

**DISSIPATIVE TRAPPED PARTICLE MODES
IN TANDEM MIRRORS**

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Abstract

The theory for the dissipative trapped particle modes is developed for a tandem mirror taking into account equilibrium rotational effects, electron temperature gradients and an axial ambipolar potential, for a broad range of collisional parameters. The electrons are treated as a Maxwellian plasma that occupies the central cell and anchor cell regions. It is assumed that the eigenfunction is piecewise constant with abrupt transitions between the anchor and central cell regions. It is found that when $\omega/\nu_p \gtrsim 1$, with ω the mode frequency and ν_p the Pastukhov loss rate, that the energy conservation structure of the collision operator produces important changes to previously developed theories. A solution to the problem is achieved by using the solution for the lifetime of an electron in an ambipolar trap, taking into account the global energy conservation. The energy conservation structure also allows a self-consistent description of dissipative instabilities when thermal gradient and electric fields are present. At very high collision frequency, a new dispersion relation is obtained, which exhibits an axially rotational shear drive coupled to radial temperature gradients producing instability. Numerical studies are presented for some parameters, with the deviation from previous theory highlighted.

I Introduction

Tandem mirrors with strongly stabilizing anchor regions are subject to trapped particle instabilities,¹ where the mode tends to isolate away from the anchor regions and be driven by the unfavorable curvature of the central cell. In the absence of dissipation, the trapped particle mode can be stabilized by a charge uncovering mechanism that relies on a difference in the passing particle populations of electrons and ions in the end-cells. However, when dissipation due to Landau damping² or collisions^{3,4,5} are included, the stable waves are often destabilized.

In the treatment of electron collisions, past descriptions are deficient in several ways. Either a Lorentz collision model,^{4,5} or a Fokker-Planck Coulomb collision operator that neglected global energy conservation,³ were used. When the ambipolar potential in the anchor region is larger than the electron temperature, so that the Pastukhov loss-rate,⁶ ν_p , of electrons in an ambipolar trap is a small fraction of a 90° collision frequency, ν_e , the Lorentz operator gives reasonable results if $\nu_e/\omega \lesssim 1$, but fallacious results when $\nu_e/\omega > 1$, where ω is the wave frequency. The neglect of global energy conservation in the collision operator as treated in Ref. 3, prevents a description with temperature gradients transverse to the magnetic field. We shall only show that the theory in Ref. 3 is quantitatively accurate only at relatively low collision frequencies. Even when there is no temperature gradients or plasma flow, at very high collision frequency the new theory predicts different growth rates and frequency shifts. At arbitrary collision frequency, or when there is equilibrium temperature gradients and equilibrium plasma flows, we will show that the global energy confinement property of the collision operator is essential to obtaining the correct linear response of the system. When $\omega/\nu_p \ll 1$, the linear response for predicting the non-dissipative response has previously been calculated.⁷ The method developed here, where the collision frequency is nearly arbitrary, reproduces the result of the high collisionality limit.

In this work we will show how the correct Fokker-Planck electron Coulomb collision operator can be included to describe trapped particle modes in tandem mirrors if electrons in the central cell and anchor can be described by a Maxwell distribution with a single tem-

perature T on a flux surface. We are then able to describe perturbed response in equilibria with transverse temperature gradients and transverse electric fields. A fundamental assumption in this analysis is that a positive ambipolar potential exists in the anchor region, so that the electrons are electrostatically trapped in the anchor region. The particle and energy lifetime in such a trap has been calculated by Pastukhov.⁶ In Ref. 3 it was shown how this solution can be used to solve the linear wave kinetic problem. In this paper the reduction of the wave kinetic problem to the Pastukhov problem is also basic to obtaining a solution. However, we note that the Pastukhov problem involves a nonlinear collision operator, whereas the wave kinetic problem involves a linear collision operator. To use the Pastukhov solution we shall first have to formulate a linearized Pastukhov problem, which can then be related to the collisional wave kinetic problem.

We consider the geometry shown in Fig. 1, to model a tandem mirror. We assume that on a given flux tube the electrons in the central cell and in the anchor are in thermal equilibrium with each other in the absence of perturbations. We will not attempt to consider a case that is relevant in many experiments, where electrons are directly heated in the anchor region with *r.f.* to produce a non-Maxwellian distribution along a flux tube.

