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REDUCED FLUID DESCRIPTIONS OF TOROIDALLY CONFINED PLASMA WITH FINITE ION TEMPERATURE EFFECTS

C.-T. Hsu
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

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APPROVED BY SUPERVISORY COMMITTEE:

Richard D Hazeltine

Philip J. Morrison

John D. Dollard

Swadesh M Mahajan

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Chi-Tien Hsu, Ph.D.

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Supervising Professors: Philip J. Morrison,

Richard D. Hazeltine

Fluid descriptions of toroidally confined plasma with FLR effects are studied, based on a generalized, energy conserving, self-consistent, nonlinear reduced fluid model (HHM). The model, derived via a fluid approach starting from moment equations, differs from Braginskii's fluid system in retaining $O(\rho_i^2)$ terms (where ρ_i is the ion gyroradius) and most of the non-ideal effects. Hence, many of the well-known reduced fluid models can be reproduced from HHM by simply specifying scales of some parameters such as ρ_i and β. On the other hand, a Padè approximation of the full FLR system, obtained from the simplified version of HHM, is also presented. This simplified model is not only energy conserving and much easier to access, but also can be shown to retain FLR effects quite accurately. We therefore remark

that this version should deserve further analytical and numerical studies.

The possible applications of HHM are discussed in a general way so that further detailed studies can readily follow. In particular, linear toroidal drift-tearing modes with finite ion temperature effects are studied. The eigenmode equations, derived from the linearized version of HHM, are analyzed both by a multi-scale variational principle for the sheared slab geometry; and by the conventional asymptotic matching process for the toroidal geometry. It is discovered that (1) without the effects of viscosity, the instability condition and the growth rate of the semicollisional drift-tearing modes are hardly affected by the finite ion temperature; (2) with the effects of viscosity, the instability condition and growth rate are characterized by the ion viscosity in a crucial manner. Since ion viscosity is sensitive to the ion temperature, we thus conclude that ion temperature could become an important parameter for controlling the drift-tearing instabilities in present and future day high temperature plasma devices.

In addition, the non-canonical Hamiltonian theory and its application to our reduced system are discussed. This fast developing theory has been useful for studying the equilibria and nonlinear instability of fluid system.

This thesis is dedecated to

my parents,

my wife, and my son.

REDUCED FLUID DESCRIPTIONS OF TOROIDALLY CONFINED PLASMA WITH FINITE ION TEMPERATURE EFFECTS

by

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CHAPTER I

INTRODUCTION

Low frequency ($\leqslant$ Shear-Alfvén frequency) activities, such as tearing modes, interchange modes and ballooning modes, are believed to play important roles in toroidally confined plasmas. Kinetic and fluid descriptions of these classes of motions are given in a vast literature. In particular, fluid plasma theory, while neglecting several kinetic effects such as Landau damping and particle trapping, takes advantage of fast growing computational techniques, the already existing nonlinear theories of fluid dynamics and noncanonical Hamiltonian field theory. Also importantly, fluid theory more explicitly provides important linear and nonlinear physics. Consequently, for the astonishing nonlinear behaviors experimentally observed in the toroidal confinement devices, fluid plasma theory evidently becomes a very useful tool to interpret the observed phenomena.

However, because the primitive fluid system is still too complicated (especially in toroidal geometry), simplification schemes are usually adopted. The main context of this dissertation is to study the so-called "reduced fluid models", based on the assumptions of large aspect ratio toroidal geometry, Shear-Alfvén time scale, and "stretched" motions. The advantages of reduced fluid models are twofold: (1) the reduced fluid equations, involving only a few field variables, are much simpler and more accessible both analytically and numerically; (2) the compressional-Alfvén motion is scaled out so that the maximum time step is much larger than the time it takes an Alfvén wave to propagate one grid space[1].

The earliest version of reduced fluid model was described in the work of Kadomtsev and Pogutse[2]. It was then studied numerically by Rosenbluth et al[3] Strauss generalized this work and produced what is now called "reduced magnetohydrodynamics" (RMHD). The low-beta version of RMHD[4] was mainly for studying global ideal kink modes in a pressureless plasma and was subsequently extended to include finite pressure, high-beta RHMD[5], for studying pressure-driven modes. These simple models and their resistive versions were then widely studied and were found to predictively describe important phenomena in tokamak discharge in agreement with the actual experiments[6]. For instance, Mirnov oscillation[7] has been associated with m/n = 2/1 (where m is poloidal mode number and n is toroidal mode number) island at the outside edge of plasma, major disruption has been interpreted as a result of 2/1 - 3/2 coupling[8], and the anomalous electron transport has been related to stochastic magnetic field lines produced by resistive ballooning modes[9]. For this reason, RMHD has become a principle tool in understanding the nonlinear processes in tokamaks and has attracted great attention from tokamak research community in the last decade.

However, RMHD excludes many potentially important kinetic and non-ideal effects, because of its MHD origin. Even in its ideal context, RMHD omits, for instance, poloidal curvature and density gradient terms which have been found to have a strong stabilizing influence on resistive modes[9,10]. Thus there have been many recent efforts[11-20] to extend reduced fluid systems. The point is that for higher temperature plasma in present and future day machines, omission of the diamagnetic drift and comparable terms, corresponding to the finite beta and the finite ion Larmor radius (FLR) effects, arising from compressibility and viscosity, is no longer appropriate.

However, different physical problems require different equations to start with. For instance, when temperature gradients matter, such as electron temperature gradients in the sawtooth effects[21] and ion temperature gradients for the η_i modes[22,23], one can include the Braginskii's[24] equations of temperature evolution. A reduced fluid model that includes the electron temperature as a dynamical variable was derived by Drake and Antonsen[11]. This model, however, is restricted to zero ion temperature (experimentally the ion temperature is usually about the same order of magnitude as electron temperature). On the other hand, when particle trapping effects become significant, one can include the neoclassical flows, such as bootstrap current into the equations of motion. For example, resistive pressure-gradient-driven modes in the Banana-Plateau collisionality regime has been studied based on a set of "Neoclassical MHD" equations in which the bootstrap current effects become dominant[12,13]. Also, to include the correct low-beta Mercier[25] stabilizing term ($\propto 1 - 1/q^2$), Strauss[14] has derived a set of finite-aspect-ratio MHD equations by including the higher order inverse aspect ratio terms. For other type of machines, we remark that similar works have been done for mirror machines in the "long-thin limit" by Newcomb[15], and for reverse field pinch (RFP) by Strauss.[16]

Our interest here is the isothermal system which has both high electron and ion temperatures, corresponding to the realistic parameter regime (semi-collisional regime). Although many attempts[17-20] have been made to include the ion diamagnetic drift to the reduced fluid equations, few have rigorously considered the FLR effects. The importance of FLR effects in the finite ion temperature semicollisional regime, in the linear context, has been pointed out by Hahm[26] and will be demonstrated in chapter 4. Therefore, the main task of this dissertation is to construct and study a generalized

FLR reduced fluid model. A recent model called the four-field model[17] keeps electron beta terms and is valid for a characteristic time as slow as the diamagnetic drift frequency ω_*. This fluid model possesses desirable features, such as a physically realistic long mean-free-path electron response (that usually requires a kinetic treatment); also, unlike conventional RMHD, the threshold for tearing instability is given by the appropriate finite critical value of Δ'. However, the four-field model's treatment of finite gyroradius terms is less complete and less self-consistent than that of the present model.

As in the four-field model, we consider slow time-scale shear-Alfvén dynamics, with constant temperature. We also allow for comparable electron and ion temperatures ($T_i \simeq T_e$), which is realistic in most of the modern day tokamaks. The inclusion of compressibility and viscosity couples the parallel flow to the usual fields of RMHD. The main distinction of the present work is that we do not scale the ion-gyroradius ρ_i or the plasma beta β in terms of the ordering parameter ε (the inverse aspect ratio); rather, we treat them as independent small parameters. Our reduced system is therefore more general: instead of imposing complicated orderings from the beginning to make the resulted equations suitable for certain problems, we only adopt the general orderings following the shear-Alfvén time scale, stretched motion, and large-aspect-ratio geometry. We thus derive a fully self-consistent and energy conserving system that includes cross-field viscosity terms, as well as electron diffusive terms provided by the Spitzer resistivity[27], and ion viscous terms due to ion-ion collisions. The latter terms have been widely used as a damping mechanism of plasma momentum in computational works.

Two distinct features of our model are: (1) It possesses a conceptually simple energy conservation law. We remark that energy conservation laws are necessary for the description of reduced fluid models as a noncanonical Hamiltonian field theory[28], a formalism that has been useful for obtaining additional constants of motion and nonlinear stability criteria for reduced system[29]. Also, energy constants can be used as a computational diagnostic. (2) Even though we do omit $O(\rho_i{}^3)$ terms for simplicity, we show that the model retains significant FLR physics even when $\rho_i^2\nabla^2 \sim 1$.

However, important kinetic effects such as Landau damping, magnetic trapping, and potentially important stabilizing effects due to the variation of temperature[30], are beyond the scope of the this thesis. Also, due to the omission of higher order inverse aspect ratio terms, the low-beta Mercier criterion[25] will not be reproduced from our model.

We organize this dissertation in the following manner. In Chapter 2, the derivation and general discussions of the generalized reduced fluid model are given. A method starting from the second-order moment equation is used to derive the gyroviscosity tensor that includes higher order ion-gyroradius terms. Then, energy conservation and the corresponding thermodynamics is discussed. The usual so-called interchange energy appearing in high-beta RMHD and other models is found to be the sum of the potential energy and the kinetic energy of parallel flow, when the incompressibility is assumed.

Next, several simplified versions of the generalized model are discussed. In particular, a Padé approximation of the full FLR fluid system is presented. This energy conserving, reasonably simple, much more numerically tractable model retains good FLR physics in a wide range of ρ_i. Hence, we claim that this model should be a reasonably good description for the isothermal high temperature fluid plasma, and therefore deserves further detailed analytical and numerical studies.

In Chapter 3, the general applications of our reduced system are briefly discussed in such a way that further detailed studies can readily follow. In particular, the linear consequences of our model are studied in a sheared slab geometry. The resulted boundary layer equations agree with the rigorous gyro-kinetic theory[26,31] in a wide range of ρ_i (with an error less than 8%). Then, the noncanonical Hamiltonian theory and its applications to the reduced fluid system is discussed. The Hamiltonian structure of a simplified reduced fluid model, which is a drift version of RHMD, is studied through an isomorphism theory.

In Chapter 4, finite ion temperature effects on linear drift-tearing modes are studied in detail yielding several new results. The eigenmode equations are derived from the linearization of the Padé approximation mentioned before, with the usual boundary layer analysis[32]. This set of equations is then analyzed by a two-scale variational principle in a sheared slab geometry and by an asymptotic matching process in the toroidal geometry. It is found that when ion collisional viscosity is negligible, finite ion temperature only mildly enhances the stabilizing ion sound and good curvature effects by a factor of $1+T_i/T_e$. On the other hand, when ion collisional viscosity becomes significant, the instability condition parameter Δ_c and drift-tearing growth rate will be characterized by ion viscosity.

Finally, in Chapter 5, conclusions, discussions, and possible future studies relating to this thesis will be given.

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CHAPTER II

REDUCED FLUID SYSTEM WITH FLR CORRECTIONS

2.1 Introduction

The term "reduced fluid model" refers to a set of simplified fluid equations that describes the nonlinear dynamics of large aspect ratio tokamak plasmas. The simplification is based on the following orderings:

(1) $\dfrac{\text{poloidal magnetic field}}{\text{toroidal magnetic field}} \sim O(\varepsilon)$

(2) $\dfrac{\text{compressional Alfvén time}}{\text{time scale of interest}} \sim O(\varepsilon)$

(3) $\dfrac{\text{transverse scale length}}{\text{parallel scale length}} \sim O(\varepsilon)$,

where the ordering parameter $\varepsilon \equiv \dfrac{a}{R_0} \ll 1$, is the inverse aspect-ratio; a is the perpendicular length scale and R_0 is the toroidal length scale (or say, the major radius of the magnetic axis). The first assumption limits the plasma safety factor to be of order unity, the second assumption eliminates compressional Alfvén dynamics, and the third assumption is appropriate to flute-like perturbations.

In addition, three other basic assumptions which are consistent with the above large aspect ratio orderings:

(1) $\underset{\sim}{A}$, the vector potential due to plasma current, $\sim O(\varepsilon)$;

(2) V, the plasma flow velocity, $\sim O(\varepsilon)$;

(3) $\dfrac{f - f_0}{f} \simeq O(\varepsilon)$, where f is a scalar variable of plasma and f_0 is the volume-averaged value of f.

The first assumption corresponds to the smallness of poloidal magnetic field when compared with the vacuum toroidal magnetic field; the second assumption keeps the convection in the shear-Alfvéen dynamics; and the third assumption apparently implies that the 0th order distribution function is Maxwellian.

Consequently, $O(\varepsilon)$ terms in the moment equations describe the compressional Alfvén equilibration, $O(\varepsilon^2)$ terms describe the shear-Alfvén dynamics, and $O(\varepsilon^3)$ terms are dropped. Moreover, for the system to be relevant for resistive modes, the electron-ion collision freguency ν_{ei} is assumed to be $O(\varepsilon)$. Similarly, in this thesis, we assume the ion-ion collision frequency ν_{ii} to be $O(\varepsilon)$ as well in order to retain the ion collisional viscosity in the finite ion temperature system. We note that, in chapter 4, when ion temperature is finite, ion viscosity will be shown to be an important mechanism in the semi-collisional drift-tearing activities.

Moreover, to retain the finite compressibility and finite ion-gyromotion, plasma beta and ion-gyroradius are assumed to be of order one during the reduction process. That is, we don't scale them in terms of ε; rather, we treat them as independent small parameters. Note that the explicit FLR terms are embedded in the stress tensor in the equation of momentum conservation. In this thesis, we adopt a more complete treatment based on the third-rank moment equation for deriving the stress tensor which retains $O(\rho_i^2)$ terms excluded by Braginskii[24] and other authors. Also, a new term in the gyroviscosity, arises from the compressibility, is obtained. We remark that this term has not been mentioned in previous works. However, for simplicity, $O(\beta^2)$ and $O(\rho_i^3)$ terms are neglected. Nevertheless, we will, in the next chapter, show that the reduced system we obtain retains significant FLR physics for a wide range of ρ_i.

Now, let's discuss the organization of this chapter. In Sec. 2.2.1, we present the derivation of the generalized reduced fluid model from the moment equations; in Sec. 2.2.2, we make some general discussions on the resultant reduced model and further simplify it to a more accessible closed four fields system. Also, we present a simple, interesting model which include the drift effects to the high-β RMHD. We name it DRMHD. In Sec. 2.3, the energy conservation and the corresponding thermodynamic process of our reduced fluid system are studied. At the same time, the internal energy with a vague form $-2hp$ is found due to the incompressibility which forces the cancellation of the parallel compression and perpendicular compression. In Sec. 2.4, we present a tractable, cleaner, energy conserving reduced fluid model with four natural field variables φ, v, ψ, p. This system is shown to be a good Padé approximation of the full FLR system. In Sec. 2.5, conclusions are given.

2.2 Reduced Fluid Models

2.2.1 Generalized Reduced Fluid Model

Before proceeding with the reduction process, we first briefly review the normalized geometry that is based on the large aspect ratio orderings. For details, we refer reader to Ref.[18]. The dimensionless coordinates (h, y, z, τ) are defined by

$$h = \frac{R - R_o}{a}, \quad y = \frac{Z}{a}, \quad z = -\zeta, \quad \tau = \frac{\varepsilon v_A t}{a}; \tag{2.1}$$

where (R, ζ, Z) are the usual cylindrical coordinates centered on the tokamak symmetry axis, the Alfvén speed $v_A^2 \equiv \frac{B_o^2}{4\pi m_i n_o}$, and B_o, n_o are the constant value of the vacuum field and plasma density, respectively. Hence, the dimensionless gradient can be written as

$$\underset{\sim}{\nabla} = \underset{\sim}{\nabla}_\perp + \hat{z}\frac{\varepsilon}{1 + \varepsilon h}\frac{\partial}{\partial z},$$

and the reduced geometry can be generally described by the metric coefficients

$$g^{ij} \equiv \nabla x^i \cdot \nabla x^j = \begin{cases} \left(\dfrac{\varepsilon}{1 + \varepsilon h}\right)^2 & i = j = z, \\ \delta_{ij} & \text{otherwise.} \end{cases} \qquad (2.2)$$

Now, due to the smallness of the poloidal magnetic field, induced by plasma current, the magnetic field can thus be written as

$$\underset{\sim}{B} = \frac{B_0}{1+\varepsilon h}\hat{z} + \underset{\sim}{\nabla} \times \underset{\sim}{A}$$

$$\simeq B_0\left((1 + \varepsilon(B_z - h))\hat{z} - \varepsilon\hat{z} \times \underset{\sim}{\nabla}_\perp\psi\right) + O(\varepsilon^2). \qquad (2.3)$$

Where $\underset{\sim}{A}$ is the vector potential induced by plasma current;

$$B_z \equiv \frac{\hat{z}\cdot(\nabla \times A)}{B_0\varepsilon}$$

is the normalized diamagnetic correction to the toroidal magnetic field;

$$\psi \equiv \frac{A\cdot\hat{z}}{aB_0\varepsilon} = \frac{\Psi_p}{2\pi B_0 a^2}(1 + O(\varepsilon))$$

is the normalized poloidal magnetic flux; and Ψ_p is the usual poloidal magnetic flux. Consequently, the unit vector along the magnetic field line $\underset{\sim}{b}$ and field line curvature $\underset{\sim}{\kappa}$, in the reduced notion, are

$$\underset{\sim}{b} = \hat{z} - \varepsilon\hat{z} \times \underset{\sim}{\nabla}_{\perp}\psi + O(\varepsilon^2), \tag{2.4}$$

$$\underset{\sim}{\kappa} = \underset{\sim}{b}\cdot\nabla\underset{\sim}{b} = \frac{1}{R_0}\underset{\sim}{\nabla}_{\perp}h\ (1 + O(\varepsilon)). \tag{2.5}$$

Also, the plasma current, due to Ampere's law, has the form

$$\underset{\sim}{J} = \frac{c}{4\pi}\underset{\sim}{\nabla} \times \underset{\sim}{B} = -\varepsilon\frac{cB_0}{4\pi a}(\hat{z}\times\underset{\sim}{\nabla}_{\perp}B_z + \hat{z}\nabla_{\perp}^2\psi) + O(\varepsilon^2). \tag{2.6}$$

Similarly, the perpendicular and parallel electric field, according to Faraday's law $\underset{\sim}{E} = -\underset{\sim}{\nabla}\Phi - \frac{1}{c}\frac{\partial}{\partial t}\underset{\sim}{A}$, are

$$\underset{\sim}{E}_{\perp} = -\varepsilon\frac{v_A B_0}{c}\underset{\sim}{\nabla}_{\perp}\varphi + O(\varepsilon^2), \tag{2.7}$$

$$\underset{\sim}{b}\cdot\underset{\sim}{E} = -\varepsilon\frac{v_A B_0}{c}(\nabla_{\parallel}\varphi + \frac{\partial}{\partial\tau}\psi) + O(\varepsilon^3), \tag{2.8}$$

respectively. Where Φ is the electrostatic potential, $\varphi \equiv \frac{c}{\varepsilon B_0 a v_A}\Phi$ is the normalized electrostatic potential,

$$\nabla_{\parallel} \equiv \underset{\sim}{b}\cdot\underset{\sim}{\nabla} = \frac{\partial}{\partial z} - [\psi,\ ] \tag{2.9}$$

is the nonlinear parallel gradient, and

$$[f,g] \equiv \hat{z}\cdot\underset{\sim}{\nabla}_{\perp}f\times\underset{\sim}{\nabla}_{\perp}g \tag{2.10}$$

defines the bracket.

For the equations of motion, we start with the exact moment equations

$$\frac{\partial}{\partial t}n_\alpha + \underset{\sim}{\nabla}\cdot n_\alpha \underset{\sim}{V}_\alpha = 0, \tag{2.11}$$

$$n_\alpha m_\alpha\left(\frac{\partial}{\partial t} + \underset{\sim}{V}_\alpha\cdot\underset{\sim}{\nabla}\right)\underset{\sim}{V}_\alpha + \underset{\sim}{\nabla}\cdot\underset{\approx}{P}_\alpha = e_\alpha n_\alpha\left(\underset{\sim}{E} + \frac{\underset{\sim}{V}_\alpha}{c}\times\underset{\sim}{B}\right) + \underset{\sim}{F}_\alpha, \tag{2.12}$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{V}_\alpha\cdot\underset{\sim}{\nabla}\right)\underset{\approx}{P}_\alpha + \nabla\cdot\underset{\approx\sim}{q}_\alpha + \left((\underset{\approx}{P}_\alpha\cdot\nabla\underset{\sim}{V}_\alpha) + \mathrm{Tr}\right) + \underset{\approx}{P}_\alpha\underset{\sim}{\nabla}\cdot\underset{\sim}{V}_\alpha$$

$$= \left(\frac{m_i e_\alpha}{m_\alpha e_i}\right)\Omega\left(\underset{\approx}{P}_\alpha\times\underset{\sim}{b} + \mathrm{Tr}\right) + \underset{\approx}{C}_\alpha, \tag{2.13}$$

where α (= i,e) is the species label, $\Omega\equiv\frac{eB}{m_i c}$ is ion gyrofrequency, and the moment tensors $\underset{\approx}{P}_\alpha$, $\underset{\approx\sim}{q}_\alpha$ and $\underset{\approx}{C}_\alpha$ are defined by

$$\underset{\approx}{P}_\alpha \equiv \int d\underset{\sim}{v}\, m_\alpha(\underset{\sim}{v} - \underset{\sim}{V}_\alpha)(\underset{\sim}{v} - \underset{\sim}{V}_\alpha)f_\alpha,$$

$$\underset{\approx\sim}{q}_\alpha \equiv \int d\underset{\sim}{v}\, m_\alpha(\underset{\sim}{v} - \underset{\sim}{V}_\alpha)(\underset{\sim}{v} - \underset{\sim}{V}_\alpha)(\underset{\sim}{v} - \underset{\sim}{V}_\alpha)f_\alpha,$$

$$\underset{\approx}{C}_\alpha \equiv \int d\underset{\sim}{v}\, m_\alpha(\underset{\sim}{v} - \underset{\sim}{V}_\alpha)(\underset{\sim}{v} - \underset{\sim}{V}_\alpha)C_\alpha(f_\alpha),$$

where $C_\alpha(f)$ is the collision operator, n_α, m_α, and $\underset{\sim}{V}_\alpha$ are density, mass, and velocity, respectively. Also we use "Tr" to denote the transpose of the preceding tensor.

For isothermal systems without the particle trapping effects, it is adequate to write the friction force $\underset{\sim}{F}_\alpha$ in the Spitzer-Härm[27] form

$$\underset{\sim}{F}_\alpha \equiv \int d\underset{\sim}{v}\ m_\alpha(\underset{\sim}{v} - \underset{\sim}{V}_\alpha)C_\alpha(f_\alpha) \simeq - n_\alpha e_\alpha \eta_S \underset{\sim}{J}, \tag{2.14}$$

where η_S is the Spitzer-Härm resistivity, which is scaled to be $O(\varepsilon)$, and $\underset{\sim}{J}$ is the plasma current. It is noted that although η_S is sensitive to the variation of plasma temperature ($\propto T_e^{-3/2}$), in this thesis, we assume them to be constant. However, we remark that there have been some studies on the "rippling modes" which originate from the variation of resistivity.[33,34]

Next, we assume quasineutrality, sum Eq. (2.12) over species, and obtain

$$nm_i\left(\frac{\partial}{\partial t} + \underset{\sim}{V}\cdot\underset{\sim}{\nabla}\right)\underset{\sim}{V} + \underset{\sim}{\nabla}\cdot(\underset{\approx}{P}_i + \underset{\approx}{P}_e) = \frac{1}{c}\underset{\sim}{J} \times \underset{\sim}{B}, \tag{2.15}$$

where $\underset{\sim}{V} \simeq \underset{\sim}{V}_i = \dfrac{J}{ne} + \underset{\sim}{V}_e$. Here we have neglected terms of $O(\dfrac{m_e}{m_i})$. Similarly upon subtraction Eq. (2.12) leads to the usual Ohm's law

$$\underset{\sim}{E} + \frac{\underset{\sim}{V}_e}{c} \times \underset{\sim}{B} = \eta_s \underset{\sim}{J} - \frac{\underset{\sim}{\nabla} n T_e}{ne}, \tag{2.16}$$

where $n = n_e = n_i$. Hence, the electron flow velocity can be written as

$$\underset{\sim}{V}_e = \left(\frac{J_\parallel}{ne} - V_\parallel\right)\underset{\sim}{b} + \frac{c}{B}\underset{\sim}{b} \times \left\{\eta_s \underset{\sim}{J} - \frac{\underset{\sim}{\nabla} n T_e}{ne} - \underset{\sim}{E}\right\}, \tag{2.17}$$

where $J_\parallel$ and $V_\parallel$ are the parallel current and the parallel plasma flow velocity, respectively. We note that Eq. (2.16) describes mainly the electron dynamics, while Eq. (2.15) describes mainly the ion dynamics. Also, the parallel component of Eq. (2.16) provides the generalized Ohm's law; $\underset{\sim}{b} \cdot (\underset{\sim}{\nabla} \times$ Eq. (2.16)) provides the diffusion equation of toroidal magnetic field and equivalently provides the particle conservation; the parallel component of Eq. (2.15) provides the equation of parallel acceleration; and $\underset{\sim}{b} \cdot (\underset{\sim}{\nabla} \times$ Eq. (2.15)) provides the important shear-Alfvén law.

Although the shear-Alfvén time scale will rule out compressional Alfvén dynamics from the equations of motion, the $O(\varepsilon)$ terms of the moment equations will describe the compressional Alfvén equilibration. From Ohm's law and the toroidal component of Faraday's law, one finds $\underset{\sim}{\nabla} \cdot \underset{\sim}{V} = O(\varepsilon^2)$.

The smallness of the compressibility is physically justified by the inability of plasma to compress the toroidal field. We also note that the assumption of $n = n_0 + O(\varepsilon)$ can be proved to be consistent with this conclusion by looking at the continuity equation. We therefore write $\underset{\sim}{V}$ as

$$\underset{\sim}{V} = \varepsilon v_A(\hat{z} \times \underset{\sim}{\nabla}_\perp F + v\hat{z}) + O(\varepsilon^2), \tag{2.18}$$

where v is the normalized parallel flow and F is the normalized stream function. Similarly, by using Eq. (2.6) and $\underset{\sim}{V}_e \simeq \underset{\sim}{V} - \frac{1}{ne}\underset{\sim}{J}$, the lowest order electron flow can be expressed as

$$\underset{\sim}{V}_e = \varepsilon v_A(\hat{z} \times \underset{\sim}{\nabla}_\perp(F+2\delta B_z) + (v+2\delta\nabla_\perp^2\psi)\hat{z}) + O(\varepsilon^2), \tag{2.19}$$

where the constant $\delta \equiv \frac{v_A}{2\Omega a}$ is a measure of FLR effects.

To find out the form of the stream function F in terms of the typical field variables, such as electrostatic potential and plasma pressure, the $O(\varepsilon)$ contribution of the equation of ion momentum conservation is required. We have

$$\underset{\sim}{\nabla}\cdot\underset{\approx}{P}_i = -en\underset{\sim}{\nabla}_\perp\Phi + m_i n\Omega\underset{\sim}{V}\times\underset{\sim}{b}. \tag{2.20}$$

The pressure tensor $\underset{\approx}{P}_i$, through which the ion gyromotion comes into our equations of motion, can in general be expressed as

$$\underset{\approx}{P}_i = P_\perp(\underset{\approx}{I} - \underset{\sim}{b}\underset{\sim}{b}) + P_\parallel \underset{\sim}{b}\underset{\sim}{b} + \hat{\underset{\approx}{P}}_i$$
$$\equiv \underset{\approx}{P}^{CGL} + \hat{\underset{\approx}{P}}_i,$$

where

$$P_\perp \equiv \int d\underset{\sim}{v}\, \frac{m_i v_\perp^2}{2} f_i,$$

$$P_\parallel \equiv \int d\underset{\sim}{v}\, m_i v_\parallel^2 f_i,$$

define the well-known Chew-Goldberger-Low[35] stress tensor $\underset{\approx}{P}^{CGL}$, and $\hat{\underset{\approx}{P}}_i$ is called the cross-field viscosity tensor. For algebraic convenience, we hereafter use " ^ " to denote the "non-CGL" portion of an arbitrary tensor; namely,

$$\hat{\underset{\approx}{A}} \equiv \underset{\approx}{A} - \underset{\approx}{A}^{CGL}$$
$$= \underset{\approx}{A} - \{\underset{\sim}{b}\underset{\sim}{b}(\underset{\approx}{A}:\underset{\sim}{b}\underset{\sim}{b}) + (\underset{\approx}{I} - \underset{\sim}{b}\underset{\sim}{b})(\frac{\underset{\approx}{I} - \underset{\sim}{b}\underset{\sim}{b}}{2}:\underset{\approx}{A})\}. \quad (2.21)$$

Eq. (2.20) therefore becomes

$$\underset{\sim}{\nabla}\cdot\hat{\underset{\approx}{P}}_i + \underset{\sim}{\nabla}_\perp P_\perp = - en\underset{\sim}{\nabla}_\perp \phi + \Omega \underset{\sim}{V}\times\underset{\sim}{b}. \quad (2.22)$$

The cross-field viscosity tensor is usually derived via kinetic theory; here, we present a method for deriving this tensor from exact moment equations. An earlier, linear application of this method is due to Lee[36]. The result differs from Braginskii's result by including higher order FLR corrections.

