta| &= \frac{1}{r} \\ |\nabla \chi| &= \frac{B}{\lambda}. \end{aligned}$$

The equilibrium Hamiltonian for the particle motion describing the particle dynamics in these coordinates is:

$$\hat{H} = \frac{r^2 B^2}{2M} p_\psi^2 + \frac{\left(p_\theta - \frac{e}{c}\psi\right)^2}{2Mr^2} + \frac{B^2}{2M\lambda^2} p_\chi^2 \tag{3}$$

where $p_\psi$, $p_\theta$, $p_\chi$ are the particle momenta conjugate to $\psi, \theta, \chi$ and $M$ the particle mass. The equations of motion are then the usual expression:

$$\begin{aligned} \dot{q}_i &= \frac{\partial \hat{H}}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial \hat{H}}{\partial q_i} \\ q_i &= (\psi, \theta, \chi) \\ p_i &= (p_\psi, p_\theta, p_\chi). \end{aligned}$$

For example

$$\dot{p}_\psi = -\frac{p_\psi^2}{2M}\frac{\partial}{\partial\psi}r^2B^2 + \frac{\left(p_\theta - \frac{e}{c}\psi\right)^2}{Mr^3}\frac{\partial r}{\partial\psi} + \frac{e}{c}\frac{\left(p_\theta - \frac{e}{c}\psi\right)}{Mr^2} - \frac{p_\chi^2}{2M}\frac{\partial}{\partial\psi}\frac{B^2}{\lambda^2}. \tag{4}$$

We are interested in low beta mirror equilibrium where the vacuum magnetic field may be approximated by a near axis expansion of the vacuum field flux function $\psi$:

$$\psi = \frac{r^2}{2}\left[\hat{B}_0 + \hat{B}_1(z) - \frac{r^2}{8}\frac{d^2}{dz^2}\hat{B}_1(z) + \cdots\right].$$

Thus

$$\begin{aligned} B_z &= \hat{B}_0 + \hat{B}_1 - \frac{r^2}{4}\frac{d^2}{dz^2}\hat{B}_1 \cdots \equiv \hat{B} - \frac{r^2}{4}\frac{d^2}{dz^2}\hat{B}_1 \\ B_r &= -\frac{r}{2}\frac{d}{dz}\hat{B}_1. \end{aligned}$$

The vacuum field curvature is

$$\boldsymbol{\kappa} \equiv \mathbf{b}\cdot\nabla\mathbf{b} \doteq -\frac{r}{2B}\frac{d^2}{dz^2}\hat{B}_1. \tag{5}$$

We assume in our subsequent analysis that the field line curvature and mirror ratio are small:

$$1 \gg \frac{r^2}{\hat{B}_0}\frac{d^2}{dz^2}\hat{B}_1 \gg \frac{r^2}{2\hat{B}_0^2}\left(\frac{d}{dz}\hat{B}_1\right)^2$$

and that $\lambda = 1$, which follows at very low beta.

The constants of motion are $H$ and $p_\theta$. In addition, we assume an adiabatic invariant arising from the rapid $\psi$ motion compared to the $\chi$ motion. From standard Hamiltonian theory the adiabatic invariant, $P_\psi$, is given by

$$\begin{aligned} P_\psi &= \oint d\psi p_\psi(H, p_\theta, p_\chi, \psi, \chi)\Big|_{H,p_\theta,\chi,p_\chi} \\ &= (2M)^{1/2}\oint\frac{d\psi}{rB}\left[H - \frac{\left(p_\theta - \frac{e}{c}\psi\right)^2}{2Mr^2} - \frac{B^2p_\chi^2}{2M}\right]^{1/2}. \end{aligned} \tag{6}$$

The equilibrium distribution can then be taken as a function of the constants of motion, i.e., $F_0 = F_0(H, p_\theta, P_\psi)$. In Sec. IV we shall develop the use of the adiabatic invariant as a canonical variable in a more formal manner in order to aid in deriving the equations for the perturbed distribution.

# III Electrostatic $m = 1$ Flute Perturbation

## III.1 General description of method of solution of dispersion relation

In order to describe the electrostatic dispersion relation for the perturbed system, we examine the structure of the variational quadratic form to the response of a rigid displacement. This procedure can be justified as follows:

(1) The electrostatic dispersion relation is valid for the threshold of instability as it occurs at such low beta that magnetic perturbations, which primarily produce large positive energy perturbations, must be negligible in order to have instability.

(2) A symmetric quadratic form in the perturbed amplitude $\tilde{\varphi}(\mathbf{r}) = \varphi(\psi, \chi)\exp(i\ell\theta)$ and its adjoint $\tilde{\varphi}^{+}(\mathbf{r}) = \varphi(\psi, \chi)\exp(-i\ell\theta)$ is known to exist, where the variation with respect to $\varphi(\psi, \chi)$ gives the exact linear eigenmode equation.[17,18,19] The substitution of a first order accurate representation of the eigenfunction gives a second order accurate expression for the eigenvalue.

(3) If the system has zero curvature, a rigid displacement with $\tilde{\varphi} \propto r\exp(i\theta)$ would be an exact eigenfunction for a zero frequency mode. Further, for a disc shaped plasma, $\partial\tilde{\varphi}/\partial\chi$ would dominate the electric response in the electric field energy unless $\varphi$ is nearly independent of $\chi$. Thus, to examine the response of a low frequency mode for a system with weak curvature, we choose $\varphi = \dfrac{\varphi_0 B^{1/2}}{B_0^{1/2}}\exp(i\theta)r/r_0$ with $r_0$ a constant, for the approximate eigenfunction in the quadratic form as it is a flute mode ($rB^{1/2} \doteqdot (2\psi)^{1/2}$ which is independent of $\chi$) and approximately a rigid displacement.

(4) Instead of forming the quadratic form for a general perturbation, and obtaining a complicated expression (see for example Ref. 18), we construct the quadratic form with the approximate eigenfunction we have suggested, thereby eliminating at the outset complicated large non-local terms that would ultimately be eliminated at the end of the analysis.

The quadratic form is obtained by considering the Poisson equation

$$\nabla^2\tilde{\varphi} = -4\pi\sum_j \int d^3p f_j q_j$$

where $f_j$ is the perturbed distribution function of species $j$ of charge $q_j$ due to the test function we have chosen. Multiplying by $\tilde{\varphi}^+$, and integrating over all space then gives the quadratic variational form

$$D(\omega) = \int d^3r \nabla\tilde{\varphi}^+ \cdot \nabla\tilde{\varphi} + 4\pi \sum_j q_j \int d^3r d^3p \varphi^+ f = 0. \tag{7}$$

A critical part of the calculation is to calculate the self-consistent $f_j$. However, as the derivation is somewhat involved, we present the final results here, and show the detailed derivation in Sec. IV. The final result is quoted immediately below. In addition we note that $\tilde{\varphi}$ must be known over all space. Hence, given $\tilde{\varphi}$ in the plasma, we must consistently solve for the integral of $\nabla\tilde{\varphi}^+ \cdot \nabla\tilde{\varphi}$ in the vacuum region. The method of solving this aspect of the problem will be described herein, and leads to an important modification of the $m = 1$ stability criterion for disc-spaced plasmas. With the terms of Eq. (17) determined one can then solve this equation for the frequency $\omega$, and determine the stability criterion.

## III.2 Evaluation of Dispersion Relation

From Sec. IV (see Eq. (69)) it is found that $D(\omega)$ is given by

$$\begin{aligned} \frac{D(\omega)}{4\pi} &\equiv \frac{1}{4\pi} \int d^3r \nabla\tilde{\varphi} \cdot \nabla\tilde{\varphi}^+ + \sum_j \int d^3r \frac{M_j N_j c^2}{r_0^2 \hat{B}_0^2} \varphi_0 \varphi_0^+ \\ &+ \left\langle F_0 \left( H_0 + \frac{\overline{p_\chi^2} \hat{B}_0^2}{2M} \right) \frac{2c^2 \kappa_0 \varphi_0 \varphi_0^+}{\omega^2 r_0^3 \hat{B}_0^2} \right\rangle \\ &- \left\langle F_0 \left( H_0 + \frac{\overline{p_\chi^2} \hat{B}_0^2}{2M} \right) \frac{2c^2 \kappa_0 \varphi_0 \varphi_0^+}{M \Omega_0 r_0^4 \hat{B}_0^2 (\omega - \omega_\kappa)} \right\rangle = 0 \end{aligned} \tag{8}$$

where $\kappa_0 = \kappa(r = r_0)$, $\omega_\kappa = -\dfrac{\kappa_0}{M\Omega_0 r_0}\left(H_0 + \dfrac{\overline{p_\chi^2}\hat{B}_0^2}{2M}\right)$, $\Omega_0 = q\hat{B}_0/Mc$ and the bracket $\langle \quad \rangle$ refers to an integration over a six-dimensional phase space and a summation over species.

One can observe that the second term of Eq. (8) is the usual cross-field inertia term, while the third term is the usual MHD drive term leading to the interchange instability. The last term is a more complicated kinetic term that would be small if $\omega \gg \omega_\kappa$. It is of the same structure that would be obtained in a zero Larmor radius kinetic theory and it is of importance at sufficiently low density near the threshold for interchange instabilities or if the energetic particles are only a small fraction of the plasma particles. Explicit Larmor radius terms do not appear in this theory because of the use of the rigid displacement perturbation discussed previously.