When perturbations are considered, we are able to solve our problem provided the perturbed electron distribution remains Maxwellian over most of phase space, albeit with different density and temperature perturbations in the central cell and anchor regions. This separate Maxwellian response in the two regions is a valid approximation because the fraction of passing to trapped electrons are small. The inhomogeneous terms driving the equations will be shown to approximately be linearized Maxwellian forms with different temperatures and amplitudes in the anchor and central cell regions. Such inhomogeneous terms allow solutions that are Maxwellian over the bulk of phase space with separate temperature and density perturbations in the central cell and anchor regions.

At the separatrix, the diffusive nature of the collision operator forces the distribution function of barely passing electrons to have the same value of the barely trapped electrons in both the central cell and anchor regions. Thus, near the separatrix, the distribution function of the barely trapped electrons in the anchor and central cell are equal to each other. The determination of this distribution function is, in general, a difficult problem. However, under several conditions the passing particle distribution is nearly equal to the

same perturbed Maxwellian function as exists in the central cell. These conditions are met for a sufficiently long central cell, or in a high collisionality regime. For these cases, which will be assumed in this paper, the technical aspects of the calculation can then be solved straightforwardly.

The structure of the paper is as follows. In Sec. II we construct the functional dispersion relation for the electrostatic response of the system. As in Ref. 3, the perturbed amplitudes are assumed to be piecewise constant functions with different constant values in the central cell and the anchor. One can show that the governing equations for the field amplitudes are obtained by taking the variations of the quadratic form with respect to the field amplitudes. By using piecewise-constant test functions, it is then straightforward to construct a global dispersion relation. However, the dispersion functional will be in terms of unknown parameters that is determined from a solution of a Fokker-Planck problem. In Sec. III we relate this Fokker-Planck problem to the linearized Pastukhov problem of an electron distribution trapped in a ambipolar potential. With this relation, the dispersion functional is completely determined in terms of the field amplitude. In Sec. IV we discuss the dispersion relation resulting from the dispersion functional.

II Derivation of Electron Contribution to Dispersion Functional

In this section we derive the quadratic variational form for the electrostatic dispersion relation with collisions. In this quadratic form the distribution function is taken to satisfy the wave kinetic Fokker-Planck equation. We construct the quadratic quantity

$$D(\omega) \equiv D_i(\omega) + D_e(\omega) \equiv \int_{-\infty}^{\infty} \frac{ds}{B} \sum_j q_j n_j^{(1)} \phi^\dagger = 0 \quad (1)$$

where $n_j^{(1)}$ is the perturbed density of species j . The $e(i)$ subscript refers to the explicit contribution from the electrons (ions). In this derivation we will treat the ion response as collisionless and the electron response at arbitrary collision frequency, but in the approximation that the electron's mean free path is long compared to the machine length and the eigenmode frequency and drift frequency much less than the electron bounce rate. The quantity ϕ^\dagger is the adjoint of ϕ , and it can be shown that if a solution for the eigenfunction

is

$$\phi(\mathbf{r}, t) = \hat{\phi}(s) \exp[i\mathbf{k}_\perp \cdot \mathbf{r} - i\omega t]$$

then $\phi^\dagger(r, t)$ satisfies

$$\phi^\dagger(\mathbf{r}, t) = \hat{\phi}(s) \exp[-i\mathbf{k}_\perp \cdot \mathbf{r} + i\omega t].$$

For simplicity we use the eikonal approximation to describe cross-field structure, i.e.,

$$\mathbf{k}_\perp = \nabla S(\alpha, \beta) = \frac{\partial S}{\partial \alpha} \nabla \alpha + \frac{\partial S}{\partial \beta} \nabla \beta \quad (2)$$

with $\mathbf{B} = \nabla \alpha \times \nabla \beta$. However, identical results for the electron response would be obtained if the eikonal approximation were not used. In this work cross-field electron diffusion is neglected.