We first define a tensor operator

$$\underset{\approx}{K}(\underset{\approx}{A}) \equiv \left((\underset{\approx}{A} \times \underset{\sim}{b}) + \mathrm{Tr}\right) \tag{2.23}$$

such that for any tensor $\underset{\approx}{A}$, we have

$$\underset{\approx}{K}(\underset{\approx}{A}^{CGL}) = 0.$$

Eq. (2.13) can thus be written as

$$\underset{\approx}{K}(\underset{\approx}{P}_i) = \frac{1}{\Omega}\,\hat{\underset{\approx}{S}}, \tag{2.24}$$

where

$$\underset{\approx}{S} \equiv \left(\frac{\partial}{\partial t}+\underset{\sim}{V}\cdot\underset{\sim}{\nabla}\right)\underset{\approx}{P}_i + \underset{\sim}{\nabla}\cdot\underset{\approx}{q}_i + \left((\underset{\approx}{P}_i\cdot\underset{\sim}{\nabla}\underset{\sim}{V})+\mathrm{Tr}\right) + \underset{\approx}{P}_i(\underset{\sim}{\nabla}\cdot\underset{\sim}{V}) - \underset{\approx}{C}_i . \tag{2.25}$$

Then, by using the following tensor identity for any symmetric second order tensor $\underset{\approx}{A}$:

$$\underset{\sim}{b}\times\underset{\approx}{A}\times\underset{\sim}{b} = \underset{\approx}{A} - (\underset{\sim}{b}\underset{\sim}{b}\cdot\underset{\approx}{A} + \mathrm{Tr}) - (\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b})(\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b}):\underset{\approx}{A} + \underset{\sim}{b}\underset{\sim}{b}(\underset{\sim}{b}\underset{\sim}{b}:\underset{\approx}{A}), \tag{2.26}$$

we find the inverse operator $\underset{\approx}{K}^{-1}$,

$$\underset{\approx}{K}^{-1}(\underset{\approx}{A}) = \frac{1}{4}\{(\underset{\sim}{b} \times \underset{\approx}{A} \cdot (\underset{\approx}{I} + 3\underset{\sim}{b}\underset{\sim}{b})) + \mathrm{Tr}\}. \tag{2.27}$$

By a simple algebra one can prove that the homogeneous solution of Eq. (2.24), i.e., solution of $\underset{\approx}{K}(\underset{\approx}{P}) = 0$, must be $\underset{\approx}{P}^{CGL}$. This agrees with the Chew-Goldberger-Low result that the lowest order (in ρ_i) pressure tensor is $\underset{\approx}{P}^{CGL}$. We thus obtain

$$\underset{\approx}{P}_i = \underset{\approx}{K}^{-1}(\hat{\underset{\approx}{S}}) + \underset{\approx}{P}^{CGL} = \frac{1}{4\Omega}\{\underset{\sim}{b}\times\hat{\underset{\approx}{S}}\cdot(\underset{\approx}{I}+3\underset{\sim}{b}\underset{\sim}{b}) + \mathrm{Tr}\} + \underset{\approx}{P}^{CGL}. \tag{2.28}$$

We also remark here that this method can be extended to derive the higher order moment tensors, such as $\underset{\approx}{q}$ (See Appendix A).

To express $\hat{\underset{\approx}{S}}$ in terms of observable quantities, it is necessary to assume the smallness of one or more ordering parameters such as ε, β, etc. In this thesis, we adopt the large aspect ratio orderings and find that

$$\hat{\underset{\approx}{S}} = \hat{\underset{\approx}{S}}^{g} - \hat{\underset{\approx}{C}}$$

and

$$\hat{\underset{\approx}{S}}^{g} = (\frac{\partial}{\partial t} + \underset{\sim}{V}\cdot\underset{\sim}{\nabla})\hat{\underset{\approx}{P}}_0 + \{(\hat{\underset{\approx}{P}}_0\cdot\underset{\sim}{\nabla}\underset{\sim}{V}) + \mathrm{Tr}\} + P_\perp \hat{\underset{\approx}{W}} + O(\varepsilon^3). \tag{2.29}$$

Where

$$\hat{\underset{\approx}{P}}_0 = \underset{\approx}{K}^{-1}\left(\frac{nT_i}{\Omega}\hat{\underset{\approx}{W}}\right)$$

$$= \frac{nT_i}{4\Omega}\left(\left(\underset{\sim}{b} \times \hat{\underset{\approx}{W}} \cdot (\underset{\approx}{I} + 3\underset{\sim\sim}{bb})\right) + Tr\right), \qquad (2.30)$$

is the first order cross-field viscosity, which is identical to Braginskii's gyroviscosity tensor. We therefore have

$$\hat{\underset{\approx}{P}}_i = \hat{\underset{\approx}{p}}^g + \hat{\underset{\approx}{p}}^c,$$

where $\hat{\underset{\approx}{p}}^g$, the gyroviscosity tensor, is

$$\hat{\underset{\approx}{p}}^g = \frac{1}{4\Omega}\left(\left(\underset{\sim}{b} \times \hat{\underset{\approx}{S}}^g \cdot (\underset{\approx}{I} + 3\underset{\sim\sim}{bb})\right) + Tr\right), \qquad (2.31)$$

and $\hat{\underset{\approx}{p}}^c$, the collisional viscosity tensor, is

$$\hat{\underset{\approx}{p}}^c = -\frac{1}{4\Omega}\left(\left(\underset{\sim}{b} \times \hat{\underset{\approx}{C}} \cdot (\underset{\approx}{I} + 3\underset{\sim\sim}{bb})\right) + Tr\right)$$

$$\simeq -\frac{3\nu_i}{10\Omega}\frac{nT_i}{\Omega}\left(\hat{\underset{\approx}{W}} + 3(\hat{\underset{\approx}{W}}\cdot\underset{\sim\sim}{bb} + Tr)\right). \qquad (2.32)$$

It is important to note that the $O(\varepsilon)$ terms of Eq. (2.31) give exactly Braginskii's[24] cross-field viscosity and the $O(\varepsilon^2)$ terms give higher order FLR corrections. Here we have neglected the contribution from $\underset{\approx}{q}$ which is $O(\rho_i^3)$. We remark that Newcomb[37] has studied an incompressible, collisionless nonlinear system with FLR corrections in the paraxial limit.

He includes terms from $\underset{\approx}{q}$; however, we note that these terms are important only when particle trapping effects or temperature variation are considered. Note that, in Appendix B, we also briefly study gyroviscosity due to particle trapping effects.

For collisional viscosity $\hat{\underset{\approx}{P}}^{c}$, arises from $\hat{\underset{\approx}{C}}$, here we have adopted Braginskii's result which is accurate enough (under our orderings on ν_{ii}) for the reduction process. However, in Appendix C, for instructional purpose, we present a method of deriving $\underset{\approx}{C}$ by manipulating the Landau collision operator[38] and expanding the distribution function in Laguerre polynomials.

Now let's go back to Eq. (2.22). By adopting Eqs. (2.29) and (2.31) and dropping $O(\varepsilon^2)$ terms, Eq. (2.22) becomes, in the reduced form,

$$(1 + a_i^2\nabla_\perp^2)F = \varphi + \delta p\frac{T_i}{T_e} + \delta\beta\frac{T_i}{T_e}\,\frac{P_\perp - nT_i}{\varepsilon n_o T_i}. \tag{2.33}$$

Where $p \equiv \frac{\beta}{\varepsilon}(\frac{n}{n_o} - 1)$ defines the normalized electron pressure for an isothermal system, and $\beta \equiv \frac{8\pi n_o T_e}{B_o^2}$ is the electron beta. It is worth mentioning here that $a_i^2 \equiv \delta^2\beta\frac{T_i}{T_e} = (\frac{\rho_i}{2a})^2$ and the operator $a_i^2\nabla_\perp^2$ corresponds to $\rho_i^2\nabla_\perp^2$, a well-known FLR operator. Recall the well-known FLR operator[26,31]

$$\Gamma_o(b_i) = e^{-b_i} I_o(b_i) \simeq 1 - b_i + \frac{3}{4}b_i^2$$

in gyrokinetic theory, where $b_i = -2a_i^2\nabla_\perp^2$ and I_o is the modified Bessell function.

The remaining unnormalized variable in Eq. (2.33) is $P_\perp - nT_i$ which usually vanishes when ion gyromotion and trapping effects are not considered. The inclusion of $P_\perp$ apparently complicates the closure system. fortunately, by observing Eq. (2.33), we see that it involves the small quantity β. This allows us to determine it from the lower order terms of moment equations. That is, by operating on Eq. (2.25) with "$(\underset{\approx}{I} - \underset{\sim\sim}{bb})$:", we find

$$(\frac{\partial}{\partial t} + \underset{\sim}{V}\cdot\underset{\sim}{\nabla})(P_\perp - nT_i) + nT_i((\underset{\approx}{I}-\underset{\sim\sim}{bb}):\underset{\sim\sim}{\nabla V}) = O(\beta). \tag{2.34}$$

Comparing this to the low beta version of the shear-Alfvén law (as will be shown later) thus suggests that

$$\frac{P_\perp - nT_i}{\varepsilon n_o T_i} = 2\delta\nabla_\perp^2 F + O(\beta). \tag{2.35}$$

Here, terms from $\underset{\sim}{\nabla}\cdot\underset{\approx}{q}$ are again neglected because of the absence of temperature variation and particle trapping effects. Eq. (2.33) thus becomes

$$(1 - a_i^2\nabla_\perp^2)F = \varphi + \delta\frac{T_i}{T_e}p. \tag{2.36}$$

Similarly, the $O(\varepsilon)$ terms of Eq. (2.15) give the reduced pressure balance law

$$B_z = -\frac{1 + \frac{T_i}{T_e}}{2}p - \frac{\delta\beta T_i}{2T_e}\nabla_\perp^2 F, \tag{2.37}$$

which describes compressional Alfvén equilibration; again, the second term of the right-hand side (RHS) is an FLR correction.

For the reduced equations of motion, we keep $O(\varepsilon^2)$ terms and drop $O(\varepsilon^3)$ terms. Consider first the electron dynamics. $\underset{\sim}{\nabla}\times$(Eq. (2.16)), particle conservation and Faraday's law lead to

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{V}_e\cdot\underset{\sim}{\nabla}\right)\underset{\sim}{B} + \underset{\sim}{B}(\underset{\sim}{\nabla}\cdot\underset{\sim}{V}_e) = \underset{\sim}{B}\cdot\underset{\sim}{\nabla}\underset{\sim}{V}_e + \eta_s c(\underset{\sim}{\nabla}\times\underset{\sim}{J}). \tag{2.38}$$

Then, by taking its parallel components and dropping $O(\varepsilon^3)$

terms, we obtain the diffusion equation of toroidal magnetic field; i.e.

$$\frac{\partial}{\partial\tau}B_Z + [F+2\delta B_Z,B_Z]$$

$$= [F+2\delta B_Z,2h] + \nabla_{\parallel}(v+2\delta J) + \frac{1}{\beta}(\frac{\partial}{\partial\tau}p+[F+2\delta B_Z,p]) + \eta\nabla_{\perp}^2 B_Z, \quad (2.39)$$

where the operator $\frac{\partial}{\partial\tau} + [F+2\delta B_Z,\ ]$ is simply a reduced form of $(\frac{\partial}{\partial t} + \underset{\sim}{V}_e\cdot\underset{\sim}{\nabla})$ and $J \equiv \nabla_{\perp}^2\psi$ corresponds to the normalized parallel current. After a rearrangement, Eq. (2.39) becomes the equation of particle conservation

$$\frac{\partial}{\partial\tau}p + [F+2\delta B_Z,p]$$

$$= -\beta[F+2\delta B_Z,2h] - \beta\nabla_{\parallel}v - 2\delta\beta\nabla_{\parallel}J$$

$$+ \beta(\frac{\partial}{\partial\tau}B_Z+[F,B_Z]) - \eta\nabla_{\perp}^2 B_Z, \quad (2.40)$$

which can be also derived through Eq. (2.11) and $\underset{\sim}{\nabla}\cdot\underset{\sim}{V}_e$, by using Eq. (2.17).

The first term on the RHS is due to the curvature drift and $\underset{\sim}{\nabla}B$ drift; the second term on the RHS corresponds to ion acoustic effects. The third term on the RHS, the semi-collisional compression, is responsible for the AC-type parallel conductivity, usually refers to as the "long mean-free-path electron response", in the semicollisional regime. Regarding the fourth term, $\beta(\frac{\partial}{\partial\tau}B_Z+[F,B_Z])$, according to the compressional Alfvén equilibration Eq. (2.37), it provides only the β corrections to the rest of terms with similar structure. Hence, it will not affect the system either qualitatively or quantitatively, provided that β is usually a small quantity.

Similarly, the parallel component of Ohm's law gives

$$\frac{\partial}{\partial\tau}\psi = -\nabla_{\parallel}F - 2\delta\nabla_{\parallel}B_Z + \eta J, \qquad (2.41)$$

where the constant

$$\eta \equiv \frac{c^2\eta_S}{4\pi a v_A \varepsilon}$$

is the normalized resistivity. The second term on the RHS not only provides the electron diamagnetic drift but also provides the ion acoustic effects and the long mean-free-path kinetic response, in cooperation with the parallel compression and the semi-collisional compression, respectively. We also note that if we allow for anisotropic resistivity[15], then

$$\underset{\sim}{F}_\alpha = n_\alpha e_\alpha \, (\eta_\perp \underset{\sim}{J}_\perp + \eta_\parallel \underset{\sim}{J}_\parallel)$$

and our model is modified by simply replacing η by a normalized $\eta_\perp$ in Eq. (2.40) and by a normalized $\eta_\parallel$ in Eq. (2.41).

The other two equations to be derived are the reduced parallel momentum equation and parallel vorticity equation. Taking $\underset{\sim}{b}\cdot$Eq. (2.15) and $\underset{\sim}{b}\cdot\underset{\sim}{\nabla}\times$Eq. (2.15) yields, respectively

$$nm_i\left(\frac{\partial}{\partial t} + \underset{\sim}{V}\cdot\underset{\sim}{\nabla}\right)(\underset{\sim}{V}\cdot\underset{\sim}{b}) + \underset{\sim}{b}\cdot(\underset{\sim}{\nabla}\cdot\underset{\approx}{\hat{P}}_i) + \underset{\sim}{b}\cdot\underset{\sim}{\nabla}(nT_i + nT_e) = O(\varepsilon^3),$$

$$nm_i\left(\frac{\partial}{\partial t} + \underset{\sim}{V}\cdot\underset{\sim}{\nabla}\right)(\underset{\sim}{b}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{V}) + \underset{\sim}{b}\cdot\underset{\sim}{\kappa}\times\underset{\sim}{\nabla}(P_\perp - nT_i) + \underset{\sim}{b}\cdot\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\cdot\underset{\approx}{\hat{P}}_i)$$

$$= -\frac{J}{c}\cdot\underset{\sim}{\nabla}B + \frac{B}{c}\cdot\underset{\sim}{\nabla}(\underset{\sim}{J}\cdot\underset{\sim}{b}) - \frac{B}{c}\,\underset{\sim}{\kappa}\cdot\underset{\sim}{J} + O(\varepsilon^3),$$

where $\underset{\sim}{\kappa}$ is the field line curvature. The difficult terms to evaluate are $\underset{\sim}{b}\cdot\underset{\sim}{\nabla}\cdot\underset{\approx}{\hat{P}}$ and $\underset{\sim}{b}\cdot\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\cdot\underset{\approx}{\hat{P}})$, since $\underset{\approx}{\hat{P}}$ is a complicated tensor. However, by using the identities

$$\underset{\sim}{b}\cdot\nabla\cdot\underset{\approx}{\hat{P}} = -\underset{\sim}{\nabla}\underset{\sim}{b}:\underset{\approx}{\hat{P}} + \underset{\sim}{\nabla}\cdot(\underset{\sim}{b}\cdot\underset{\approx}{\hat{P}})$$

$$= -\underset{\sim}{\nabla}\underset{\sim}{b}:\underset{\approx}{\hat{P}}_o + \underset{\sim}{\nabla}\cdot\left(\frac{1}{\Omega}\underset{\sim}{b}\cdot\underset{\approx}{\hat{S}}\times\underset{\sim}{b}\right) + O(\varepsilon^3),$$

$$\underset{\sim}{b}\cdot\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\cdot\underset{\approx}{\hat{P}}) = (\underset{\sim}{\nabla}\times\underset{\sim}{b})\cdot\underset{\sim}{\nabla}\cdot\underset{\approx}{\hat{P}} - \nabla_k(\underset{\approx}{\hat{P}}\cdot\underset{\sim}{\nabla}\underset{\sim}{b})_{ij}\varepsilon_{ijk} + \frac{1}{2}\underset{\sim}{\nabla}\underset{\sim}{\nabla}:(\underset{\approx}{\hat{P}}\times\underset{\sim}{b} + \mathrm{Tr})$$

$$= (\underset{\sim}{\nabla}\times\underset{\sim}{b})\cdot\underset{\sim}{\nabla}\cdot\underset{\approx}{\hat{P}}_0 - \nabla_k(\underset{\approx}{\hat{P}}_0\cdot\underset{\sim\sim}{\nabla b})_{ij}\varepsilon_{ijk} + \frac{1}{2}\underset{\sim\sim}{\nabla\nabla}:(\frac{1}{\Omega}\underset{\approx}{\hat{S}}) + O(\varepsilon^3),$$

as well as

$$\underset{\sim}{\nabla}\times\underset{\sim}{b} = -\varepsilon(\hat{y} + J\hat{z}) + O(\varepsilon^2),$$

$$\underset{\sim\sim}{\nabla b} = -\varepsilon(\hat{z}\hat{x} + \underset{\sim}{\nabla}_\perp(\hat{z}\times\underset{\sim}{\nabla}_\perp\psi)) + O(\varepsilon^2),$$

$$\underset{\approx}{\hat{P}}_0 = \delta n_0 T_i \varepsilon(\nabla_\perp^2 F(\underset{\approx}{I}-\hat{z}\hat{z}) - 2\underset{\sim}{\nabla}_\perp\underset{\sim}{\nabla}_\perp F) + O(\varepsilon^2),$$

the calculation becomes much easier. Here, $\underset{\approx}{\hat{S}}$ is provided by Eqs. (2.29) and (2.30). Also, most of the calculations in $\underset{\sim}{\nabla}\cdot(\frac{1}{\Omega}\underset{\sim}{b}\cdot\underset{\approx}{\hat{S}}\times\underset{\sim}{b})$ and $\frac{1}{2}\underset{\sim\sim}{\nabla\nabla}:(\frac{1}{\Omega}\underset{\approx}{\hat{S}})$ are straightforward. However, special care is needed for keeping the correct curvature terms when calculating $\underset{\sim}{\nabla}\cdot(\underset{\sim}{b}\cdot\underset{\approx}{\hat{W}}\times\underset{\sim}{b})$ and $\underset{\sim\sim}{\nabla\nabla}:(\underset{\approx}{\hat{W}})$. We have

$$\begin{aligned}\underset{\sim}{\nabla}\cdot(\underset{\sim}{b}\cdot\underset{\approx}{\hat{W}}\times\underset{\sim}{b}) &= \underset{\sim}{\nabla}\cdot(2\underset{\sim}{b}\cdot(\underset{\sim\sim}{\nabla V})\times\underset{\sim}{b} + (\underset{\approx}{I}-\underset{\sim\sim}{bb})\cdot(\underset{\sim\sim}{\nabla V}))\\ &= 2\underset{\sim\sim}{\nabla b}:\underset{\sim}{\nabla}(\underset{\sim}{V}\times\underset{\sim}{b}) + 2\underset{\sim}{b}\cdot\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot(\underset{\sim}{V}\times\underset{\sim}{b})) - \underset{\sim}{b}\cdot\underset{\sim}{\nabla}(\underset{\sim}{b}\cdot(\underset{\sim}{\nabla}\times\underset{\sim}{V})) + O(\varepsilon^3)\end{aligned}$$

and

$$\underset{\sim\sim}{\nabla\nabla}:(\underset{\approx}{\hat{W}}) = \nabla^2(\underset{\sim}{\nabla}\cdot\underset{\sim}{V} + \underset{\sim}{b}\cdot\underset{\sim}{\nabla}(\underset{\sim}{b}\cdot\underset{\sim}{V}) - \underset{\sim}{\kappa}\cdot\underset{\sim}{V}) + O(\varepsilon^3).$$

Regarding the parallel collisional viscosity, we can simply adopt Braginskii's result and find it yields no contribution to our reduced model.

Finally, we obtain

$$\frac{\partial}{\partial\tau}(1-2a_i^2\nabla_\perp^2)v + [(1-a_i^2\nabla_\perp^2)F-\delta\frac{T_i}{T_e}(p+\beta(4h-B_z)),v] + \frac{1+\frac{T_i}{T_e}}{2}\nabla_\parallel p$$

$$= \delta\beta\frac{T_i}{T_e}\{\nabla_\parallel\nabla_\perp^2 F + [\underset{\sim}{\nabla}_\perp F;\underset{\sim}{\nabla}_\perp\psi] + 2\delta\underset{\sim}{\nabla}_\perp\cdot[F,\underset{\sim}{\nabla}_\perp v]\} - 4\beta\mu\nabla_\perp^2 v, \qquad (2.42)$$

and

$$\frac{\partial}{\partial\tau}(1-\frac{1}{2}a_i^2\nabla_\perp^2)\nabla_\perp^2 F + [F,(1-\frac{1}{2}a_i^2\nabla_\perp^2)\nabla_\perp^2 F] + \nabla_\parallel J + 2[B_z,h] - \beta\mu\nabla_\perp^4 F$$

$$= -\ \delta\beta\frac{T_i}{T_e}\{\underset{\sim}{\nabla}_\perp\cdot([\underset{\sim}{\nabla}_\perp F,\frac{p}{\beta}-B_z+2h] + [\underset{\sim}{\nabla}_\perp\psi,v])\}$$

$$-\ \frac{1}{2}\delta\beta\frac{T_i}{T_e}\{\nabla_\perp^2(\nabla_\parallel v - \frac{1}{\beta}(\frac{\partial}{\partial\tau}p + [F,p]))\}, \qquad (2.43)$$

where the notation

$$[\underset{\sim}{A};\underset{\sim}{B}] = \sum_k [A_k,B_k],$$

is used, and

$$\mu \equiv \frac{3\nu_i\delta T_i}{10\varepsilon\Omega T_e}$$

is the normalized viscosity coefficient.

The second term on the left-hand side (LHS) of Eq. (2.42) is due to the $\underset{\sim}{E}\times\underset{\sim}{B}$, curvature, and gradient B drift; the third term on the LHS of Eq. (2.42) is apparently responsible for the ion acoustic effects. Also, terms on the RHS of Eq. (2.42) are the FLR terms originate from gyroviscosity.

Regarding equation (2.43), the third term on the LHS, $\nabla_{\parallel}J$, includes the stabilizing line bending term and the kink term; also, its nonlinear part is responsible for the magnetic island formation. The fourth term on the LHS, $2[B_z,h]$, is the curvature term responsible for the return equilibrium flows and the interchange force. The first term on the RHS, $-\delta\beta\frac{T_i}{T_e}\underset{\sim}{\nabla}_{\perp}\cdot[\underset{\sim}{\nabla}_{\perp}F,\frac{p}{\beta}]$, is responsible for the ion-diamagnetic drift. The last term on the RHS of Eq. (2.43), $\nabla_{\perp}^{2}(\frac{\partial}{\partial\tau}p + [F,p])$, is responsible for the FLR corrections to the line bending and the curvature terms. These FLR corrections will become important in the semi-collisional regime where the ion gyroradius become larger than the layer width centered at the resonant surfaces. It is also worth mentioning here that this compressible portion of gyroviscosity is the a crucial term in a Hamiltonian FLR-fluid model.[39]

Although Eqs. (2.40)-(2.43) was derived carefully to retain correct FLR physics to $O(\rho_i^2)$, they along will fail to conserve energy. An intuitive reason is that our FLR system involves an implicit variable $P_\perp - nT_i$, arises from ion gyromotion, which has been replaced according to Eq. (2.35). Therefore, it should not be surprised that a residual $O(\beta^2)$ will survive the equation of energy evolution obtained from Eqs. (2.40)-(2.43). It will also becomes clear in Sec. 2.3, where the energy conservation of the reduced fluid system is discussed in detailed, that ion gyromotion will generate an internal energy with form $(P_\perp - nT_i)^2/(2n_0T_i)$. Therefore, it is suggested that we should self-consistently include the evolution of "$P_\perp - nT_i$". That is, adding "$-\frac{a_i^2}{\delta}\nabla_\perp^2$(Eq. (2.34))" to Eq.(2.43) and yields

$$\frac{\partial}{\partial\tau}(1-\frac{5}{2}a_i^2\nabla_\perp^2)\nabla_\perp^2 F + [F,(1-\frac{5}{2}a_i^2\nabla_\perp^2)\nabla_\perp^2 F] + \nabla_\parallel J + 2[B_z,h] - \beta\mu\nabla_\perp^4 F$$

$$= -\delta\beta\frac{T_i}{T_e}\{\underset{\sim}{\nabla}_\perp\cdot[\underset{\sim}{\nabla}_\perp F,\frac{p}{\beta}-B_z+4h-4\delta\nabla_\perp^2 F] + (\underset{\sim}{\nabla}_\perp\cdot[\underset{\sim}{\nabla}_\perp\psi,v]+\nabla_\perp^2\nabla_\parallel v)\}$$

$$-\frac{1}{2}\delta\beta\frac{T_i}{T_e}\{\nabla_\perp^2(\nabla_\parallel v + \frac{1}{\beta}(\frac{\partial}{\partial\tau}p + [F,p]))\}. \tag{2.44}$$

We will show that, without the dissipation, Eqs (2.40)-(2.42) and (2.44) do conserve energy exactly.

It is interesting to note here that in a system without $O(\rho_i{}^2)$ terms and with constant magnetic field, that is, in which $\hat{\underset{\approx}{P}}$ is given by Eq. (2.30), $\underset{\sim}{\nabla}\psi$ and B_z are constant and curvature is neglected, Eqs. (2.42) and (2.43) can be derived alternatively by using

$$m_i n\left(\frac{\partial}{\partial t}+\underset{\sim}{V}\cdot\underset{\sim}{\nabla}\right)\underset{\sim}{V}_D + \underset{\sim}{\nabla}\cdot\underset{\approx}{\hat{P}}_0 = -\frac{1}{2}\underset{\sim}{\nabla}_\perp\left(\frac{nT_i}{\Omega}\,\underset{\sim}{b}\cdot\underset{\sim}{\nabla}_\perp\times\underset{\sim}{V}_\perp\right) - \underset{\sim\sim}{bb}\cdot\underset{\sim}{\nabla}\left(\frac{nT_i}{\Omega}\,\underset{\sim}{b}\cdot\underset{\sim}{\nabla}_\perp\times\underset{\sim}{V}_\perp\right)$$

$$+ \underset{\sim\sim}{bb}\cdot\left(\underset{\sim}{\nabla}_\perp V_\parallel\times\underset{\sim}{\nabla}_\perp\frac{nT_i}{\Omega}\right) - \frac{nT_i}{2\Omega}\,\underset{\sim}{b}\times\underset{\sim}{\nabla}_\perp(\underset{\sim}{\nabla}_\perp\cdot\underset{\sim}{V}_\perp) + O(\varepsilon^3), \qquad (2.45)$$

where $\underset{\sim}{V}_D = \frac{1}{m_i n\Omega}\,\underset{\sim}{b} \times \underset{\sim}{\nabla} nT_i$ is the diamagnetic velocity. The first two terms on the RHS of Eq. (2.45), involving respectively the perpendicular and parallel gradients of the parallel vorticity, display anisotropy in that the coefficients differ by a factor of two. This factor can be related to the dimensionality of the system. To our knowledge this anisotropy has not previously been noticed. We also note that the above equation differs from the usual "gyroviscosity cancellation"[40], because of the last three terms, which are due to parallel gradients, parallel flow, and compressibility. It implies that when the compressibility is considered, the usual gyroviscosity cancellation, which has been adopted in many previous work studying the gyroviscous effects, will not be adequate.

Furthermore, one might expect that the linear version of Eqs.(2.42) and (2.43) can be derived through the ion gyro-kinetic equation to order of $\rho_i^{\,2}$, but the calculation is complicated. The linear, sheared-slab version of our model will be compared with that of the gyrokinetic theory[31]. Here, we only remark that the second term on the LHS of Eq. (2.42),

$$[(1-a_i^2\nabla_\perp^2)F-\delta\frac{T_i}{T_e}(p+\beta(4h-B_z)),v] = [\varphi-\delta\frac{T_i}{T_e}\beta(4h-B_z),v]$$

is the reduced form of

$$\int d\underset{\sim}{v}\; v_\parallel\; \underset{\sim}{v}_d\cdot\underset{\sim}{\nabla}f,$$

where $\underset{\sim}{v}_d$ is the particle drift velocity which includes $\underset{\sim}{E}\times\underset{\sim}{B}$, curvature, and gradient B drifts.