In addition there is the electric field integration in the first term of Eq. (8). We note that the electric field must be integrated over all space, and if the plasma is a thin disc, the predominant contribution to the integral cones from the region outside the plasma. Then the problem to be addressed is the solution for $\tilde{\varphi} = \varphi(r,z)e^{-i\omega t+i\theta}$ in the vacuum region outside the plasma. This solution must match to the assumed form of $\tilde{\varphi}$ inside the plasma

$$\tilde{\varphi}(\mathbf{r}) \longrightarrow \frac{\varphi_0 r}{r_0}e^{i\theta} \quad \text{at} \quad z \to 0, \quad r < r_0. \tag{9}$$

To obtain this solution, we consider the plasma to occupy a volume in the form of a thin disk of width $\Delta z \to 0$ and radius $r_0$ which we now choose as the radial limit of the plasma. We then imagine that the vacuum solution $\varphi(r,z)$ is generated by an equivalent charge density $\tilde{\rho}_v$

$$\tilde{\rho}_v = \sigma(r)\delta(z)S(r_0 - r)e^{-i\omega t+i\theta} \tag{10}$$

where

$$S(r_0 - r) = \begin{cases} 1 & r_0 < r \\ 0 & r_0 < r \end{cases}$$

$\delta(z)$ is the Dirac delta function. $\sigma(r)$ is the equivalent surface charge density consistent with Eq. (9).

The solution of Poisson's equation, given $\sigma(r)$, is:

$$\varphi(r,z) = \int_{-\infty}^{+\infty} dz' \int_0^{2\pi} d\theta \int_0^{r_0} r'dr' \frac{\sigma(r')\delta(z')e^{i(\theta'-\theta)}}{R'} \tag{11}$$

where

$$R'^2 = (z-z')^2 + r^2 + r'^2 - 2rr'\cos(\theta' - \theta).$$

It may be noted that

$$\begin{aligned} \varphi(r,z) \quad &\to 0 \quad \{r,z\} \to \infty \\ \frac{\partial\varphi}{\partial z} \quad &\to 0 \quad z = 0, r > r_0\,. \end{aligned}$$

If we impose the boundary condition at $z = 0$, $r < r_0$, namely Eq. (9), we obtain

$$\frac{\varphi_0 r}{r_0} = \int_0^{2\pi} d\zeta \int_0^{r} dr' \frac{r'\sigma(r')\cos\zeta}{[r^2 + r'^2 - 2rr'\cos\zeta]^{1/2}}.$$

Defining $z = rr'\exp(i\zeta)$, then performing contour deformations in the $z$-plane to the real axis, encircling branch points at $z = 0$ and $z = \min(r, r')$, leads to the following integral when we choose $s = z^2$,

$$\begin{aligned}\frac{\varphi_0 r}{r_0} &= \int_0^{r_0} dr'\frac{4}{r}\sigma(r')\int_0^{\min(r,r')} \frac{ds s^2}{[(r^2-s^2)(r'^2-s^2)]^{1/2}} \\ &= \frac{4}{r}\int_0^r ds\frac{s^2}{(r^2-s^2)^{1/2}}\int_s^{r_0}\frac{dr'\sigma(r')}{(r'^2-s^2)^{1/2}}.\end{aligned}$$

This equation determines $\sigma(r)$.

Let

$$W(s) = \int_s^{r_0}\frac{dr'\sigma(r')}{(r'^2-s^2)^{1/2}}.$$

By Abel inversion:

$$s^2 W(s) = \frac{\varphi_0}{2\pi r_0}\frac{\partial}{\partial s}\int_0^s ds'\frac{s'^3}{(s^2-s'^2)^{1/2}} = \frac{\varphi_0 s^2}{r_0\pi}$$

and

$$\sigma(r) = -\frac{2}{\pi}\frac{\partial}{\partial r}\int_r^{r_0} ds\frac{sW(s)}{(s^2-r^2)^{1/2}} = \frac{2\varphi_0}{\pi^2 r_0}\frac{r}{(r_0^2-r^2)^{1/2}}.$$

Equation (11) then determines $\varphi(r, z)$ outside the plasma. The volume integral $\int d^3r\nabla\tilde{\varphi}\cdot\nabla\tilde{\varphi}^+$ is most conveniently evaluated by integration by parts:

$$\begin{aligned}\int d^3r\nabla\tilde{\varphi}\cdot\nabla\tilde{\varphi}^+ &\approx \int_{\text{vacuum}} d^3r\nabla\tilde{\varphi}\cdot\nabla\tilde{\varphi}^+ \\ &= 4\pi\int d^3r\varphi^+\sigma(r)S(r_0-r)\delta(z) = \frac{32r_0}{3}\varphi_0\varphi_0^+.\end{aligned}$$

We then obtain from Eq. (14), the following dispersion relation

$$\begin{aligned}Zr_0 &+ \sum_i\frac{4\pi q_i^2}{M_i\Omega_i^2}\int_0^{r_0}\frac{drr}{r_0^2}\int_{-\Delta z/2}^{\Delta z/2} dzN_i(r,z) \\ &+ \left\langle F_{0h}\left(H_0+\frac{\overline{P_X^2}\hat{B}_0^2}{2m}\right)\frac{2c^2}{\omega}\frac{\kappa_0}{r_0^3\hat{B}_0^2}\frac{1}{(\omega-\omega_\kappa)}\right\rangle = 0 \qquad (12)\end{aligned}$$

with $Z = 8/3\pi$. $\Omega_i = q_i\hat{B}_0/M_ic$.

We also observe that if we are to consider a more standard case of an extended plasma, where $\Delta z \gg r_0$, the factor $Z$ is easily calculated. We use

$$\begin{aligned}\tilde{\varphi} &= \varphi_0\frac{r}{r_0}\exp(i\theta),\ r < r_0 \\ \tilde{\varphi} &= \varphi_0\frac{r_0}{r}\exp(i\theta),\ r > r_0\end{aligned}$$

and then integrate

$$\int d^3r \nabla\tilde{\varphi}\cdot\nabla\tilde{\varphi}^+$$

over all $r$ and $\theta$, and $-\dfrac{\Delta z}{2} < z < \dfrac{\Delta z}{2}$ (axial fringing field for $|z| > \Delta z/2$ are neglected). Then we find

$$Z(\text{cylinder}) = \Delta z/r_0.$$

As an interpolation formula we can use $Z(\text{interpolation}) \approx 8/3\pi + \Delta z/r_0$.

For further analysis we shall assume that only the hot ion species contribute to the last term, as the pressure of the other species are assumed negligible. We also model the hot ion distribution function, $F_{0h}$, by an ideal Migma distribution,

$$F_{0h} \propto \delta(H_0 - \hat{H}_0)\delta(p_\theta)$$

and we assume $\hat{p}_\chi^2 \hat{B}_0^2/2M \ll \hat{H}_0$, which must be the case for a low beta disc-like plasma. The dispersion relation then simplifies to,

$$\frac{2Zr_0\Omega_i^2}{\Delta z\omega_{pi}^2} + 1 + \delta_H \frac{\hat{\omega}_\kappa \Omega_h}{(\omega - \hat{\omega}_\kappa)\omega} = 0 \tag{13}$$

where

$$\begin{aligned}
\frac{\omega_{pi}^2}{\Omega_i^2} &= \sum_j \frac{4M_i c^2}{B_0^2 r_0^2 \Delta z}\int d^3r N_i(r,z) \\
\delta_h &= \frac{M_h \int d^3r N_h(r,z)}{\sum_i M_i \int d^3r N_i(r,z)}
\end{aligned}$$

where the sum is over ion-species.

Now solving Eq. (13) for $\omega$ yields

$$\omega = \frac{\hat{\omega}_\kappa}{2} \pm \left[\frac{\hat{\omega}_\kappa^2}{4} - \frac{\delta_h \hat{\omega}_\kappa \Omega_h}{1 + 2Zr_0\Omega_i^2/(\omega_{pi}^2\Delta z)}\right]^{1/2}. \tag{14}$$

We note that stability requires

$$\hat{\omega}_\kappa > \frac{4\delta_h \Omega_h}{1 + 2Zr_0\Omega_i^2/\omega_{pi}^2\Delta z}. \tag{15}$$

If there are no cold ions present $\delta_h = 1$ and stability requires $\left(\text{assuming} \frac{2Zr_0\Omega_i^2}{\omega_{pi}^2 \Delta z} \gg 1\right)$

$$\frac{\omega_{pi}^2 \Delta z}{2Zr_0\Omega_i^2} \equiv \frac{2M_h c^2 \int d^3r N_h(r,z)}{\hat{B}_0^2 r_0^3 Z} < \frac{\omega_\kappa}{4\Omega_h} = \frac{v_{\perp h}^2 \kappa_0}{8\Omega_h^2 r} = \frac{\kappa_0 r_0}{32} \tag{16}$$

where we used that $r_0 = 2v_{\perp h}/\Omega_h$ in an ideal Migma distribution. We note that the use of $Z = 8/3\pi$, rather than the $Z$ for the the cylindrical limit ($Z = \Delta z/r_0$) gives a higher density threshold for instability. Using the $Z$ of the cylindrical limit would produce previously derived dispersion relations as found in Refs. (1) and (15).