The equilibrium configuration is shown in Fig. 1. The equilibrium potential is taken with values $\Phi_c(\alpha)$ and $\Phi_a(\alpha)$ in the central cell and anchor regions, and these potentials are nearly constant along a field line in their respective regions. The rotational frequency is also taken as nearly piecewise constant with values ω_{Ec} and ω_{Ea} in the central cell and anchor regions, with

$$\omega_{Ej} = cm \frac{\partial \Phi}{\partial \alpha} \quad (3)$$

with m an integer and c the velocity of light (we now choose units where $c = 1$). In addition we include a cross-field electron temperature gradient, and magnetic field curvature along the field line. The test functions for the perturbation will be taken as piecewise constant in each region of the tandem mirror, with the magnitude of the discontinuity of the test function between the regions determined by the extremization of the quadratic form. The equilibrium electron distribution is given by

$$F_e = \frac{n_0(\alpha)}{[2\pi T(\alpha)/m_e]^{3/2}} \exp \left[\frac{-m_e v^2/2 - e\Phi(s, \alpha)}{T(\alpha)} \right] \quad (4)$$

with e the electron charge, m_e the electron mass and T the electron temperature. The electrons will be considered in thermal equilibrium in the central cell and anchor region. However, it is possible to have an additional hot electron component in the anchor region, which we shall treat as a collisionless component which can be added to the collisionless ion response. For the derivation of the explicit ion response, the reader is referred to the literature as in Refs. 2, 8 and 9.

The perturbed electron distribution, f_e , is given by

$$f_e = \frac{-e\phi}{T} F_e + h F_e \quad (5)$$

where $h = h(E, \mu, \alpha)$ satisfies the bounce-averaged kinetic equation

$$-i(\omega - \bar{\omega}_d)h - \bar{C}_2(h) = -ie \frac{(\omega - \omega_T^*)\bar{\phi}}{T(\alpha)} \quad (6)$$

with

$$\begin{aligned} \bar{q} &= \frac{1}{\tau} \int_{-s_0}^{s_0} \frac{dsq(s)}{|v_{\parallel}(E, \mu, s)|} \\ \tau(E, \mu) &= \int_{-s_0}^{s_0} \frac{ds}{|v_{\parallel}(E, \mu, s)|} \equiv \text{electron bounce time} \\ \pm s_0 &\equiv \text{turning points of particle} \\ \omega_T^* &= \omega_e^* \left[1 + \eta \frac{(m_e v^2/2 + e\Phi)}{T} - \frac{3}{2}\eta \right] \\ \omega_e^* &= \frac{Tm}{en_0} \cdot \frac{\partial n_0}{\partial \alpha} \\ \eta &= \left(\frac{T \partial n_0}{n_0 \partial \alpha} \right)^{-1} \frac{\partial T}{\partial \alpha} \\ \omega_{en}^* &= \omega_e^* - \omega_E + \eta \omega_e^* \frac{e\phi}{T} = \frac{Tm}{en_e} \frac{\partial}{\partial \alpha} \left[n_0(\alpha) \exp \left(\frac{-e\phi(s, \alpha)}{T(\alpha)} \right) \right] \\ \omega_d &= m \frac{\partial \Phi}{\partial \alpha} + \frac{m\kappa}{\omega_{ce} r m_e} (2E - \mu B - 2e\Phi) \equiv \omega_E + \omega_{\kappa} \\ \omega_{ce} &= eB/m_e \\ \mu &= \frac{m_e v_{\perp}^2}{2B}, \quad \kappa = \mathbf{b} \cdot \nabla \mathbf{b} \cdot \nabla \alpha / |\nabla \alpha|, \end{aligned} \quad (7)$$

where n_e is the equilibrium electron density. Note that ω_{en}^* is proportional to the density gradient while ω_e^* is a parameter that is constant on a flux surface.

In the expression for the curvature drift we have neglected geodesic curvature for convenience of presentation, and we assumed sufficiently small β so that $\nabla_{\perp} B/B \simeq \bar{\kappa}$. However, at higher β the correction of the diamagnetic component of the grad- B drift, is usually compensated by magnetic compressibility corrections. In most regimes these two effects cancel. By neglecting both effects, which we do in this work, one usually obtains the correct result.