2.2.2 Discussions

We have derived a closed reduced fluid system; namely, the particle conservation law

$$\frac{\partial}{\partial\tau}p + [F+2\delta B_Z,p] + \beta\{[F+2\delta B_Z,2h] + \nabla_{\parallel}(v+2\delta J)\}$$

$$= \beta(\frac{\partial}{\partial\tau}B_Z+[F,B_Z]) - \eta\beta\nabla_{\perp}^2 B_Z, \tag{2.46}$$

the generalized Ohm's law

$$\frac{\partial}{\partial\tau}\psi + \nabla_{\parallel}F = -2\delta\nabla_{\parallel}B_Z + \eta J, \tag{2.47}$$

the parallel acceleration law

$$\frac{\partial}{\partial\tau}(1-2a_i^2\nabla_{\perp}^2)v + [(1-a_i^2\nabla_{\perp}^2)F-\delta\frac{T_i}{T_e}(p+\beta(4h-B_Z)),v] + \frac{1+\frac{T_i}{T_e}}{2}\nabla_{\parallel}p$$

$$= \delta\beta\frac{T_i}{T_e}\{\nabla_{\parallel}\nabla_{\perp}^2F + [\underset{\sim}{\nabla}_{\perp}F;\underset{\sim}{\nabla}_{\perp}\psi] + 2\delta\underset{\sim}{\nabla}_{\perp}\cdot[F,\underset{\sim}{\nabla}_{\perp}v]\} + 4\beta\mu\nabla_{\perp}^2 v, \tag{2.48}$$

and the shear-Alfvén law

$$\frac{\partial}{\partial\tau}(1-\frac{5}{2}a_i^2\nabla_{\perp}^2)\nabla_{\perp}^2F + [F,(1-\frac{5}{2}a_i^2\nabla_{\perp}^2)\nabla_{\perp}^2F] + \nabla_{\parallel}J + 2[B_Z,h] - \beta\mu\nabla_{\perp}^4F$$

$$= -\delta\beta\frac{T_i}{T_e}\{\underset{\sim}{\nabla}_{\perp}\cdot[\underset{\sim}{\nabla}_{\perp}F,\frac{p}{\beta}-B_Z+4h-4\delta\nabla_{\perp}^2F] + (\underset{\sim}{\nabla}_{\perp}\cdot[\underset{\sim}{\nabla}_{\perp}\psi,v]+\nabla_{\perp}^2\nabla_{\parallel}v)$$

$$+ \frac{1}{2}\nabla_\perp^2(\nabla_\parallel v + \frac{1}{\beta}(\frac{\partial}{\partial\tau}p + [F,p]))\}. \tag{2.49}$$

Also, the electrostatic potential and toroidal magnetic field are given by

$$\varphi = (1 - a_i^2\nabla_\perp^2)F - \delta\frac{T_i}{T_e}p, \tag{2.50}$$

$$B_z = -\frac{1+\frac{T_i}{T_e}}{2}p - \frac{\delta\beta T_i}{2T_e}\nabla_\perp^2 F. \tag{2.51}$$

Although Eqs. (2.46)-(2.51) appear to form a system of five fields, only four of these are independent. This is because B_z (or p) can be straightforwardly eliminated through the simple relation of Eq. (2.51). Nevertheless, the resultant set of equations will be much more complicated unless some further simplifications, such as dropping $O(\beta^2)$ terms, are made. However, terms with small quantities, such as β, ρ_i, are kept mainly for the reason to retain the non-idea physics which can resolve the singularities; therefore, if there is term which only gives a small correction to other terms with the same mathematical form, there is no reason the keep it to complicate our equations.

By a quick survey on terms with "$p-\beta B_z$", we easily see that B_z in those terms only give β corrections to other terms with exactly the same mathematical forms. We therefore suggest that by taking $p - \beta B_z \dashrightarrow p$ in those terms, our system will still retain all the desirable physics as in Eqs. (2.46)-(2.51). After some manipulations and dropping $O(\beta^2)$ terms, we obtain, the particle conservation

$$\frac{\partial}{\partial\tau}p + [(1-a_i^2\nabla_\perp^2)F,p]$$

$$= -\beta\{[(1-a_i^2\nabla_\perp^2)F-\delta(1+\frac{T_i}{T_e})p,2h]+\nabla_\parallel(v+2\delta J)\} + \eta_\perp\beta\frac{1+\frac{T_i}{T_e}}{2}\nabla_\perp^2 p, \quad (2.52)$$

the generalized Ohm's law

$$\frac{\partial}{\partial\tau}\psi = -\nabla_\parallel((1-a_i^2\nabla_\perp^2)F-\delta(1+\frac{T_i}{T_e})p) + \eta_\parallel J, \quad (2.53)$$

the parallel acceleration law

$$\frac{\partial}{\partial\tau}(1-2a_i^2\nabla_\perp^2)v + [(1-a_i^2\nabla_\perp^2)F-\delta\frac{T_i}{T_e}(p+4\beta h),v] + \frac{1+\frac{T_i}{T_e}}{2}\nabla_\parallel p - 4\beta\mu\nabla_\perp^2 v$$

$$= \delta\beta\frac{T_i}{T_e}\{\nabla_\parallel\nabla_\perp^2 F + [\underset{\sim}{\nabla}_\perp F;\underset{\sim}{\nabla}_\perp\psi] + 2\delta\underset{\sim}{\nabla}_\perp\cdot[F,\underset{\sim}{\nabla}_\perp v]\}, \quad (2.54)$$

and the shear-Alfvén law

$$\frac{\partial}{\partial\tau}(1-\frac{5}{2}a_i^2\nabla_\perp^2)\nabla_\perp^2 F + [F,(1-\frac{5}{2}a_i^2\nabla_\perp^2)\nabla_\perp^2 F] + (1-a_i^2\nabla_\perp^2)(\nabla_\parallel J-(1+\frac{T_i}{T_e})[p,h])$$

$$= -\delta\frac{T_i}{T_e}\{\underset{\sim}{\nabla}_\perp\cdot[\underset{\sim}{\nabla}_\perp F, p+2\beta h-4\delta\beta\nabla_\perp^2 F] + \beta(\underset{\sim}{\nabla}_\perp\cdot[\underset{\sim}{\nabla}_\perp\psi, v]+\nabla_\perp^2\nabla_\parallel v)\}$$

$$+ \beta\mu\nabla_\perp^4 F. \qquad (2.55)$$

Here, Eqs. (2.52)-(2.55) by themselves form a "closed" reduced fluid system: four equations with four variables providing self-consistency and energy conservation.

We summarize here that both models provide the same physics and both are exact to $O((\frac{\rho_i}{a})^2)$. The only difference is that the first model, described by Eqs. (2.46)-(2.51), retains the evolution of the toroidal magnetic field B_z, which, in our compressible system, has been suggested to be negligible; nonetheless, the second model, described by Eqs. (2.52)-(2.53), apparently is much more accessible.

By further specifying the scale of the small parameters, e.g., δ or β. our system can be reduced to many well-known reduced fluid models. For instance, by neglecting every terms involving βT_i, our system reduces to the "four-field model" derived by HKM[17]. The point is that they keep the compressibility in equation of particle conservation by retaining the electron beta terms; but for ion viscous tensor, they basically follow the usual gyroviscosity cancellation mentioned in the end of last subsection. Also, it is of interest that a high-β version of RMHD[5] with both ion and electron drift corrections can be derived by simply setting $\beta \to 0$, and redefining $p \equiv \frac{8\pi T_e}{\varepsilon B_o^2}(n-n_o)$. We thus obtain a three-field model, given by

$$\frac{\partial}{\partial \tau}p + [\varphi, p] = 0, \tag{2.56}$$

$$\frac{\partial}{\partial \tau}\psi + \nabla_{\|}\varphi - \delta\nabla_{\|}p = \eta_{\|}J, \tag{2.57}$$

$$\frac{\partial}{\partial \tau}\nabla_{\perp}^2\varphi + [\varphi, \nabla_{\perp}^2\varphi]$$

$$= - \nabla_{\|}J + (1+\frac{T_i}{T_e})[p,h] - \delta\frac{T_i}{T_e}\, \underset{\sim}{\nabla}_{\perp}\cdot[p, \underset{\sim}{\nabla}_{\perp}\varphi] = 0. \tag{2.58}$$

The stream function and the toroidal magnetic field are, respectively,

$$F = \varphi + \delta\frac{T_i}{T_e}p,$$

$$B_z = -\frac{1+\frac{T_i}{T_e}}{2}p.$$

The parallel flow is decoupled from the other fields and satisfies

$$\frac{\partial}{\partial \tau} v + [\varphi, v] + \frac{(1+\frac{T_i}{T_e})}{2} \nabla_{\parallel} p = 0; \tag{2.59}$$

also, the compressibility is given by

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{V} = \varepsilon^2 \frac{v_A}{a} \{ [\varphi - \delta p, 2h] + \nabla_{\parallel} (v + 2\delta J) \}. \tag{2.60}$$

This model arises since when β is of order ε, compressibility and viscosity are both discarded from the system. Hence, the equation of adiabatic compression acts the same as the equation of isothermal particle conservation, during the course of shear-Alfvén motion. We hereafter call this model as "Drift-RMHD" (DRMHD). Note that this model agrees with the previous derived incompressible drift fluid model for large aspect ratio plasma by Hinton and Horton.[40] It is also worth mentioning here that this simple model not only conserves energy but also lead to a non-canonical Hamiltonian formalism which contains a good "Poisson Bracket"; we will discuss that detailedly in Chapter 4.

Moreover, since our reduction orderings are basically the same as those in RHMD, it is not surprised that by setting $\delta \to 0$, we will get the usual high-β version of RMHD[5]; and by further setting $p \to 0$ clearly leads to the low-β version of RMHD[4].

2.3 Energy conservation

In a dynamical system, if a quantity Q satisfies

$$\frac{\partial}{\partial t}Q + \nabla\cdot U = 0,$$

where $\langle\ \rangle$ denotes the fixed volume average (all the surface terms are omitted), then $\langle Q\rangle$ is a constant of motion. For a general discussion of the constants of motion in RMHD, we refer reader to Ref.[28]. Here we study the most common constant of motion - the energy.

The energy conservation law for our primitive fluid system, Eqs. (2.15), and (2.16), can be determined by calculating $\langle V \cdot \text{Eq.}(2.15)\rangle$, $\langle J \cdot \text{Eq.}(2.16)\rangle$, and $\langle B\cdot\frac{\partial}{\partial t}B\rangle$. With the aid of Faraday's law and Ampere's law, we derive

$$\frac{\partial}{\partial t}\left\langle\frac{m_i nV^2}{2} + \frac{B^2}{8\pi}\right\rangle = -\eta_s\langle|J|^2\rangle + \langle\nabla V : (P_i + P_e)\rangle$$

$$= \langle -\eta_s|J|^2 + \nabla V : \hat{P}^c\rangle + \langle n(T_e+T_i)(\nabla\cdot V)\rangle$$

$$+ \langle((P_\perp - nT_i)I + (P_\parallel - P_\perp)bb) + \hat{P}^g) : \nabla V\rangle. \qquad (2.61)$$

Recall that the omission of the electron anisotropic stress tensor is due to the smallness of m_e. We note here that the

RHS of this equation corresponds to the rate of change of the internal energy of the system. Hence, the equation simply represents the conservation of the total of kinetic energy, magnetic energy, and internal energy.

From the thermodynamics point of view, the change of the internal energy is due to the entropy heat production and work done on the system; i.e.,

$$du = Tds + P\frac{dn}{n},$$

where u and s are the internal energy and entropy per unit volume, respectively. The term involving Tds corresponds to the collisional terms on the RHS of Eq. (2.61); while the second and the third terms on the RHS of Eq. (2.61) represent the generalized work done due to isotropic stress and anisotropic stress, respectively. Thus, Eq. (2.61) is equivalent to

$$\frac{\partial}{\partial t} \left\langle \frac{m_i n V^2}{2} + \frac{B^2}{8\pi} + u \right\rangle = 0.$$

However, except for the entropy production, which must be a positive quantity due to the well-known H-theorem, all forms of energy are expected to merge into an energy functional $\langle H \rangle$ such that

$$\frac{\partial}{\partial t}\langle H \rangle = - \langle T\frac{ds}{dt}\rangle$$

$$= \text{"the dissipation of energy"} \leqslant 0$$

To absorb the work done into the energy functional, we use Eq. (2.28) to derive the equality

$$\langle \underset{\sim}{\nabla}\underset{\sim}{V} : \underset{\approx}{\hat{P}} \rangle = - \frac{\Omega}{2nT_i} \langle \underset{\approx}{\hat{P}}_o : \underset{\approx}{\hat{S}} \rangle,$$

where, again, $\underset{\approx}{\hat{P}}_o$ and $\underset{\approx}{\hat{S}}$ are given by Eqs. (30) and (29), respectively. For the present paper, we also adopt the large-aspect-ratio scalings; that is, we use Eqs. (A4) and (B1) to obtain the energy conservation law

$$\frac{\partial}{\partial t} \langle H \rangle = - \eta_s \langle |\underset{\sim}{J}|^2 \rangle - \frac{3\nu_i}{10\Omega}\frac{nT_i}{\Omega} \langle 4|\underset{\sim}{\nabla}V_\parallel|^2 + \sum_{i,j} (\nabla_i V_{\perp j})^2 \rangle, \quad (2.62)$$

where the energy functional $\langle H \rangle$ has the form

$$\langle H \rangle = \langle \frac{m_i n V^2}{2} + \frac{B^2}{8\pi} + n(T_i+T_e)(\ln\frac{n}{n_o}) + \frac{(P_\perp - nT_i)^2}{2n_o T_i} \rangle$$

$$+ \frac{nT_i}{8\Omega^2} \langle 4 | \nabla V_\parallel |^2 + \sum_{i,j} (\nabla_i V_{\perp j})^2 \rangle . \tag{2.63}$$

Notice that the stress induced by ion-gyromotion tends to expand the plasma, the same behavior as that indicated by conventional isotropic pressure. It now becomes clear that how the energy generated by the gyromotion through $(P_\perp - nT_i)$ comes into the total internal energy. Again, the RHS of Eq. (2.62) contains the Ohmic and viscous entropy heat production.

Equation (2.62) involves the large-aspect-ratio approximation but is not expressed in terms of reduced field variables. We next compute the reduced energy functional in terms of p, ψ, F, and v from the reduced fluid equations (2.46)-(2.49). First note that, without surface terms, we have the identity $\langle [f,g]h \rangle = \langle f[g,h] \rangle$. Then, we calculate

$$\langle v\frac{\partial}{\partial \tau}(1-2a_i^2\nabla_\perp^2)v\rangle - \langle F\frac{\partial}{\partial \tau}(1-\frac{5}{2}a_i^2\nabla_\perp^2)\nabla_\perp^2 F\rangle - \langle J\frac{\partial}{\partial \tau}\psi\rangle + \langle \frac{1+\frac{T_i}{T_e}}{2\beta} p\frac{\partial}{\partial \tau}p\rangle ,$$

and find that

$$\frac{\partial}{\partial \tau}\langle \hat{H}\rangle = -\eta\langle J^2\rangle - \beta\mu\langle (\nabla_\perp^2 F)^2 + 4\,|\nabla_\perp v|^2\rangle , \tag{2.64}$$

where the reduced energy functional has the form

$$\langle \hat{H} \rangle = \langle \frac{v^2 + |\nabla_\perp \underset{\sim}{F}|^2}{2} + \frac{B_z^2 + |\nabla_\perp \underset{\sim}{\psi}|^2}{2} + \frac{1 + \frac{T_i}{T_e}}{4\beta} p^2 \rangle$$

$$+ a_i^2 \langle (\frac{5}{4} \nabla_\perp^2 F)^2 + |\nabla_\perp \underset{\sim}{v}|^2 \rangle. \qquad (2.65)$$

By noticing that direct reduction from Eqs. (2.61) and (2.62) leads exactly to Eqs. (2.64) and (2.65), the self-consistency of our model is further indicated.

Similarly, Eqs. (2.52)-(2.55) lead to an energy functional $\langle \hat{H}' \rangle$ which differs from Eq. (2.65) only in the absence of B_z term which has been suggested to be negligible when compared with p/β. That is,

$$\langle \hat{H}' \rangle = \langle \frac{v^2 + |\nabla_\perp \underset{\sim}{F}|^2}{2} + \frac{|\nabla_\perp \underset{\sim}{\psi}|^2}{2} + \frac{1 + \frac{T_i}{T_e}}{4\beta} p^2 \rangle$$

$$+ a_i^2 \langle (\frac{5}{4} \nabla_\perp^2 F)^2 + |\nabla_\perp \underset{\sim}{v}|^2 \rangle. \qquad (2.66)$$

Where the evolution of $\langle \hat{H}' \rangle$ is also governed by Eq. (2.64). We now can conclude that the reduced model described by Eqs. (2.52)-(2.55) prevails over the other one with B_z. Since, it clearly retains the same physics the other has, while it is obviously more accessible.

On the other hand, to understand the appearance of $\langle 2hp \rangle$ in the energy functional of the high-β version of RMHD[5], we first note that this system satisfies

$$\left\langle \frac{\partial}{\partial t} \frac{P}{\gamma - 1} \right\rangle = \langle \underset{\sim}{V} \cdot \nabla P \rangle ,$$

where γ is the ratio of specific heats. This describes $\frac{2}{\gamma - 1}$ dimensional adiabatic compression. From Eqs. (2.56)-(2.58), with $\delta \to 0$, we find that the above time rate of change of the internal energy is reduced to

$$- \varepsilon^3 \frac{v_A^3}{a} m_i n \langle p[F,2h] + p \nabla_{\parallel} v \rangle .$$

This energy involves v because of parallel compressibility. However, the evolution of the kinetic energy due to parallel flow is

$$\frac{\partial}{\partial \tau} \left\langle \frac{v^2}{2} \right\rangle = - \langle v \nabla_{\parallel} p \rangle .$$

Thus the term $\langle -2hp \rangle$ which appears in the conserved energy functional of the high-β version of RMHD comes from

$$\left\langle \frac{\partial}{\partial t}\frac{P}{\gamma - 1} + \frac{\partial}{\partial t} m_i n \frac{V_\parallel^2}{2} \right\rangle \dashrightarrow \frac{\partial}{\partial \tau}\langle -2hp\rangle . \tag{2.67}$$

We then conclude that whenever the incompressibility is imposed, the kinetic energy due to parallel flow will combine with the internal energy, and will lead to a conserved energy functional with form of $\langle -2hp\rangle$. This also implies that even though the parallel flow is decoupled from the reduced system within the scalings of the high-β version of shear-Alfvén dynamics, it still implicitly exchanges energy with the thermal field.

We now show that the simplified incompressible model DRMHD, in Eqs. (2.56)-(2.58), is energy conserving, too. By calculating

$$- \langle (\varphi-\delta p)\frac{\partial}{\partial \tau}\nabla_\perp^2\varphi\rangle - \langle J\frac{\partial}{\partial \tau}\psi\rangle + \langle \{-(1+\frac{T_i}{T_e})h+\delta\nabla_\perp^2(\varphi+\delta\frac{T_i}{T_e}p)\}\frac{\partial}{\partial \tau}p\rangle ,$$

we obtain

$$\langle \hat{H}\rangle = \langle \delta p\nabla_\perp^2\varphi + \frac{|\nabla_\perp\varphi|^2}{2} + \frac{|\nabla_\perp\psi|^2}{2}\rangle$$

$$- \langle (1+\frac{T_i}{T_e})hp + \frac{T_i}{T_e}\delta^2\frac{|\nabla_\perp p|^2}{2}\rangle , \tag{2.68}$$

with

$$\frac{\partial}{\partial\tau}\langle\hat{H}\rangle = -\,\eta\langle J^2\rangle .$$

Note that this energy functional along with Eqs. (2.56)-(2.58) will be taken as an example for a non-canonical Hamiltonian formalism in chapter 3.

2.4 A Padé Approximation

It is always instructive to have the equations of motion in terms of the electrostatic potential φ, rather than the stream function F, as in most of the kinetic theories. By using Eq. (2.50) and neglecting $O(\beta^2)$ terms, Eqs. (2.52)-(2.55) yield

$$\frac{\partial}{\partial\tau}p + [\varphi,p] = -\beta\{[\varphi-\delta p,2h] + \nabla_{\parallel}(v+2\delta J)\} + \eta_{\perp}\beta\frac{1+\frac{T_i}{T_e}}{2}\nabla_{\perp}^2 p; \quad (2.69)$$

$$\frac{\partial}{\partial\tau}\psi = -\nabla_{\parallel}(\varphi-\delta p) + \eta_{\parallel}J; \quad (2.70)$$

$$\frac{\partial}{\partial\tau}(1+b_i)v + [\varphi-4\delta\frac{T_i}{T_e}\beta h\ ,\ v] + \frac{1+\frac{T_i}{T_e}}{2}\nabla_{\parallel}p - 4\beta\mu\nabla_{\perp}^2 v$$

$$= \delta\beta\frac{T_i}{T_e}\{\nabla_{\parallel}\nabla_{\perp}^2(\varphi+\delta\frac{T_i}{T_e}p) + [\underset{\sim}{\nabla}_{\perp}(\varphi+\delta\frac{T_i}{T_e}p);\underset{\sim}{\nabla}_{\perp}\psi]\}$$

$$+ 2\delta^2\beta\frac{T_i}{T_e}\underset{\sim}{\nabla}_{\perp}\cdot[(\varphi+\delta\frac{T_i}{T_e}p),\underset{\sim}{\nabla}_{\perp}v]; \quad (2.71)$$

$$\frac{\partial}{\partial\tau}(1+\frac{b_i}{4})\nabla_{\perp}^2\varphi + [(\varphi+\delta\frac{T_i}{T_e}p),(1+\frac{b_i}{4})\nabla_{\perp}^2\varphi]$$

$$+ \left(1+\frac{b_i}{2}\right)\left(1+\frac{b_i}{2}\right)\left(\nabla_{\|} J-\left(1+\frac{T_i}{T_e}\right)[p,h]\right)$$

$$= \delta\frac{T_i}{T_e}\{[\nabla_\perp\varphi;\nabla_\perp p] - \beta\nabla_\perp\cdot[\nabla_\perp\psi,v]\} + \beta\mu\nabla_\perp^4\left(\varphi+\delta\frac{T_i}{T_e}p\right). \qquad (2.72)$$

Here, $b_i = -2a_i^2\nabla_\perp^2$ is the usual notation of FLR operator in the gyrokinetic theory. Since we only neglect $O(\beta^2)$ terms in deriving this set of equations from the more general one, we should retain most of the desirable physics. In fact, Eqs. (2.69)-(2.72), after being linearized in a sheared slab geometry, will be investigated and compared with the gyrokinetic theory in the next chapter. The comparisons show that our system, although keeps only to $O(\rho_i^2)$, retains good FLR effects over a wide range of b_i. On the other hand, the omission of $O(\beta^2)$ terms raises residual of order β^2 in the equation of energy evolution, obtained from this set of equations. Even though this problem can be fixed by artificially adding some "harmless" $O(\beta^2)$ terms to the equations of motion, it will apparently complicate our system. Moreover, the operator $\frac{b_i}{4}$ and $(1+\frac{b_i}{2})(1+\frac{b_i}{2})$ in Eq. (2.72) apparently complicate the nonlinear analysis.

In this subsection, we thus present an energy conserving model through a minor simplification of Eqs. (2.69)-(2.72). A commonly used simplification, which approximates the real functional operator into a more accessible functional with simpler polynomial form and still retains good physics in a wide range of parameter regimes, is adopted. The resulted system usually refers to as a "Padé approximation" of the real system. In this sense, the system of Eqs. (2.69)-(2.72), according to the comparison made in the next chapter, is a Padé approximation of the full FLR system; while it still suffers the disadvantages described in the last paragraph. Nevertheless, it motivates us to further simplify Eqs. (2.69)-(2.72) by neglecting $\frac{b_i}{4}$ term and taking $(1+\frac{b_i}{2})(1+\frac{b_i}{2}) \longrightarrow (1+b_i)$. We thus obtain

$$\frac{\partial}{\partial\tau}p + [\varphi,p] = -\beta\{[\varphi-\delta p,2h] + \nabla_{\|}(v+2\delta J)\} + \eta_{\perp}\beta\frac{1+\frac{T_i}{T_e}}{2}\nabla_{\perp}^2 p, \quad (2.73)$$

$$\frac{\partial}{\partial\tau}\psi = -\nabla_{\|}(\varphi-\delta p) + \eta_{\|}J, \quad (2.74)$$

$$\frac{\partial}{\partial\tau}v + [\varphi-4\delta\frac{T_i}{T_e}\beta h\ ,\ v] + \frac{1+\frac{T_i}{T_e}}{2}\nabla_{\|}p - \beta\mu\nabla_{\perp}^2 v$$

$$= \delta\beta\frac{T_i}{T_e}\{\nabla_\parallel\nabla_\perp^2(\varphi+\delta\frac{T_i}{T_e}p) + [\underset{\sim}{\nabla}_\perp(\varphi+\delta\frac{T_i}{T_e}p);\underset{\sim}{\nabla}_\perp\psi], \qquad (2.75)$$

$$\frac{\partial}{\partial\tau}\nabla_\perp^2\varphi + [\varphi,\nabla_\perp^2\varphi] + (1+b_i)(\nabla_\parallel J-(1+\frac{T_i}{T_e})[p,h]) - \beta\mu\nabla_\perp^4(\varphi+\delta\frac{T_i}{T_e}p)$$

$$= \delta\frac{T_i}{T_e}\{\underset{\sim}{\nabla}_\perp\cdot[\underset{\sim}{\nabla}_\perp\varphi,p] - \beta\underset{\sim}{\nabla}_\perp\cdot[\underset{\sim}{\nabla}_\perp\psi,v]\}. \qquad (2.76)$$

The linear consequence of this system leads to an FLR functional operator

$$\gamma(b_i) \equiv \frac{1}{1+b_i} \qquad (2.77)$$

which is a good Padé approximation of both $h(b_i)$ (FLR operator of the full FLR system) and $g(b_i)$ (FLR operator of Eqs. (2.69)-(2.72)). It is also worth mentioning that $\gamma(b_i)$, although is not as accurate as $g(b_i)$ for b_i smaller than one, is even more accurate than $g(b_i)$ for b_i larger than one. Note that this consequence coincides with a rather rigorous kinetic FLR analysis by Hahm[26], in which $h(b_i)$ is also taken an approximation to be $\gamma(b_i)$.

More importantly, this simpler model conserves energy exactly in the non-dissipative case. That is, by calculating

$$\langle v\frac{\partial}{\partial\tau}v - (\varphi+\delta\frac{T_i}{T_e}p)\frac{\partial}{\partial\tau}\nabla_\perp^2\varphi - J\frac{\partial}{\partial\tau}\psi + \{\frac{1+\frac{T_i}{T_e}}{2\beta}p-\delta\frac{T_i}{T_e}\nabla_\perp^2(\varphi+\delta\frac{T_i}{T_e}p)\}\frac{\partial}{\partial\tau}p\rangle ,$$

we obtain

$$\langle\hat{H}\rangle = \langle \frac{1+\frac{T_i}{T_e}}{4\beta}p^2 + \frac{|\nabla_\perp(\varphi+\delta\frac{T_i}{T_e}p)|^2}{2} + \frac{|\nabla_\perp\psi|^2}{2} + \frac{v^2}{2}\rangle , \tag{2.78}$$

which changes only due to dissipation.

Finally, we note that equations (2.73)-(2.76) will be used in chapter 4 for studying finite ion temperature effects on the linear drift-tearing modes. The results recover many previously derived results in certain limiting cases. Therefore, we conclude that this simpler Padé approximation deserves further nonlinear analysis.

2.5 Conclusions

In this chapter, we have derived and discussed several reduced fluid models. First, a generalized reduced fluid model, given in Eqs. (2.46)-(2.51), is derived from moment equations to carefully retain FLR terms to $O(\rho_i^2)$. The conserved energy functional of this generalized model is exactly the same as that directly derived from moment equations. Although our system contains only four independent field variables, the equations of motion explicitly involves six variables. Therefore, by using the fact that terms with βB_z gives only β corrections to other terms with similar structure, we then took $p-\beta B_z \to p$ and omitted $O(\beta^2)$ terms to obtain a closed four fields (F, v, ψ, p) model, given in Eqs. (2.52)-(2.57). This model is also proved to be energy conserving. Finally, we have also present a simplified, more accessible, energy conserving model, which is a Padé approximation of the full FLR system. This model will be used for investigating the linear drift-tearing modes in chapter 4. In certain limit, the results agree with previously derived results through rigorous gyrokinetic treatments. Hence, we should emphasize that this simpler model deserve more detailed nonlinear investigation both numerically and analytically. In the next chapter, we will discuss some general applications of our reduced fluid system.

Although some potentially important effects arises from temperature variations are omitted in this thesis, we remark that our system can be easily extended to include those effects. We note that a work in progress has indeed included the electron temperature variations.

CHAPTER III

APPLICATIONS

3.1 Introduction

As mentioned in chapter 1, the reduced fluid model is primarily constructed for studying the nonlinear dynamical phenomena observed in toroidally confined plasma. Nonetheless, the reduction orderings based upon shear-Alfvén time scale and large aspect-ratio geometry allow us to apply our system to study many of the low frequency activities, such as equilibrium and transport, in the toroidal devices. Most importantly, the inclusion of the FLR terms allow us to study many of the drift-type instabilities in high temperature plasma, both linearly and non-linearly. In this chapter, we discuss some of these applications.

In sections 3.2 and 3.3, we briefly discuss the local equilibrium in an isolated system and review low frequency shear-Alfvén activities in the toroidally confined plasma. Note that these topics have been the main interest since the early day of plasma research and have been detailedly reviewed in a recent review article by Hazeltine and Meiss[41]. We therefore discuss them only briefly. We use the reduced system obtained in the last chapter, in a self-contained manner so that the subsequent discussions can readily follow.

In section 3.4, a set of linearized equations is derived through the boundary layer analysis and several important non-ideal effects are discussed. Also, the comparison of the resulted linear eigenmode equations to the gyrokinetic theory is given.

In section 3.5, the toroidal Pfirsch-Schlüter particle transport is studied by using our reduced model and the usual result with toroidal enhancement is reproduced.

