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Bremsstrahlung from Channeled Charged Particles: Application to a Crystal X-ray Accelerator

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Abstract

The ultimate acceleration structure for a very high energy accelerator may be provided by the solid state crystal lattice. We evaluate the interaction between high-energy particles and lattice particles. The mean free path of electrons due to bremsstrahlung is less severe when they are channeled. Even when particles are channeled, however, the bremsstrahlung losses are proportional to the energy of the particle. For a muon or heavier particles this may be tolerable in practical application.

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I. Introduction: the crystal X-ray accelerator

In the pursuit of ultra-high energies, higher accelerating fields and frequency, and thus a finer structure, have to be considered.[1] In the particular case of plasma accelerators, this means also higher densities of the accelerating medium. One may say that an ultimate step in pursuit of finer structure and higher densities is to use an ordinary condensed matter $\left(n \sim 10^{22}/\mathrm{cm}^3\right)$ as an accelerating medium. This is the main focus of the present paper.

Indeed it is well known that a beam of positively charged particles can propagate between the rows of atoms of an ordinary crystal with reduced losses for ionization and bremsstrahlung. (This is particularly important for positrons.) Similarly, negatively charged particles may be trapped around any single row of atoms, but with higher losses.

The crystal lattice is a natural iris structure for X-rays propagating along the rows of atoms—X-rays develop a longitudinal field component, which may be used to accelerate particles. This idea was proposed and analyzed in detail in Ref. 2. Regardless of the acceleration mechanism, there are general features in the processes of bremsstrahlung losses, focalization, and possible emittance increase that are due to the channeled particle interaction with the crystal lattice. The ionization losses are simply proportional to the electronic density of the medium in which high energy particles travel. For ordinary solid matter (say, two or three times the density of water) the ionization losses are about 5 MeV/cm.

The luminosity attainable with this acceleration scheme[2] is unclear at present. On the one hand, the channeling particles are confined by a very strong ($> 1V/\text{Å}$) electrostatic field. Thus the beam is well collimated in microscopic areas and can be focused into a very tiny interacting region. On the other hand, incoherent scattering due to the lattice thermal vibrations could increase the emittance and leads to beam losses.

The inverse of this accelerator, an electron beam injected into a crystal to lase X-rays,[3] may be the power source for this and other accelerators.

II. Channeling of high energy particles

There are well-documented studies, both theoretical and experimental, of channeling of low energy particles (ions and electrons of a few MeV) in a crystal.[4] Here we discuss the features for high energy particles ($\gamma \gg 1$). In our simple idealization, the crystal is a static and fixed distribution of charge, producing an electrostatic field which perturbs the motion of high energy particles.

To make it concrete, we consider an hexagonal lattice, where $2b$ is the distance of two atoms lying in the same $x - y$ plane, and a is the distance between planes (see Fig. 1). Let us normalize a, b to the Bohr radius a_B:

$$a = \check{a} a_B; \qquad b = \check{b} a_B; \qquad a_B = \alpha^{-2} r_e, \tag{2-1}$$

where $\alpha \sim 1/137$ is the fine structure constant and r_e is the classical radius of an electron $r_e = e^2/m_e c^2$. Typical values of $\check{a}$ and $2\check{b}$ are around 5.

As a model for the charge density, we assume that Z_{eff} electrons per atom are uniformly distributed in a sphere of radius d around the nucleus $\vec{g}_i$ of atom number i:

$$n(\vec{x}) \equiv \frac{\rho(x)}{e} = \frac{Z_{\text{eff}}}{\pi b^2 a} - \sum_i Z\delta\left(\vec{x} - \vec{g}_i\right) + \sum_i \frac{Z - Z_{\text{eff}}}{\frac{4\pi}{3} d^3} \theta\left(d^2 - (\vec{x} - \vec{g}_i)^2\right). \tag{2-2}$$

The electrostatic potential φ, measured in unit of $m_e c^2/e$ to make it nondimensional, satisfies the Poisson equation

$$\nabla^2 \varphi = -4\pi r_e n(\vec{x}), \tag{2-3}$$

while the vector potential is zero. It is convenient to measure the particle momenta in units of $m_e c$ also to make them nondimensional.