$C_\ell(h)$ is the linearized electron collision operator and it is given by,

$$C_\ell(h) = \sum_j \frac{b_j}{m_e} \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3 \mathbf{v}' \left(\frac{\mathbf{I}g^2 - \mathbf{g}\mathbf{g}}{g^3} \right) F_e(v) F_j(v') \cdot \left(\frac{1}{m_e} \frac{\partial h(\mathbf{v})}{\partial \mathbf{v}} - \frac{1}{m_j} \frac{\partial h_j(\mathbf{v}')}{\partial \mathbf{v}'} \right) \quad (8)$$

where $b_j = 2\pi e^2 q_j^2 \Lambda_{ej}$, Λ_{ej} is the Coulomb logarithm between electrons and species j , F_j is a Maxwellian of species j , $\mathbf{g} = \mathbf{v} - \mathbf{v}'$. We observe that, $C_\ell(\text{constant}) = 0$ and $C_\ell(v^2) = 0$. We observe that, if we neglect electron-ion energy exchange, $\bar{C}_\ell(m_e v^2/2 + e\Phi) = 0$, as $C_\ell(m_e v^2/2 + e\Phi) = 0$ and $\frac{m_e v^2}{2} + e\Phi$ is independent of position.

We now define h_c , h_a and h_p as the electron distribution functions in the central cell, anchor regions and passing regions respectively. We define the amplitude ψ by,

$$\psi = \phi/(\omega - \omega_E) \quad (9)$$

and g_s through the relation,

$$h_s \frac{e(\omega - \omega_T^*)}{T} \frac{\overline{\psi(\omega - \omega_E)}}{\omega - \bar{\omega}_E} + g_s. \quad (10)$$

We notice that the first term in Eq. (10) is nearly independent of E and μ for particles trapped in the central cell or the anchor regions if they sample a nearly constant ψ and ω_E . For passing particles, the first term generally depends on E and μ . However, if one has a long central cell compared to anchor region, the first term on the right-hand side of Eq. (10), is also nearly independent of E and μ . Now taking a test function where ψ is piecewise constant, and an equilibrium model where ω_E is piecewise constant, we find that g is given by,

$$\begin{aligned} (\omega - \omega_{Ec})g_c &= i\bar{C}_\ell(g_c) + \bar{\omega}_{\kappa c} \left[\frac{e(\omega - \omega_T^*)}{T} \psi_c + g_c \right] \\ (\omega - \omega_{Ea})g_a &= i\bar{C}_\ell(g_a) + \bar{\omega}_{\kappa a} \left[\frac{e(\omega - \omega_T^*)}{T} \psi_a + g_a \right] \\ (\omega - \bar{\omega}_E)g_p &= i\bar{C}_\ell(g_p) + i\bar{C}_\ell \left(\frac{e(\omega - \omega_T^*)\overline{\psi(\omega - \omega_E)}}{(\omega - \bar{\omega}_E)} \right) \\ &+ \bar{\omega}_\kappa \left[\frac{e(\omega - \omega_T^*)\overline{\psi(\omega - \omega_E)}}{T(\omega - \bar{\omega}_E)} + g_p \right] \end{aligned} \quad (11)$$

where we have used $\bar{C}_\ell(\text{const}) = 0$ and $\bar{C}_\ell(E) = 0$, where E is the total energy ($E = \frac{1}{2}m_e v^2 + e\Phi$) of a particle. Roughly, we estimate

$$g_{c,a} \approx \frac{\bar{\omega}_{\kappa c,a}}{\omega - \bar{\omega}_{E c,a}} \frac{e(\omega - \omega_T^*)}{T} \psi_{c,a} \approx \varepsilon h$$

where $\varepsilon \approx \bar{\omega}_{\kappa c,a}/(\omega - \bar{\omega}_{E c,a})$. Thus, we treat $g_{c,a} \approx \mathcal{O}(\varepsilon)$. We shall take the ratio of passing to trapped particles to be $\mathcal{O}(\varepsilon)$.