# IV Derivation of Equations

We now derive the perturbed response of the distribution $f$, to the rigid displacement perturbation. To derive the response it is convenient to define new canonical variables that make use of the adiabatic invariant $P_\psi$ as one of the canonical momenta. We therefore consider the transformation

$$p_\psi, \psi, p_\theta, \theta, p_\chi, \chi \longrightarrow P_\chi, Q_\psi, P_\theta, Q_\theta, P_\chi, Q_\chi$$

defined by the generating function

$$G(P_\psi, P_\theta, P_\chi, \psi, \theta, \chi) = \int^\psi \hat{p}_\psi d\psi + P_\theta \theta + P_\chi \chi \tag{17}$$

where

$$\hat{p}_\psi^2 = \frac{2M}{r^2 B^2}\left[H_0(P_\psi, P_\theta, P_\chi, \chi) - \frac{\left(P_\theta - \frac{e}{c}\psi\right)^2}{2mr^2} - B^2 \frac{P_\chi^2}{2m}\right] \tag{18}$$

$$P_\psi = \oint \hat{p}_\psi(H_0, P_\theta, P_\chi, \psi, \chi) d\psi. \tag{19}$$

$P_\psi$ is an adiabatic invariant defined by integrating $\hat{p}_\psi$ over one period in $\psi$ holding $\chi$ constant. It is explicitly evaluated in Eq. (A.7) of the Appendix. Equation (19) determines the function $H_0 = H_0(P_\psi, P_\theta, P_\chi, \chi)$. The equilibrium magnetic field varies slowly in $\chi$.

The transformation equations are:

$$\begin{aligned}
p_\psi &= \frac{\partial G}{\partial \psi} = \hat{p}_\psi\left(H_0, P_\theta, P_\chi, \psi\right) \\
p_\theta &= \frac{\partial G}{\partial \theta} = P_\theta \\
p_\chi &= \frac{\partial G}{\partial \chi} = P_\chi + \frac{\partial H_0}{\partial \chi}\frac{\partial}{\partial H_0}\int^\psi d\psi \hat{p}_\psi + \frac{\partial}{\partial \chi}\int^\psi d\varphi \hat{p}_\psi \\
Q_\psi &= \frac{\partial G}{\partial P_\psi} = \frac{\partial H_0}{\partial P_\psi}\frac{\partial}{\partial H_0}\int^\psi d\psi \hat{p}_\psi \\
Q_\theta &= \frac{\partial G}{\partial P_\theta} = \theta + \frac{\partial H_0}{\partial p_\theta}\frac{\partial}{\partial H_0}\int^\psi d\psi \hat{p}_\psi + \frac{\partial}{\partial P_\theta}\int^\psi d\psi \hat{p}_\psi \\
&\equiv \theta - \tilde{Q}_\theta \\
Q_\chi &= \frac{\partial G}{\partial P_\chi} = \chi + \frac{\partial H_0}{\partial P_\chi}\frac{\partial}{\partial H_0}\int^\psi d\psi \hat{p}_\psi + \frac{\partial}{\partial P_\chi}\int^\psi d\psi \hat{p}_\psi \\
&= \chi - \tilde{Q}_\chi
\end{aligned} \tag{20}$$

The new Hamiltonian $H(P_i, Q_i)$ is obtained by using the transformation to express $\hat{H}$ in terms of $P_i$, $Q_i$:

$$H(P_i, Q_i) = \hat{H} = H_0\left(P_\psi, P_\theta, P_\chi, Q_\chi\right) + \cdots \tag{21}$$

where higher order terms involving derivatives with respect to $Q_\chi$ are neglected.

The equation of motion may be approximated by:

$$\begin{aligned}
\dot{Q}_\psi &= \frac{\partial H_0}{\partial p_\psi} = \frac{1}{\oint \dfrac{d\psi}{r^2 B^2 \hat{p}_\psi}} \equiv \Omega_\psi \\
\dot{Q}_\theta &= \frac{\partial H_0}{\partial P_\theta} = \Omega_\psi \oint \frac{d\psi}{r^2 B^2 \hat{p}_\psi}\frac{\left(P_\theta - \dfrac{e}{c}\psi\right)}{r^2} \\
\dot{Q}_\chi &= \frac{\partial H_0}{\partial P_\chi} = \Omega_\psi \oint \frac{d\psi}{r^2 B^2 \hat{p}_\psi}\frac{P_\chi B^2}{M} \\
\dot{P}_\psi &= 0 \\
\dot{P}_\theta &= 0 \\
\dot{P}_\chi &= -\frac{\partial H_0}{\partial Q_\chi}
\end{aligned} \tag{22}$$

where $\dot{Q}_\psi = \dfrac{d}{dt}Q_\psi$, etc.

The equilibrium distribution function $F_0 \equiv F_0(H_0, P_\theta, P_\psi)$ now satisfies the equation

$$[F_0, H_0] = 0 \tag{23}$$

where we use the standard Poisson bracket notation.

Now we consider the linearized equation for $f$, the perturbed distribution, with

$$\tilde{\varphi} = \varphi_0(\chi)\frac{r}{r_0}e^{i\theta - i\omega t} \tag{24}$$

where we choose $r_0$ as the plasma radius at $z = 0$ and $\varphi_0(\chi) \approx (\hat{B}/\hat{B}_0)^{1/2}$.

The linearized perturbed Hamiltonian is $H_1 = e\tilde{\varphi}$. The perturbed distribution function $f$ is determined by the linearized Liouville equation:

$$-i\omega f + [f, H] + [F_0, H_1] = 0. \tag{25}$$

The solution for $f$ may be obtained by integrating the perturbed Hamiltonian $H_1$ along the unperturbed trajectories. To do this, we note that

$$H_1 = \frac{d}{dt}i\frac{e\varphi_0(\chi)re^{-i\omega t + i\theta}}{\omega r_0} - \frac{i}{\omega}\frac{B^2}{m\lambda^2}p_\chi\frac{\partial}{\partial\chi}\frac{e\varphi_0 re^{-i\omega t + i\theta}}{r_0} - W_1 \tag{26}$$

where

$$W_1 \equiv \frac{i}{\omega}\left\{\frac{r^2B^2}{M}p_\psi\frac{e\varphi_0}{r_0}\frac{\partial r}{\partial\psi} + i\frac{\left(p_\theta - \frac{e}{c}\psi\right)}{mr^2}\frac{e\varphi_0 r}{r_0}\right\}e^{-i\omega t + i\theta}$$

and $\frac{d}{dt}$ is time differentiation along the unperturbed trajectories.

The virtue of this procedure is to note that $W_1$ can be integrated in time, over the unperturbed orbits, nearly exactly, with corrections of $\mathcal{O}(\kappa r)$ and $\mathcal{O}\left(\frac{\omega}{\Omega}\right)$ with $\Omega$ the ion cyclotron frequency. To see clearly that $W_1$ is nearly a total derivative, we note that for a spatially homogeneous field $B_0$, that

$$W_1 = \frac{i}{\omega}\left(\dot{r} + \dot{r}\dot{\theta}\right)e^{-i\omega t + i\theta}.$$

Using the equations of motion

$$\begin{aligned}\dot{r} &= r\dot{\theta}\Omega + \dot{\theta}^2 r\\ r\dot{\theta} &= -\dot{r}\Omega - 2\dot{r}\dot{\theta},\end{aligned}$$

we find

$$(\dot{r} + ir\dot{\theta})e^{i\omega t + i\theta} = \frac{d}{dt}\left[\frac{(r + ir\dot{\theta})e^{-i\omega t + i\theta}}{\Omega}\right] + \mathcal{O}\left(\frac{\omega}{\Omega}\right). \tag{27}$$

As the right-hand side is a total convective derivative, it can be integrated exactly in the Vlasov equation.

In an inhomogeneous magnetic field, the relation in Eq. (27) still approximately holds, but additional corrections must be obtained due to the magnetic field variation. We also note that the assumption that $H_1 \propto \psi^{1/2}e^{i\theta}$ (a displacement mode) is necessary for $W_1$ to be nearly a total derivative.

Now taking into account the variations of magnetic field, we find

$$\begin{aligned}
W_1 = & -\frac{d}{dt}\left\{\frac{cp_\psi}{\omega}\frac{r\varphi_0}{r_0} + \frac{ic}{\omega}\left(p_\theta - \frac{e}{c}\psi\right)\frac{\varphi_0}{r_0}\frac{\partial r}{\partial \psi}\right\}e^{-i\omega t + i\theta} \\
& -\frac{c}{\omega}\frac{r\varphi_0 e^{-i\omega t + i\theta}}{r_0}\left\{\frac{1}{2}\left[\frac{r^2B^2}{M}p_\psi^2 + \frac{\left(p_\theta - \frac{e}{c}\psi\right)^2}{Mr^2}\right]\frac{1}{B}\frac{\partial B}{\psi} + \frac{p_\chi^2}{M}B\frac{\partial}{\partial \psi}B\right\} \\
& -W_2 - \frac{ic}{\omega}\frac{r\varphi_0 e^{i\theta - i\omega t}p_\psi}{r_0}\frac{\left(p_\theta - \frac{e}{c}\psi\right)^2}{Mr^2}\left\{rB\left(\frac{\partial r}{\partial \psi} - \frac{1}{rB}\right) - r^3B^2\frac{\partial}{\partial \psi}\left(\frac{\partial r}{\partial \psi} - \frac{1}{rB}\right)\right\}
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
W_2 = & \frac{c}{\omega}\frac{r\varphi_0 e^{i\theta - i\omega t}}{r_0}\left\{\frac{1}{2}\left[\frac{r^2B^2p_\psi^2}{M} - \frac{\left(p_\theta - \frac{e}{c}\psi\right)^2}{Mr^2}\right]\frac{1}{B}\frac{\partial B}{\partial \psi} + \frac{ip_\psi\left(p_\theta - \frac{e}{c}\psi\right)}{M}\frac{\partial B}{\partial \psi}\right\} \\
= & \frac{c}{\omega}\left(-i\omega + \frac{B^2p_\chi}{M}\frac{\partial}{\partial \chi}\right)\left\{\frac{p_\psi r\varphi}{r_0} + i\left(p_\theta - \frac{e}{c}\psi\right)\frac{\varphi_0}{r_0}\frac{\partial r}{\partial \psi}\right\}e^{i\theta - i\omega t}.
\end{aligned}$$

The second, third, and fourth terms on the right-hand side of Eq. (28) are all of order curvature. This is a direct consequence of choosing nearly rigid perturbation of the form prescribed by Eq. (24). Since

$$\frac{\partial r}{\partial \psi} - \frac{1}{rB} = \frac{1}{rB_z} - \frac{1}{rB} = \frac{1}{2rB_z}\frac{B^2r}{B_z^2} + \frac{B_r}{B_z} \sim \kappa r\frac{L_p}{r} \ll 1$$

we will neglect the last term in Eq. (28).