Classical mechanics is suitable enough for discussing the motion of channeled particles. This is true even for the transverse motion, except for the case of electrons. Indeed the order of magnitude of the electrostatic field φ (difference) is $Z_{\text{eff}} \alpha^2$ over a length scale $\ell \sim a_B$. To apply classical mechanics to the longitudinal motion, we need $p \gg \ell/\lambda_e \sim \alpha$, where $\lambda_e = \hbar/m_e c$ is electron Compton length and p is a normalized momentum (by mass times c). As p_z is about $\gamma \frac{m_c}{m_e}$ with m_c being the mass of the channeled particle, this criterion becomes $\gamma \alpha^{-1} \frac{m_c}{m_e} \gg 1$. This is well satisfied.

For the transverse degrees of freedom, the transverse momenta p_x is bounded by the channel acceptance:

$$p_x^2 \frac{m_e}{m_c} < O(\varphi) \sim \alpha^2 Z_{\text{eff}}. \tag{2-4}$$

The applicability criterion of classical mechanics is

$$\frac{p_x \ell}{\lambda_e} = \sqrt{\frac{m_c}{m_e} Z_{\text{eff}}} \gg 1. \tag{2-5}$$

This is well satisfied for a proton $m_c = 1840 m_e$. On the other hand, if the channeled particle is an electron, we expect a few bounded levels $\left(\sqrt{Z_{\text{eff}}} \sim 5\right)$, whose detailed description requires quantum mechanics.

Now we will prove that a particle moving almost parallel to the z-axis has average trajectory determined by an effective potential which is uniform but represented approximately by

$$\bar{\varphi}(x,y) = \frac{1}{a} \int_0^a dz \varphi(x,y,z).$$

The non-uniform components of the electrostatic potential $\tilde{\varphi} = \varphi - \bar{\varphi}$ oscillates with frequency $2\pi \frac{c}{a} \gamma$ and its harmonics in the rest frame of the particles. This is too high a frequency to change the average motion of the particle; its only effect is to make the particle radiate.

We start from the covariant Hamiltonian equation of motion[5]

$$\frac{d\xi}{d\lambda} = [h, \xi], \tag{2-6}$$

where ξ is any function of x^ν, p^ν, λ is the proper time, and the Hamiltonian is given by

$$h = -\frac{1}{2} \left[(p_4 + \varphi + \tilde{\varphi})^2 - p_x^2 - p_y^2 - p_z^2 \right]. \tag{2-7}$$

h and p_4 are exact invariants of the motion and

$$I_z = \langle p_z \rangle = \frac{1}{a} \int_0^a dz p_z \tag{2-8}$$

is an adiabatic invariant of the motion. Recovering p_z from (2-6) and expanding it in powers of $\tilde{\varphi}$ in the integral (2-8), we get

$$I = \hat{I} - \frac{\langle \bar{\varphi}^2 \rangle}{2\hat{I}^3} \left(p_x^2 + p_y^2 - 2h \right) \tag{2-9}$$

with

$$\hat{I} = \sqrt{2h + (p_4 + \bar{\varphi})^2 - (p_x^2 + p_y^2)}. \tag{2-10}$$

Solving for h up to second power in the fields gives

$$h = h_{\rm tr} - \frac{1}{2} p_4^2 + \frac{I^2}{2},$$

where the transverse covariant Hamiltonian is

$$h_{\rm tr} = \frac{1}{2} \left(p_x^2 + p_y^2 \right) - p_4 \bar{\varphi} + \bar{\varphi}^2 + \langle \tilde{\varphi}^2 \rangle \frac{1 + p_x^2 + p_y^2}{I^2}. \tag{2-11}$$

It is now evident that only $\bar{\varphi}$ matters for relativistic particles ($p_4 \gg 1$; $I \gg 1$).