We substitute the electron contribution into Eq. (1), using Eqs. (5) and (10), to find,

$$\begin{aligned} D_e(\omega) &\equiv e \int_{-\infty}^{\infty} \frac{ds}{B} n_e^{(1)}(\omega - \omega_E) \psi = \frac{4\pi e}{m_e^2} \int_{-\infty}^{\infty} ds (\omega - \omega_E) \psi \int \frac{dE d\mu f_e}{|v_{\parallel}|} \\ &= -\frac{4\pi e^2}{T m_e^2} \int_{-\infty}^{\infty} ds \psi^2 (\omega - \omega_E)^2 \int \frac{dE d\mu}{|v_{\parallel}|} F_e \\ &\quad + \frac{4\pi e}{m_e^2} \int_{-\infty}^{\infty} ds \psi (\omega - \omega_E) \int \frac{dE d\mu}{|v_{\parallel}|} F_e h \\ &= -\frac{4\pi e^2}{T m_e^2} \int dE d\mu \tau F_e \left[\overline{\psi^2 (\omega - \omega_E)^2} - (\omega - \omega_T^*) \frac{\overline{\psi (\omega - \bar{\omega}_E)^2}}{(\omega - \bar{\omega}_E)} \right] \\ &\quad + \frac{4\pi e}{m_e^2} \int dE d\mu \tau F_e \overline{g(\omega - \omega_E) \psi}. \end{aligned} \quad (12)$$

Now we write $\psi = \psi_c + \delta\psi$, and use Eq. (11) and the particle conservation property of the collision operator which implies for arbitrary $q(v)$ that

$$\int dE d\mu \tau F_e \bar{C}(q) = 0. \quad (13)$$

Then using Eqs. (11) and (13), we exactly find,

$$\begin{aligned} \delta D_e &\equiv \frac{4\pi e}{m_e^2} \int dE d\mu \tau F_e \overline{g(\omega - \omega_E) (\psi_c + \delta\psi)} \\ &= \frac{4\pi}{m_e^2} \int dE d\mu \tau F_e \bar{\omega}_\kappa \left[\frac{e^2 \overline{(\omega - \omega_E) \psi^2}}{T (\omega - \bar{\omega}_E)^2} (\omega - \omega_T^*) + e \frac{\overline{(\omega - \omega_E) \psi g}}{(\omega - \bar{\omega}_E)} \right] \\ &\quad + \frac{4\pi e i}{m_e^2} \int dE d\mu \tau F_e \delta\psi \bar{C}_\ell(h). \end{aligned} \quad (14)$$

In the last term we note that there is no contribution from the particles trapped in the central cell where $\delta\psi = 0$. Also, in this term, we now neglect the contribution from passing particles as their contribution is smaller than from the anchor region by the ratio of passing to trapped particle density which is of order to be $\mathcal{O}(\varepsilon)$. Then note that $C_\ell(h_a) = C_\ell(g_a)$

as $C_\ell(h_a - g_a) = 0$ as $h_a - g_a = \alpha + \beta E$ with α and β constant. We also set $g = 0$ in the square bracket of Eq. (14) and the expression is still correct to $\mathcal{O}(\varepsilon^2)$. We note in passing that this term can be somewhat larger than $\mathcal{O}(\varepsilon^2)$ in a very high collisionality limit, but will normally be small compared to the terms retained. At moderately high collisionality one need not discard this term to carry through this calculation, although we do discard this term for the sake of simplification.

Now we use that the charge density $\rho = \sum_j 4\pi q_j/m_j^2 \int dE d\mu B F_j/|v_{||}| = 0$, to obtain the following identities that follow after appropriate parts integrating,

$$\begin{aligned} m \int ds \psi^2(\omega - \omega_E) B \frac{\partial \rho / B^2}{\partial \alpha} &\equiv \sum_j \frac{4\pi q_j^2}{m_j^2} \int dE d\mu \tau \overline{(\hat{\omega}_s^* - \omega_E - \omega_\kappa)} \psi^2(\omega - \omega_E) \frac{\partial F_j}{\partial E} = 0 \\ m \int \frac{ds}{B} \psi^2(\omega - \omega_E) \frac{\partial B}{\partial \alpha} &\equiv \sum_j \frac{4\pi q_j^2}{m_j^2} \int dE d\mu \tau \frac{\partial F_j}{\partial E} \overline{\omega_{\kappa||}(\omega - \omega_E)} \psi^2 = 0 \end{aligned} \quad (15)$$