The method of integration can be continued, and if terms higher order in the smallness parameters $\frac{\omega}{\Omega}$, $\frac{\omega_\chi}{\Omega}$, $\kappa r$, $\frac{\Delta z}{r}$ are neglected, where $\omega_\chi$ is the axial bounce frequency, it may also be verified that

$$W_2 = \frac{d}{dt}\left\{\frac{ic^2}{4e\omega}\left(p_\psi^2 r^2 B^2 - \frac{\left(p_\theta - \frac{e}{c}\psi\right)^2}{r^2}\right)\frac{r\varphi_0}{r_0 B^2}\frac{\partial B}{\partial\psi}e^{i\theta - i\omega t}\right.$$

$$- \frac{1}{2}\frac{c^2}{e\omega}p_\psi\left(p_\theta - \frac{e}{c}\psi\right)\frac{\varphi_0 r}{r_0 B}\frac{\partial B}{\partial\psi}e^{i\theta - i\omega t}$$

$$\left.+ \frac{Mc^2}{e\omega}\left(-i\omega + B^2\frac{p_\chi}{M}\frac{\partial}{\partial\chi}\right)\left[\frac{\left(p_\theta - \frac{e}{c}\psi\right)}{r^2B^2}\frac{\varphi_0 r}{r_0} - ir^2 p_\psi\frac{\varphi_0}{r_0}\frac{\partial r}{\partial\psi}\right]e^{i\theta - i\omega t}\right\} - W_3 \quad (29)$$

$$W_3 \equiv \frac{1}{4}\frac{c^2}{e\omega}\frac{r\varphi_0}{r_0 B}\frac{\partial B}{\partial\psi}e^{i\theta - i\omega t}$$

$$\left\{-\frac{\left(p_\theta - \frac{e}{c}\psi\right)^3}{Mr^4B} + ip_\psi\frac{\left(p_\theta - \frac{e}{c}\psi\right)^2}{Mr^2} + \frac{ir^2B^2}{M}p_\psi^3 - \frac{B}{M}p_\psi^2\left(p_\theta - \frac{e}{c}\psi\right)\right\}$$

and that

$$W_3 = -\frac{d}{dt}\left\{\frac{c^3}{4e^2\omega}\frac{\varphi_0}{r_0B^2}\frac{\partial B}{\partial\psi}e^{i\theta - i\omega t}\left[\frac{p_\psi}{r}\left(p_\theta - \frac{e}{c}\psi\right)^2 + r^3B^2p_\psi^3\right.\right.$$

$$\left.\left.+ irBp_\psi^2\left(p_\theta - \frac{e}{c}\psi\right) + \frac{i\left(p_\theta - \frac{e}{c}\psi\right)^3}{r^3B}\right]\right\} + \cdots . \quad (30)$$

We may therefore combine Eqs. (26), (30) to obtain:

$$H_1 = \frac{d}{dt}\mathcal{H}_1^{(0)} + \mathcal{H}_1^{(1)} = -i\omega\mathcal{H}_1^{(0)} + \left[\mathcal{H}_1^{(0)}, H\right] + \mathcal{H}_1^{(1)} \quad (31)$$

where

$$\mathcal{H}_1^{(0)} = \left\{i\frac{e}{\omega}\frac{r}{r_0}\varphi_0 + \frac{c}{\omega}\frac{r}{r_0}\varphi_0\left[p_\psi + \frac{i\left(p_\theta - \frac{e}{c}\psi\right)}{r}\frac{\partial r}{\partial\psi}\right] + \hat{\mathcal{H}}_2^{(0)}\right\}e^{-i\omega t + i\theta} \quad (32)$$

$$\hat{\mathcal{H}}_2^{(0)} = \frac{c^2M}{e\omega}\left(-i\omega + B^2\frac{p_\chi}{M}\frac{\partial}{\partial\chi}\right)\left[\frac{\left(p_\theta - \frac{e}{c}\psi\right)\varphi_\theta}{rr_0B^2} - ip_\psi\frac{r^2}{r_0}\frac{\partial r}{\partial\psi}\varphi_0\right]$$

$$- \frac{1}{2}\frac{c^2}{e\omega}p_\psi\left(p_\theta - \frac{e}{c}\psi\right)\frac{\varphi_0 r}{r_0B}\frac{\partial B}{\partial\psi} + \frac{ic^2}{4e\omega}\left[p_\psi^2r^2B^2 - \frac{\left(p_\theta - \frac{e}{c}\psi\right)^2}{r^2}\right]\frac{r\varphi_0}{r_0B^2}\frac{\partial B}{\partial\psi}$$

$$+\quad \frac{c^3}{4e^2\omega}\frac{1}{B^3}\frac{\partial B}{\partial\psi}\frac{\varphi_0}{r_0}\left[\frac{B}{r}p_\psi\left(p_\theta-\frac{e}{c}\psi\right)^2+r^3B^3p_\psi^3+irB^2p_\psi^2\left(p_\theta-\frac{e}{c}\psi\right)+i\frac{\left(p_\theta-\frac{e}{c}\psi\right)^3}{r^3}\right]$$

$$\begin{aligned}\mathcal{H}_1^{(1)} &= \left\{-\frac{ie}{\omega}B^2\frac{p_\chi}{M}\frac{\partial}{\partial\chi}\frac{\varphi_0 r}{r_0}+\frac{c}{2\omega}\left[p_\psi^2r^2B^2+\frac{\left(p_\theta-\frac{e}{c}\psi\right)^2}{r^2}\right]\frac{1}{MB}\frac{\partial B}{\partial\psi}\frac{r\varphi_0}{r_0}\right.\\ &+\left.\frac{c}{\omega}B\frac{p_\chi^2}{M}\left(\frac{\partial}{\partial\psi}B\right)\right\}e^{i\theta-i\omega t}\equiv\hat{\mathcal{H}}_1^{(1)}e^{i\theta-i\omega t}.\end{aligned} \tag{33}$$

This equation is valid for arbitrary Larmor radii.

We also note that as we discuss only flute-like perturbations, for which $\frac{\partial}{\partial\chi}\frac{r\varphi_0(\chi)}{r_0}=0$, we avoid a large contribution to the variational form.

Let

$$f=g-\left[F_0,\mathcal{H}_1^{(0)}\right]. \tag{34}$$

Substituting this expression for $f$ in Eq. (25), we obtain

$$-i\omega g+[g,H]+\left[F_0,\mathcal{H}_1^{(1)}\right]=0 \tag{35}$$

where we used the Jacobi identity

$$[A,[B,C]]+[C,[A,B]]+[B,[C,A]]=0$$

and the equilibrium condition

$$[F_0,H]=0.$$

This equation for $g$ is most conveniently solved by evaluating the Poisson brackets in the transformed canonical variables $P_i$, $Q_i$.

The equilibrium Hamiltonian $H$ is approximated by

$$H=H_0\left(P_\psi,P_\theta,P_\chi,Q_\chi\right)+\cdots$$

while

$$F_0 = F_0\left(H_0,P_\psi,P_\theta\right)$$

$$g = \hat{g}\left(H_0,P_\psi,P_\theta,Q_\psi,Q_\chi\right)e^{-i\omega t+iQ_\theta}+\frac{\partial F_0}{\partial H_0}\mathcal{H}_1^{(1)} \tag{36}$$

$$\mathcal{H}_1^{(1)} = \hat{\mathcal{H}}_1e^{-i\omega t+iQ_\theta+i\tilde{Q}_\theta} \tag{37}$$

where $\hat{\mathcal{H}}_1^{(1)}$ expressed in terms of the transformed variables may be approximated by

$$\begin{aligned}\hat{\mathcal{H}}_1^{(1)} &= \frac{c}{2\omega}\left[2MH_0 - B^2P_\chi^2\left(H_0, P_\psi, P_\theta, \chi\right)\right]\frac{1}{MB}\frac{\partial B}{\partial\psi}\frac{r\varphi_0}{r_0} \\ &+ \frac{c}{\omega}\frac{B}{M}P_\chi^2\left(H_0, P_\psi, P_\theta, \chi\right)\frac{\partial B}{\partial\psi}\frac{r\varphi_0}{r_0}.\end{aligned} \tag{38}$$