In section 3.6, the non-canonical Hamiltonian formalism is constructed based on the DRMHD given in Eqs. (2.56)-(2.58). The basic concept of the non-canonical Hamiltonian theory is briefly discussed; and the generalized Poisson bracket of that simple model is derived. Also, the Casimir invariants are derived by utilizing an isomorphism theory.

Finally, the conclusions are given in section 3.7.

3.2 Local equilibrium

Local equilibrium in an isolated magnetohydrodynamic system is best characterized by the lowest order steady state solution of equation (2.75), described by

$$\nabla_{\parallel} p = 0, \tag{3.1}$$

which is an immediate consequence of the MHD force balance. The omission of inertia terms is due to the smallness of the ion-gyroradius when compared with the global scale length. Eq. (3.1) implies that the confinement of the toroidally confined plasma is characterized by a set of smooth, nested, closed surfaces of constant pressure wound by helical magnetic field lines. Specialist usually refer to them as "flux surfaces" and label each of them with a flux function, F, which satisfies

$$\nabla_{\parallel} F = 0. \tag{3.2}$$

Several physical equilibrium quantities are natural flux functions in lowest order. For instance, the pressure is naturally an approximate flux function; and from the lowest order of the generalized Ohm's law, one sees that the electrostatic potential is also a flux function. It is convenient to construct flux coordinates (r, ϑ, z), where

$$r \equiv r(F) \tag{3.3}$$

is a normalized "radial" coordinate which labels the flux surface; z coincides with the usual toroidal angle; and ϑ, a poloidal angle, is defined through a very important flux function q such that

$$q(F) \equiv \frac{\nabla_{\parallel} z}{\nabla_{\parallel} \vartheta}. \tag{3.4}$$

q, which measures the field-line pitch on each flux surface, is usually called "safety factor" due to the prediction from ideal MHD theory that the most dangerous modes are likely to happen near the flux surfaces having small integral safety factor.

The change in field-line pitch from one flux surface to another refers to as the magnetic "shear" which tends to localize the instability to the region where $\nabla_{\parallel} \simeq 0$. The existence of magnetic shear implies the structure of flux surfaces in the plasma: the singular periodic surfaces amongst a background of quasi-periodic surfaces. For quasi-periodic surfaces, wound by a single helical field line, small disturbances are usually harmless due to its ergodicity. On the other hand, in the vicinity of rational surfaces, on which q is rational and thus each field line close to itself after sufficient number of circuits, disturbances can result in the island formation and local destruction of the flux-surface topology, similar to those happening near the rational tori in Hamiltonian theory[42].

The importance of rational surfaces in the shear-Alfvén dynamics can be further understood through the lowest order shear-Alfvén law without plasma inertia, i.e.

$$\nabla_{\parallel} J = \left(1 + \frac{T_i}{T_e}\right) [h,p], \tag{3.5}$$

where $\nabla_{\parallel}$ can be written as

$$\nabla_{\parallel} = (\nabla_{\parallel})\vartheta\left(\frac{\partial}{\partial\vartheta} + q\frac{\partial}{\partial z}\right). \tag{3.6}$$

From a Fourier decomposition, one finds that the solution of equation (3.5) is in general singular on the rational surfaces unless the solubility condition[41,43]

$$\left(\frac{[h,p]}{\nabla_{\parallel}\vartheta}\right)_{mn}(r_{mn}) = 0 \tag{3.7}$$

is satisfied for each (m,n), where r_{mn} is the reduced radial coordinate which labels the rational surface with safety factor $\frac{m}{n}$. In other words, singularity occurs unless the above solubility condition is satisfied on each rational surface. Also,

$$(A)_{mn}(r) \equiv \oint \frac{dz d\vartheta}{(2\pi)^2} A(\underset{\sim}{x})\, e^{-i(m\vartheta - nz)} \tag{3.8}$$

defines the corresponding Fourier component. Unfortunately, the solubility condition is rarely satisfied except for the rigorously axisymmetric equilibrium system. Failure to satisfy the solubility condition leads generally to a large surface current, field-line annihilation, field-line reconnection, local changes in the magnetic field topology and island formation. Note that the size of the island can effectively determine the degree the solubility condition is violated. This argument also implies that the existence of a smooth, well-behaved flux function really depends on whether the solubility condition is satisfied.

In this thesis, we will restrict ourself to the axisymmetric case. For axisymmetric system, Eq. (3.2) implies that the equilibrium poloidal magnetic flux ψ_0 is a

3.3 Shear-Alfvén law

Regarding the non-equilibrium situation, in an axisymmetric system, one can write the shear-Alfvén law as

$$\nabla_{\parallel 0}\tilde{J} = -\tilde{\nabla}_{\parallel}J_0 + (1+\frac{T_i}{T_e})[h,p] - \tilde{\nabla}_{\parallel}\tilde{J} + I, \tag{3.12}$$

where f_0 ($\tilde{f}$) represents the equilibrium (perturbed) quantity, and $\nabla_{\parallel 0}$ ($\tilde{\nabla}_{\parallel}$) apparently represents the parallel gradient due to the equilibrium (perturbed) magnetic field. The term on the LHS, $\nabla_{\parallel 0}\tilde{J}$, is the usual stabilizing line bending term; note that the stabilization essentially comes from the tendency of magnetic field to straighten the field lines and relax to its lowest free-energy state. The first term on the RHS, $\tilde{\nabla}_{\parallel}J_0$, is the kink term which correspond to the "current driven" modes. There are generally two types of kink activities; namely, internal kink and external kink, corresponds to the fixed plasma boundary and free plasma boundary, respectively. The second term on the RHS, involving [h,p], is the interchange term and is responsible for the "pressure driven" modes.

The interchange driving term involves the interchange of fluid elements in the Rayleigh-Taylor instability, which occurs when a dense fluid is supported by a fluid with less density. The point is that field line curvature, acting like gravity, tends to draw plasma fluid toward the $-\underset{\sim}{\kappa}$ direction, and interchange activity is activated when the curvature is in the same direction of the pressure gradient. Therefore, $\underset{\sim}{\kappa}\cdot\underset{\sim}{\nabla}p>0$ ($\underset{\sim}{\kappa}\cdot\underset{\sim}{\nabla}p<0$) is usually referred to as unfavorable (favorable) curvature. It was first suggested by CGJ[47] that in devices like reversed field pinch and Spheromak, where the average curvature is unfavorable, the interchange modes become unstable. On the other hand, it has been shown by GGJ[32] that the favorable average curvature, in device such as tokamak, provides a stabilizing effect to the tearing modes. Another unstable mechanism corresponds to the unfavorable curvature localized in certain range of poloidal angle. It is thus called "ballooning instability".

The third term represents the nonlinear effects due to magnetic fluctuations and is responsible for nonlinear island evolution and the local changes of the magnetic field topology. The fourth term, I, refers to plasma inertia which is generally responsible for resolving the singularity and forming a narrow current layer on the resonant surface.

When the nonlinear terms are assumed to be small, the first two terms on the RHS together with the LHS simply describe the marginal stabilities of the usual linear MHD activities, such as the ideal kink modes and the ideal interchange modes. In this case, the inertia terms are presumably small as long as the Fourier components of the disturbances satisfy the solubility condition on each corresponding rational surface. On the other hand, the success in experimentally achieving the ideal MHD stabilities in most of the present day tokamaks leads to the importance of plasma inertia. The point is that the remaining important perturbations are those which do not satisfy the solubility condition in Eq. (3.5), therefore a complete stability study requires the inclusion of plasma inertia which can resolve the singularity near the rational surfaces, at least in the linear context. Even at this stage, plasma motions far from the rational surfaces, where inertia terms are unimportant, are still mainly governed by the MHD descriptions. This then leads to a usual "boundary-layer problem": one solves the "layer interior" equations including the non-ideal effects while one uses the well-known MHD solutions as the boundary conditions. In other word, one asymptotically matches the two solutions.

Although the non-linear behaviors are most likely to be the realistic descriptions of the phenomena observed in the magnetic fusion devices, thorough understandings of linear behaviors can always provide clear physical insight and is always of great interests. In the next subsection and chapter 4, we will devote ourself to the linear studies near the resonant surfaces. For nonlinear approach, we will discuss a recent developed non-canonical Hamiltonian formalism and its application for studying the nonlinear stability.

3.4 Linear consequence

In this subsection and the next chapter, we restrict ourselves to the linear problem where the distance between two distinct mode rational surfaces is much larger than the boundary layer width. Therefore the boundary layer analysis becomes appropriate. In this case, linearization can be achieved by expressing the field variable as

$$f \longrightarrow f_0(r,\vartheta) + \hat{f}(r,\vartheta)e^{i(m\vartheta-nz)}, \tag{3.13}$$

where f_0 and f represent the equilibrium and perturbed quantity, m and n are the poloidal and toroidal mode number. The ϑ dependence of f_0 and $\hat{f}$ is mainly due to toroidicity which we will discuss detailedly in Chapter 4. Here, for simplicity, we consider the cylindrical geometry. That is, by neglecting the curvature and assuming both f_0 and $\hat{f}$ depend on r only, we can reduce our system to 1-D problem.

The reduced operators therefore have the forms

$$[F,G] = ik_\perp(F_{0r}\hat{G} - G_{0r}\hat{F}) \tag{3.14}$$

and

$$\nabla_\parallel F = ik_\parallel\hat{F} + ik_\perp F_{0r}\hat{\psi}; \tag{3.15}$$

where

$$k_\perp \equiv \frac{m}{\sqrt{g}} \tag{3.16}$$

and

$$k_\parallel \equiv \frac{m}{q} - n. \tag{3.17}$$

The relevance of layer interior problem can be further specified by the "boundary layer ordering":

$$k_\perp,\ \frac{\partial f_0}{\partial r} \sim 1; \quad \frac{\partial \hat{f}}{\partial r} \sim w^{-1} \gg 1. \tag{3.18}$$

Hence, the bracket with inner product form becomes

$$2[\underset{\sim}{\nabla}_\perp F \ ; \ \underset{\sim}{\nabla}_\perp G] = \nabla_\perp^2([F,G]) - [\nabla_\perp^2 F, G] - [F, \nabla_\perp^2 G] \simeq O(w).$$

Regarding the equilibrium current gradient, which is responsible for the kink instability outside the boundary layer, we assume it to be negligible inside the boundary layer, for simplicity. Actually, for tearing modes, the kink term has been shown[41] to be unimportant even when it is not negligible. Moreover, we assume that the lowest order equilibrium flow is solely due to diamagnetic drift; i.e. $\varphi_0(r)$, $v_0(r) \sim 0$. The linearization of Eqs. (2.69)-(2.72) thus yields

$$-i\omega p - ik_\perp p_{or}\varphi = -ik_\parallel\beta(v+2\delta\psi_{rr}) + \eta_\perp\beta\frac{1+\frac{T_i}{T_e}}{2}p_{rr}, \tag{3.19}$$

$$-i\omega\psi = -ik_\parallel(\varphi-\delta p) + i\delta k_\perp p_{or}\psi + \eta_\parallel\psi_{rr}, \tag{3.20}$$

$$-i\omega v = -ik_\parallel\frac{1+\frac{T_i}{T_e}}{2}p - ik_\perp\frac{1+\frac{T_i}{T_e}}{2}p_{or}\psi + O(\beta) \tag{3.21}$$

$$-i(\omega-k_\perp\delta\frac{T_i}{T_e}p_{or})(1+\frac{b_i}{4})\varphi_{rr} - \mu\beta(\varphi+\delta\frac{T_i}{T_e}p)_{rrrr}$$

$$= -(1+\frac{b_i}{2})(1+\frac{b_i}{2})ik_\parallel\psi_{rr}. \tag{3.22}$$

Here we have omitted the "^" notation for perturbed fields and assumed $|\nabla \tilde{r}|^2 \simeq 1$. The omission of the $O(\beta)$ terms on the RHS of Eq. (3.21) is due to the fact that v involves with other fields only through the parallel compressibility in Eq. (3.19). This simply means that our system is a three-field system in the linear context. Also note that $b_i \simeq -2\delta^2\beta\frac{T_i}{T_e}\frac{\partial^2}{\partial r^2}$. This implies that even if we assume $|2\delta^2\beta\frac{T_i}{T_e}k_\perp^2| \ll 1$, $|b_i|$ could still be large inside the boundary layer.

In addition to the boundary layer orderings and the assumptions on the equilibrium quantities, certain ancillary orderings for various problem are also needed. For resistive modes, we follow the most intuitive choice of orderings, first introduced in the work of GGJ[32]:

$$\omega \sim w; \quad k_\parallel \sim w; \quad \varphi,\ \psi \sim w; \quad p,\ v \sim 1. \tag{3.23}$$

It is easily understood why the FLR operator, b_i, only appears in the shear-Alfvén law, Eq. (3.22), in the linear context. The point is that Eqs. (3.19)-(3.20) mainly describe the electron dynamics while the parallel velocity involves only the ion acoustic effects through the parallel compressibility. Moreover, the relevance of the FLR effects in the boundary layer problem is that the ion-gyroradius becomes comparable to the layer width; i.e. $b_i \sim 1$. This can be achieved by a further ordering that

$$\delta \sim w. \tag{3.24}$$

We also note here that this ordering is consistent with the drift modes ordering; i.e.,

$$\omega_e = -\delta k_\perp p_{or} \sim \omega \sim w, \tag{3.25}$$

where ω_e is the electron diamagnetic drift frequency.

Equations. (3.19)-(3.22) can thus be written as

$$(1 - \alpha\beta\frac{k_\parallel^2}{\omega^2})p_* - \varphi = -\alpha\beta\frac{k_\parallel}{\omega}\psi + 2\delta^2\beta\frac{k_\parallel}{\omega_e}\psi_{rr} + \alpha\beta\frac{\eta_\perp}{-i\omega}p_{*rr}, \tag{3.26}$$

$$(1 - \frac{\omega_e}{\omega})\psi = \frac{k_\parallel}{\omega}(\varphi - \frac{\omega_e}{\omega}p_*) + \frac{\eta_\parallel}{-i\omega}\psi_{rr}, \tag{3.27}$$

$$(1- \frac{\omega_i}{\omega})\{(1+\frac{b_i}{4})\varphi_{rr} - \frac{\mu\beta}{-i\omega}\varphi_{rrrr}\} = (1+\frac{b_i}{2})(1+\frac{b_i}{2})(\frac{k_\parallel}{\omega}\psi_{rr}); \tag{3.28}$$

where p has been replaced by $p_* \equiv -\frac{\delta w}{\omega_e}p$ for convenience and $\omega_i \equiv \delta\frac{T_i}{T_e}k_\perp p_{0r}$ is the ion-diamagnetic frequency. Note that the three variables now has the same scale; i.e., p_*, ψ, $\varphi \sim$ w.

It is worth mentioning here that Eq. (3.28) retains pretty accurate FLR effects over a wide range of b_i, in comparison with the gyrokinetic theory. In the absence of ion collisional viscosity, Eq. (3.28) can be written as

$$(1- \frac{\omega_i}{\omega})g(b_i)\varphi = 2\delta^2\beta\frac{T_i k_\parallel}{T_e\,\omega}\psi_{rr}, \tag{3.29}$$

where $g(b_i)$ is an FLR operator with form

$$g(b_i) \equiv \frac{(1+\frac{b_i}{4})b_i}{(1+\frac{b_i}{2})(1+\frac{b_i}{2})}. \tag{3.30}$$

While, from the ion gyrokinetic equation, one finds that the full FLR operator[31] has the form

$$h(b_i) \equiv 1 - \Gamma_0(b_i). \tag{3.31}$$

The difference is apparently minor, since

for $b_i < 1$

$$g(b_i) \sim h(b_i) \sim b_i(1-\frac{3}{4}b_i);$$

$$(3.33)$$

and for $b_i \longrightarrow \infty$

$$g(b_i) \sim h(b_i) \sim 1.$$

Actually, the error of $g(b_i)$ will not exceed 8% for all value of b_i. In Sec. 2.4, this argument has been extended a priori to motivate an approach to the derivation of a further simplified version, "A Padé Approximation", of the generalized reduced fluid model.

Now, let's consider the electron responses described by Eqs. (3,26)-(3,28). In Eq. (3.26), $\alpha\beta(\frac{k_\parallel}{\omega})^2$ on the LHS and the first term on the RHS refer to the ion acoustic effects;

the third term on the RHS is the semi-collisional compression..the last term on the RHS is the radial particle diffusion. In Eq. (3.27), the first term on the RHS is sometimes referred to as the non-adiabatic response; and the last term on the RHS is due to the usual parallel resistivity.

After some manipulations Eqs. (3.26) and (3.27) can be combined to form a generalized Ohm's law

$$\sigma(\omega,k_\parallel)\cdot(\psi - \frac{k_\parallel}{\omega}\varphi) = \psi_{rr} - D_\perp(\omega,k_\parallel)\frac{k_\parallel}{\omega}p_{*rr}. \tag{3.34}$$

Where $\psi - \frac{k_\parallel}{\omega}\varphi$ corresponds to the parallel electric field;

$$\sigma(\omega,k_\parallel) = \frac{1 - \frac{\omega_e}{\omega} - \alpha\beta\frac{k_\parallel^2}{\omega^2}}{\frac{\eta_\parallel}{-i\omega}(1-\alpha\beta\frac{k_\parallel^2}{\omega^2}) - 2\delta^2\beta\frac{k_\parallel^2}{\omega^2}} \tag{3.35}$$

is the generalized parallel conductivity including the semi-collisional effects[48] and ion acoustic effects[49] which is usually derived through kinetic approach; and

$$D_\perp(\omega,k_\parallel) = \frac{\alpha\beta\dfrac{\eta_\perp \omega_e}{-i\omega\,\omega}}{\dfrac{\eta_\parallel}{-i\omega}\left(1-\alpha\beta\dfrac{k_\parallel^{\,2}}{\omega^2}\right) - 2\delta^2\beta\dfrac{k_\parallel^{\,2}}{\omega^2}} \tag{3.36}$$

corresponds to the perpendicular particle diffusion.

Before further studying the significance of the generalized Ohm's law, we note that $k_\parallel$ is usually conveniently approximated to be linear near the rational surface. Due to the fact that $k_\parallel = 0$ at $r=r_{mn}$, we thus write

$$k_\parallel \simeq k_\parallel' x; \tag{3.37}$$

where $x \equiv r-r_{mn}$ is the distance from the resonant surface and $k_\parallel'$ is constant characterizing the magnetic shear. Also note that the "shear length" can be written as

$$L_S \simeq \frac{qR_o}{r}\,\frac{q}{q'} = R_o\left|\frac{k_\perp}{k_\parallel}\right|. \tag{3.38}$$

Now, for the generalized Ohm's law, let's first neglect the perpendicular diffusion. It becomes clear that the term corresponding to the semi-collisional compression causes a drastic drop of the parallel conductivity when away from the rational surface. That is, the effective conductivity becomes small at large distance from the rational surface, rather than staying constant over the space. If the width of the generalized conductivity becomes smaller than the width of the parallel electric field, then the parallel current drops before the parallel electric field drop when away from the rational surface. This leads to a current-channelling: the surface current tends to concentrate inside a narrow channel. This case is sometimes referred to as "semicollisional", as opposed to the opposite collisional case in which the classical resistive layer width is much smaller than the width of conductivity.

On the other hand, the ion acoustic effects can now be viewed as a mechanism that keeps the effective conductivity to be a certain nonzero value at a distance much larger than the width of the current channel. This long tail will be shown in the next chapter to stabilize the drift-tearing modes by dragging the drift wave out of the resistive layer characterized by the parallel resistivity $\eta_{\parallel}$.

Regarding the perpendicular diffusion, one can see from Eqs. (3.26) and (3.27) that, in semicollisional regime, it is not important unless

$$|(2\delta^2\beta\frac{k_\parallel^2}{\omega^2})(\alpha\beta\frac{\eta_\perp}{\eta_\parallel})| > 1. \tag{3.39}$$

The point is when the above inequality is satisfied, the semi-collisional compression is balanced against the perpendicular diffusion. The usual semicollisional current channel thus disappears and the singularity is resolved by the $\eta_\perp$ term.

Note that all of the above discussion will be studied in detail in chapter 4, for case of tearing modes. We also note that when ion acoustic effects and perpendicular diffusion are neglected, Eqs. (3.26)-(3.28) can by combined into one single eigenmode for studying the resistive shear-Alfvén mode with FLR effects. The result agrees with the previous derived result, from gyrokinetic theory, by Hahm.[26]

3.5 Particle transport

Plasma transport in axisymmetric toroidal devices was first studied by Pfirsch and Schlüter[46] and later on extensively studied in more rigorous ways that combine both kinetic and fluid approaches[50]. In this section we show how we can reproduce, from our model, the particle transport coefficient in the isothermal Pfirsch-Schlüter regime, where the trapped population is negligibly small.

We first choose the transport ordering in the reduced system; that is,

$$\frac{\partial}{\partial \tau} \sim O(\eta\beta); \; F, \; v \sim O(\delta\beta); \; p \sim O(\beta); \; J, \psi \sim O(1).$$

One thus finds that, in axisymmetric system, pressure, electrostatic potential, and parallel flows are approximately equal to the equilibrium quantities described in Sec. 3.2.

$$p = \bar{p}(\psi) + O(\delta^2\beta^2), \tag{3.40}$$

$$\varphi = \bar{\varphi}(\psi) + O(\eta\beta), \tag{3.41}$$

and that the equilibrium parallel flows are given by

$$v = \bar{v}(\psi) + 2h\frac{\partial}{\partial\psi}(\bar{\varphi} + \delta\frac{T_i}{T_e}\bar{p}) + O(\eta\beta), \tag{3.42}$$

$$J = \bar{J}(\psi) - h(1 + \frac{T_i}{T_e})\frac{\partial}{\partial\psi}\bar{p} + O(\delta^2\beta^2). \tag{3.43}$$

Next, we extract the radial particle transport by flux surface averaging, $\langle\ \rangle_\psi$, the equation of particle conservation, just as in the usual neoclassical transport theory. Before doing so, we recall two important identities relating to flux surface averaging:

$$\langle \underset{\sim}{B}\cdot\underset{\sim}{\nabla}F\rangle_\psi = 0,$$

$$\langle \underset{\sim}{\nabla}\cdot\underset{\sim}{A}\rangle_\psi = \frac{d}{dV}\langle \underset{\sim}{A}\cdot\underset{\sim}{\nabla}V\rangle_\psi,$$

where $V(\psi)$ is the volume contained within the flux surface labelled by ψ. We therefore obtain the lowest order reduced consequence of these relations in axisymmetric system:

$$\langle \nabla_\parallel f\rangle_\psi = 0, \tag{3.44}$$

$$\langle [F,G]\rangle_\psi = \frac{1}{q}\frac{\partial}{\partial\psi}(q\langle G\nabla_\parallel F\rangle_\psi), \tag{3.45}$$

$$\langle \nabla^2 F\rangle_\psi = \frac{1}{q}\frac{\partial}{\partial\psi}(q\langle \underset{\sim}{\nabla}F\cdot\underset{\sim}{\nabla}\psi\rangle_\psi), \tag{3.46}$$

where q is the plasma safety factor. Then, by calculating $\langle$Eq. (2.73)$\rangle_\psi$, $\langle$Eq. (2.74)$\rangle_\psi$ and $\langle h\cdot$Eq. (3.43)$\rangle_\psi$, we obtain

$$\langle\frac{\partial}{\partial\tau}p\rangle = -\frac{1}{q}\frac{\partial}{\partial\psi}(q\Gamma^\psi),$$

$$\Gamma^\psi = -\eta_\perp\beta\,\frac{1+\frac{T_i}{T_e}}{2}\,(\frac{\partial}{\partial\psi}\bar{p})\langle|\nabla\psi|^2\rangle_\psi + 2\beta\,\langle h\nabla_\parallel(\varphi-\delta p)\rangle_\psi,$$

$$\langle h\nabla_\parallel(\varphi-\delta p)\rangle_\psi \sim \eta_\parallel\langle hJ\rangle_\psi,$$

$$\langle hJ\rangle_\psi \sim -\langle h^2\rangle_\psi(1+\frac{T_i}{T_e})\frac{\partial}{\partial\psi}\bar{p}.$$

Therefore, the radial flux has the form

$$\Gamma^\psi = -\eta_\perp\beta\,\frac{1+\frac{T_i}{T_e}}{2}\,(\frac{\partial}{\partial\psi}\bar{p})\{\langle|\nabla\psi|^2\rangle_\psi + \frac{4\eta_\parallel}{\eta_\perp}\langle h^2\rangle_\psi\};$$

where the simplest version of the last term in $\{\cdots\}$, due to the circular cross section shape of flux surface, reduces to the usual toroidal enhancement $(\frac{1}{q^2}+\frac{2\eta_\parallel}{\eta_\perp})$. Of course, ∇T effects are omitted as mentioned before.

3.6 Noncanonical Hamiltonian Formalism

3.6.0 Introduction

Although there has been a long history of noncanonical Hamiltonian theory in finite-degree-of-freedom systems, due to Lie, Dirac, and others, it was not until mid 1960's that this formalism was extended to continuous media, by Arnold.[51] In the past few years, the capability of this fast growing theory, of generally providing the nonlinear stability criteria, has drawn the attention of the plasma community. In particular, several reduced fluid descriptions, such as RMHD, Charney-Hasegawa-Mima equations (CHM)[52,53] and compressible RMHD (CRMHD)[17], have been proven to possess Hamiltonian structure of this type. Most significantly, the usual non-dissipative version of Hamiltonian theory has been extended to include the dissipation. In this section, we will study the Hamiltonian structure of DRMHD given in Eqs. (2.56)-(2.58).

The general concept of this formalism has been detailedly documented in many previous works.[54,56] We first briefly review the fundamental concept which leads to this formalism in subsection 3.6.1. Then, in subsection 3.6.2, we study the Hamiltonian structure and determine the Casimir invariants of

DRMHD. An isomorphism theory is adopted to simplify the procedure. Further detailed instability study of DRMHD, based on the discussions presented here, is subject to future research.

3.6.1 Generalized Hamiltonian Filed Theory

Conventional Hamiltonian description of a physical system of field variables χ^i (i=1,..,2N) is obtained by Legendre transforming a Lagrangian functional, which is constructed on physical bases. The dynamical system is thus governed by a Hamiltonian functional H and a set of differential equations with the following form

$$\frac{\partial}{\partial \tau}\chi^i = \{\chi^i, H\} = J^{ij}\frac{\delta}{\delta \chi^i}H, \qquad i = 1,\ldots,2N \tag{3.52}$$

where the Poisson bracket is given, for any functions F and G of the variables χ^i, by

$$\{F,G\} \equiv \langle F_i J^{ij} G_j \rangle . \tag{3.53}$$

Here, $\langle .. \rangle$ is a volume integral, F_i defines a functional derivative of F over the variable χ^i, i.e.,

$$F_i \equiv \frac{\delta}{\delta\chi^i}F; \tag{3.54}$$

and the 2N×2N matrix J^{ij} is

$$J^{ij} = \begin{cases} \delta_{i,2N+1-j}\,, & i \leqslant N \\ -\delta_{i,2N+1-j}\,. & i \geqslant N+1 \end{cases} \tag{3.55}$$

This is due to the splitting of 2N dynamical variables into N configuration components and their canonically conjugate momenta. It is well-known that this Poisson bracket satisfies the following algebraic relations, which are usually called the "Poisson structure",

(1) bilinearity,

(2) antisymmetry,

(3) Jacobi's identity : $\{F, \{G,E\}\} + \circlearrowleft$

$$\equiv \{F, \{G,E\}\} + \{G, \{E,F\}\} + \{E, \{F,G\}\} = 0, \tag{3.56}$$

(4) derivation : $\{F,GE\} = G\{F,E\} + \{F,G\}E$,

for arbitrary functionals F, G, E. Note that (1), (2) and (3) define a Lie algebra.

By definition, a transformation of coordinates that preserves the form of J^{ij} is called canonical. An arbitrary transformation, while preserving the above algebraic relations, can change the form of the Poisson bracket and lead to an obscure form of the Hamiltonian equations. Actually, equations of state of most of the continuous media described by means of Eulerian variables do not possess the canonical form. This therefore motivates the generalization of the Hamiltonian theory, which is defined in terms of the generalized Poisson brackets. That is, a dynamical system is Hamiltonian, in a generalized sense, if it is described by a set of equations which can be cast into the form of Eqs. (3.52)-(3.53) with J^{ij} which defines a generalized Poisson bracket, satisfying the Poisson structure, but which need not have the form of Eq. (3.55).

Another important feature of the noncanonical J^{ij}, distinct from the canonical one, is that it allows for a reducible singular phase space structure. That is, it can contain null eigenvectors which correspond to a special kind of kinematic invariants: Casimir invariants. This leads to changes of the phase space structure arising from the noncanonical transformation. The existence of the Casimir invariants confines the phase space trajectory to the hypersurfaces labeled by the Casimir invariants.

Note that Casimir invariants have been useful for finding the generalized free energy which is a good Liapunov functional; constructing the global nonlinear stability criteria; examining symmetry breaking for general equilibria; and formulating the generalized nonlinear energy principles. For detailed discussions, we refer reader to Ref.[29]. Here we only remark that the ambiguity of the usual energy principle, which approaches thermodynamic equilibrium by minimizing energy subject to some constant entropy or helicity, becomes clearer according to the noncanonical Hamiltonian theory. The point is, a more general class of equilibria can be obtained by minimizing the generalized free energy F which is the sum of the Hamiltonian functional and Casimir invariants. That is,

$$\frac{\delta}{\delta \chi^i} F = 0 \quad \text{for} \tag{3.57}$$

$$\frac{\partial}{\partial \tau}\chi^i = 0, \text{ where } F = H + C, \tag{3.58}$$

where H is energy functional and C is Casimir constant. Also important is that the definiteness of the second variation of this free energy, $\delta^2 F$, has been shown to be a sufficient condition for stability[29].