where $\hat{\omega}_j^* \partial F_j / \partial E = -(m/q_j) \partial F_j / \partial \alpha$ and $\omega_{\kappa||} = \frac{2m\kappa(E - \mu B - q_j \Phi)}{\omega_{c_j} r m_j}$ and at low beta $\kappa = r \partial B / \partial \alpha$. These identities follow exactly from charge neutrality. We add these terms into our quadratic form, and for the electron contribution we have,

$$\hat{\omega}_e^* \frac{\partial F}{\partial E} = -\frac{\omega_T^* F}{T}.$$

Now substituting Eqs. (14) and (15) into Eq. (12), yields,

$$\begin{aligned} D_e(\omega) &= -\frac{4\pi e^2}{T m_e^2} \int dE d\mu \tau F_e \\ &\left\{ \left[\overline{\psi^2(\omega - \omega_E)(\omega - \omega_T^*)} - \left(\overline{\psi(\omega - \omega_E)} \right)^2 \frac{(\omega - \omega_T^*)}{(\omega - \bar{\omega}_E)} \right] \right. \\ &+ \left[-\bar{\omega}_\kappa \frac{(\omega - \omega_E) \psi^2}{(\omega - \bar{\omega}_E)} + \overline{\omega_\kappa(\omega - \omega_E) \psi^2} \right] \\ &\left. + \frac{(\omega - \omega_E) \psi^2}{(\omega - \bar{\omega}_E)^2} \bar{\omega}_\kappa (\omega_T^* - \bar{\omega}_E) - \frac{i}{e} T_e \bar{C}_\ell(g) \delta \psi \right\} + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (16)$$

Now substituting $\psi = \psi_c$ in the central cell, $\psi = \psi_a$ in the two anchor regions, define $\Delta\psi = \psi_a - \psi_c$ and we find,

$$D_e(\omega) = -\frac{8\pi e^2}{m_e^2 T} \int_p dE d\mu F_e \frac{\tau_a \tau_c}{\tau} \frac{\Delta\psi^2(\omega - \omega_{E_c})(\omega - \omega_{E_a})(\omega - \omega_T^*)}{\omega - (\omega_{E_c} \tau_c + \omega_{E_a} \tau_a)/\tau} \left(1 + \mathcal{O}\left(\frac{\omega_\kappa}{\omega - \omega_E}\right) \right)$$

$$\begin{aligned}
& -2m^2 \int_c \frac{ds}{B^2 r} \kappa \frac{\partial P_e}{\partial \alpha} \psi_c^2 - 4m^2 \int_a \frac{ds}{B^2 r} \kappa \frac{\partial P_e}{\partial \alpha} \psi_a^2 \\
& + \frac{8\pi}{m_e^2} e i \Delta \psi \int dE d\mu \tau \bar{C}_\ell(g_a) F_e + \mathcal{O}(\varepsilon^2)
\end{aligned} \tag{17}$$

where P_e is the electron pressure. The “c” and “a” refer to integration over the central cell and anchor regions. The “p” refers to integration over the passing particle distribution. In Eq. (17) the integrals are over just one anchor region, and a factor of two has been added to account for both anchor regions. In obtaining this result we used

$$\frac{4\pi}{m_e^2} \int dE d\mu \tau_{c,a} F_e \bar{\omega}_\kappa (\omega_T^* - \omega_{E_{c,a}}) = 2 \int_{c,a} \frac{ds}{B^2 r} \kappa \frac{\partial p_e}{\partial \alpha} (1 + \mathcal{O}(\varepsilon)).$$

With $g = 0$, Eq. (17) is the correct response for collisionless theory. For collisional theory we need to calculate g_a and g_c . To determine these functions we modify and correct a method developed in Ref. 3.