Thus, Eq. (34) may be written as follows:

$$\begin{aligned}&-i\omega\hat{g} + \dot{Q}_\chi\frac{\partial\hat{g}}{\partial Q_\chi} + i\dot{Q}_\theta\hat{g} + \Omega_\psi\frac{\partial\hat{g}}{\partial Q_\psi} \\ &= \left(i\omega\frac{\partial F_0}{\partial H_0} + i\frac{\partial F_0}{\partial P_0}\right)\hat{\mathcal{H}}_1^{(1)}e^{i\tilde{Q}_\theta} + \frac{\partial F_0}{\partial P_\psi}\frac{\partial}{\partial Q_\psi}\hat{\mathcal{H}}_1^{(1)}e^{i\tilde{Q}_\theta}\end{aligned} \tag{39}$$

where

$$\dot{Q}_\chi = \Omega_\psi P_\chi\left(H_0, P_\psi, P_\theta, \chi\right)\oint\frac{d\psi}{r^2B^2\hat{p}_\psi}\frac{B^2}{M} \doteq P_\chi.$$

Let

$$\psi = \psi_0 + \delta\psi$$

where

$$\begin{aligned}\psi_0 &= \frac{r^2}{2}\hat{B}_0 \\ \delta\psi &= -\frac{r^4}{16}\frac{d^2\hat{B}_1}{dz^2}.\end{aligned}$$

Expanding about $\psi = \psi_0$ and $B = \hat{B}_0$, we obtain

$$rB\hat{p}_\psi = rB\hat{p}_{0\psi} + rB\delta\hat{p}_{0\psi} + \cdots$$

where

$$\begin{aligned}r^2B^2\hat{p}_{0\psi}^2 &\equiv 2M\left[H_0 - \frac{\left(P_\theta - \frac{e}{c}\psi_0\right)^2}{2Mr^2} - \frac{\hat{B}^2P_\chi^2}{2M}\right] \\ rB\delta\hat{p}_{0\psi} &\equiv \left(\delta\psi\frac{\partial}{\partial\psi_0} + \frac{a4\delta\psi}{r^2}\frac{\partial}{\partial\hat{B}}\right)rB\hat{p}_{0\psi}.\end{aligned}$$

$\dot{Q}_\theta$ may then be expressed as follows:

$$\dot{Q}_\theta = -\Omega_\psi\frac{\partial}{\partial P_\theta}\oint d\psi\hat{p}_\psi = \Omega_\psi\bar{\theta} = \Omega_\psi\left(\bar{\theta}_0 + \bar{\theta}_1 + \cdots\right) \tag{40}$$

$$
\begin{aligned}
\bar{\theta}_0 &= -\frac{\partial}{\partial P_\theta}\oint dr\; rB\hat{p}_{0\psi} \\
\bar{\theta}_1 &= \frac{\partial^2}{\partial P_\theta^2}\oint dr\frac{e\delta\psi}{c}rB\hat{p}_{0\psi} - \frac{\partial}{\partial P_\theta}P_\chi\frac{\partial}{\partial P_\chi}\oint dr\frac{4\delta\psi}{r^2\hat{B}}rB\hat{p}_{0\psi} \\
&\doteq -\frac{\pi}{B}\frac{d^2\hat{B}_1}{dz^2}\left[\frac{r_L^2}{2}+\frac{\hat{B}^2P_\chi^2}{M^2\Omega^2}\right].
\end{aligned}
\tag{41}
$$

where $r_L^2 = \dfrac{2\Omega^2}{M}H - B\hat{B}^2P_\chi^2/M^2 \equiv$ square of Larmor radius. $\bar{\theta}_0$ is essentially the change in the azimuthal angle, neglecting the effects of curvature, in one "radial" bounce period. Thus, one can expect that $\bar{\theta}_0 = 0$ or $2\pi$ depending on whether the particle trajectory does or does not encircle the axis of symmetry (see Appendix A for a rigorous demonstration). The expression for $\bar{\theta}_1$ is evaluated in Appendix A (see Eq. (A.10)).

Let

$$
\hat{g}e^{i\bar{\theta}_0 Q_\psi} = h + \frac{1}{\Omega_\psi}\frac{\partial F_0}{\partial P_\psi}\hat{\mathcal{H}}_1^{(1)}e^{i\bar{\theta}_0 Q_\psi + i\tilde{Q}_\theta}. \tag{42}
$$

Substituting in Eq. (39), we obtain

$$
\begin{aligned}
&-i\omega h + i\bar{\theta}_1\Omega_\psi h + \dot{Q}_\chi\frac{\partial h}{\partial Q_\chi} + \Omega_\psi\frac{\partial h}{\partial Q_\psi} \\
&= \left\{ i\omega\left(\frac{\partial F_0}{\partial H_0} + \frac{1}{\Omega_\psi}\frac{\partial F_0}{\partial P_\psi}\right) + i\left(\frac{\partial F_0}{\partial P_\theta} - \frac{\dot{Q}_\theta}{\Omega_\psi}\frac{\partial F_0}{\partial P_\psi}\right)\right\}\hat{\mathcal{H}}_1^{(1)}e^{i\theta_0(\psi)+i\delta\tilde{Q}_\theta} \\
&= -\dot{Q}_\chi\frac{\partial}{\partial Q_\chi}\frac{1}{\Omega_\psi}\frac{\partial F_0}{\partial P_\psi}\hat{\mathcal{H}}_1^{(1)}e^{i\theta_0(\psi)+i\delta\tilde{Q}_\theta}
\end{aligned}
\tag{43}
$$

where

$$
\begin{aligned}
\bar{\theta}_0 Q_\psi + \tilde{Q}_\theta &= \theta_0(\psi) + \delta\tilde{Q}_\theta \\
\theta_0(\psi) &\equiv -\frac{\partial}{\partial P_\theta}\int^\psi d\psi\hat{p}_{0\psi} \\
\delta\tilde{Q}_\theta &\equiv -\frac{\partial}{\partial P_\theta}\int^\psi d\psi\delta\hat{p}_{0\psi} + \frac{\left(\dfrac{\partial}{\partial P_\theta}\oint d\psi\delta\hat{p}_{0\psi}\right)\dfrac{\partial}{\partial H_0}\displaystyle\int^\psi d\psi\hat{p}_\psi}{\dfrac{\partial}{\partial H_0}\displaystyle\oint d\psi\hat{p}_\psi} \ll \theta_0(\psi).
\end{aligned}
\tag{44}
$$

We shall now neglect $\delta\tilde{Q}_\theta/\theta_0(\psi)$ in the evaluation of Eq. (43).

We consider the limit where

$$
\left|\Omega_\psi\frac{\partial h}{\partial Q_\psi}\right| \gg \left|\dot{Q}_\chi\frac{\partial h}{\partial Q_\chi}\right| \gg \left|-i\omega h + i\bar{\theta}_1\Omega_\psi h\right|,
$$

that is, the radial bounce frequency is larger than the parallel bounce frequency, and the perturbation frequency and curvature drift frequency are less than both. We note that $\frac{\dot{Q}_\theta}{\Omega_\psi}\frac{\partial F_0}{\partial P_\psi}\hat{\mathcal{H}}_1^{(1)}$ to leading order is independent of $\chi$, and subsequently we shall neglect the small $\chi$ dependent terms, including the last term in Eq. (43).

We can therefore average Eq. (43) first in the variable $Q_\psi$ and then in $Q_\chi$ to obtain

$$\begin{aligned} h &= \bar{h} + \cdots \\ (-i\omega + i\omega_\kappa)\bar{h} &= \left\{ i\omega \left[\frac{\partial F_0}{\partial H_0} + \frac{1}{\bar{\Omega}_\psi}\frac{\partial F_0}{\partial P_\psi}\right] + i\left[\frac{\partial F_0}{\partial P_\theta} - \bar{\bar{\theta}}\frac{\partial F_0}{\partial P_\psi}\right]\right\} I \end{aligned} \tag{45}$$

where

$$I = \frac{\oint \frac{dQ_\chi}{\dot{Q}_\chi}\oint \frac{dQ_\psi}{\dot{Q}_\chi}\mathcal{H}_1^{(1)} e^{i\theta_0(\psi)}}{\left(\oint \frac{dQ_\chi}{\dot{Q}_\chi}\right)\left(\oint \frac{dQ_\psi}{\dot{Q}_\psi}\right)} \tag{46}$$

$$\bar{\Omega}_\psi = \frac{\oint \frac{dQ_\chi}{\dot{Q}_\chi}\Omega_\psi}{\oint \frac{dQ_\chi}{\dot{Q}_\chi}} \tag{47}$$

$$\bar{\bar{\theta}} = \frac{\oint \frac{dQ_\chi}{\dot{Q}_\chi}\bar{\theta}}{\oint \frac{dQ_\chi}{\dot{Q}_\chi}} \tag{48}$$

$$\omega_\kappa = \oint \frac{dQ_\chi}{\dot{Q}_\chi}\Omega_\psi\bar{\theta}_1. \tag{49}$$

In summary, the perturbed distribution function $f$ is:

$$\begin{aligned} f &= g - \left[F_0, \mathcal{H}_1^{(0)}\right] \\ g &= \frac{\partial F_0}{\partial H_0}\mathcal{H}_1^{(1)} + \frac{1}{\Omega_\psi}\frac{\partial F_0}{\partial P_\psi}\mathcal{H}_1^{(1)} + \bar{h}e^{-i\bar{\theta}_0 Q_\psi + iQ_\theta - i\omega t}. \end{aligned} \tag{50, 51}$$

The perturbed charge density $\tilde{\rho}$ is:

$$\tilde{\rho} = \sum e \int d^3v f. \tag{52}$$

Substitution in Poisson's equation yields the self-consistent equation for the perturbed electrostatic potential $\tilde{\varphi}$:

$$\nabla^2\tilde{\varphi} = -4\pi \sum e \int d^3v f. \tag{53}$$

As already discussed, instead of evaluating the perturbed charge density directly, it is more convenient to construct a quadratic variational since $f$ will need to be known only to first order in the curvature.