3.6.2 Hamiltonian Structure of DRMHD

The generalized Hamiltonian theory has special significance for reduced fluid models, mainly because many it can help preserve Hamiltonian structure which might otherwise be lost during the reduction process. Moreover, the understanding of the Hamiltonian structure of a reduced system can enable one to design numerical algorithms with superior accuracy. It is therefore the main task of this subsection to first prove that DRMHD possess a generalized Poisson bracket (GPB) and then determine its Hamiltonian structure and Casimir invariants. Further study on the stability condition of the system of DRMHD is subject to future research.

Before considering DRMHD, we first briefly study the Poisson structure of the reduced fluid system. Note that bilinearity and derivation are always true; while antisymmetry is equivalent to anti-self-adjointness of J^{ij} which is trivial. Hence, the remainder of the proof will be checking the Jacobi's identity.

As mentioned in Ref. [28], the Jacobi's identity Eq. (3.56) can be reduced to

$$\langle E_i J^{ij} \langle F_l Q_j^{lm} G_m \rangle\rangle + \uparrow = 0. \tag{3.59}$$

Here, higher order functional derivative terms are omitted due to the anti-self-adjointness of J^{ij} and the self-adjointness of F_{ij}. For most of the reduced fluid systems, such as RMHD, CRMHD, CHM, etc., J^{ij} can be written in a generic form

$$J^{ij} = a_k^{ij}[\ ,\chi^i] + b_k^{ij}D^k, \tag{3.60}$$

where a_k^{ij}'s and b_k^{ij}'s are constant, D^k's are spatial operators, such as $\frac{\partial}{\partial z}$ and $[h,\ ]$, which do not involve field variables. It is important to note here that the inner bracket [ ], defined in Eq. (2.10), and D^k themselves satisfy the Poisson structure. Also note that antisymmetry requires that

$$a_k^{ij} = a_k^{ji}, \qquad \text{and} \qquad b_k^{ij} = b_k^{ji}, \tag{3.61}$$

which are usually true for energy conserving reduced system. Therefore, after some straightforward manipulations, Jacobi's identity becomes

$$T_k^{ijl} \equiv a_k^{ij}a_m^{kl} = a_k^{lj}a_m^{ki}, \tag{3.62}$$

and

$$a_k^{ij}b_m^{kl} = a_k^{lj}b_m^{ki}, \tag{3.63}$$

for any j and m. Also note that Eqs. (3.61) and (3.62) together mean (i,j,l) are permutable in T_k^{ijl} for any k.

Let's now go back to DRMHD. From Eqs. (2.56)-(2.58) and (2.68), we can rewrite DRMHD in the form of Eq. (3.52) with the following J^{ij} :

$$J^{13} = J^{31} = [\ ,\chi^1], \qquad J^{23} = J^{32} = \nabla_{\|},$$

$$J^{33} = [\ ,\chi^3] + \delta\frac{T_i}{T_e}\underset{\sim}{\nabla}_\perp\cdot[\chi^1,\underset{\sim}{\nabla}_\perp\ ], \tag{3.64}$$

$$J^{ij} = 0, \qquad \text{otherwise.}$$

Here

$$\chi^i = (p,\ \psi,\ \nabla_\perp^2\varphi),$$

$$H_1 = -\ (1+\frac{T_i}{T_e})h + \delta\nabla_\perp^2(\varphi+\delta\frac{T_i}{T_e}p), \tag{3.65}$$

$$H_2 = -\ J, \quad \text{and} \quad H_3 = -\ \varphi + \delta p.$$

Except for the last term of J^{33}, which provides the ion diamagnetic drift effects, J^{ij} can be expressed in a generic form as in Eq. (3.60). We thus have

$$a_1^{13} = a_1^{31} = a_2^{23} = a_2^{32} = a_3^{33} = 1,$$

$$a_k^{ij} = 0, \qquad \text{otherwise.}$$

Also, by the observation that $\frac{\partial}{\partial z}$ always comes with $[\chi^2,\ ]$ in $\nabla_\parallel$, one finds that $b_2^{ij}=a_2^{ij}$, and $b_k^{ij}=0$, otherwise. It is then not difficult to prove that Eqs. (3.62)-(3.63) are satisfied. Hence, the generic portion of DRMHD possess a GPB.

Although it is possible to further prove the Jacobi's identity, with the inclusion of the last term of J^{33}, by a conventional but rather lengthy manipulation, we here present an isomorphism theory which not only straightforward proves the Jacobi's identity, but also makes the searching of Casimir invariants much easier.

For a mapping M of field variables $\chi \to \chi'$ with

$$\chi'^i = M^i_j \chi^j, \tag{3.66}$$

we have

$$F_i = \frac{\delta\chi'^j}{\delta\chi^i}\cdot\frac{\delta F}{\delta\chi'^j} = M^j_i F'_j. \tag{3.67}$$

The generic portion of the Poisson bracket thus becomes

$$\{F,G\}_g = \langle a^{ij}_k M^l_i F'_l [M^m_j G'_m, (M^{-1})^k_s\ \chi'^k] \rangle. \tag{3.68}$$

Here we have omitted the field independent part since, for reduced fluid models, it only due to $\nabla_\parallel$ and $[\ ,\ ]$.

The theorem is : If a system with generic bracket satisfies Jacobi's identity, then after an isomorphism, the system will still satisfy Jacobi's identity. This can be seen from Eqs. (3.59) and (3.66)-(3.68), which yield

$$\{E,\{F,G\}\} + \circlearrowleft$$
$$= \langle T^{ijk}_{s}(M^{-1})^{s}_{t}\, q^{\prime t}\, [M^{l}_{k}E^{\prime}_{l},[M^{m}_{i}F^{\prime}_{m},M^{n}_{j}G^{\prime}_{n}]]\rangle + \circlearrowleft . \qquad (3.69)$$

Therefore, if the original bracket is good, it means (i, j, k) are permutable for any s, then, by using the Poisson structure of the inner bracket, one can easily prove that Eq. (3.69) vanishes. That is, the new bracket is good, too. Note that similar isomorphism has been adopted to prove the Poisson structure of the 2-D gyroviscous MHD.

Our purpose is to eliminate the last term of J^{33} in Eq.(3.64); whence, the appropriate choice shall be

$$M^{i}_{j} = \delta^{i}_{j} + \frac{1}{2}\delta\frac{T_i}{T_e}\nabla^2_{\perp}\delta_{j3}\delta^{i1},$$

$$(M^{-1})^{i}_{j} = \delta^{i}_{j} - \frac{1}{2}\delta\frac{T_i}{T_e}\nabla^2_{\perp}\delta_{j3}\delta^{i1}.$$

We thus obtain the new bracket

$$J^{13} = J^{31} = [\ ,\chi^1], \qquad J^{23} = J^{32} = \nabla_{\parallel}, \qquad J^{33} = [\ ,\chi^3]$$

$$J^{ij} = 0, \qquad \text{otherwise}, \tag{3.70}$$

and new field variables

$$\chi^i = (p,\ \psi,\ \nabla_\perp^2(\varphi+\frac{1}{2}\delta\frac{T_i}{T_e}p)),$$

$$H_1 = -\ (1+\frac{T_i}{T_e})h + \delta\nabla_\perp^2(\varphi+\delta\frac{T_i}{T_e}p) + \frac{1}{2}\delta\frac{T_i}{T_e}\nabla_\perp^2(\varphi-\delta p), \tag{3.71}$$

$$H_2 = -\ J, \quad \text{and} \quad H_3 = -\ \varphi + \delta p.$$

Note that the energy functional is not changed, since it is a scalar.

We have shown that DRMHD is indeed a Hamiltonian system, the next step is thus to determine the Casimir invariants, which have vanishing GPB with any functional. That is, C_i's are null eigenvectors of J^{ij}. We start with Eqs. (3.70)-(3.71) and find that

$$[\chi^1, C_3] = 0, \qquad \nabla_\| C_3 = 0, \quad \text{and} \tag{3.72}$$

$$[C_1, \chi^1] + \nabla_\| C_2 + [C_3, \chi^3] = 0. \tag{3.73}$$

Let's define the general flux function ψ_x which satisfies

$$\nabla_\| \psi_x = 0$$

at all time. Note that in an axisymmetric system, $\psi_\chi = \psi$, while in a helical symmetry system, $\psi_\chi = \psi + (r^2)/(2q_0)$.

The only nontrivial solution of Eq. (3.72) is that both $p = \chi^1$ and C_3 are general functions. However, this nontrivial solution is not physical since it leads to the vanishing Hall effects. Hence, C_3 must be zero. We therefore obtain the only Casimir

$$C = \langle L(p,\psi_\chi)\rangle, \tag{3.74}$$

where L is arbitrary function with argument (p,ψ_χ).

Finally, we note here that after the isomorphism, the bracket of DRMHD reduces exactly to that of high-β RMHD. Hence, they have the same Casimir and the same bracket, but the different field variables and energy different functional.

3.7 Conclusions

In this chapter, several introductory topics have been discussed in a general manner so that further detailed study can readily follow. In particular, linear consequence of our model has been studied through a boundary layer analysis. It was shown that our model provides good FLR effects for a wide range of ρ_i. In section 3.6, the noncanonical Hamiltonian theory has been studied and applied to DRMHD, which is given in Eqs. (2.56)-(2.58). An isomorphism theory has been adopted to first show that DRMHD is Hamiltonian, in a generalized sense; and then the Casimir invariants has been found. Further instability study of DRMHD will be interesting. However, it is believed that incompressible description of drift-modes is not accurate. We remark that a drift, compressible, Hamiltonian model has been obtained by using the concept of isomorphism (work in progress).

CHAPTER IV

FINITE ION TEMPERATURE EFFECTS ON LINEAR TEARING MODES

4.1 Introduction

Magnetic field line tearing, which transfers the magnetic shear energy into the kinetic and thermal energy, has been a potential candidate for interpreting some of the astonishing phenomena observed in both magnetic fusion plasma (such as major disruption)[8] and astrophysical plasma (such as solar flares).[57]

After the pioneering work of Furth, Killeen and Rosenbluth[33] (FKR), in which incompressibility, vanishing ρ_i, and sheared slab geometry were imposed, there has been a vast literature studying the fusion oriented tearing modes. In particular, in Coppi, Greene and Johnson[47] (CGJ), plasma compression (due to finite β) has been found give stabilizing influences; while bad averaged curvature in cylindrical geometry is found destabilize the resistive interchange modes. Glasser, Greene and Johnson[32] (GGJ) then subsequently extent this work to the toroidal geometry and found that the good average curvature can stabilize the tearing modes.

However, for high temperature plasma, as in most of the present and future day machines, many non-ideal effects, such as diamagnetic drift effects, ion sound effects, and FLR effects, can importantly modify the dynamics inside the boundary layer. For instance, as found by Coppi and collaborators[58] in the low beta limit, diamagnetic drift can enlarge the resistive layer width and drastically reduce the tearing growth rate when the usual pure growing mode becomes the so-called drift-tearing mode with a real frequency near the electron diamagnetic drift frequency. Also, semi-collisional effects, as first termed by Drake and Lee,[48] induce a rather thin current channel centered on the rational surface by drastically reduce the parallel conductivity at x away from the rational surface; when ρ_i (or the effective ion gyroradius ρ_s) becomes larger than the resistive layer width. At the same time the drift effects and semi-collisional regime become significant, ion sound effects, arises from parallel compression, induce a long tail to the parallel conductivity and a finite fluctuation of the electrostatic potential (usually refers to as the adiabatic response), and hence provide stabilizing influence to the drift-tearing modes as found by Bussac[49].

Although there have been extensive studies on the drift tearing modes in the semi-collisional regime, yet, to our knowledge, there is a complete investigation on the toroidal semi-collisional drift-tearing modes with finite ion temperature. Recently, Hahm[59] has studied the semi-collisional drift-tearing mode with both perpendicular resistivity and ion sound effects, in the toroidal geometry;

while cold ion is assumed. Note that in practical, ion temperature is about the same as the electron temperature. As for the FLR effects to semi-collisional drift-tearing modes, as pointed out in another recent work by Hahm,[26] the treatment of FLR effects to tearing modes by Drake and Lee was incomplete due to the omission of the second order FLR term which is actually the same size as the cold ion semi-collisional term. Nevertheless, in this work, Hahm has assumed sheared slab geometry and neglect ion sound effects and ion viscosity. We remark that when $T_i \simeq T_e$, the viscous skin-depth becomes the same size as the resistive skin-depth.

In this Chapter, we therefore investigate the toroidal drift-tearing modes with finite ion temperature. The eigenmode equations, derived from the linearization of the reduced fluid model we obtained in the last chapter, retain ion sound, FLR, ion viscosity, and perpendicular resistivity effects. It is also noted that both the usual and the new FLR-modified geometrical factors are obtained. Hence, it is not surprising that this set of equations are found to agree with Hahm's eigenmode equations in both sheared slab geometry with FLR effects and toroidal geometry with cold ion, in the relevant limit. However, several important effects, such as temperature gradient[30], particle trapping[60], has already been omitted from the beginning of the last chapter. Also, the purely toroidal term H, will be ignored for simplicity. By solving the eigenmode equations, it is found that ion viscosity will importantly modify the tearing dynamics and thus the instability conditions as well as the tearing growth rate.

We now discuss the organization of this chapter. In Sec. 4.2, derivation of our eigenmode equations is present and their physical implications are discussed. Note that the linearization process, which starts with a two-dimensional eikonal, is basically following Ref.[41].

In Sec. 4.3, these eigenmode equations are analyzed by three conventional methods: (1) variational principle, (2) asymptotic matching, and (3) dimensional analysis (some time refers to as "dominant balance"). In 4.3.0, the general features of these methods are discussed. In 4.3.1, the system is investigated via the variational principle in sheared slab geometry. We note here that, since about a decade ago, variational principle has been widely used in studying tearing modes.[61,64] It has not only unified[61] many of the already existed tearing theories in various parameter regimes, but also explored many new modes due to complicated effects such as temperature gradients[62]. However, all of these variational treatments are restricted to the "single-scale" problems; while in this chapter, we try to extend the scheme to be able to deal with a two-scale problem, as when ion sound effects are comparable. In 4.3.2, the intuitive dimensional analysis is briefly studied, and carry out many of the physical insights related. In 4.3.3, the toroidal semi-collisional drift-tearing modes with perpendicular resistivity is investigated. Finally, the conclusions are given in Sec. 4.4.

4.2 Derivation of the linearized eigenmode equations

Linearization for boundary layer problem, in an axisymmetric toroidal system, can generally start with the expression described in Eq. (3.75); i.e.,

$$f \to f_0(r,\vartheta) + \hat{f}(r,\vartheta)\, e^{i(m\vartheta - nz - \omega t)},$$

which creates a two dimensional problem. Where the coordinates (r,ϑ,z) are defined in Sec. 3.3.1, such that r is a equilibrium flux label. The point is that when the interchange driving term, due to the toroidal curvature, becomes comparable to the line bending term in the shear-Alfvén law. Regarding the local equilibrium, we assume, as in section 3.3.3, that equilibrium flows are solely due to the diamagnetic effects. In a toroidal geometry with $\varphi_0 = 0$, this means,

$$v_0 = 2h\frac{q(r)}{\sqrt{g}}\frac{\partial}{\partial r}\, \delta\frac{T_i}{T_e}p_0,$$

$$J_0 = \nabla_\perp^2\psi_0(r) = h\frac{q(r)}{\sqrt{g}}(1 + \frac{T_i}{T_e})\frac{\partial}{\partial r}p_0.$$

Of course, the equilibrium pressure p_0 and toroidal component of vector potential ψ_0 are still flux functions, or say, dependent of r only.

Because of the ϑ-dependence of f_0 and $\hat{f}$, the linearized forms of the reduced operators can no longer be described by Eq. (3.76). Instead we have

$$[F,G] \rightarrow ik_\perp(F_{0r}\hat{G} - G_{0r}\hat{F}) + [\hat{F},G_0] + [F_0,\hat{G}], \tag{4.1}$$

$$\nabla_\parallel F \rightarrow ik_\parallel\hat{F} + ik_\perp F_{0r}\hat{\psi} + [F_0,\hat{\psi}] + \frac{\hat{F}_\vartheta}{q}, \tag{4.2}$$

$$[h,F] \rightarrow ik_\perp h_r\hat{F} + [h,\hat{F}]; \tag{4.3}$$

where $k_\perp$, $k_\parallel$ are as in Eq. (3.77), the safety factor q is given in Eq. (3.71), and the symbol "$\rightarrow$" simply means linearization. We note that the eikonal factor has already been suppressed on the RHS of the above relations; therefore, $[f,g]$ on the RHS means

$$\frac{1}{\sqrt{g}}\,(f_r g_\vartheta - f_\vartheta g_r). \tag{4.4}$$

The boundary layer orderings as well as the ancillary orderings described in section 3.3.3 are also adopted here, so that

$$\nabla_\perp^2\hat{F} \simeq |\nabla r|^2\hat{F}_{rr}\ (1 + O(w)). \tag{4.5}$$

The bracket with inner product form is no longer negligible, mainly due to ϑ-dependence of the equilibrium flows. In particular, we have

$$[\underset{\sim}{\nabla}_\perp(\varphi+\delta\frac{T_i}{T_e}p);\underset{\sim}{\nabla}_\perp\psi] \ \rightarrow\ -\frac{1}{2\sqrt{g}}(\hat{\varphi}+\delta\frac{T_i}{T_e}\hat{p})_r J_{o\vartheta} + O(w),$$

$$\underset{\sim}{\nabla}_\perp\cdot[\underset{\sim}{\nabla}_\perp\psi,v] \ \rightarrow\ -\frac{1}{\sqrt{g}}\ (v_{o\vartheta}\hat{J}_r - \frac{1}{2}\hat{v}_r J_{o\vartheta}) + O(w).$$

However, these terms are later found not to contribute to our final eigenmode equations.

We then suppress the carets from the perturbed amplitude and obtain a set of linearized equations

$$\begin{aligned}
&-i\omega p - ik_\perp p_{or}\varphi + [\varphi,p_o] \\
&\quad = -\beta\Big(ik_\parallel(v+2\delta J) + [(v_o+2\delta J_o),\psi] + \frac{(v+2\delta J)_\vartheta}{q}\Big) \\
&\qquad + 2\beta(ik_\perp h_r(\varphi-\delta p) + [h,(\varphi-\delta p)]) + \eta_\perp\beta\alpha|\nabla r|^2 p_{rr},
\end{aligned} \tag{4.6}$$

$$-i(\omega-\omega_e)\psi = -ik_\parallel(\varphi-\delta p) - \frac{(\varphi-\delta p)_\vartheta}{q} + \delta[p_o,\psi] + \eta_\parallel J, \tag{4.7}$$

$$-i\omega v + [\varphi,v_o] = -ik_\parallel\Upsilon - ik_\perp\alpha p_{or}\psi - [\alpha p_o,\psi] - \frac{\Upsilon_\vartheta}{q}$$

$$+\ 4\mu\beta|\nabla r|^2 v_{rr}\ -\ \frac{\delta\beta\frac{T_i}{T_e}}{2\sqrt{g}}(\varphi+\delta\frac{T_i}{T_e}p)_r J_{o\vartheta}, \tag{4.8}$$

$$-i(\omega-\omega_i)|\nabla r|^2\varphi_{rr}\ +\ \delta\frac{T_i}{T_e}[p_o,|\nabla r|^2\varphi_{rr}]\ -\ \mu\beta|\nabla r|^4(\varphi+\delta\frac{T_i}{T_e}p)_{rrrr}$$

$$=\ -\ (1+b_i)(ik_\parallel J\ +\ ik_\perp J_{or}\psi\ +\ [J_o,\psi]\ +\ \frac{J_\vartheta}{q})$$

$$-\ 2\alpha(1+b_i)(ik_\perp h_r p\ +\ [h,p])\ -\ \frac{1}{\sqrt{g}}\delta\beta\frac{T_i}{T_e}(v_{o\vartheta}J_r-\frac{1}{2}v_r J_{o\vartheta}); \tag{4.9}$$

where $\alpha\ \equiv\ \dfrac{1+\dfrac{T_i}{T_e}}{2}$ and Υ is defined as

$$\Upsilon\ \equiv\ \alpha p\ -\ \delta\beta\frac{T_i}{T_e}\nabla_\perp^2(\varphi+\delta\frac{T_i}{T_e}p),$$

which, after a rearrangement, can be written as

$$\Upsilon\ =\ \alpha(1+b_i)p\ -\ \delta\beta\frac{T_i}{T_e}\nabla_\perp^2(\varphi-\delta p). \tag{4.10}$$

Equation (4.10) will be useful for the calculation of the second term on the RHS of Eq. (4.9).

Equations (4.6)-(4.9) form a two dimensional problem. It is desirable to further simplify the mode equations to a set of coupled ordinary differential equations. In general, this can be achieved by a further Fourier decomposition of $\hat{f}(r,\vartheta)$ over poloidal angle. This represents a set of poloidal modes couplings for each fixed helical mode number n. However, this approach is complicated.

One important feature of the boundary layer problem is that the disturbances on each helical field line can be treated independently. This leads to a rather simple approach: taking the ϑ-average of the mode equations. The point is, for each fixed helical field line, the resonant Fourier component of $\hat{f}(r,\vartheta)$ corresponds to the eigenmode which minimizes the stabilizing line bending effects near the rational surface; therefore, the mode structure described by the equations of the resonant component dominates the shear-Alfvén dynamics in the layer interior. This component is obviously equal to the ϑ-average of $\hat{f}$, i.e.,

$$\bar{f} \equiv \langle f \rangle \equiv \oint \frac{d\vartheta}{2\pi} f,$$

$$\hat{f} = \bar{f}(r) + \tilde{f}(r,\vartheta).$$

The point is that, for each fixed helical number n, $\bar{f}$ corresponds to the disturbance which has minimum parallel gradient on the rational surface.

Before taking the ϑ-average, we first look at the linearized equations (4.6)-(4.9) and find that the lowest order non-resonant field variables are given by the lowest order terms of those equations; i.e.,

$$v_\vartheta = -2\frac{q}{\sqrt{g}}(\bar{\varphi}+\delta\frac{T_i}{T_e}\bar{p})_r h_\vartheta + O(w), \tag{4.11}$$

$$J_\vartheta = 2\frac{q}{\sqrt{g}}\alpha\bar{p}_r h_\vartheta + O(1), \tag{4.12}$$

$$\tilde{T} \sim O(w), \qquad \text{and therefore } \tilde{p} \sim O(w),\ \tilde{\varphi} \sim O(w^2)\cdot \tag{4.13}$$

$$\tilde{\varphi}-\delta\tilde{p} \sim O(w^2),$$

Here Eqs. (4.11)-(4.12) give the perturbed Pfirsch-Schluter flows.

Now, by using Eqs. (4.11)-(4.13), the fact that

$$\langle[\tilde{f},g]\rangle = 0,$$

and the equilibrium described in the early part of this section, the ϑ-average of Eqs. (4.6)-(4.9) becomes

$$-i\omega\bar{p} - ik_\perp p_{0r}\bar{\varphi}$$

$$= \beta\{2ik_\perp\bar{h}_r(\bar{\varphi}-\delta\bar{p}) + 2\langle[h,((\varphi-\delta p)-2\frac{q}{\sqrt{g}}\delta p_{0r}\psi)]\rangle - ik_\parallel(\bar{v}+2\delta\bar{J})\}$$

$$+ \eta_\perp \beta\alpha \langle |\nabla r|^2 \rangle \bar{p}_{rr} + O(w^2), \tag{4.14}$$

$$-i(\omega-\omega_e)\bar{\psi} = -ik_\parallel(\bar{\varphi}-\delta\bar{p}) + \eta_\parallel \bar{J} + O(w^3), \tag{4.15}$$

$$-i\omega\bar{v} = - ik_\parallel \alpha\bar{p} - ik_\perp \alpha p_{0r}\bar{\psi} + O(\beta) + O(w^2), \tag{4.16}$$

$$-i(\omega-\omega_i)\langle |\nabla r|^2 \rangle \bar{\varphi}_{rr} - \mu\beta \langle |\nabla r|^4 \rangle (\bar{\varphi}+\delta\frac{T_i}{T_e}\bar{p})_{rrrr}$$

$$= - (1+\langle |\nabla r|^2 \rangle \langle |\nabla r|^{-2} \rangle b_0)(ik_\parallel \bar{J}+2\alpha ik_\perp \bar{h}_r\bar{p})$$

$$- b_0 \langle |\nabla r|^{-2} \rangle (ik_\parallel \langle |\nabla r|^2 \tilde{J} \rangle)$$

$$- 2\alpha \langle [h \ , \ ((1+b_i)p+\frac{q}{\sqrt{g}}p_{0r}\psi)] \rangle + O(w), \tag{4.17}$$

where

$$b_0 \equiv - \frac{2\delta^2\beta}{\langle |\nabla r|^{-2} \rangle} \frac{T_i}{T_e} \frac{\partial^2}{\partial r^2}.$$

One notices that the toroidicity comes into the eigenmode equations through the terms involving nonvanishing $\langle \tilde{f}\tilde{g} \rangle$ which explicitly appear in Eqs. (4.14) and (4.17).

Let's first consider the nonresonant term in Eq. (4.14). By using the identity

$$\langle [\tilde{h}, \tilde{f}] \rangle = \frac{1}{\sqrt{g}} \langle \tilde{h}\tilde{f}_\vartheta \rangle_r$$

and the nonresonant part of Eq. (4.7), we find that

$$\langle [h\ ,\ ((\varphi-\delta p)-2\frac{q}{\sqrt{g}}\delta p_{or}\psi)]\rangle = \frac{q}{\sqrt{g}}\{i(\omega-\omega_e)\langle \tilde{h}\psi\rangle_r + \eta_{\parallel}\langle \tilde{h}J\rangle_r\}.$$

Then, by using Eqs. (4.5) and (4.12), we further find that

$$\bar{J} = \frac{1}{\langle|\nabla r|^{-2}\rangle}(\bar{\psi}_{rr} - 2\frac{q}{\sqrt{g}}\langle\frac{\tilde{h}}{|\nabla r|^2}\rangle\alpha\bar{p}_r),$$

$$\langle \tilde{h}\psi\rangle_r = 2\frac{q}{\sqrt{g}}\langle\frac{\tilde{h}^2}{|\nabla r|^2}\rangle\alpha\bar{p} + \langle\frac{\tilde{h}}{|\nabla r|^2}\rangle\frac{1}{\langle|\nabla r|^{-2}\rangle}(\bar{\psi}_r - 2\frac{q}{\sqrt{g}}\langle\frac{\tilde{h}}{|\nabla r|^2}\rangle\alpha\bar{p})$$

and

$$\langle \tilde{h}J\rangle_r = 2\frac{q}{\sqrt{g}}\langle\tilde{h}^2\rangle\alpha\bar{p}_{rr}.$$

Eq. (4.14) thus becomes

$$-i\omega\bar{p} - ik_{\perp}p_{or}\bar{\varphi} = -\beta ik_{\parallel}(\bar{v}+2\delta\bar{J}) + 2\beta ik_{\perp}\bar{h}_r(\bar{\varphi}-\delta\bar{p}) +$$

$$i(\omega-\omega_e)\beta\{\frac{4q^2}{g}(\langle\frac{\tilde{h}^2}{|\nabla r|^2}\rangle - \frac{\langle\frac{\tilde{h}}{|\nabla r|^2}\rangle^2}{\langle|\nabla r|^{-2}\rangle})\alpha\bar{p} + 2\frac{q}{\sqrt{g}}\langle\frac{\tilde{h}}{|\nabla r|^2}\rangle\frac{1}{\langle|\nabla r|^{-2}\rangle}\}\bar{\psi}_r$$

$$+ \alpha\beta(\eta_{\parallel}\langle\tilde{h}^2\rangle\frac{4q^2}{g}+\eta_{\perp}\langle|\nabla r|^2\rangle)\bar{p}_{rr} + O(w^2). \tag{4.18}$$

For the curvature term on the RHS of Eq. (4.17), by using Eq. (4.10), we have

$$\alpha\langle[h,((1+b_i)p+\frac{q}{\sqrt{g}}p_{or}\psi)]\rangle$$

$$=\frac{1}{\sqrt{g}}\langle h(\Upsilon+\alpha\frac{q}{\sqrt{g}}p_{or}\psi+\delta\beta\frac{T_i}{T_e}\nabla_\perp^2(\varphi-\delta p))_\vartheta\rangle_r. \qquad (4.19)$$

Then, by using the nonresonant parts of Eqs. (4.7)-(4.8) as well as Eq. (18), together with the identity

$$\delta\frac{T_i}{T_e}(\omega p-k_\perp p_{or}\varphi) = \omega(\varphi+\delta\frac{T_i}{T_e}p) - (\omega-\omega_i)\varphi,$$

the curvature term can be determined. The calculation is straightforward but rather lengthy, we therefore only give the resulted eigenmode equations

$$(1+\frac{2\beta\bar{h}_r}{p_{or}})(p_*-\varphi) = -\alpha\beta\frac{k_\parallel}{\omega}\psi - \frac{k_\parallel}{\omega_i}x_i^2\psi_{rr} + \alpha\beta\frac{k_\parallel{}^2}{\omega^2}p_* + \alpha\beta\frac{\eta_\parallel}{-i\omega}M_\perp p_{*rr}$$

$$- (1-\frac{\omega_e}{\omega})\beta K p_* + 2\delta\alpha\beta H(\frac{1-\frac{\omega}{\omega_e}}{1-\frac{\omega_i}{\omega_e}}\psi_r - \frac{k_\parallel}{\omega}p_{*r}), \qquad (4.20)$$

$$(1 - \frac{\omega_e}{\omega})\psi = \frac{k_\parallel}{\omega}(\varphi - \frac{\omega_e}{\omega}p_*) + \frac{\eta_\parallel}{-i\omega}\frac{\psi_{rr}-H\frac{\delta\omega}{\omega_e}p_{*r}}{\langle|\nabla r|^{-2}\rangle}, \qquad (4.21)$$

$$(1-\frac{\omega_i}{\omega})(M\varphi_{rr}-\frac{\mu\beta}{-i\omega}M_\mu\varphi_{rrrr})$$

$$= (1+M_J b_o)\frac{k_\parallel}{\omega}\psi_{rr} + \alpha\frac{k_\perp^2 p_{or}^2}{\omega^2}(1+(N+\frac{K_b}{K})b_o)Kp_*$$

$$+ \alpha\frac{\omega_e}{\delta\omega}H\{(1+Nb_o)\psi_r - (1+(M_J-\frac{H_b}{H})b_o)\frac{k_\parallel}{\omega}p_{*r}\}. \quad (4.22)$$

Where the overbars, which denote the resonant fields, are suppressed and p has been replaced by $p_* \equiv \frac{\delta\omega}{\omega_e}p$.