The "velocity" $dx^4/d\lambda$ is not a constant; however, we can compute the ratio of the time elapsed $\Delta\lambda$ during one period:[5,6]

$$\begin{aligned} \frac{\Delta x^4}{\Delta \lambda} = \frac{\frac{\partial I}{\partial p^4}}{\frac{\partial I}{\partial(-2h)}} = \frac{\frac{\partial I}{\partial \hat{I}^2} \frac{1}{(p_4 + \bar{\varphi})}}{\frac{\partial I}{\partial \hat{I}^2} - \frac{\langle \tilde{\varphi}^2 \rangle}{2I^3}} &= \frac{1}{p_4 + \bar{\varphi}} \left(1 + \frac{\langle \tilde{\varphi}^2 \rangle}{2I^3} \left(\frac{\partial I}{\partial \hat{I}^2} \right)^{-1} \right) \\ &= \frac{1}{p_4 + \bar{\varphi}} \left(1 + \frac{\langle \tilde{\varphi}^2 \rangle}{I^2} \right). \end{aligned} \tag{2-12}$$

We can change the parameter of the motion from λ to the real time average t_1, defined

$$t_1 = \lambda \frac{\Delta x^4}{\Delta \lambda}. \tag{2-13}$$

The Hamiltonian equation (2-6) changes into

$$\frac{d\xi}{dt_1} = \left[H_{\rm tr}, \xi \right], \tag{2-14}$$

where the noncovariant Hamiltonian $H_{\rm tr}$ is defined as

$$H_{\rm tr} = h_{\rm tr} \frac{\Delta x^4}{\Delta \lambda}. \tag{2-15}$$

Because real laboratory time and real time average are almost the same, we drop the index herewith.

We obtain the final form

$$\frac{d\xi}{dt} = [H_{\text{tr}}, \xi], \tag{2-16}$$

where ξ is any function of p_x, p_y, x, y and from (2-15) and (2-12) the transverse Hamiltonian, with the desired accuracy, is

$$H = \frac{m_e}{m_c}\frac{p_x^2 + p_y^2}{2\gamma} - \bar{\varphi} + O\left(\bar{\varphi}^2, \langle\tilde{\varphi}^2\rangle\right). \tag{2-17}$$

Here we recover the explicit mass dependency. When needed, the quantization of (2-16), (2-17) is trivial.

A negatively charged channeled particle is attracted in the region near a row of atoms $r = \sqrt{x^2 + y^2} < b$. In this region we can neglect the asymmetric contribution of 6 nearby rows of atoms; setting the boundary condition $\bar{\varphi}(b) = 0$ we get

$$\begin{aligned}\bar{\varphi}(r) =& \frac{r_e}{a}\left[Z_{\text{eff}}\left(\frac{b^2 - r^2}{b^2} - 2\ln\frac{b}{r}\right)\right.\\ &\left. - (Z - Z_{\text{eff}})\left\{2\frac{r^2 + 2d^2}{3d^3}\sqrt{d^2 - r^2} - 2\ln\frac{d}{r}\right\}\theta(d - r)\right]\end{aligned} \tag{2-18}$$

for the model (2-2).

Appropriate values of angular momentum $L = xp_y - yp_x$ and the transverse energy $\mathcal{E} = H_{\text{tr}}$ are required to channel the negative particle. Indeed, from (2-16, 17, 18), enforcing the conditions

$$\left.\frac{\partial H_{\text{tr}}}{\partial r}\right|_{r=b} > 0 \qquad \text{and} \qquad p_r^2(r = b) < 0,$$

we get

$$2\mathcal{E} < \frac{m_e}{m_c}\frac{L^2}{b^2} < b\left.\frac{\partial\bar{\varphi}}{\partial r}\right|_{r=b}. \tag{2-19}$$

A positively charged particle may be channeled near the center of every triangle, whose vertices are atoms in a $x - y$ plane. Let us call r_i the distance of the particle from this center

$$r_i^2 = (x - b)^2 + \left(y - \frac{b}{\sqrt{3}}\right)^2$$

from Fig. 1.