We can construct a quadratic variational form if we multiply Eq. (59) by the adjoint potential $\tilde{\varphi}^+$ and integrate over all space.

With the boundary terms set equal to zero because of the boundary conditions, we obtain:

$$\frac{1}{4\pi}\int d^3r\nabla\tilde{\varphi}\cdot\nabla\tilde{\varphi}^+ = \left\langle fH_1^+\right\rangle = \left\langle F_0\left[H_1^+,\mathcal{H}_1^{(0)}\right]\right\rangle + \left\langle gH_1^+\right\rangle \tag{54}$$

where $\langle\quad\rangle$ implies integration over all phase space and summation over particle species.

The adjoint equation for $H_1^+$ analogous to Eq. (31) is:

$$H_1^+ = i\omega\mathcal{H}_1^{(0)+} + \left[\mathcal{H}_1^{(0)},\hat{H}\right] + \mathcal{H}_1^{(0)+}. \tag{55}$$

Thus

$$\left\langle gH_1^+\right\rangle = \left\langle g\mathcal{H}_1^{(0)+}\right\rangle + \left\langle F_0\left[\mathcal{H}_1^{(1)},\mathcal{H}_1^{(0)+}\right]\right\rangle \tag{56}$$

and Eq. (54) may be expressed as follows:

$$\begin{aligned}\frac{1}{4\pi}\int d^3r\nabla\tilde{\varphi}\cdot\nabla\tilde{\varphi}^+ &= \left\langle F_0\left[H_1^+,\mathcal{H}_1^{(0)}\right]\right\rangle + \left\langle F_0\left[\mathcal{H}_1^{(0)},\mathcal{H}_1^{(0)+}\right]\right\rangle \\ &+\left\langle\left(\frac{\partial F_0}{\partial H_0}+\frac{1}{\Omega_\psi}\frac{\partial F_0}{\partial P_\psi}\right)\mathcal{H}_1^{(0)}\mathcal{H}_1^{(1)+}\right\rangle \\ &-\left\langle\frac{\left\{\omega\left(\dfrac{\partial F_0}{\partial H_0}+\dfrac{1}{\Omega_\psi}\dfrac{\partial F_0}{\partial P_\psi}\right)+\left(\dfrac{\partial F_0}{\partial P_\theta}-\dot{\theta}\dfrac{\partial F_0}{\partial P_\psi}\right)\right\}II^+}{(\omega-\omega_\kappa)}\right\rangle.\end{aligned} \tag{57}$$

Substituting for $H_1^+$, $\mathcal{H}_1^{(0)}$, $\mathcal{H}_1^{(0)+}$ and $\mathcal{H}_1^{(1)}$, we obtain

$$\begin{aligned}\left\langle F_0\left[H_1^+,\mathcal{H}_1^{(0)}\right]\right\rangle &= -2\left\langle F_0\frac{Mc^2\varphi_0\varphi_0^+}{r_0^2B^2}\right\rangle - \left\langle F_0\frac{c^2}{\omega}\frac{\left(p_\theta-\frac{e}{c}\psi\right)}{r_0^2B^2}\frac{\partial B}{\partial\psi}\varphi_0\varphi_0^+\right\rangle \\ &+\left\langle F_0\frac{c^3}{e\omega}\left(r^2B^2p_\psi^2+\frac{\left(p_\theta-\frac{e}{c}\psi\right)^2}{r^2}\right)\frac{1}{r_0^2B^3}\frac{\partial B}{\partial\psi}\varphi_0\varphi_0^+\right\rangle+\cdots\end{aligned} \tag{58}$$

$$\left\langle F_0\left[\mathcal{H}_1^{(1)},\mathcal{H}_1^{(0)+}\right]\right\rangle = \left\langle F_0\frac{c^2}{\omega^2}\frac{\varphi_0^+\varphi_0}{r_0^2}\frac{p_\chi^2}{M}\frac{\partial}{\partial\psi}r^2B\frac{\partial}{\partial\psi}B\right\rangle$$

$$+\left\langle F_0 \frac{c^2}{\omega^2}\frac{\varphi^+\varphi}{r_0^2}\frac{1}{2}\left[\frac{r^2B^2}{M}+\frac{\left(p_\theta-\frac{e}{c}\psi\right)^2}{Mr^2}\right]\frac{\partial}{\partial\psi}\frac{r^2}{B}\frac{\partial B}{\partial\psi}\right\rangle$$

$$+\cdots . \tag{59}$$

Adding,

$$\left\langle F_0\left[H_1^+,\mathcal{H}_1^{(0)}\right]\right\rangle+\left\langle F_0\left[\mathcal{H}_1^{(1)},\mathcal{H}_1^{(0)+}\right]\right\rangle\approx-2\left\langle F_0\frac{mc^2\varphi_0\varphi_0^+}{r_0^2B^2}\right\rangle$$

$$+\left\langle F_0\frac{c^2}{\omega^2}\frac{\varphi_0^+\varphi_0}{r_0^2}\left\{H_0+\frac{P_\chi^2B^2}{2M}\right\}^2\frac{\partial B}{\partial\psi}\right\rangle \tag{60}$$

where we neglect the last two terms in Eq. (58) since they are higher order in $\frac{\omega}{\Omega_0}\ll 1$.

In evaluating the integrals to determine $I$, we neglect $\delta\tilde{Q}_\theta$ in the exponent and axial inhomogeneities in the magnetic field $B$. We then obtain (see Appendix A, Eq. (A.12)).

$$I=\frac{\oint\frac{dQ_\chi}{\dot{Q}_\chi}\oint\frac{d\psi}{r^2B^2\hat{p}_\psi}\mathcal{H}_1^{(1)}e^{i\theta_0(\psi)}}{\left(\oint\frac{dQ_\chi}{\dot{Q}_\chi}\right)\left(\oint\frac{d\psi}{r^2B^2\hat{p}_\psi}\right)}\approx-\frac{c}{\omega}\frac{\kappa_0\varphi_0}{r_0^2\hat{B}_0}\left(H_0+\frac{\hat{B}_0\overline{P_\chi^2}}{2M}\right)\bar{r}_G \tag{61}$$

where

$$\kappa_0 = \frac{r_0}{2\hat{B}_0}\frac{d^2\hat{B}_1}{dz^2} \tag{62}$$

$$\bar{r}_G^2 = r_L^2+\frac{2P_\theta}{M\Omega_0} \tag{63}$$

$$\bar{r}_L^2 = \frac{2}{M\Omega_0^2}\left(H_0-\frac{\hat{B}_0^2}{2M}P_\chi^2\right) \tag{64}$$

$$\Omega_0 = \frac{e\hat{B}_0}{Mc},\ \overline{P_\chi^2}=\int\frac{d\chi}{\dot{Q}_\chi}P_\chi^2\Big/\int\frac{d\chi}{\dot{Q}_\chi}.$$

$\kappa_0$ is the curvature at $r_0$, $r_L$ is the Larmor radius, and $r_G$ is the distance of the guiding center from the axis of symmetry.

The curvature drift frequency $\omega_\kappa$ is (see Appendix A, Eq. (A.9))

$$\omega_\kappa=-\frac{\kappa_0}{M\Omega_0r_0}\left(H+\frac{\hat{B}_0^2\overline{P_\chi^2}}{2M}\right). \tag{65}$$

From Eq. (19), we deduce that

$$\frac{\partial P_\chi(H_0, P_\theta, P_\psi, \chi)}{\partial P_\theta} = -\frac{\oint d\psi \dfrac{\partial \hat{p}_\psi}{\partial P_\theta}}{\oint d\psi \dfrac{\partial \hat{p}_\psi}{\partial P_\chi}}$$

$$\frac{\partial P_\chi}{\partial P_\psi} = \frac{1}{\oint d\psi \dfrac{\partial \hat{p}_\psi}{\partial P_\chi}}$$

and therefore

$$\frac{\partial P_\chi}{\partial P_\psi} - \bar{\theta}\frac{\partial P_\chi}{\partial P_\theta} = 0. \tag{66}$$

Similarly,

$$\frac{\partial P_\chi}{\partial H_0} + \frac{1}{\Omega_\psi}\frac{\partial P_\chi}{\partial P_\psi} = 0. \tag{67}$$

Thus, the kinetic term in Eq. (63) may be integrated by parts to obtain

$$\begin{aligned}
&\left\langle \left\{ \omega \left( \frac{\partial F_0}{\partial H_0} + \frac{1}{\Omega_\psi}\frac{\partial F_0}{\partial P_\psi} \right) + \left( \frac{\partial F_0}{\partial P_\theta} - \bar{\theta}\frac{\partial F_0}{\partial P_\psi} \right) \right\} \frac{II^+}{(\omega - \omega_\kappa)} \right\rangle \\
&= -\left\langle \omega F_0 \frac{\partial}{\partial H_0} \left\{ \frac{II^+}{\omega - \omega_\kappa} \right\} \Big|_{P_\theta, P_\chi} \right\rangle - \left\langle F_0 \frac{\partial}{\partial P_\theta} \left\{ \frac{II^+}{\omega - \omega_\kappa} \right\} \Big|_{H_0, P_\chi} \right\rangle \\
&\approx -\left\langle F_0 \frac{c^2}{\omega^2} \left( \frac{\kappa_0}{r_0 \hat{B}_0} \right)^2 \frac{2H_0^2}{M\Omega_0 r_0^2} \frac{\varphi\varphi^+}{(\omega - \omega_\kappa)} \right\rangle
\end{aligned} \tag{68}$$

where the term involving $\dfrac{\partial}{\partial H_0}\left\{\dfrac{II^+}{\omega - \omega_\kappa}\right\}\Big|_{P_\theta, P_\chi}$ may be neglected since it is higher order in $\omega/\Omega_0$. For the same reason, the third term in Eq. (57) is negligible.