The geometrical factors are defined as follows:

$$K \equiv \frac{4q^2}{g}\alpha[\langle\frac{\tilde{h}^2}{|\nabla r|^2}\rangle\langle|\nabla r|^{-2}\rangle - \langle\frac{\tilde{h}}{|\nabla r|^2}\rangle^2] - \frac{2\bar{h}_r}{p_{or}}\langle|\nabla r|^{-2}\rangle;$$

$$K_b \equiv [\frac{1-\frac{\omega}{\omega_e}}{1-\frac{\omega_i}{\omega_e}}\frac{4q^2}{g}\langle\tilde{h}^2\rangle\alpha - \frac{2\bar{h}_r}{p_{or}}\langle|\nabla r|^2\rangle]\langle|\nabla r|^{-2}\rangle;$$

$$N \equiv \frac{1-\frac{\omega}{\omega_e}}{1-\frac{\omega_i}{\omega_e}}\frac{\frac{4q^2}{g}\langle\tilde{h}^2\rangle}{1+\frac{2\beta\bar{h}_r}{p_{or}}}\langle|\nabla r|^{-2}\rangle;$$

$$M \equiv [\langle|\nabla r|^2\rangle + \frac{4q^2}{g}\langle\tilde{h}^2\rangle]\langle|\nabla r|^{-2}\rangle]\langle|\nabla r|^{-2}\rangle;$$

$$M_J \equiv [\langle|\nabla r|^2\rangle + \frac{\frac{4q^2}{g}\langle\bar{h}^2\rangle}{1+\frac{2\beta\bar{h}_r}{p_{or}}}]\langle|\nabla r|^{-2}\rangle;$$

$$M_\mu \equiv [\langle|\nabla r|^4\rangle + \frac{4q^2}{g}\langle\bar{h}^2|\nabla r|^2\rangle]\langle|\nabla r|^{-2}\rangle;$$

$$M_\perp \equiv [\langle\bar{h}^2\rangle\frac{4q^2}{g} + \frac{\eta_\perp}{\eta_\parallel}\langle|\nabla r|^2\rangle]\langle|\nabla r|^{-2}\rangle;$$

$$H \equiv 2\frac{q}{\sqrt{g}}\langle\frac{\bar{h}}{|\nabla r|^2}\rangle; \qquad H_b \equiv 2\frac{q}{\sqrt{g}}\langle\bar{h}|\nabla r|^2\rangle\langle|\nabla r|^{-2}\rangle.$$

Here, K, measuring the averaged curvature, corresponds to the interchange driving term; its first term comes from the diamagnetic correction while the second term comes from the vacuum curvature. It is also important to note here that

$$\bar{h}_r = \frac{d\psi_o}{dr}\langle\frac{dR}{d\psi_o}\rangle\psi_o$$

actually measures the shift of the equilibrium flux surface. Recall that r is radial-like coordinate which labels the equilibrium flux surface. In the low beta Shafranov[65] geometry -- the shifted circular shape, this shift is known to be $O(\varepsilon)$. On the other hand, in high beta equilibrium, the flux surface is distorted from the circular shape, and the shift becomes larger than that in the low beta case. Also from

$$|p_{or}| \simeq \beta \frac{R_o}{L_n},$$

we see that vacuum curvature term becomes much larger than the diamagnetic correction in the low beta case in which $\beta \sim O(\varepsilon^2)$; while in high beta case, where $\beta \sim O(\varepsilon)$, it is still larger than the diamagnetic correction. This also implies that, for devices such as tokamaks, $K < 0$. In addition, for drift-type modes, the diamagnetic correction to K_b is apparently unimportant in either high beta or low beta case.

One also notices that the term $\frac{2\beta \bar{h}_r}{p_{or}}$ is much smaller than one and is thus negligible. We therefore hereafter take $1 + \frac{2\beta \bar{h}_r}{p_{or}} \to 1$. M_J thus reduces to M, the usual toroidal enhancement. The factor 16 in M_μ is due to the ratio of the parallel viscosity to the perpendicular viscosity.

The significance of the purely toroidal factor H, which measures the variation of B on the magnetic surface, is first found by GGJ to modify the usual tearing mode growth rate scaling from $\eta^{3/5}$ into $\eta^{(3-2H)/(5+2H)}$. Also, for resistive modes, it modifies the usual interchange driving term, with respect to K, into $K_R = K + \alpha H^2$.[32] However, since H is usually small, we hereafter neglect it to avoid the complications. We therefore obtain the simplified eigenmode equations

$$(1+\alpha\beta(\frac{x}{x_A})^2)p_* - \varphi = i\alpha\beta\frac{x}{x_A}\psi + i\frac{\omega}{\omega_i}x_i^2\frac{x}{x_A}\psi_{xx} + \alpha\beta x_R^2 M_\perp p_{*xx}, \qquad (4.23)$$

$$(1 - \frac{\omega_e}{\omega})\psi = -i\frac{x}{x_A}(\varphi - \frac{\omega_e}{\omega}p_*) + x_R^2\psi_{xx}, \qquad (4.24)$$

$$(1- \frac{\omega_i}{\omega})(\varphi_{xx} - x_\mu^2\varphi_{xxxx}) = -i(\frac{1}{M}+b_o)\frac{x}{x_A}\psi_{xx} - \frac{1}{x_A^2}(D+b_oD_i)p_*. \qquad (4.25)$$

Here

$$D \equiv \alpha\frac{k_\perp^2 p_{or}^2}{k_\parallel'^2}\frac{K}{M},$$

$$D_i \equiv \alpha\frac{k_\perp^2 p_{or}^2}{k_\parallel'^2}\frac{NK+K_b}{M},$$

and

$$x \equiv r - r_{mn}$$

is the normalized radial distance from mode rational surface.

Here, the normalized characteristic scale lengths are defined, basically following Ref.[41], as:

the shear-Alfven width

$$x_A \equiv \frac{-i\omega}{k_{\parallel}'};$$

the normalized ion gyroradius

$$x_i^2 \equiv \frac{2\delta^2\beta}{\langle|\nabla r|^{-2}\rangle}\frac{T_i}{T_e},$$

and the FLR operator b_o thus becomes

$$b_o \equiv -\, x_i^2 \frac{\partial^2}{\partial x^2}; \tag{4.26}$$

the normalized resistive skin-depth

$$x_R^2 \equiv \frac{\eta_{\parallel}}{-i\omega\langle|\nabla r|^{-2}\rangle} = \frac{m_e T_e}{2\beta m_i T_i}\frac{\nu_{ei}}{-i\omega}x_i^2; \tag{4.27}$$

similarly, the viscous skin-depth with toroidal modification

$$x_\mu^2 \equiv \frac{\mu\beta}{-i\omega}\frac{M_\mu}{M} = \frac{3M_\mu}{10M}\langle|\nabla r|^{-2}\rangle\frac{\nu_{ii}}{-i\omega}x_i^2. \tag{4.28}$$

We have derived a set of eigenmode equations which retains desirable non-ideal terms such as ion sound terms (terms involving $\alpha\beta$), FLR terms (terms involving x_i^2 or b_o), and interchange driving terms (terms involving D or D_i), all with toroidal modification. However, as mentioned before, some potentially important effects, such as which arises from temperature gradients and particle trapping, are omitted.

4.3 Derivations of the dispersion relation

4.3.0 Preliminary discussions

There are basically three methods for solving the eigenmode equations in the boundary layer problems.

(1) Variational principle: The basic concept of the variational formulation is based upon the self-adjoint property, with respect to the natural inner product

$$(f,g) \equiv \int_V d\underset{\sim}{x}\, fg,$$

of the eigenmode equations. That is, if the eigenmode equations can be expressed as

$$Lf = 0,$$

where f is the solution to the field equations, then the linear operator L is self-adjoint in the sense that

$$(h,Lg) = (g,Lh)$$

for any well behaved h and g. For the boundary layer problem, the integration is taken from $-\infty$ to ∞ in radial direction. We can then define a bilinear functional

$$K[h,g] \equiv (h,Lg),$$

so that

$$K[g,g] = 0 \quad \text{and} \quad \delta K[g,g] = 0 \qquad \text{at} \quad g = f.$$

Therefore, a proper trial function can be adopted for deriving the dispersion relations by taking the extremal value of the functional with respect to the variational parameters.

(2) Asymptotic matching: By separating the radial space (or the Fourier space) into several regions with respect to the characteristic scale lengths, mode equations can be simplified into analytically tractable differential equations in each region and the boundary conditions can be matched asymptotically at the boundary between two neighboring regions.

(3) Heuristic analysis: The mode widths may usually be approximately estimated via a simple dimensional analysis, or say, dominant balance; and then, by adopting the constant-ψ approximation, the dispersion relations can be qualitatively derived.

An obvious restriction on the methods (2) and (3) is that the so-called "subsidiary orderings"[32] concerning the relative magnitudes of ω, ν_{ei}, ρ_i etc., are needed before analytic solutions can be achieved. Note that these orderings usually relate to various classifications of modes such as "collisional" and "semi-collisional" regimes. By an appropriate choice of subsidiary orderings, complicated effects such as ion sound effects, perpendicular resistivity and interchange effects can be included simultaneously by methods (2) and (3). However, the procedure of method (2) is still rather complicated and usually requires further assumptions on the behavior of solutions, such as "nearly hydromagnetic" and "nearly adiabatic",[59] a priori.

On the other hand, the variational scheme allows for uniform analytical treatment over a relatively wide range of the radial variable x (or k_r), and the rather general and accurate dispersion relations can be obtained without a priori subsidiary orderings. Also, the procedure of the variational technique is much simpler than the rather lengthy and intricate matching process, and thus minimizes the opportunity for error.

Let's now go back to equations (4.23)-(4.25). We can see that the inclusion of the ion sound and perpendicular resistivity terms in Eq. (4.23), as well as the FLR terms, the viscosity term and interchange term in Eq. (4.25), make it impossible for these three equations to be combined into one equation. Through a Fourier transformation, the problem arises from the higher order differentiation can be obviated; nevertheless, new complication of the higher order differentiation in the Fourier space will appear, due to the ion sound terms with x^2 form in real space. In this case, semi-collisional resistive interchange modes, in which the ion sound effects are not important[59], can be studied.

On the other hand, for tearing modes, the main interest of this chapter, ion sound effects are known to be crucial[49] and responsible for the nearly adiabatic responses; and even the curvature effects can come into action only through the coupling with ion sound waves, in both collisional and semi-collisional regimes. It will be shown later that, without the perpendicular resistivity, the ion sound effects can be included in a variational formulation in sheared slab geometry. Therefore, in the next subsection, the sheared slab tearing modes are studied via the variational scheme.

For effects due to toroidicity, curvature, and perpendicular resistivity, which we ignore in subsection 4.3.1, they will be included in subsection 4.3.3, in which the asymptotic matching process will be adopted. However, due to the complication of the matching process, we shall restrict ourselves in the semi-collisional regime with strong ion-viscosity effects.

Finally, we must note here that method (3), although is rather rough, it always, by a proper choice of the subsidiary orderings, leads to qualitatively accurate solutions which usually differ from the exact solutions in only the numerical factors. Also note that this method is usually more physically insightful, and therefore more instructive. Hence, in subsection 4.3.2, we will briefly discuss many cases via the utilization of method (3).

4.3.1 Sheared-slab tearing modes with finite ion temperature

As has been pointed out by many authors[49], the ion sound effects are crucial to the drift-tearing modes. In this subsection, we therefore neglect toroidal curvature terms while emphasizing ion acoustic effects.

By using Eqs. (4.23) and (4.24) to eliminate p_*, and by ignoring the perpendicular resistivity, which is usually unimportant, we obtain a more generalized Ohm's law

$$\sigma(x)\cdot(\psi + i\frac{x}{x_A}\varphi) = \psi_{xx}, \tag{4.28}$$

Here,

$$\sigma(x) = \frac{1}{x_R^2} \frac{1 - \frac{\omega_e}{\omega} + \alpha\beta\frac{x^2}{x_A^2}}{1 + \alpha\beta\frac{x^2}{x_A^2} + \frac{x^2}{x_\sigma^2}} \tag{4.29}$$

is the generalized parallel conductivity, and

$$x_\sigma^2 = \frac{T_i}{T_e} \frac{x_R^2 x_A^2}{x_i^2} \tag{4.30}$$

is the width of the conductivity.

Moreover, through a Fourier transformation, Eq. (4.28) and Eq. (4.25), without the interchange driving terms, can merge into one self-adjoint differential equation, appropriate to the variational formulation. Now let's apply a Fourier transformation

$$\bar{f}(y) \equiv \int \frac{dx}{x_A} \; f(x) \; \exp(-i\frac{x}{x_A}y)$$

on Eq. (4.25) and Eq. (4.28). We find

$$\bar{\psi} - \frac{d}{dy}\bar{\varphi} = \int dy' \; \Lambda(y,y') \; \bar{J}(y'), \tag{4.31}$$

$$\bar{\varphi} = - G(y) \frac{d}{dy}\bar{J}; \tag{4.32}$$

where

$$G(y) \equiv \frac{1}{1- \dfrac{\omega_i}{\omega}} \frac{1 + \dfrac{x_i^2}{x_A^2}y^2}{y^2(1+ \dfrac{x_\mu^2}{x_A^2}y^2)}, \tag{4.33}$$

$$\wedge(y,y') \equiv \int \frac{dx}{2\pi x_A} \frac{1}{x_A^2\sigma(x)} \exp[-i\frac{x}{x_A}(y-y')], \tag{4.34}$$

and all the integrations are taken from $-\infty$ to ∞. Also,

$$\bar{J}(y) \equiv \int \frac{dx}{x_A} x_A^2\psi_{xx} \exp(-i\frac{x}{x_A}y) \tag{4.35}$$

corresponds to the Fourier transformed parallel current. Then, by eliminating $\bar{\varphi}$, Eqs. (4.31) and (4.32) yield

$$\bar{\psi} + \frac{d}{dy}(G(y) \frac{d}{dy}\bar{J}) = \int dy' \wedge(y,y') \bar{J}(y'), \tag{4.36}$$

which is self-adjoint due to Eqs. (4.35) and (4.34) as well as the fact that $G(y)$ and $\sigma(x)$ are even functions.

It is important to note that the boundary condition, which asymptotically matches the exterior (or ideal) solution and the interior solution, is embedded in $\bar{\psi}(y)$ due to the discontinuity of the slope of $\psi(x)$ on the boundary between interior and exterior regions; and it is usually defined by

$$\Delta' \equiv \left(\frac{d}{dx} \ln\psi(x)\right)_{0_-}^{0_+}, \tag{4.37}$$

when view from the exterior region. The explicit form of $\bar{\psi}(y)$ was generally derived by Hazeltine et.al[63] through Ampere's law and the boundary condition Eq. (4.37) with finite $k_\perp w$; here we only give the consequence of their result, in the small mode-width limit $k_\perp w \ll 1$,

$$\int dy\ \bar{\psi}(y)\bar{J}(y) = \frac{2\pi}{\Delta' x_A} - \int dy\ \frac{J^2(y)}{y^2}. \tag{4.38}$$

Therefore, we obtain the variational functional, from $\int dy\ (\bar{J}$ Eq. (4.36)),

$$K[f] = K_1[f] + K_2[f] + K_3[f], \tag{4.39}$$

where

$$K_1[f] \equiv \int dy\ \frac{f^2}{y^2}, \tag{4.40}$$

$$K_2[f] \equiv \int dy\ G(y)\ \left(\frac{d}{dy}f\right)^2, \tag{4.41}$$

$$K_3[f] \equiv \iint dy dy' \, \Lambda(y,y') \, f(y) f(y'); \tag{4.42}$$

so that

$$\delta K = 0 \quad \text{at } f = \bar{J},$$

$$\text{and } K(\bar{J}) = \frac{2\pi}{\Delta' x_A} \tag{4.43}$$

provides the dispersion relations. We note here that the variational quantity K here is constructed basically following Ref.[63]; however, it is more generalized in the sense that the FLR and viscosity effects are retained in $G(y)$ and the more generalized version of conductivity is given by $\Lambda(y,y)$.

For tearing-parity modes, the perturbed, Fourier transformed current can be represented by the trial function

$$f(\lambda) = \exp(-\tfrac{1}{2}\lambda y^2). \tag{4.44}$$

Here the variational parameter λ, which approximately measures the mode width

$$w \simeq x_A |\lambda|^{1/2},$$

must satisfy the consistency condition

$$\mathrm{Re}(\lambda) > 0. \tag{4.45}$$

Note that the magnitude of $f(\lambda)$ have been normalized against $\bar{J}(0)$.

By using Eq. (4.44) and Eq. (4.34), Eq. (4.42) becomes

$$K_3[f] = \frac{1}{\lambda x_A^2} \int \frac{dx}{x_A} \frac{1}{\sigma(x)} \exp(-\frac{x^2}{\lambda x_A^2}), \tag{4.46}$$

with which we can allow for even more complicated $\sigma(x)$, even involving the plasma dispersion function Z, as long as it is an even function. After a straightforward calculation, we obtain

$$K = -2(\pi\lambda)^{1/2}$$

$$+ \frac{x_R^2}{x_A^2}(1+\frac{x_A^2}{\alpha\beta x_\sigma^2})(\pi\lambda)^{1/2}\lambda^{-2}\{\lambda + (\frac{1}{\alpha\beta+\frac{x_A^2}{x_\sigma^2}} - \frac{1-\frac{\omega_e}{\omega}}{\alpha\beta})\cdot F(i(\frac{1-\frac{\omega_e}{\omega}}{\alpha\beta\lambda})^{1/2})$$

$$+ \frac{x_i^2}{(1-\frac{\omega_i}{\omega})x_\mu^2}(\pi\lambda)^{1/2}\lambda\{1 + \lambda(\frac{x_A^2}{x_i^2} - \frac{x_A^2}{x_\mu^2})\cdot F(i\frac{x_A}{x_\mu}\lambda^{1/2})\}. \tag{4.47}$$

Where

$$F(ix) \equiv \frac{1}{\pi^{1/2}} \int dy \, \frac{1}{y^2+x^2} \exp(-y^2)$$

$$= \frac{1}{ix}\left(Z(ix) - \pi^{1/2} i \exp(x^2) \right)$$

$$= \begin{cases} -2\sum_0^{\infty} \dfrac{(2x^2)^n}{(2n+1)!!} & \text{for } |x| < 1, \\[2ex] \dfrac{1}{x^2} \sum_0^{\infty} \dfrac{(2n-1)!!}{(-2x^2)^n} & \text{for } |x| > 1. \end{cases}$$

Note that equation (4.47) describes tearing modes of various regimes, corresponding to the relative magnitude of scale lengths x_i, x_R, x_A and x_μ, etc., except for the collisionless tearing modes which arise from the collisionless parallel conductivity.[61] In the rest of this subsection, we will study the dispersion relations of tearing modes (A) without viscosity and (B) with viscosity, in both collisional and semi-collisional regimes.

A. Inviscid plasma

The viscosity can be ignored whenever the inequality

$$\left|\lambda\frac{x_A^2}{x_\mu^2}\right| >> 1$$

is satisfied. We can further simplify K by first considering the limiting case

$$\left|\alpha\beta\lambda/(1-\frac{\omega_e}{\omega})\right| << 1,$$

which corresponds to the neglect of the ion sound effects. We then obtain

$$\frac{K_2+K_3}{\pi^{1/2}} = \frac{\lambda^{3/2}}{1-\frac{\omega_i}{\omega}} + \frac{\alpha\frac{T_i}{T_e}\frac{x_i^2}{x_A^2}}{(1-\frac{\omega_i}{\omega})(1-\frac{\omega_e}{\omega})}\lambda^{1/2} + \frac{\frac{x_R^2}{x_A^2}}{1-\frac{\omega_e}{\omega}}\lambda^{-1/2}. \qquad (4.48)$$

One finds that for

$$\text{(i)} \quad \left|\alpha\frac{T_e}{T_i}x_i^2\right| << \left|x_R x_A(1-\frac{\omega_i}{\omega})^{1/2}(1-\frac{\omega_e}{\omega})^{1/2}\right|, \qquad (4.49)$$

Eqs. (4.48) and (4.43) yield

$$\lambda = \frac{x_R}{x_A}\left(\frac{1-\frac{\omega_i}{\omega}}{3(1-\frac{\omega_e}{\omega})}\right)^{1/2} \qquad (4.50)$$

and the dispersion relation

$$\frac{\Delta'}{2\pi^{1/2}} = \frac{3^{3/4}x_A^{1/2}}{4x_R^{3/2}}\left(1-\frac{\omega_i}{\omega}\right)^{1/4}\left(1-\frac{\omega_e}{\omega}\right)^{3/4}, \tag{4.51}$$

which reproduces the classical collisional tearing mode. We note here that in order to satisfy the condition Eq. (4.44), the unstable root of Eqs. (4.50) and (4.51) must satisfy $\omega \gg \omega_e$ and therefore becomes a purely growing mode. This is consistent with Eq. (4.49) which corresponds to the low temperature regime where the collisionality is high and the diamagnetic frequency is low. Also noted here is the validity condition for Eqs. (4.50) and (4.51):

$$\left|x_\mu^2\right|, \ \left|\alpha\frac{T_e}{T_i}x_i^2\right| \ll \left|x_R x_A\right|; \tag{4.52}$$

where the first term on the LHS implies that the viscous skin depth is thinner than the classical resistive layer width. Finally, we remark that the irrelevance of drift-tearing mode in the high collisionality regime has been pointed out first by Rutherford and Furth[66], and then by many other authors.[64]

(ii) For $\left|\alpha\frac{T_e}{T_i}x_i^2\right| \gg \left|x_R x_A(1-\frac{\omega_i}{\omega})^{1/2}(1-\frac{\omega_e}{\omega})^{1/2}\right|$, (4.53)

there is one self-consistent root

$$\lambda = \frac{T_i}{\alpha T_e}\frac{x_R^2}{x_i^2}(1-\frac{\omega_i}{\omega}). \tag{4.54}$$

The dispersion relation thus becomes

$$\frac{\Delta'}{2\pi^{1/2}} = \frac{x_A}{2x_R x_i}(1-\frac{\omega_i}{\omega})^{1/2}(1-\frac{\omega_e}{\omega})\left(\frac{2\frac{T_i}{T_e}}{1+\frac{T_i}{T_e}}\right)^{1/2}, \tag{4.55}$$

with the validity conditions

$$\frac{|x_\mu^2|}{|x_A^2|} \ll \left|\frac{x_R^2}{\alpha x_s^2}(1-\frac{\omega_i}{\omega})\right| \ll \left|\frac{x_R^2(1-\frac{\omega_i}{\omega})}{x_A^2(1-\frac{\omega_e}{\omega})}\right|^{1/2},\ \frac{|1-\frac{\omega_e}{\omega}|}{\alpha\beta}. \tag{4.56}$$

Eq. (4.55) confirms Hahm's[26] result, obtained from a gyro-kinetic treatment in the semi-collisional regime, which is different from Drake and Lee's[48] (DL) result. This is due to the fact that DL kept FLR terms only to the first order while, for $\frac{T_i}{T_e} \simeq 1$, the second order FLR terms (due to x_i) are comparable to the cold-ion semi-collisional terms (due to the finite value of x_s). Here

$$x_s^2 \equiv \frac{T_e}{T_i} x_i^2 \tag{4.57}$$

refers to the effective ion gyroradius

$$\rho_s^2 \equiv \frac{T_e}{2T_i} \rho_i^2 .$$

However, we shall note here that for drift-type modes, $\omega \simeq \omega_e$, Eq. (4.56) is not really different from its cold-ion counter part.

It is also noticed that when the temperature increases from the collisional regime, the collision frequency become lower, the ion (or the effective ion) gyroradius become larger, while the drift wave broadening the resistive layer w_η characterized by

$$w_\eta \sim \left| \frac{x_R x_A}{1 - \frac{\omega_e}{\omega}} \right|^{1/2}$$

for drift-resistive modes. Consequently, we see that, also from Eq. (4.53), the semi-collisional regime, which occurs when the ion (or ion-sound) gyroradius becomes larger than the resistive layer, is much easier reached for drift-type mode.

Regarding the ion-sound effects on the resistive modes, perturbative treatment has been adopted in most of the previous variational calculations[64]. That is,

$$K(\lambda,\omega) = K^0(\lambda,\omega) + K^1(\lambda,\omega)$$

$$\omega = \omega^0 + \omega^1, \qquad \lambda = \lambda^0 + \lambda^1,$$

where (4.58)

$$\frac{\partial}{\partial\lambda}K^0(\lambda^0,\omega^0) = 0, \qquad K^0(\lambda^0,\omega^0) = 0$$

and $\frac{\omega^1}{\omega^0}, \frac{\lambda^1}{\lambda^0}, \frac{K^1}{K^0} \ll 1;$

then, the dispersion relation becomes

$$K(\lambda,\omega) = \omega^1\frac{\partial}{\partial\omega}K^0(\lambda^0,\omega^0) + K^1(\lambda^0,\omega^0) = 0.$$

Therefore,

$$\omega^1 = -\frac{K^1(\lambda^0,\omega^0)}{\frac{\partial}{\partial\omega}K^0(\lambda^0,\omega^0)}. \tag{4.59}$$

For our problem, we can retain the next order $(\alpha\beta\lambda)/(1-\frac{\omega_e}{\omega})$

expansion of K_3 and use the perturbation theory to calculate the correction of the mode frequency due to the ion-sound terms. Therefore, sufficient information about whether the ion-sound effects are stabilizing or destabilizing will be obtained, although the smallness of the ion-sound terms is presumed.

However, it will be more interesting to derive the general dispersion relations which retain the comparable ion-sound effects; therefore, the critical stability condition parameter Δ_c due to the ion-sound effects can be derived. It is also noted here that, according to Eq. (4.29), the ion-sound effects are negligible in the collisional regime where x_σ is very large while the drift type modes are irrelevant. We therefore consider the ion sound effects only in the semi-collisional regime.

The point is, in semi-collisional regime, one finds another consistent root

$$\lambda = \frac{1-\frac{\omega_i}{\omega}}{3\alpha\beta}\,\frac{x_s}{x_A} \tag{4.60}$$

from determining K_3 in the limit

$$\left|\frac{\alpha\beta\lambda}{1-\frac{\omega_e}{\omega}}\right| >> 1.$$

It implies that the real solution of Eq. (4.36) has two distinct length scales λ_1 and λ_2 which refer to Eq. (4.54) and Eq. (4.60), respectively. We remark here that these two length scales actually correspond to the so-called nearly hydromagnetic solution and nearly adiabatic solution, which will be discussed in the next two subsections where the dimensional analysis and the asymptotic matching process is utilized.

Conventional variational scheme will not be sufficient for dealing with this kind of multi-scale problem. However, with a minor modification, we find a two-scale variational scheme, leading to the dispersion relation

$$\frac{\Delta' x_A}{2\pi^{1/2}} = \frac{1}{K(\lambda_1)} + \frac{1}{K(\lambda_2)}. \tag{4.61}$$

This relation simply represents that the total discontinuity is the sum of those corresponding to the two disparate length scales. The derivation is as follows. If we assume a trial function which involves two distinct length scales,

$$f = A\exp(-\tfrac{1}{2}\lambda_1 y^2) + (1-A)\exp(-\tfrac{1}{2}\lambda_2 y^2), \tag{4.62}$$

where A and 1-A are due to the normalization against $\bar{J}(0)$. We thus have

$$K[f] = A^2 K(\lambda_1) + (1-A)^2 K(\lambda_2) + 2A(1-A) K(\lambda_1,\lambda_2). \tag{4.63}$$

If the two length scales are decoupled from each other, i.e., if the coupling term $K(\lambda_1,\lambda_2)$ is negligible, then, we find the best value of K[f], with respect to the variation of A,

$$K[f] = \frac{K(\lambda_1)K(\lambda_2)}{K(\lambda_1) + K(\lambda_2)}. \tag{4.64}$$

Hence, from Eqs. (4.43), this yields Eq. (4.61). It is consistent with Eq. (4.43) also provided that

$\dfrac{\partial K[f]}{\partial \lambda_i} = 0$ leads to

$\dfrac{\partial K(\lambda_i)}{\partial \lambda_i} = 0 \qquad i=1,2;$

Now consider the coupling term. Due to the fact that

$\lambda_2 >> \lambda_1$,

Eqs. (4.41) and (4.46) yield

$$K_2(\lambda_1,\lambda_2) = \frac{4\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2} K_2(\frac{\lambda_1+\lambda_2}{2}) \simeq \frac{4\lambda_1}{\lambda_2} K_2(\frac{\lambda_2}{2}),$$

$$K_3(\lambda_1,\lambda_2) = [\frac{4\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2}]^{1/2} K_3(\frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2}) \simeq 2[\frac{\lambda_1}{\lambda_2}]^{1/2} K_3(2\lambda_1).$$

We also note that K_1 is much smaller than K_2 and K_3. Therefore, the omission of the coupling term is appropriate for our problem.