For $r_i \ll b$ the potential may be approximated as

$$\bar{\varphi}(r_i) = 2Z_{\text{eff}}\frac{r_e}{a}\left(\frac{r_i}{b}\right)^2. \tag{2-20}$$

III. Bremsstrahlung

We compute bremsstrahlung losses based on what is called the virtual photon method. The virtual photon method offers the advantage of being directly applicable to a high energy classical particle moving with $r = \sqrt{x^2 + y^2}$ being constant or slowly changing. As discussed earlier, this is a perfectly legitimate description of a muon or a proton. For an electron the resulting bremsstrahlung must be weighted with $|\psi(r_c)|^2$, where ψ is the electron eigenstate of (2-17).

We expand the computation in Ref. 7, looking for whether the periodicity of the crystal may eliminate the typical γ-dependence of bremsstrahlung losses, which makes the bremsstrahlung losses so intolerable for a high energy electron in ordinary matter. Consider a particle moving parallel to the z-axis with velocity βc, with $\beta \sim 1$ and $\gamma = \left(1 - \beta^2\right)^{-1/2}$ as usual. The method is based on the fact that in the rest frame of this electron, the crystal electromagnetic fields may be written as

$$\begin{aligned} A_4'\left(x^{4'}, \vec{x}'\right) &= \gamma\varphi\left(x, y, \gamma(z' + \beta ct')\right), \\ A_z' &= \beta A_4'. \end{aligned} \tag{3-1}$$

Near the electron these fields are almost equal to the fields of a plane wave that propagates in the opposite direction in $\hat{z}$:

$$\begin{aligned} A_4' &= \gamma\varphi\left(r_e, \varphi, \gamma(z' + ct')\right), \\ A_z' &= A_4'. \end{aligned} \tag{3-2}$$

Let us compute the spectral density of this plane wave. We need to know φ. Let us make a Fourier series expansion of φ:

$$\begin{aligned} \hat{\varphi}_n(x, y) &= \frac{1}{a}\int_0^a dz e^{-ik_n z}\varphi(x, y, z) \\ \varphi(x, y, z) &= \sum_{n=-\infty}^{+\infty} e^{-ik_n z}\hat{\varphi}_n(x, y), \end{aligned} \tag{3-3}$$

where $k_n = 2\pi n/a$. The Poisson equation (2-3) becomes

$$\left(\left(\partial_x^2 + \partial_y^2\right) - k_n^2\right)\hat{\varphi}_n = -4\pi r_0 \hat{n}_n. \tag{3-4}$$

To make the calculation easier, we take the limit $d \to 0$ in the density expression (2-2), obtaining

$$n(\vec{x}) = Z_{\text{eff}} \left[\frac{1}{\pi b^2 a} - \sum_i (\vec{x} - \vec{g}_i) \right].$$

This yields

$$n_n = \frac{1}{a} Z_{\text{eff}} \delta(x) \delta(y) \qquad \text{for} \qquad n_z \neq 0. \tag{3-5}$$

Imposing the approximate boundary condition $\varphi_n \to 0$ for $r \to \infty$, we obtain

$$\varphi = 2 \frac{r_0}{a} Z_{\text{eff}} K_0 (k_n r), \tag{3-6}$$

where K_0 is the modified Bessel function[8] diverging like $-\ln r$ for $r \to 0$ and damping as $\sqrt{\pi} e^{-r} / \sqrt{2r}$ for $r \to \infty$.