Equation (57) may therefore be approximated by:

$$\begin{aligned}
&\frac{1}{4\pi}\int d^3r \nabla\tilde{\varphi}\cdot\nabla\tilde{\varphi}^+ + \sum \int d^3r \frac{2MNc^2}{r_0^2 \hat{B}_0^2}\varphi_0\varphi_0^+ + \left\langle F_0 \left( H_0 + \frac{\overline{P_\chi^2}\hat{B}_0^2}{2M} \right) \frac{2c^2}{\omega^2} \frac{\kappa_0 \varphi_0 \varphi_0^+}{r_0^3 \hat{B}_0^2} \right\rangle \\
&- \left\langle F_0 \left( H_0 + \frac{\overline{P_\chi^2}\hat{B}_0^2}{2Mm} \right)^2 \frac{2c^2}{\omega^2} \frac{\kappa_0^2}{M\Omega_0 r_0^4 \hat{B}_0^2} \frac{\varphi_0\varphi_0^+}{(\omega - \omega_\kappa)} \right\rangle = 0.
\end{aligned} \tag{69}$$

Equation (69) is the desired form of the dispersion relation which was analyzed in the previous section.

# V Conclusions

We have rigorously derived the dispersion relation for the low beta flute interchange mode to a perturbed rigid displacement for a plasma system where the Larmor radius is of arbitrary size and the curvature radius is large compared to the plasma radius. The dispersion relation is given in Eq. (13). It is characterized by being independent of finite Larmor radius terms, and thus is valid for any low beta system where the curvature is weak. This dispersion closely resembles those derived for the displacement mode of small[15] and finite Larmor radius systems,[10] except for how the electric field energy couples into the problem. We note that this energy term can significantly differ in disc shape and cylindrically shaped plasmas. The coupling term, denoted by the symbol $Z$, is given by $Z = 8/3\pi$ for a disc shape and $Z = \dfrac{\Delta z}{r_0}$ for a cylindrical shape if $\Delta z \gg r_0$ where $r_0$ is the radius and $\Delta z$ the axial extent.

The stability condition is given by Eq. (16). To interpret this relation easily let us assume that the cold and hot ions in the system have the same mass $M$, and the pressure of the cold component is negligible compared to the hot component. Stability then requires

$$\frac{\kappa_0 a_h^2}{r_0} > \frac{8\delta_h}{1 + 2Z\dfrac{r_0}{\Delta z}\dfrac{\Omega^2}{\omega_{pi}^2}} \tag{70}$$

where $a_h = v_{\perp h}/\Omega \equiv$ hot ion Larmor radius,

$$\frac{\omega_{pi}^2}{\Omega^2} = \frac{4\pi \bar{N} M c^2}{B^2} \quad , \quad \bar{N} = \sum_i \int d^3r N_i(\mathbf{r}) \Big/ \pi r_0^2 \Delta z$$

$\equiv$ mean ion density of all species $\delta_h = \int d^3r N_h(\mathbf{r}) \Big/ \sum_i \int d^3r N_i(\mathbf{r}) \equiv$ hot ion density fraction, $\dfrac{\kappa_0}{r_0} = \dfrac{1}{2B(0)} \dfrac{d^2B(0)}{dz^2}$ with "0" referring to the on axis mid-plane value, and $Z$ is approximately given by an interpolation formula $Z = \dfrac{8}{3\pi} + \dfrac{\Delta z}{r_0}$ which goes over to the correct limits if either $\Delta z/r_0 \gg 1$ or $\Delta z/r_0 \ll 1$. Note that for a disc shaped plasma, the stability criterion is independent of $\Delta z$.

Results for the Migma experiments are reported in Ref. 14. A disc shaped deuterion plasma was formed with a hot particle energy of 650 keV in a 3.17$T$ magnetic field. The plasma radius is $r_0 = 9.3$ cm and the axial length is $\Delta z \sim 1.0$ cm. It is noted that $\omega_\kappa/\Omega =$

.04 from which it can be inferred, by using $r_0 \approx 2a_h$, that $\kappa_0 r_0 \approx 0.3$. Assuming $\delta_h = 1$, $Z = 8/3\pi$, we find that the stability condition is $\mathcal{N}_h \equiv \int d^3r N_h(\mathbf{r}) < \mathcal{N}_{cr} \equiv 1.1 \times 10^{12}$ particles. A maximum of $3.2 \times 10^{11}$ particles are reported to have been stored. Hence the lack of experimental observation of interchange mode is consistent with theory. Note that the theory for a long cylinder would predict instability if $\mathcal{N} > 10^{11}$ particles. Thus, using the correct coupling to the electric field energy is important in interpreting why the Migma experiment does not exhibit an interchange instability.

Above the particle threshold, $\mathcal{N} > \mathcal{N}_{cr}$, stability can be obtained by introducing cold ions. Alternatively feedback stabilization may be effective; at lower densities successful capacitive feedback techniques have been developed.

The former method of stabilization, that of using cold plasma, has successfully been used in hot electron experiments, and the stability criterion given by Eq. (69) when $Z\omega_{pi}^2 r_0/2\Omega^2\Delta z > 1$ is essentially the same as mechanism needed to explain the stability of these experiments. At such densities, stability requires

$$\delta_h \equiv \frac{\mathcal{N}_h}{\mathcal{N}_t} < \frac{1}{8}\frac{\kappa_0 a_h^2}{r_0} \tag{71}$$

where $\mathcal{N}_t$ is the total number of ions (cold and hot). More specifically, in the Migma configuration of Ref. 14, where we take all the same parameters except the number of particles, stability would require,

$$\delta_h < .01 + \frac{\mathcal{N}_{cr}}{\mathcal{N}_t}. \tag{72}$$

The main penalty with this method of stabilization is that for the same stored plasma energy the collisional slowing down time of the electrons is enhanced by a factor of 100 (due to the increased factor in electron density that must be introduced to charge neutralize the system), thereby increasing the input power requirements needed to achieve a given stored plasma energy.

Feedback stabilization[20] should be effective in stabilizing the interchange mode, especially as the interchange instability is expected to only affect the displacement mode, while shorter wavelength modes should be stabilized by finite Larmor radius effects. However, it has been noted[21,22] that there are unstable density bands present even with strong finite Larmor radius effects. Feedback by capacitive coupling should be effective if $\Omega^2/\omega_{pi}^2 > 1$, but the plasma may shield external electric potential signals when $\Omega^2/\omega_{pi}^2 \lesssim 1$. Further studies of this problem are needed.

Conducting wall stabilization, through the magnetic interaction of the plasma with its image current arising from the plasma perturbation, can stabilize[7] the interchange mode and other negative energy modes,[8] such as the precessional mode. However, to be effective a substantial plasma beta is required. Very roughly, stabilization from a conducting wall situated just outside the plasma radius requires

$$\beta > \alpha \kappa_0 r_0 \equiv \beta_{cr}$$

with $\alpha$ a numerical factor of $\mathcal{O}(1)$, for which a careful calculation still needs to be performed. For definitiveness, let us consider $\beta_{cr} = 0.2$. Then an estimate of the number of hot particles, $\mathcal{N}_h$, needed to achieve wall stabilization from a magnetic interaction is roughly,

$$\mathcal{N}_h \geq \frac{\beta_{cr} B_0^2 r_0^2 \Delta z}{8 E_h}$$

where $E_h$ is the energy of the hot particles. For the Migma parameters considered here we require

$$\mathcal{N}_h > 3 \times 10^{16} \frac{\Delta z}{r_0} \quad \text{particles} .$$

Note that there is several orders of magnitude in the hot particle number where straightforward stabilization of the interchange could be difficult, as the plasma will tend to shield signals from active capacitive feedback, while passive magnetic feedback with conducting walls is insufficient to provide stabilization. It would appear that it is then necessary to develop magnetic feedback techniques to stabilize the interchange mode in this intermediate particle number regime.

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# Appendix A – Evaluation of Various Phase Space Integrals

The equilibrium particle trajectories may be approximated by the Hamiltonian $H_0(P_\psi, P_\theta, P_\chi, \chi)$ defined by Eq. (19). The "radial" momentum is $\hat{p}_\psi(H_0, P_\theta, P_\chi, \psi, \chi)$ defined by Eq. (18).