Now, from Eqs. (4.55), (4.57) and (4.61), we finally have the dispersion relation for the drift-tearing modes with comparable ion-sound effects

$$\frac{\Delta'}{2\pi^{1/2}}$$

$$= \frac{x_A}{2x_R x_S}(\frac{1-\frac{\omega_i}{\omega}}{\alpha})^{1/2}(1-\frac{\omega_e}{\omega}) + \frac{(3\alpha\beta)^{3/4}x_A^{1/2}}{4x_S^{3/2}}(1-\frac{\omega_i}{\omega})^{1/4}, \quad (4.65)$$

with the validity condition Eq. (4.56) and

$$\left|\frac{1-\frac{\omega_e}{\omega}}{1-\frac{\omega_i}{\omega}}\right|,\quad \left|\frac{\alpha\beta x_\mu^2/x_A^2}{(1-\frac{\omega_i}{\omega})}\right|,\quad \left|\frac{\alpha\beta x_R/x_A}{(1-\frac{\omega_i}{\omega})^{1/2}}\right| \ll \left|\frac{\alpha\beta x_s^2}{(1-\frac{\omega_i}{\omega})x_A^2}\right|^{1/2} \ll \frac{T_e}{T_i}. \tag{4.66}$$

It is noticed that Eq. (4.66) prefers a colder ion, and in that limit, Eq. (4.65) reduces to Hahm's[59] result.

By using a Nyquist technique, one easily obtain the tearing modes instability condition

$\Delta' > \Delta_c$, where

$$\Delta_c = \frac{\pi^{1/2}(3\alpha\beta)^{3/4}x_A^{1/2}}{2^{1/2}x_s^{3/2}}(1-\frac{\omega_i}{\omega})^{1/4}. \tag{4.67}$$

For $\omega \simeq \omega_e$, it reduces to

$$\Delta_c = [\frac{27\pi^2}{2}]^{1/4}\ [\frac{L_s}{L_n}]^{1/2}\ \frac{\alpha\beta}{\rho_s}; \tag{4.68}$$

hence, the growth rate becomes

$$\gamma \sim \alpha\ [\frac{L_n m_e \nu_{ei}}{L_s m_i \omega_e}]^{1/2}\ \omega_e. \tag{4.69}$$

We then notice that parameter α, measuring the finite ion temperature, enhances the ion sound stabilizing effects by increasing Δ_C, while it also increases the growth rate when unstable region is reached.

However, the more important effects arise from the finite ion temperature are the ion viscous effects. According to Eqs. (4.27) and (4.28), with the maximal ordering for typical machines,

$$\frac{\nu_i}{\nu_{ei}} \sim \left[\frac{m_e}{m_i}\right]^{1/2} \sim \beta,$$

viscous skin-depth is found comparable to the resistive skin-depth, i.e.,

$$x_R \simeq x_\mu,$$

when $T_i \simeq T_e$. As a result, the validity conditions imposed by Eqs. (4.52), (4.56) and (4.66) become unlikely to be satisfied at the same time except for the case of very weak shear. It is therefore relevant to investigate the viscosity effects when T_i is comparable to T_e.

B. Viscous plasma

Let's now consider the case that ion viscosity has strong effects on the tearing modes. That is

$$\left|\frac{x_\mu^2}{x_A^2}\lambda\right| \gg 1, \qquad \text{which yields}$$

$$K_2 = \frac{\pi^{1/2}}{1-\frac{\omega_i}{\omega}}\frac{x_i^2}{x_\mu^2}\lambda^{3/2}\left(1-2\lambda\frac{x_A^2}{x_i^2}\right).$$

(i) For collisional regime, we find a consistent root

$$\lambda = \left(\frac{x_R^2 x_\mu^2\left(1-\frac{\omega_i}{\omega}\right)}{-10x_A^4\left(1-\frac{\omega_e}{\omega}\right)}\right)^{1/3}, \tag{4.70}$$

which yields the dispersion relation

$$\frac{\Delta'}{2\pi^{1/2}} = \frac{5x_A}{6x_R^2}\left(1-\frac{\omega_e}{\omega}\right)\left(\frac{x_R^2 x_\mu^2\left(1-\frac{\omega_i}{\omega}\right)}{-10x_A^4\left(1-\frac{\omega_e}{\omega}\right)}\right)^{1/6} \tag{4.71}$$

with the validity conditions

$$\frac{|x_s^2|}{|x_A^2|} \ll \left|\frac{x_R^4\left(1-\frac{\omega_e}{\omega}\right)}{x_A^2 x_\mu^2\left(1-\frac{\omega_i}{\omega}\right)}\right|^{1/3} \ll \left|\frac{x_R^2\left(1-\frac{\omega_e}{\omega}\right)}{x_A^2\left(1-\frac{\omega_i}{\omega}\right)}\right|^{1/2}. \tag{4.72}$$

This result agrees with CGJ when viscosity dominates inertia; in addition, we have retained the drift effects. One also notices that, unlike the inviscid case, the drift-type modes are possible in the strong-viscous collisional regime. However, ion sound effects are still negligible in this regime.

(ii) For the semi-collisional regime, as in the inviscid case, we find two consistent roots,

$$\lambda_1 = 2\frac{x_R^2}{x_S^2} \tag{4.73}$$

$$\lambda_2 = \left(\frac{T_e x_\mu^2}{3\alpha\beta T_i x_A^2}\left(1-\frac{\omega_i}{\omega}\right)\right)^{1/2}, \tag{4.74}$$

which yield the dispersion relation

$$\frac{\Delta'}{2\pi^{1/2}} = \frac{x_A}{2^{1/2}x_R x_S}\left(1-\frac{\omega_e}{\omega}\right) + \frac{(3\alpha\beta)^{3/4}x_A^{1/2}x_\mu^{1/2}}{4x_S^{3/2}x_i^{1/2}}\left(1-\frac{\omega_i}{\omega}\right)^{1/4}. \tag{4.75}$$

Here, the validity conditions are

$$\left|\frac{x_\mu^2}{x_A^2}\right|,\ \left|\frac{x_i^2}{x_A^2}\right| \quad \gg \quad \left|\frac{T_e x_\mu^2}{\alpha\beta T_i x_A^2}\left(1-\frac{\omega_i}{\omega}\right)\right|^{1/2}$$

$$>> \left|\frac{1-\frac{\omega_e}{\omega}}{\alpha\beta}\right| >> \left|\frac{x_R^2}{x_s^2}\right|. \qquad (4.76)$$

It is important to note that the first term on the RHS of Eq. (4.75), referring to the nearly hydromagnetic response, is not affected by the ion dynamics, such as ion viscosity and the finite ion temperature; while the second term, referring to the nearly adiabatic response, is modified by a factor $(x_\mu/x_i)^{1/2}$, which implies the enhancement of ion-sound stabilizing effects for the increase of ion viscosity. Therefore, the instability condition parameter Δ_c becomes

$$\Delta_c = \frac{(3\alpha\beta)^{3/4} x_A^{1/2} x_\mu^{1/2}}{4x_s^{3/2} x_i^{1/2}} (1-\frac{\omega_i}{\omega})^{1/4} \frac{\pi^{1/2}}{\sin(\pi/8)}$$

$$= \cos(\pi/8)\left[\frac{27\pi^2}{2}\right]^{1/4} \left[\frac{L_s}{L_n}\right]^{1/2} \frac{\alpha\beta}{\rho_s} \left[\frac{\nu_{ii}}{\omega_e}\right]^{1/4}. \qquad (4.77)$$

It is interesting to note here that the stability condition is now independent of the electron resistivity, as in the cold ion case. Also, the growth rate is proportional to $\nu_{ii}^{1/4}$.

In summary, the viscosity modifies both Δ_c and growth rate by a factor $(\nu_{ii}/\omega_e)^{1/4}$; therefore, ion temperature (and thus ion viscosity) could become a crucial control mechanism in high temperature plasma devices.

We have derived the dispersion relations for collisional and semi-collisional tearing modes in both viscous and inviscid cases, in the sheared slab geometry with the neglect of the perpendicular resistivity. Except for the strongly viscous case in the semi-collisional regime, the results agree with previously known results derived via quite different approaches. This further confirms the advantage and accuracy of the variational scheme.

4.3.2 Heuristic analysis

Before proceeding to the next subsection, in which a more rigorous calculation retaining the toroidal effects is made, it is instructive to briefly discuss method (3) which can easily lead to the same results we have derived, and help us understand some of the questions. For example, how do ion sound effects enter the mode action and how do the FLR effects mix with their cold-ion counter parts (the usual semi-collisional effects arises from the semi-collisional compression)? In addition, the concept of dominant balance we utilize here will also be the crucial key for simplifying the complicated differential equations, even in the deep-resistive region, in the next subsection.

We first begin with the Ohm's law Eq. (4.28), through which the dispersion relation is determined. That is,

$$\Delta' = \frac{1}{\psi(0)} \int_{-\infty}^{\infty} dx\ \sigma(x)\cdot(\psi + i\frac{x}{x_A}\varphi). \tag{4.78}$$

The integral is usually estimated by the "constant ψ" approximation along with the fact that the width of current channel is the minimum of width of parallel electric field $(\psi + i\frac{x}{x_A}\varphi)$ and width of parallel conductivity; i.e.

$$\Delta' \sim w(J)\sigma(w(J)), \quad \text{with} \tag{4.79}$$

$$w(J) \sim \mathrm{Min}(w(E),\ w(\sigma)). \tag{4.80}$$

The usual semi-collisional effects arises from the largeness of x_S, and therefore the smallness of x_σ which yields

$$w(J) \sim x_\sigma .$$

The result is the usual semi-collisional tearing mode as derived by DL.

However, the above estimation is apparently over simplifying in two senses:

(1) It ignores the ion sound contribution to the long tail of $\sigma(x)$. The point is, although $\alpha\beta x_\sigma^2/x_A^2$ is a very small quantity, for drift type modes, $(1-\frac{\omega_e}{\omega})$ is also a very small quantity. This is precisely the reason why ion sound effects is important only for drift type modes.

(2) It assumes the smallness of the electrostatic potential fluctuation φ. Note that for the hydromagnetic response, which has length scale not larger than the resistive layer, φ_A is indeed very small except for the inviscid case with ρ_i larger than the scale length. However, there is another type of response which has a larger φ with length scale larger than the resistive layer. This can be seen from a simple dominant balance study of mode equations.

Let's take example of the strong viscous case with $\rho_i >>$ w. We have, from the vorticity equation,

$$(1-\frac{\omega_i}{\omega})x_\mu^2\varphi_{xx} \sim -ix_i^2\frac{x}{x_A}\psi_{xx}. \qquad (4.81)$$

Then, by inserting it into the Ohm's law Eqs. (4.24), with the subsidiary ordering

$x_i \sim x_R \sim x_\mu >> x_A$, we have

$$\frac{x^2}{x_A^2}(\varphi - \frac{\omega_e}{\omega}p_*) \sim \frac{x_R^2}{x_i^2}(1- \frac{\omega_i}{\omega})x_\mu^2\varphi_{xx}. \tag{4.82}$$

With a quick observation on this equation, we find that two distinct length scales w_A, w_B, are possible: w_A, which is thinner than the resistive layer, leads to $\varphi_A << p_A$; while w_B, which is thicker than the resistive layer, leads to $\varphi_B \sim \frac{\omega_e}{\omega}p_B$. Then, by using this information, obtained from the vorticity equation and Ohm's law, we can easily estimate the two length scales, from Eq. (4.23). The results agree with Eqs. (4.73) and (4.74) perfectly. Note that the first one refers to as hydromagnetic response; while the second one refers to as adiabatic response.

It is also clear from the calculation that, in semi-collisional regime, the adiabatic response is due to the dominant balance of the semi-collisional compression and the ion sound term. It is important to note here that, from dominant-balance point of view, the semicollsional regime refers to the parameter regime in which the semi-collisional compression is dominating the equation of density evolution.

To understand why Eq. (4.55) is different from DL's result, let's also briefly discuss the semi-collisional inviscid case, in which

x_{μ}, $x_i << w$;

therefore, vorticity equation becomes

$$(1- \frac{\omega_i}{\omega})\varphi = ix_i^2 \frac{x}{x_A}\psi_{xx}. \tag{4.83}$$

It implies that semi-collisional compression is now comparable to the convective term in Eq. (4.23). The increase of the fluctuation of electrostatic potential φ is apparently due to the finite ion gyromotion during which ion will see the change of φ. (For viscous case, however, φ is again smoothed out by the viscosity). Then, by inserting this relation into Eqs. (4.23) and (4.24), one easily finds that

$$\psi_{xx} \simeq \frac{1}{x_R^2} \frac{1- \frac{\omega_e}{\omega}}{1 + \frac{2x^2}{x_A^2 \lambda}} \psi; \tag{4.84}$$

where λ is as in Eq. (4.54). Therefore, w(J) is now

$[(1-(\omega_i)/(\omega))/2\alpha]x_\sigma$ rather than x_σ. Hence the resulting dispersion relation differs from Eq. (4.55) only in the numerical factor. Also note that the validity of Eq. (4.65) depends upon the inequality $\lambda_1 \ll x_i^2 \ll \lambda_2$; therefore, when estimating the length scale of the adiabatic response in that regime, one must adopt

$$(1-\frac{\omega_i}{\omega})\varphi_{Bxx} \sim -i\frac{x}{Mx_A}\psi_{xx}, \tag{4.84}$$

rather than Eq. (4.83).

We can now write down the more accurate form of the dispersion relation, for semi-collisional regime,

$$\Delta' \sim w_A\frac{1-\frac{\omega_e}{\omega}}{x_R^2} + \frac{\sigma(w_B)}{X_A}\int_0^\infty dx\,\frac{ix\varphi_B^1}{\psi(0)}. \tag{4.85}$$

It is important to note here that in Eq. (4.85), φ_B^1 means the "nearly adiabatic response"; because, according to Eq. (4.24), the exact adiabatic response will not contribute to the dispersion relation. The "nearly adiabatic response" can be obtained from the vorticity equation. Let's consider the viscous semi-collisional regime, with the inclusion of average curvature term D. Provided that

$$\varphi_B^0 \sim \frac{\omega_e}{\omega} p_B^0 \sim \frac{x_A^2 x_S^2}{\alpha\beta w_B^2}[-i\frac{x}{x_A}\psi_{xx}],$$

the vorticity equation becomes

$$(1-\frac{\omega_i}{\omega})\frac{x_\mu^2}{w_B^2}\varphi \sim -ix_i^2\frac{x}{x_A}\psi_{xx}[1 - D_i\frac{\omega x_S^2}{\omega_e\alpha\beta w_B^2}]$$

$$\sim -ix_i^2\frac{x}{x_A}[1 - D_i\frac{\omega x_S^2}{\omega_e\alpha\beta w_B^2}]\cdot\sigma(w_B)(\psi + i\frac{x}{x_A}\varphi).$$

w_B then is estimated from

$$w_B = [(1-\frac{\omega_i}{\omega})\frac{x_\mu^2 x_A^2}{x_i^2\sigma(w_B)}]^{1/4},$$

where

$$\sigma(w_B) \simeq \frac{\alpha\beta}{x_S^2}.$$

Eq. (4.85) thus yields

$$\Delta' \sim w_A\frac{1-\frac{\omega_e}{\omega}}{x_R^2} + \sigma(w_B)w_B[1 - D_i\frac{\omega x_S^2}{\omega_e\alpha\beta w_B^2}]. \qquad (4.86)$$

This result will be found in agreement with the result from a rigorous calculation in the next subsection.

Furthermore, we remark that Eq. (4.85) is also relevant for other cases such as those which include perpendicular resistivity and curvature effects. For instance, one finds that the importance of the perpendicular resistivity, in the semi-collisional regime, arises when semi-collisional compression is balanced against perpendicular diffusion. In this case, the narrow width of conductivity disappears, and the length scale becomes

$$w_A \sim x_\sigma \left[\alpha\beta M_\perp \frac{x_s^2}{x_A^2}\right]^{1/4}.$$

Finally, we remark that the similar procedure can be taken to estimate the length scales for each parameter regime, as has been discussed by many other authors.[41] Here, we only write down, without showing the procedure, the general expression of the dispersion relation for collisional regime, in which, there is only one length scale w and the ion sound effects are negligible. That is

$$\Delta' \sim w\sigma + \frac{D}{w}.$$

With this expression, for both inviscid and viscous cases, the results agree with wellknown results derived by CGJ[47].

4.3.3 Toroidal semi-collisional drift-tearing modes with perpendicular resistivity and ion viscosity effects

Although the dispersion relation in this parameter regime has been approximately estimated in the last subsection, it is always desirable to solve it in a more rigorous way. We have shown that it is very difficult for a variational principle to include the ion sound effects together with the perpendicular resistivity and the interchange driving force at the same time. Therefore, in this subsection, we use the usual asymptotic matching process to solve for viscous semi-collisional tearing modes. For inviscid semi-collisional case, we remark that finite T_i will only enhances the usual cold ion curvature effects by a foator of 2α. This can be understood from Eqs. (4.65)-(4.69) and Eq. (4.86).

Due to the higher order differential form of the eigenmode equations (4.23)-(4.25), it is obviously easier to study them in the Fourier space. Through the same Fourier transformation in subsection 4.3.1, we obtain

$$(1+\alpha\beta\frac{x_R^2}{x_A^2}M_\perp y^2)p_* - \alpha\beta p'' = \varphi - \alpha\beta\psi' + \frac{\omega x_i^2}{\omega_i x_A^2}(y^2\psi)', \tag{4.87}$$

$$(1 - \frac{\omega_e}{\omega})\psi = (\varphi - \frac{\omega_e}{\omega}p_*)' - \frac{x_R^2}{x_A^2}y^2\psi, \tag{4.88}$$

$$(1- \frac{\omega_i}{\omega})y^2(1+\frac{x_\mu^2}{x_A^2}y^2)\varphi = (\frac{1}{M}+\frac{x_i^2}{x_A^2}y^2)(y^2\psi)' + (D+\frac{x_i^2}{x_A^2}y^2D_i)p_*. \tag{4.89}$$

Before proceeding to solve the above equations, we shall note here that the appropriate subsidiary orderings for semi-collisional drift-tearing modes, with strong ion viscosity, are chosen to be

$$x_\mu^2 \sim x_R^2 \sim x_i^2 > \frac{x_A^2}{\beta}.$$

This is suggested by Eqs. (4.75) and (4.76), with $\omega \sim \omega_e$ and the assumption that the two terms on the RHS of Eq. (4.75) are comparable. In addition, to keep the finite interchange driving force, we further assume

$$D \sim D_i \sim \frac{x_\mu x_A}{x_i^2}\beta^{1/2}.$$

We now can solve the eigenmode equations by conveniently separating the Fourier space into three asymptotic regions:

(1) ideal MHD region, where $y^2 \ll \frac{x_A^2}{x_R^2}(1-\frac{\omega_e}{\omega})$;

(2) intermediate region, where $y^2 \sim \frac{x_A^2}{x_R^2}(1-\frac{\omega_e}{\omega})$;

(3) deep resistive region, where $y^2 \gg \frac{x_A^2}{x_R^2}(1-\frac{\omega_e}{\omega})$.

For region (1), the eigenmode equations reduce to

$$p_*'' = \psi', \qquad (y^2\psi)' = 0, \qquad \text{and}$$

$$(1-\frac{\omega_e}{\omega})\psi = (\varphi - \frac{\omega_e}{\omega}p_*)'.$$

The large-y behavior is then given by[hahm46]

$$\varphi = p_* = 1 + \frac{\Delta' x_A}{\pi}\frac{1}{y},$$

where Δ', related to the gradient of current profile, is a parameter given by the outside ideal MHD solutions.

For region (2), the resistive effects appear, we have

$$p_*'' = \psi', \qquad (y^2\psi)' = 0, \qquad \text{and}$$

$$(1 - \frac{\omega_e}{\omega} + y^2\frac{x_R^2}{x_A^2})\psi = (\varphi - \frac{\omega_e}{\omega}p_*)'.$$

By eliminating ψ, these equations further reduce to two ordinary differential equations which has the general solutions

$$\varphi = (-\frac{1}{y} + \frac{x_R^2}{x_A^2}y)C_1 - \frac{\omega_e}{\omega}yC_2 + C_3, \tag{4.90}$$

$$p_* = -\frac{C_1}{y} - yC_2 + C_4; \tag{4.91}$$

where C_i's are constants to be determined by asymptotically matching the small-y behaviors of the solutions in region (2) with the large-y behaviors of the solutions in region (1). We easily obtain,

$$C_1 = -\frac{\Delta' x_A}{\pi}, \qquad \text{and } C_3 = C_4 = 1. \tag{4.92}$$

For region (3), where the inertial, perpendicular resistivity, and other desirable non-ideal effects appear, the eigenmode equations become

$$(1 + \alpha\beta\frac{x_R^2}{x_A^2}M_\perp y^2)p_* - \alpha\beta p_*'' = \frac{\omega x_\mu^2}{\omega_i x_A^2}(1 - \frac{\omega_i}{\omega})y^2\varphi, \tag{4.93}$$

$$(\varphi - \frac{\omega_e}{\omega}p_*)'' = \frac{x_R^2 x_\mu^2}{x_A^2 x_i^2}(1 - \frac{\omega_i}{\omega})y^2\varphi - \frac{x_R^2}{x_A^2}D_i p_*, \tag{4.94}$$

Where ψ has been eliminated through the line bending term of the reduced vorticity equation.

We note that for semi-collisional modes, the term on the RHS of Eq. (4.93), which arises from the semi-collisional compression, is expected to dominate in the deeply resistive region. Also, by defining

$$\lambda^2 \equiv 4\frac{T_e x_\mu^2}{T_i x_A^2}\frac{1-\frac{\omega_i}{\omega}}{\alpha\beta} \gg \frac{1}{\beta^2},$$

Eq. (4.94) can be rewritten as

$$(\varphi - \frac{\omega_e}{\omega}p_*)'' = \alpha\beta\frac{x_R^2}{4x_S^2}\lambda^2 y^2\varphi - \frac{x_R^2}{x_A^2}D_i p_*. \tag{4.95}$$

We therefore find two types of solutions, with respect to two different length scales as found in the variational treatment.

(1) Nearly hydromagnetic solution $\quad (y^4 > \frac{1}{\beta\lambda^2})$

Therefore,

$$\varphi \sim -\frac{4}{\alpha\beta\lambda^2 y^2}\frac{x_S^2}{x_R^2}\frac{\omega_e}{\omega}p_*'' \ll p_*,$$

which, with Eq. (4.93), yields

$$(1 + \alpha\beta\frac{x_R^2}{x_A^2}M_\perp y^2)p_* = \frac{x_S^2}{x_R^2}p_*''. \tag{4.96}$$

The solution of Eq. (4.96) which decays as $y \to \infty$ is parabolic cylinder function,

$$p_A = U(\Lambda_\perp, \lambda_\perp^{1/2} y), \tag{4.97}$$

which has the small-y behavior

$$p_A = p_A(0)\left(1 - \frac{\Gamma(3/4 + \Lambda_\perp/2)}{\Gamma(1/4 + \Lambda_\perp/2)}(2\lambda_\perp)^{1/2} y\right); \tag{4.98}$$

where Γ is the Gamma function,

$$\lambda_\perp^2 \equiv 4\alpha\beta \frac{x_R^4}{x_A^2 x_S^2} M_\perp, \tag{4.99}$$

$$\Lambda_\perp^2 \equiv \frac{x_A^2}{4\alpha\beta x_S^2 M_\perp}. \tag{4.100}$$

(2) Nearly adiabatic solution $(y^4 \sim \lambda^{-2} \ll 1)$

Therefore,

$$\alpha\beta \frac{x_R^2}{x_i^2} \lambda^2 y^4 \sim \beta \ll 1,$$

which, together with Eq. (4.95), implies an adiabatic response

$$\varphi \sim \frac{\omega_e}{\omega} p_* .$$

Hence, Eq. (4.93) becomes

$$4p_*'' = \lambda^2 y^2 p_*, \tag{4.101}$$

which has solution

$$p_B^0 = U(0, \lambda^{1/2} y).$$

That the dispersion relation will be derived via the asymptotic matching of the small-y behavior of the solution in region (3) with the large-y behavior of solution in region (2) that

$$(\varphi - \frac{\omega_e}{\omega} p_*)' \simeq - \frac{\Delta' x_A x_R^2}{\pi \; x_A^2}. \tag{4.102}$$

We thus require the higher order solution, i.e., the nearly adiabatic response, which can be determined from Eq. (4.95). We have

$$(\varphi_B^1 - \frac{\omega_e}{\omega} p_B^1)'' = \alpha\beta \frac{x_R^2}{4x_S^2} \lambda^2 y^2 \varphi_B^0 - \frac{x_R^2}{x_A^2} D_i p_B^0.$$

Through a direct integration, the small-y behavior is given by

$$(\varphi_B^1 - \frac{\omega_e}{\omega} p_B^1)' = [- \frac{\alpha\beta\omega_e x_R^2}{4\omega x_S^2} \lambda^{1/2} I_2 + \frac{x_R^2}{x_A^2} D_i I_0 \lambda^{-1/2}] p_B^0(0); \tag{4.103}$$

where

$$I_n \equiv \int_0^\infty dy\, y^n \frac{U(0,y)}{U(0,0)}.$$

By using the identities

$$U(0,x) = \left(\frac{x}{2\pi}\right)^{1/2} K_{1/4}(x^2/4),$$

$$\int_0^\infty dx\, x^\mu K_\nu(x) = 2^{\mu-1}\, \Gamma\left(\frac{\mu+\nu+1}{2}\right) \Gamma\left(\frac{\mu-\nu+1}{2}\right),$$

and the recursion relations of parabolic cylinder function, we find

$$I_2 = \frac{4}{\pi}\,(\Gamma(3/4))^2 = (32)^{1/2}\,\frac{\Gamma(3/4)}{\Gamma(1/4)},$$

$$I_0 = \left(\frac{\pi}{2}\right)^{1/2};$$

where K_ν is the modified Bessel function.

Now, the general solutions in the deeply resistive region are

$$p_* = p_A + p_B^0 + p_B^1,$$

$$\varphi = \frac{\omega_e}{\omega} p_B^0 + \varphi_B^1;$$

hence, $p_A(0)$ and $p_B^0(0)$ can be determined by matching with Eqs. (4.90) - (4.92), we have

$$p_A(0) = 1 - \frac{\omega}{\omega_e},$$

$$p_B^0(0) = \frac{\omega}{\omega_e}.$$

Eqs. (4.100), (4.102) and (4.103) thus yield

$$\frac{\Delta'}{\pi} = (1- \frac{\omega_e}{\omega})\frac{x_A \Gamma(3/4 + \Lambda_\perp/2)}{x_R^2 \Gamma(1/4 + \Lambda_\perp/2)}(2\lambda_\perp)^{1/2}$$

$$+ [\frac{\alpha\beta x_A}{4x_S^2}\lambda^{1/2} I_2 - \frac{\omega}{\omega_e x_A} D_i I_0 \lambda^{-1/2}]. \qquad (4.104)$$

The first term on the RHS refers to the nearly hydromagnetic response, while the second and the third terms refer to the nearly adiabatic response. Also note that this expression agrees with Eq. (4.86) in the last subsection, where w_A corresponds to $\lambda_\perp$ while w_B corresponds to λ. Also, since the first term is a purely electron response, it is not surprising that it agrees with Hahm's cold ion result.

The importance of perpendicular resistivity apparently depends on the magnitude of the parameter $\Lambda_\perp$. Let's consider two limiting cases:

(1) For small perpendicular diffusion, $\Lambda_\perp \gg 1$, i.e.,

$$M_\perp \simeq \left(\frac{\eta_\perp}{\eta_\parallel} + 2q^2\right) \ll \frac{x_A^2}{4\alpha\beta x_S^2} \simeq \frac{L_S^2}{4\alpha L_n^2}, \qquad \text{for } \omega \sim \omega_e$$

Eq. (4.104) reduces to

$$\frac{\Delta'}{\pi} = \left(1-\frac{\omega_e}{\omega}\right)\frac{x_A}{x_R x_S} + 2\frac{(\alpha\beta)^{3/4} x_A^{1/2} x_\mu^{1/2}}{x_S^{3/2} x_i^{1/2}}\left(1-\frac{\omega_i}{\omega}\right)^{1/4}\frac{\Gamma(3/4)}{\Gamma(1/4)}$$

$$- \frac{\omega}{\omega_e} D_i \left[\frac{\pi}{4x_A x_\mu}\right]^{1/2}\left[\frac{\alpha\beta}{1-\frac{\omega_i}{\omega}}\right]^{1/4}. \qquad (4.105)$$

As expected, this expression agrees with the variational calculation Eq. (4.75) in the sheared slab system, and the intuitive dimensional analysis result Eq. (4.86).