In the comoving frame the power flux is $S = cE' \times B'/4\pi$, where the (dimensional) field is given by

$$E'_r = -\frac{mc^2}{e} \frac{\partial A'_4}{\partial r}, \qquad B_v = -mc^2 \frac{\partial A'_z}{\partial r}. \tag{3-7}$$

Therefore,

$$S = \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} S_n$$

with

$$S_n = \frac{c}{4\pi} \frac{mc^2}{r_e} \frac{\partial A'_{4n}}{\partial r} \frac{\partial A'_{z;-n}}{\partial r} = \frac{c}{\pi} Z^2_{\text{eff}} \frac{mc^2}{a^2} r_e \gamma^2 k_n^2 K'^2_0 (k_n r). \tag{3-8}$$

The frequency of the nth component is $\omega' = c\gamma k_n = 2\pi c\gamma n/a$.

We can now use the formulae for the scattering of a photon on a rest particle.[9] The parameter f_n, expresses the hardness of the photon

$$f_n = \frac{\hbar\omega'}{m_c c^2} = 2\pi\gamma n \left(\frac{\lambda_e}{a} \right) \left(\frac{m_e}{m_c} \right). \tag{3-9}$$

When $f_i \gg 1$, the full Klein-Nishima formula must be used.[10] This ultrarelativistic regime corresponds to $\varepsilon > 60$ MeV for electrons, and $\varepsilon > 2400$ GeV for muons. Calling ω''

the scattered photon frequency, ϑ the angle of scattering of the photon, we express the differential cross section as[9]

$$d\sigma = \frac{1}{2} r_c^2 \left(\frac{\omega''}{\omega'}\right)^2 \left(\frac{\omega''}{\omega'} + \frac{\omega'}{\omega''} - \sin^2\vartheta\right) d\Omega, \tag{3-10}$$

where the frequencies satisfy

$$\frac{1}{\omega''} - \frac{1}{\omega'} = \frac{\hbar}{m_c c^2}(1 - \cos\vartheta). \tag{3-11}$$

From (3-11)

$$\frac{\omega''}{\omega'} = \frac{1}{1 + f_n x} \qquad \text{with} \qquad x = 1 - \cos\vartheta. \tag{3-12}$$

The force F_n exerted on the electron is the sum of the photon scattered momenta

$$\Delta p_z = \frac{1}{c}(h\omega' - h\omega'' \cos\vartheta)$$

on all the photons belonging to the nth component scattered in the unit time

$$\begin{aligned} F_n = \Sigma \frac{h\omega'}{c}\left(1 - \frac{\omega''}{\omega'}\cos\vartheta\right) &= \int d\sigma \Sigma \frac{h\omega'}{c}(f_n - 1)\frac{x}{1 + f_n x} \\ &= \frac{S_n}{c}\int d\sigma\,(f_n - 1)\frac{x}{1 + f_n x}, \end{aligned} \tag{3-13}$$

where the first summation is over photons scattered in one second and the second summation is over photons incoming in a second in one cm^2. In order to obtain the expression in the leading power in f_n, corresponding to the ultrarelativistic electron case, we approximate the differential cross section as

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_c^2 \frac{1}{(1 + f_n x)^2}\left\{\frac{1}{(1 + f_n x)} + (1 + f_n x) - \left(2x - x^2\right)\right\} \simeq \frac{1}{2} r_c^2 \frac{1}{1 + f_n x}, \tag{3-14}$$

so that

$$F_n = \frac{S_n}{c}\pi r_c^2 \int_0^2 dx\,(f_n - 1)\frac{x}{(1 + f_n x)^2} = \frac{S_n}{c}\pi r_c^2 g(f_n), \tag{3-15}$$

where

$$g(f) = \frac{f - 1}{f^2}\left(\ln(1 + 2f) + \frac{1}{1 + 2f} - 1\right) \simeq \frac{1}{f}\ln 2f.$$