If the field line curvature is neglected and the magnetic field is approximated by $B = \hat{B}(z)a$ and $\psi = \dfrac{\hat{B}(z)r^2}{2}$, the radial momentum is

$$r\hat{B}\hat{p}_{0\psi} = \left\{ 2MH_0 - \frac{\left(P_\theta - \dfrac{e\hat{B}r^2}{2c}\right)^2}{r^2} - \hat{B}^2 P_\chi^2 \right\}^{1/2} . \tag{A.1}$$

The turning points (where $\hat{p}_{0\psi} = 0$ in Eq. (A.1)) of the trajectories occur at

$$r = r^{\pm} \equiv \left(r_L^2 + \frac{2P_\theta c}{e\hat{B}}\right)^{1/2} \pm r_L \,, \qquad P_\theta > 0$$
$$r = r^{\pm} \equiv r_L \pm \left(r_L^2 - \frac{2|P_\theta| c}{e\hat{B}}\right)^{1/2} \,, \quad P_\theta < 0$$

where the Larmor radius $r_L$ is

$$R_L = \frac{c}{e\hat{B}} \left(2MH_0 - P_\chi^2 \hat{B}^2\right)^{1/2} .$$

Particles with $P_\theta < 0$ encircle the axis of symmetry while particles with $P_\theta < 0$ do not. The radial position of the particle guiding center $r_G$ is

$$r_G = \left(r_L^2 + \frac{2P_\theta c}{e\hat{B}}\right)^{1/2} .$$

In order to evaluate integrals over a closed particle trajectory, it is convenient to introduce the Larmor phase angle $\varphi$ given by

$$r^2 = r_G^2 + r_L^2 + 2r_G r_L \cos\varphi. \tag{A.2}$$

Thus, for example, the integral for the lowest order adiabatic invariant, Eq. (19), with use of (A.1)-(A.3), is given by

$$\oint d\psi \hat{p}_{0\psi} = \oint dr\, r\hat{B}\hat{p}_{0\psi} = \oint d\varphi \frac{M\Omega r_G^2 r_L^2 \sin^2\varphi}{[r_G^2 + r_L^2 + 2r_G r_L \cos\varphi]} \tag{A.3}$$

where $\Omega = \dfrac{e\hat{B}}{Mc}$.

Let $Z = e^{i\varphi}$. The integral may be transformed to a contour integral $C$ along the unit circle in the $Z$-plane:

$$\oint d\psi \hat{p}_{0\psi} = -\frac{M\Omega r_G r_L}{4i} \oint_C dz \frac{(Z^2-1)^2}{Z^2\left(Z+\frac{r_G}{r_L}\right)\left(Z+\frac{r_L}{r_G}\right)}. \tag{A.4}$$

If $\dfrac{r_L}{r_G} < 1$ ($> 1$), the poles at $Z = 0$ and $Z = \dfrac{r_L}{r_G}\left(\dfrac{r_G}{r_L}\right)$ contribute to the contour integral:

$$\oint d\psi \hat{p}_{0\psi} = \begin{cases} M\Omega\pi r_L^2 \,, & \dfrac{r_L}{r_G} < 1 \\ M\Omega\pi r_G^2 \,, & \dfrac{r_L}{r_G} > 1 \,. \end{cases}$$

Note, that if $r_L/r_G < 1$, this is the usual expression for the adiabatic invariant, but a different form from the conventional one arises if $r_L/r_G > 1$.

The change in the azimuthal angle $\bar{\theta}_0$ (Eq. (41)), neglecting curvature, in one "radial" bounce period is

$$\bar{\theta}_0 = -\frac{\partial}{\partial P_\theta} \oint d\psi \hat{p}_{0\psi} \tag{A.5}$$

$$= \begin{cases} 0 & P_\theta > 0 \\ 2\pi & P_\theta < 0 \end{cases} \,. \tag{A.6}$$

We have thereby explicitly confirmed the intuitive result of the text.

Now let us consider the change in the azimuthal angle $\bar{\theta}_1$ induced by the field line curvature in one "radial" bounce. From Eq. (41) we have,

$$\bar{\theta}_1 = \frac{\partial^2}{\partial P_\theta^2} \oint dr \frac{e}{c} \delta\psi r \hat{B} \hat{p}_{0\psi} - \frac{\partial}{\partial P_\theta} P_x \frac{\partial}{\partial P_x} \oint dr \delta B r \hat{p}_{0\psi} \tag{A.7}$$

where $\delta\psi = -\dfrac{r^4}{16}\dfrac{d^2\hat{B}_1}{dz^2}$, $\delta B = -\dfrac{r^2}{4}\dfrac{d^2\hat{B}}{dz^2}$.

Since

$$\begin{aligned} \oint dr r^5 \hat{B} \hat{p}_{0\psi} &= \oint d\varphi M\Omega r_G^2 r_L^2 \sin^2\varphi \left\{ r_G^2 + r_L^2 + 2 r_G r_L \cos\varphi \right\} \\ &= \pi M\Omega r_G^2 r_L^2 (r_G^2 + r_L^2) \end{aligned}$$

and

$$\oint drr^3 \hat{B}\hat{p}_{0\psi} = \pi M\Omega r_G^2 r_L^2.$$

$\bar{\theta}_1$ may be evaluated:

$$\begin{aligned}
\bar{\theta}_1 &= -\frac{\pi e M\Omega}{16c}\frac{d^2\hat{B}_1}{dz^2}\frac{\partial^2}{\partial P_\theta^2} r_G^2 r_L^2 \left(r_G^2 + r_L^2\right) \\
&+ \frac{\pi M\Omega}{4\hat{B}}\frac{d^2\hat{B}_1}{dz^2}\frac{\partial}{\partial P_\theta} P_\chi \frac{\partial}{\partial P_\chi} r_G^2 r_L^2 \\
&= -\frac{\pi}{\hat{B}}\frac{d^2\hat{B}_1}{dz^2}\left\{\frac{r_L^2}{2} + \frac{\hat{B}^2 P_\chi^2}{M^2\Omega^2}\right\}.
\end{aligned} \tag{A.8}$$

The local curvature drift frequency $\omega_{\text{curv}}$ is

$$\omega_{\text{curv}} = \frac{\Omega}{2\pi}\dot{\bar{\theta}}_1 = -\frac{\kappa_0}{M r_0 \Omega}\left(H_0 + \frac{P_\chi^2 \hat{B}^2}{2M}\right)$$

where $\kappa_0 = \dfrac{r_0}{2\hat{B}}\dfrac{d^2\hat{B}_1}{dz^2}$ is the local curvature at $r = r_0$.

The local drift averaged over the axial motion is

$$\omega_\kappa = \frac{\oint \dfrac{d\chi}{\dot{Q}_\chi}\dfrac{\Omega}{2\pi}\bar{\theta}_1}{\oint \dfrac{d\chi}{\dot{Q}_\chi}} = -\frac{\kappa_0}{M r_0 \Omega_0}\left(H_0 + \frac{\overline{P_\chi^2}\hat{B}_0^2}{2M}\right) \tag{A.9}$$

where

$$\overline{P_\chi^2} = \frac{\oint \dfrac{d\chi}{\dot{Q}_\chi} P_\chi^2}{\oint \dfrac{d\chi}{\dot{Q}_\chi}}.$$

The radial integral in Eq. (61) may be evaluated by using the approximate particle trajectory in the absence of curvature

$$\begin{aligned}
&\oint \frac{d\psi}{r^2 B^2 \hat{p}_\psi}\mathcal{H}_1^{(1)} e^{i\theta_0(\psi)} \Big/ \oint \frac{d\psi}{r^2 B^2 \hat{p}_\psi} \\
&= -\frac{\varphi_0 c}{4\omega r_0 M \hat{B}^2}\frac{d^2\hat{B}_1}{dz^2}\left(2MH_0 + \hat{B}^2 P_\chi^2\right)\frac{\oint \dfrac{dr}{r\hat{B}\hat{p}_{0\psi}} r e^{i\theta_0}}{\oint \dfrac{dr}{r\hat{B}\hat{p}_{0\psi}}}
\end{aligned}$$

$$= -\frac{\varphi_0 \kappa_0 c}{\omega r_0^2 \hat{B}} \left( H_0 + \frac{\hat{B}^2 P_\chi^2}{2M} \right) \oint \frac{d\varphi}{2\pi} \left( r_G + r_L \cos\varphi - i r_L \sin\varphi \right)$$

$$= -\frac{\varphi_0 \kappa_0 c}{\omega r_0^2 \hat{B}} \left( H_0 + \frac{\hat{B}^2 P_\chi^2}{2M} \right) r_G \tag{A.10}$$

If $\hat{B}$ and $r_G$ are approximated by

$$\begin{aligned} \hat{B} &= \hat{B}_0 + \cdots \\ r_G &= \bar{r}_G + \cdots \end{aligned}$$

where

$$\begin{aligned} \bar{r}_G &= \left( \overline{r_L^2} + \frac{2 P_\theta c}{e \hat{B}_0} \right)^{1/2} \\ \overline{r_L^2} &= \frac{c}{e \hat{B}_0} \left( 2 M H_0 - \overline{P_\chi^2} \hat{B}_0^2 \right), \end{aligned}$$

the average over the axial motion yields

$$I = \frac{\oint \frac{d\chi}{\dot{Q}_\chi} \oint \frac{d\chi}{r^2 B^2 \hat{p}_\psi} \mathcal{H}_1 e^{i\theta_0(\psi)}}{\oint \frac{d\chi}{\dot{Q}_\chi} \oint \frac{d\chi}{r^2 B^2 \hat{p}_\psi}} = -\frac{\varphi_0 \kappa_0 c}{\omega r_0^2 \hat{B}_0} \left( H_0 + \frac{\hat{B}_0^2 \overline{P_\chi^2}}{2M} \right) \bar{r}_G. \tag{A.11}$$