The instability condition with both ion sound and curvature effects is

$$\Delta_c = \pi\cos(\pi/8)\frac{(4\alpha\beta)^{3/4} x_A^{1/2} x_\mu^{1/2}}{x_S^{3/2} x_i^{1/2}}\left(1-\frac{\omega_i}{\omega}\right)^{1/4}\frac{\Gamma(3/4)}{\Gamma(1/4)}$$

$$- \pi\sin(\pi/8)\frac{\omega}{\omega_e} D_i \left[\frac{\pi}{2x_A x_\mu}\right]^{1/2}\left[\frac{\alpha\beta}{1-\frac{\omega_i}{\omega}}\right]^{1/4}$$

$$\sim \left(\frac{L_s}{L_n}\right)^{1/2} \frac{\alpha\beta}{\rho_s} \left(\frac{\nu_{ii}}{\omega_e}\right)^{1/4} + (-D_i)\left(\frac{L_n}{L_s}\right)^{1/2} \left(\frac{M\omega_e}{M_\mu \nu_{ii}}\right)^{1/4} \frac{1}{\rho_s}. \quad (4.106)$$

As in the cold ion case, the interchange term will be stabilizing when the curvature is good; i.e., D_i is negative. However, it is worth mentioning here that the usual interchange driving term D is now replaced by D_i, due to the FLR effects. From the definition of D_i and K_b in section 4.2, we notice that the diamagnetic correction to the effective average curvature, which is usually destabilizing disappears, when $\omega \simeq \omega_e$. Also noticed is the α factor in the definition of D_i. Finally, one notices that Δ_{cD} can become very large for small viscosity, which is likely to happen when ion temperature gets higher. On the other hand, when ion viscosity increases, the ion sound stabilizing effects are enhanced. Hence, we conclude that the finite ion temperature effect enhances both the ion sound and curvature effects, and it can become a crucial control mechanism in the future day high temperature plasma devices. Moreover, it is also implied that the increase of $|\Delta_{cD}|$ due to ion viscosity can also drastically enhance the interchange instability for system with bad average curvature.

(2) For large perpendicular diffusion, $\Lambda_\perp \ll 1$, the dispersion relation becomes

$$\frac{\Delta'}{\pi} = 2(1-\frac{\omega_e}{\omega})\frac{x_A^{1/2}}{x_R x_S^{1/2}}(\alpha\beta M_\perp)^{1/4}\frac{\Gamma(3/4)}{\Gamma(1/4)} - \frac{\omega}{\omega_e}D_i\left[\frac{\pi}{4x_A x_\mu}\right]^{1/2}\left[\frac{\alpha\beta}{1-\frac{\omega_i}{\omega}}\right]^{1/4}$$

$$+ 2\frac{(\alpha\beta)^{3/4}x_A^{1/2}x_\mu^{1/2}}{x_S^{3/2}x_i^{1/2}}(1-\frac{\omega_i}{\omega})^{1/4}\frac{\Gamma(3/4)}{\Gamma(1/4)}. \tag{4.107}$$

In this case, the center current channel width is enlarged, due to the perpendicular particle diffusion, by a factor

$$(\frac{2}{\Lambda_\perp})^{1/2} \sim \left[\frac{\alpha L_n^2}{L_S^2}(\frac{\eta_\perp}{\eta_\parallel} + 2q^2)\right]^{1/4};$$

while the growth rate are reduced by $(\Lambda_\perp/2)^{1/2}$. Also, the instability condition is the same as in case (1). Therefore, as in the cold ion limit, pointed out by Hahm, the perpendicular resistivity is rarely important for drift tearing modes.

4.4 Conclusions

In this chapter, drift-tearing modes with finite ion temperature have been investigated. The eigenmode equations, derived from the linearization of the reduced fluid model we derived in chapter 2, recover Hahm's in both toroidal cold ion limit[59] and sheared slab FLR limit.[26] These equations are analyzed by both variational scheme, in sheared slab geometry, and asymptotic matching process, in toroidal geometry. New dispersion relations, which describe the semi-collisional drift-tearing modes with finite T_i effects, has been obtained.

For inviscid plasma, finite ion temperature is found only mildly enhances the ion sound and average curvature effects by a factor of $2\alpha = 1 + T_i/T_e$. Nonetheless, in the viscous semi-collisional regime, in addition to the 2α enhancement, two potentially important effects of finite ion temperature, when coupling with toroidal curvature term, are found: (1) it enhances the good curvature effects by avoiding the destabilizing diamagnetic correction to the average curvature; (2) when T_i gets higher, therefore ν_{ii} gets lower, the instability condition parameter,

$$|\Delta_{cD}| \propto (1/\nu_{ii})^{1/4},$$

can become very large. On the other hand, ion sound stabilizing effects are enhanced for increasing of the ion viscosity. Therefore, ion temperature could become a crucial parameter in controlling the linear tearing modes in the future day high temperature plasma devices.

Since the inclusion of comparable ion sound effects induces two responses with distinct scales, a two-scale variational scheme is utilized. In its cold ion limit, the general dispersion relation derived via the variational scheme recovers many of the previously derived results in various parameter regimes. The success of the two-scale variational calculation could imply the extended applications of the variational principle to the more complicated system which includes temperature gradients effects, which, to our knowledge, has yet to be studied in the viscous semi-collisional regime.

Finally, we remark that, before a rigorous, complicated calculation is carried out, the intuitive dimensional analysis of dominant balance, described in 4.3.2, is always a good starting point in dealing with boundary layer problems. It can easily provide not only the profound physical insight, but also the qualitatively accurate result.

CHAPTER V

CONCLUSIONS

We have studied the fluid descriptions of toroidally confined plasma with finite ion temperature effects. This work is motivated by noticing that despite the successes of RMHD in interpretation of many nonlinear phenomena experimentally observed in the devices of toroidally confined plasma, such simple description will become inadequate for the hot plasma encountered in the present and future machines where many non-ideal effects can significantly modify the dynamics. In particular, the diamagnetic drift-type frequencies and finite ion temperature semicollisional regime become the realistic descriptions of the plasma low frequency activities. In this thesis, we have constructed a reduced fluid description pertains to this parameter regime.

In chapter 2, a generalized reduced fluid model is derived, through a moment approach, to retain accurate $O(\rho_i^2)$ FLR terms. It is generalized in a sense that, instead of imposing complicated orderings from the beginning to make the resulted equations suitable for restricted problems, we have adopted the general orderings for the low frequency activities in large aspect ratio toroidally confined plasma. That is, shear-Alfvén time scale, stretched motion, and small poloidal magnetic field. This generalized model is not only self-consistent, energy conserving, but also provides good FLR effects for a wide range of ρ_i.

Several simplified versions of this model have also been obtained. In particular, a Padé approximation of the full FLR fluid system, which lead to an FLR operator $\frac{1}{1+b_i}$, has been presented. This simpler, energy conserving, numerically tractable reduced fluid model has been claimed deserving further detailed studies both analytically and numerically. Moreover, although the present work has been restricted to the isothermal system, we remark that temperature gradients effects can be easily included (work in progress).

In chapter 3, several general applications of our reduced fluid system have been briefly studied in such a way that further detailed studies can readily follow. In particular, the noncanonical Hamiltonian theory and its applications to the reduced fluid system has been discussed. The Hamiltonian structure of the drift-RMHD has been studied. The difficulty arises from ω_i term has been simplified via an isomorphism theory. Further study of the instability condition of DRMHD based on this result should be interesting. Nonetheless, incompressibility is unlikely an appropriate description of drift-type activities. We remark that an extended work retaining both ω_i and compressibility is in progress. This new Hamiltonian reduced fluid system is obtained partially through the similar isomorphism theory presented in this thesis.

In chapter 4, the finite ion temperature effects on linear drift-tearing modes have been investigated in detailed, based on the linearization of the model of Padé approximation. The resulted eigenmode equations have been found consistent with Hahm's sheared slab system and cold ion toroidal system. Due to the second boundary layer interior length scale, arises from the adiabatic response, a two-scale variational scheme has been developed to derive the dispersion relations in the semi-collisional regime in the sheared slab geometry. For toroidal geometry, we adopted the asymptotic matching process for solving the eigenmode equations.

It has been found that when ion viscous skin-depth is thinner than the current width, the existence of finite ion temperature can only mildly modify the semicollisional drift-tearing modes through enhancing the stabilizing ion sound effects and good curvature effects by a factor of 2α. On the other hand, when ion viscous skin depth becomes larger than the current width, ion temperature significantly modifies the semicollisional drift-tearing modes by replacing good average curvature term ($D \to D_i$), and characterizing the instability condition through ion viscosity. Therefore, ion temperature could be a crucial parameter in controlling the drift tearing modes in the future high temperature plasma.

We also remark that further studies which include the temperature gradients effects, particle trapping effects, will be interesting. The success of the two-scale variational scheme should make the extended approaches relatively easier.

Finally, we hope that the reduced model of Padé approximation will be useful in the future stability analysis of magnetically confined high temperature plasma.

APPENDIX A

DERIVATION OF $\underset{\underset{\sim}{\approx}}{q}$

Here we present a derivation which yields the general expression of $\underset{\underset{\sim}{\approx}}{q}$ as that of $\underset{\approx}{P}$ given in Eq. (2.28). The point is, similar to Eqs. (2.24) and (2.25), the third rank moment equation can be written as,

$$[(\underset{\underset{\sim}{\approx}}{q}\times\underset{\sim}{b}) + \mathrm{TTr}] = \hat{\hat{\underset{\underset{\sim}{\approx}}{T}}}, \tag{A1}$$

where

$$\underset{\underset{\sim}{\approx}}{T} \equiv \frac{1}{\Omega}\Big\{\frac{d}{dt}\underset{\underset{\sim}{\approx}}{q} + \underset{\sim}{\nabla}\cdot\underset{\underset{\sim}{\approx}}{r} + [(\underset{\underset{\sim}{\approx}}{q}\cdot\underset{\sim}{\nabla}\underset{\sim}{V})+\mathrm{TTr}] + \underset{\underset{\sim}{\approx}}{q}(\underset{\sim}{\nabla}\cdot\underset{\sim}{V})$$

$$- [(\frac{1}{mn}\underset{\approx}{P}\underset{\sim}{\nabla}\cdot\underset{\approx}{P})+\mathrm{TTr}] - \underset{\underset{\sim}{\approx}}{C}\Big\}, \tag{A2}$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underset{\sim}{V}\cdot\underset{\sim}{\nabla},$$

$$(\underset{\underset{\sim}{\approx}}{A} + \mathrm{TTr})_{ijk} \equiv A_{ijk} + A_{ikj} + A_{kji}, \tag{A3}$$

and "$\hat{\hat{}}$" of a third-ranked tensor is defined by

$$\hat{\hat{\underset{\underset{\sim}{\approx}}{A}}} \equiv \underset{\underset{\sim}{\approx}}{A} - \underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b}(\underset{\underset{\sim}{\approx}}{A}\,\vdots\,\underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b}) - \frac{1}{2}[(\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b})\underset{\sim}{b}+\mathrm{TTr}]\{\underset{\underset{\sim}{\approx}}{A}\,\vdots\,(\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b})\underset{\sim}{b}\}. \tag{A4}$$

Hence, we have

$$\underset{\underset{\sim}{\approx}}{q} = q_1\underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b} + q_2[(\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b})\underset{\sim}{b}+\mathrm{TTr}] + \hat{\hat{\underset{\underset{\sim}{\approx}}{q}}}, \tag{A5}$$

where

$$q_1 = \int d\underset{\sim}{v}\; m v_{\parallel}^3\, f, \tag{A6}$$

$$q_2 = \int d\underset{\sim}{v}\; \frac{m}{2} v_{\parallel} v_{\perp}^2\, f. \tag{A7}$$

Here, v is particle velocity in CM frame. The main task of this appendix is thus to solve for $\underset{\approx}{\hat{q}}$ from equation (A1). Before doing so, we first present the derivation which leads to Eqs. (2.26) and (2.27).

It is obvious that any antisymmetric second-rank tensor $\underset{\approx}{B}$ can be expressed in term of a vector $\underset{\sim}{E}$ such that

$$B_{ij} = -B_{ij} = \varepsilon_{ijk} E_k,$$

which, along with the identity

$$\frac{1}{2}\varepsilon_{ijk}\varepsilon_{jkm} = \delta_{im},$$

yield

$$\underset{\sim}{E} = -(\underset{\sim}{b}\times\underset{\approx}{B})\cdot\underset{\sim}{b} + \frac{1}{2}(\underset{\sim}{b}\times\underset{\approx}{B}):\underset{\approx}{I}\underset{\sim}{b}.$$

Hence, for any anti-symmetric second-rank tensor, we have

$$\underset{\sim}{b}\times\underset{\approx}{B} = \frac{1}{2}[(\underset{\sim}{b}\times\underset{\approx}{B}):\underset{\approx}{I}](\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b}) + (\underset{\sim}{b}\times\underset{\approx}{B})\cdot\underset{\sim}{b}\underset{\sim}{b}. \tag{A8}$$

Then, from the fact that $(\underset{\approx}{A}\times\underset{\sim}{b}-\mathrm{Tr})$ is an anti-symmetric tensor, we find that, for any second rank tensor $\underset{\approx}{A}$,

$$\underset{\sim}{b}\times\underset{\approx}{A}\times\underset{\sim}{b} = \underset{\approx}{A}^{T} - \underset{\sim}{b}\underset{\sim}{b}\cdot\underset{\approx}{A}^{T} - \underset{\approx}{A}^{T}\cdot\underset{\sim}{b}\underset{\sim}{b} - (\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b})(\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b}):\underset{\approx}{A} + \underset{\sim}{b}\underset{\sim}{b}(\underset{\sim}{b}\underset{\sim}{b}:\underset{\approx}{A}), \qquad \text{(A9)}$$

where $\underset{\approx}{A}^{T}$ is the transpose of $\underset{\approx}{A}$. For symmetric $\underset{\approx}{A}$, Eq. (A9) reduces to Eq. (2.26).

From Eqs. (2.23)-(2.26), we find

$$\underset{\sim}{b}\times\underset{\approx}{K}(\underset{\approx}{A}) = \underset{\sim}{b}\times\underset{\approx}{A}\times\underset{\sim}{b} + (\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b})\cdot\underset{\approx}{A}$$

$$= 2\hat{\underset{\approx}{A}} - (2\underset{\sim}{b}\underset{\sim}{b}\cdot\hat{\underset{\approx}{A}} + \hat{\underset{\approx}{A}}\cdot\underset{\sim}{b}\underset{\sim}{b}), \qquad \text{and} \qquad \text{(A10)}$$

$$\underset{\sim}{b}\times\underset{\approx}{K}(\underset{\approx}{A})\cdot\underset{\sim}{b} = (\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b})\cdot\underset{\approx}{A}\cdot\underset{\sim}{b}. \qquad \text{(A11)}$$

One can then obtain Eq. (2.27).

Similarly, from Eqs. (A1) and (2.27), we have

$$\underset{\approx}{T}\times\underset{\sim}{b} + \mathrm{TTr}$$

$$= -\,9\underset{\approx}{q} + \{[2(\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b})\underset{\approx}{q}:(\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b}) - 2\underset{\sim}{b}\underset{\sim}{b}\underset{\approx}{q}:\underset{\sim}{b}\underset{\sim}{b} + 5\underset{\approx}{q}\cdot\underset{\sim}{b}\underset{\sim}{b}] + \mathrm{TTr}\},$$

$$\underset{\sim}{b}\cdot\underset{\approx}{T} = \underset{\sim}{b}\cdot\underset{\approx}{q}\times\underset{\sim}{b} + \mathrm{Tr} = \underset{\approx}{K}(\underset{\sim}{b}\cdot\underset{\approx}{q}),$$

$$\underset{\sim}{b}\underset{\sim}{b}:\underset{\approx}{T} = \underset{\sim}{b}\underset{\sim}{b}:\underset{\approx}{q}\times\underset{\sim}{b}.$$

Therefore, we obtain

$$\hat{\underset{\approx}{q}} = \frac{1}{9}\,(\underset{\approx}{D} + TTr), \tag{A12}$$

where

$$\underset{\approx}{D} = -\underset{\approx}{T}\times\underset{\sim}{b} - 2[(\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b})(\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b}) - \underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b}]:\underset{\approx}{T}\times\underset{\sim}{b} + 5\{\underset{\approx}{K}^{-1}(\underset{\approx}{T}\cdot\underset{\sim}{b})\}\underset{\sim}{b}. \tag{A13}$$

Recall that $\underset{\approx}{T}$ is defined by Eq. (A2) and $\underset{\approx}{K}^{-1}$ is given by Eq. (2.27).

Eqs. (A12)-(A13) are useful for deriving the part of higher order cross-field stress tensor which arises from $\underset{\sim}{\nabla}\cdot\underset{\approx}{q}$. More importantly, they can be used to determine the perpendicular heat flux. Note here that the heat flux,

$$\underset{\sim}{q} = q_{\parallel}\underset{\sim}{b} + \underset{\sim}{q}_{\perp} \tag{A14}$$

$$\equiv \frac{1}{2}\,\underset{\approx}{I}:\underset{\approx}{q},$$

can be determined by

$$q_{\parallel} = q_1 + 2q_2, \tag{A15}$$

$$\underset{\sim}{q}_{\perp} = \frac{1}{2}\,\underset{\approx}{I}:\hat{\underset{\approx}{q}}. \tag{A16}$$

As an example, in Pfirsch-Schluter regime, the lowest order of Eq. (A2) yields

$$\underset{\approx\atop\sim}{T} = \frac{nT}{m\Omega}\,(\underset{\approx}{I}\underset{\sim}{\nabla}T + TTr);$$

therefore,

$$\underset{\approx\atop\sim}{\hat{q}} = \frac{nT}{m\Omega}\,(\underset{\approx}{I}\underset{\sim}{b}\times\underset{\sim}{\nabla}T + TTr). \qquad \text{(A17)}$$

One can then easily show that Eqs. (A16)-(A17) lead to the classical perpendicular heat flux due to gyromotion.

APPENDIX B

A MOMENT APPROACH TO GYROVISCOSITY IN BANANA-PLATEAU REGIME

As mentioned in chapter one, the main context of this thesis is restricted to the collisionality regime where trapped population is negligible. However, in this appendix, we present an approach to obtain the gyroviscous tensor in banana-plateau regime where particle trapping effects are significant.

We start with Eqs. (2.24)-(2.25) and adopt the Maxwellian ansatz that the lowest order distribution function is a moving-Maxwellian. Our goal here is to evaluate gyroviscosity to $O(\rho_i^2)$, therefore, we need only to $O(\rho_i)$ of $\hat{\underset{\approx}{S}}^g$. We have

$$\begin{aligned}
\hat{\underset{\approx}{S}}^g \simeq\ & \frac{d}{dt}\hat{\underset{\approx}{P}}_o + P_\perp \hat{\underset{\approx}{W}} + (P_\parallel - P_\perp)[\underset{\sim}{b}(\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b})\cdot(\underset{\sim}{b}\cdot\nabla\underset{\sim}{V}+\frac{d}{dt}\underset{\sim}{b}) + Tr] + \hat{\underset{\approx}{P}}_o(\nabla\cdot\underset{\sim}{V}) \\
& + \{[(q_1-2q_2)\underset{\sim}{b}\underset{\sim}{\kappa} + q_2(\underset{\sim}{\nabla}_\perp\underset{\sim}{b}) + \underset{\sim}{b}\underset{\sim}{\nabla}_\perp q_2 + (\underset{\sim}{b}\times\nabla\underset{\sim}{T})\underset{\sim}{\nabla}\frac{nT}{m\Omega} \\
& + \frac{nT}{m\Omega}(\underset{\sim}{b}\times\underset{\sim}{\nabla}\underset{\sim}{\nabla}T + (\underset{\sim}{\nabla}\underset{\sim}{b})\times(\underset{\sim}{\nabla}T))] + TTr\} - 2\underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b}\times\underset{\sim}{\kappa}\cdot\underset{\sim}{\nabla}T) \\
& - (\underset{\approx}{I}-\underset{\sim}{b}\underset{\sim}{b})[q_2\underset{\sim}{\nabla}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{\nabla}T\times\underset{\sim}{\nabla}\frac{nT}{m\Omega} + \frac{nT}{m\Omega}((\underset{\sim}{\nabla}\times\underset{\sim}{b})\cdot\nabla T - \underset{\sim}{b}\times\underset{\sim}{\kappa}\cdot\underset{\sim}{\nabla}T)]. \quad (B1)
\end{aligned}$$

Note that terms which involve $\underset{\sim}{\nabla}T$ arise from the first order $\hat{\underset{\approx}{q}}$ obtained in Eq. (A17).

It is obvious that the gyroviscosity determined from Eq. (B1) will have a form even more complicated than Eq. (B1). However, further scaling can drastically simplify the result. For instance, when large aspect-ratio (or long-thin limit) is assumed, the first five terms will reduce to Eq. (2.29); or, when $\underset{\sim}{V}$ is assumed to be $O(\rho_i)$, the first five terms will reduce to nT $\hat{\underset{\approx}{W}}$.

Our main interest here is the particle trapping contribution to gyroviscosity, therefore, let's concentrate only on terms which involve q_1 and q_2. From terms with q_1 and q_2 in Eq. (B1), Eq. (2.27) thus yields

$$\frac{1}{\Omega}\{\underset{\sim}{b}[\underset{\sim}{b}\times\underset{\sim}{\nabla}q_2 + (q_1-q_2)\underset{\sim}{b}\times\underset{\sim}{\kappa}] + \frac{q_2}{4}[\underset{\sim}{b}\times(\underset{\sim}{\nabla}_\perp\underset{\sim}{b}+Tr)\} + Tr. \qquad \text{(B2)}$$

Note that similar results have been obtained by Newcomb[37] and Siebert[67] for the long-thin mirror configuration. However, we claim that our result should be more exact since we start with the exact moment equations and have imposed only the gyroradius ordering. Also note that further detailed study based upon Eq. (B1) should be interesting.

APPENDIX C

COLLISIONAL CROSS-FIELD VISCOSITY

In this appendix, we present a semi-moment approach of deriving collisional cross-field viscosity via determining $\hat{\underset{\approx}{C}}$. The detailed information of distribution function is not required.

Starting with the Landau collision operator[38], for like-species, $\underset{\approx}{C}$, after integration by part and some manipulations, can be written as

$$\underset{\approx}{C} = \frac{m_i \Delta}{2n} \int d\underset{\sim}{v} \int d\underset{\sim}{v}' \, \frac{1}{u^3}(\underset{\sim}{u}\underset{\sim}{u} - \frac{u^2}{3}\underset{\approx}{I}) \; f(\underset{\sim}{v}) f(\underset{\sim}{v}'). \qquad \text{(C1)}$$

Here, $\underset{\sim}{u} = \underset{\sim}{v} - \underset{\sim}{v}'$ and

$$\Delta = -\frac{9}{2}\sqrt{2\pi}\;\nu_i. \qquad \text{(C2)}$$

By assuming the lowest order $f(\underset{\sim}{v})$ is Maxwellian $f_M(\underset{\sim}{v})$, Eq. (C1) becomes

$$\underset{\approx}{C} \simeq \frac{m_i \Delta}{n} \int d\underset{\sim}{v} \int d\underset{\sim}{v}' \, \frac{1}{u^3}(\underset{\sim}{u}\underset{\sim}{u} - \frac{u^2}{3}\underset{\approx}{I}) \; f(\underset{\sim}{v}) f_M(\underset{\sim}{v}').$$

Then, by using the facts that

$$\int d\underset{\sim}{v} \int d\underset{\sim}{v}' \; \frac{\underset{\sim}{u}\underset{\sim}{u}}{u^3} \; f(\underset{\sim}{v}) f_M(\underset{\sim}{v}')$$

$$= \int d\underset{\sim}{v} \; f(\underset{\sim}{v}) \; [\frac{\partial}{\partial \underset{\sim}{v}}(\int d\underset{\sim}{v}' \frac{1}{u} \underset{\sim}{v}' f_M(\underset{\sim}{v}')) - \underset{\sim}{v} \frac{\partial}{\partial \underset{\sim}{v}}(\int d\underset{\sim}{v}' \frac{1}{u} f_M(\underset{\sim}{v}'))],$$

$$\frac{1}{u} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{v_<^{\ell}}{v_>^{\ell+1}} Y_{\ell m}(\vartheta,\varphi) \; Y_{\ell m}(\vartheta',\varphi'),$$

and

$$\underset{\sim}{v} = 2\sqrt{\pi/3} \; v[Y_{10}\underset{\sim}{b} - \sqrt{2}(\hat{e}_1 \mathrm{Re}(Y_{11}) + \hat{e}_2 \mathrm{Im}(Y_{11}))],$$

we have

$$\underset{\approx}{C} \simeq m_i \Delta \int d\underset{\sim}{v} \; f_i(\underset{\sim}{v}) \; (\underset{\sim}{v}\underset{\sim}{v} - \frac{v^2}{3}\underset{\approx}{I}) \; \Theta(x). \tag{C3}$$

Here $x^2 = \dfrac{m_i v^2}{2T_i}$,

$$\Theta(x) = (x^2 - \frac{3}{2}) \frac{\mathrm{erf}(x)}{x^5} + \frac{3}{\sqrt{\pi}} \frac{\exp(-x^2)}{x^4},$$

erf(x) is the error function, $Y_{\ell m}$ is spherical harmonic and $\hat{e}_{1,2}$ are unit vectors perpendicular to $\underset{\sim}{b}$.

To evaluate Eq. (C3), it is convenient to expand the distribution function in spherical harmonic and Laguere polynomials[68,69]

$$\tilde{f}(\underset{\sim}{v}) = f(\underset{\sim}{v}) - f_M(\underset{\sim}{v})$$

$$\simeq \frac{m_i}{T_i}[\underset{\sim}{v}\cdot\sum_k \underset{\sim}{u}_k L_k^{3/2}(x^2) + \frac{1}{2nT_i}(\underset{\sim}{v}\underset{\sim}{v}-\frac{1}{5}v^2\underset{\approx}{I}):\sum_k \underset{\approx}{P}_k L_k^{5/2}(x^2)]f_M(\underset{\sim}{v}), \quad \text{(C4)}$$

where L_k^ms are the generalized Laguere polynomials,

$$\underset{\sim}{u}_k \equiv \frac{3}{2}\,\frac{\int d\underset{\sim}{v}\;\underset{\sim}{v}L_k^{3/2}(x^2)\;\tilde{f}(\underset{\sim}{v})}{\int d\underset{\sim}{v}\;(xL_k^{3/2}(x^2))^2\;f_M(\underset{\sim}{v})}, \quad \text{(C5)}$$

and

$$\underset{\approx}{P}_k \equiv \frac{15nm_i}{4}\,\frac{\int d\underset{\sim}{v}\;\underset{\sim}{v}\underset{\sim}{v}L_k^{5/2}(x^2)\;\tilde{f}(\underset{\sim}{v})}{\int d\underset{\sim}{v}\;(x^2L_k^{5/2}(x^2))^2\;f_M(\underset{\sim}{v})}$$

$$= \sum_{j=0}^{k} \frac{(5/2)!k!}{(j+(5/2))!j!(k-j)!}\left(-\frac{m_i}{T_i}\right)^j \underset{\approx}{M}_{2j+2}, \quad \text{(C6)}$$

$$\underset{\approx}{M}_{2j+2} \equiv \int d\underset{\sim}{v}\; m_i\underset{\sim}{v}\underset{\sim}{v}\;\left(\frac{v^2}{2}\right)^j\;\tilde{f}.$$

In Eq. (C4), we have neglected the velocity space spherical harmonic components with $\ell > 2$, for simplicity. It is also understood that the first term on the RHS of Eq. (C4) corresponds to the odd ranked moments such as particle flux, heat flux, etc.; while the second term corresponds to the even ranked moments.

From Eqs. (C3)-(C6), we have

$$\underset{\approx}{\hat{C}} = -\frac{6\nu_i}{5}\sum_k \underset{\approx}{\hat{P}}_k\, 4\sqrt{2}\int dx\, x^6 e^{-x^2}\,\Theta(x)L_k^{5/2}(x^2). \tag{C7}$$

Note that our interest is to evaluate $\underset{\approx}{\hat{C}}$ to $O(\rho_i)$, therefore $O(\rho_i)$ of $\underset{\approx}{M}_{2j+2}$. By noting $O((\rho_i)^0)$ of the 2kth moment equation

$$\underset{\approx}{K}(\underset{\approx}{M}_{2k}) = \frac{1}{\Omega}\, nm_i\left(\frac{T_i}{m_i}\right)^k \frac{(k+\frac{3}{2})!}{(\frac{5}{2})!}\,\underset{\approx}{\hat{W}}, \tag{C8}$$

one finds that

$$\underset{\approx}{\hat{P}}_k = 0 \qquad \text{for } k \neq 0. \tag{C9}$$

Hence,

$$\underset{\approx}{\hat{C}} \simeq -\frac{6}{5}\nu_i\,\underset{\approx}{\hat{P}}_0. \tag{C10}$$

where $\underset{\approx}{\hat{P}}_0$ is given in Eq. (2.30). One can then use

$$\underset{\approx}{\hat{P}}^c = \frac{1}{\Omega}\,\underset{\approx}{K}^{-1}(\underset{\approx}{\hat{C}})$$

and obtain the collisional cross-field viscosity which agrees with Braginskii's result.

We remark that the above technique is very useful in deriving the cross-field moment tensor in the magnetized plasma regime, because of the small factor $\frac{1}{\Omega}$ in front of $\hat{\underset{\approx}{S}}$. This approach has been used to reproduce the usual neoclassical cross-field viscosity (unpublished).

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VITA

Chi-Tien Hsu was born on April 20, 1955 in Taipei, Taiwan, The Republic of China, the son of Tsi-Ching Hsu and Pi-Ing Lin Hsu. After graduating from Cheng Kong High School, he entered the National Tsing Hwa University in Hsing Tsu, Taiwan during the fall of 1973. In June of 1977, he was awarded the Bachelor of Science degree, majoring in Nuclear Engineering. After the graduation, he entered the Atomic Energy Council of China as a research assistant. In 1981, he entered the Graduate School of the University of Texas at Austin and has been employed by the Institute for Fusion Studies.

Permanent Address: 1-4, Lane 38, Tai Ho Street
Chung Ho City, Taiwan 23530
R.O.C.