The total drag force acting on the particle in its rest frame is $F' = \sum_n F_n$, parallel to $\hat{z}$. The magnitude of this force is invariant, i.e., it is the same as in the laboratory frame. The energy loss (stopping power $d\mathcal{E}/dx$) is simply due to the work done by this force

$$\frac{d\mathcal{E}}{dx} = -F = -F' \simeq -2F_{n_1}. \tag{3-16}$$

In the last step in (3-16) we kept only the two terms $n = 1, -1$, because $g(f_n) \simeq 1/n$ and

$$S_n \propto \left(k_n K_0' \left(k_n r\right)\right)^2 \propto e^{-n\frac{4\pi}{a}r}.$$

Finally, collecting all γ-dependencies, we find that energy losses due to bremsstrahlung in the channel are roughly proportional to the energy:

$$\frac{d\mathcal{E}}{dx} = -\frac{1}{\lambda_R(r)}\mathcal{E}. \tag{3-17}$$

The radiation length $\lambda_R(r)$ depends on the distance r between the particle and nearest rows of atoms in a dramatic way:

$$\lambda_R^{-1}(r) = 4Z_{\text{eff}}^2 \frac{r_c^2}{r_e^3} \breve{a}^{-3} \alpha^7 K_0' \left(\frac{2\pi}{a} r\right) \ln\left(\frac{2\gamma\alpha m_e}{m_c}\right), \tag{3-18}$$

where K_0' is the derivative of the modified Bessel function, a is the distance between two adjacent $x - y$ planes, r_c the classical radius of channeled particle, and r_e the classical electron radius. Equation (3-18) is a crude result, which may be modified when different lattices, or thermal effect, or imperfection are considered. We see that the bremsstrahlung losses are increasingly severe as the energy of the particle goes up and still unsustainable for high energy electrons. This is the case in spite of the fact that the bremsstrahlung losses have been reduced for channeled electrons.

IV. Conclusions

We considered bremsstrahlung losses from charged particles passing through the periodic lattice structure of a crystal. To understand this process accurately is of importance for many reasons, one of which is that this process heavily influences the choice of particles and method of acceleration for the crystal X-ray accelerator. We obtained the Hamiltonian (2-17) which governs the channeling process in a rigorous way. The quadratic term $\langle\tilde{\varphi}^2\rangle$ in (2-17) repels electrons away from nuclei, limiting bremsstrahlung. We employed the virtual photon method to evaluate the lattice electrostatic fields interacting with high energy particles: In the frame of the particle the electrostatic fields behave almost as if they are electromagnetic fields of a photon in the very high energy regime.

The resultant bremsstrahlung power off the particle by the lattice fields is roughly proportional to the energy of the particle, which is consistent with the usual result. However, the constant of proportionality is reduced in our calculation. Thus the channeling bremsstrahlung is somewhat more reduced than the nonchanneling case but it is still an increasing function of the energy. The sign of charge also affects the power losses drastically through the size of the distance from the lattice row r—the negatively charged particles radiate much more severely. (For a given energy, the more massive the particle is, the longer the radiation length is.) The bremsstrahlung losses are intolerably high for electrons in ultra-high energies for the purpose of the crystal X-ray accelerator. For muons or heavier particles they are tolerable. For this reason the electron is in fact a good candidate for channeling radiator instead of a candiate to be accelerated to very high energies in a crystal.

For a negatively charged muon, the typical distance from the nucleus of a lattice site may be $r = \frac{b}{2} \sim \frac{a}{4}$; therefore taking $Z_{\text{eff}} \sim 20$ we find the ionization length

$$\ell_R\left(\frac{b}{2}\right) \sim 100km.$$

For example, if we put bremsstrahlung losses comparable with the ionization losses, the energy for this corresponds to a 50 TeV beam. The same requirement sets an electron beam at 1.25 GeV.

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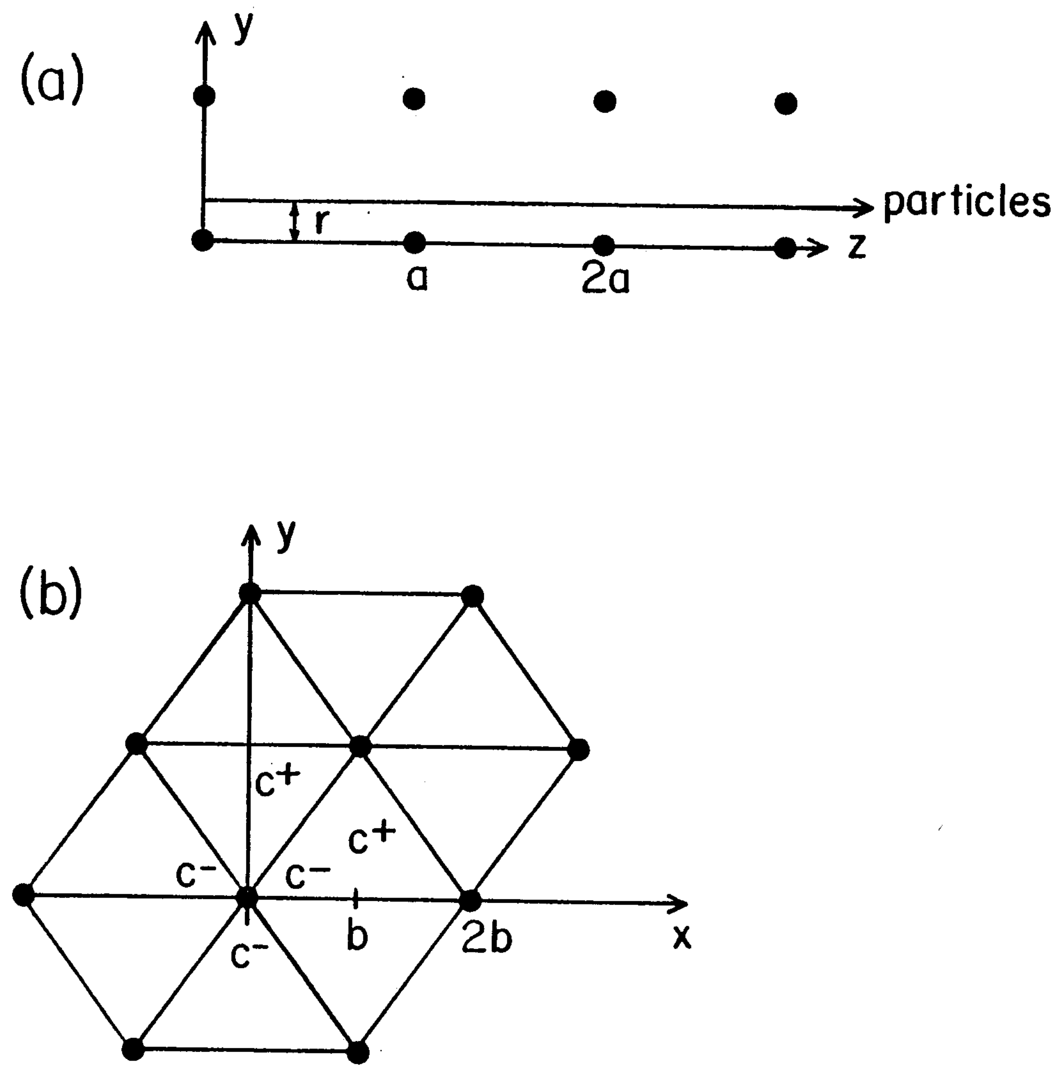
Figure Captions

Fig. 1 Crystal lattice and channels.

a) Coordinate system and the beam in the longitudinal direction.

b) Coordinate system and channels for positively and negatively charged particles.



(a)
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particles
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a
2a
(b)
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c+
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2b
x