We observe that the boundary condition for g is that it is continuous on the separatrix between the trapped and passing particles, i.e.,

$$h_a = h_p \quad ; \quad h_c = h_p.$$

Thus we have,

$$h_{a,c} \equiv g_{a,c} + \frac{e \psi_{a,c} (\omega - \omega_T^*)}{T} = h_p \equiv g_p + \frac{e (\psi_c \Omega_c \tau_c + \psi_a \Omega_a \tau_a) (\omega - \omega_T^*)}{(\Omega_c \tau_c + \Omega_a \tau_a)}$$

where $\Omega_{c,a} = (\omega - \omega_{E_{c,a}})$. Thus,

$$\begin{aligned}
g_a - g_p &= \frac{-e \Omega_c \tau_c \Delta \psi (\omega - \omega_T^*)}{T (\Omega_c \tau_c + \Omega_a \tau_a)} \\
g_c - g_p &= \frac{e \Omega_a \tau_a \Delta \psi (\omega - \omega_T^*)}{T (\Omega_c \tau_c + \Omega_a \tau_a)}.
\end{aligned} \tag{18}$$

As there is only one separatrix contour of passing particles separating the trapped anchor region and the trapped central cell region, the E and μ dependences of Ω_c and Ω_a are at the same energy and magnetic moment. Then, subtracting the two equations yields the boundary condition

$$g_a - g_c = -\frac{e}{T} \Delta \psi (\omega - \omega_T^*). \tag{19}$$

To solve for g_a and g_c we have from Eqs. (11)

$$\begin{aligned}\Omega_c g_c &= i \bar{C}_\ell(g_c) + \mathcal{O}(\varepsilon) \\ \Omega_a g_a &= i \bar{C}_\ell(g_a) + \mathcal{O}(\varepsilon).\end{aligned}\tag{20}$$

If $\nu > \Omega_{Ea,c}$, where ν is the electron collision frequency, the solution is dominated by the collisional term, so that g has the form over most of phase space

$$\begin{aligned}g_a &= \frac{e\Delta\psi\omega_e^*}{T} \left[\lambda_{1a} + \lambda_{2a} \left(\frac{1}{2} \frac{m_e v^2}{T} - \frac{3}{2} \right) \right] \\ g_c &= \frac{e\Delta\psi\omega_e^*}{T} \left[\lambda_{1c} + \lambda_{2c} \left(\frac{1}{2} \frac{m_e v^2}{T} - \frac{3}{2} \right) \right],\end{aligned}\tag{21}$$

where λ_{1a} , λ_{2a} , λ_{2c} and λ_{1c} are constants that need to be determined. In fact, the validity of Eq. (21) may be larger than our estimation. For example, in Ref. 3 the valid range of collisional frequency was found to be $\frac{\nu}{\Omega} > \frac{1}{(|e\Phi|/T + B_{\max}/B_a)}$. Equation (21) is not necessarily satisfied in a boundary layer near the separatrix. However, we shall assume that passing particle equilibrate primarily with central cell particles, a condition which implies

$$\frac{L_a}{L_c} \left(\frac{T}{|e\Phi|} \right)^2 \exp \left(\frac{|e\Phi|}{T} \right) \ll 1.\tag{22}$$

This condition limits the validity of the calculation if $|e\Phi|/T$ is too large. However, when Eq. (22) is satisfied, we can expect (21b) to be valid near the separatrix (as the combined passing and central cell trapped distribution are described to lowest order by the same Maxwellian distribution function) but (21a) to break down near the separatrix as the distribution of particles trapped near the separatrix in the anchor has a distribution that is strongly affected by both the central cell distribution and the anchor distribution, which are distributions of different functional forms, than in the bulk of the anchor region.

We can obtain relationships among the four unknowns if we use particle and energy conservation properties of the collision operator. We take the zeroth moment and an energy moment $\left(\frac{2}{3}(E - T) = \frac{mv^2}{3} + \frac{2}{3}e\Phi - T \right)$ of Eqs. (20a) and (20b) and sum the expression (thereby annihilating the collisional contribution) and obtain,

$$\Omega_c N_c \lambda_{1c} + 2\Omega_a N_a \lambda_{1a} = 0\tag{23}$$

$$N_c \Omega_c \left(\frac{2}{3} \lambda_{1c} e \Phi_c / T + \lambda_{2c} \right) + 2 N_a \Omega_a \left(\frac{2}{3} \lambda_{1a} e \Phi_a / T + \lambda_{2a} \right) = 0$$

or with the use of Eq. (23),

$$\frac{2}{3} N_c \Omega_c \tilde{\Phi} \lambda_{1c} + N_c \Omega_c \lambda_{2c} + 2 N_a \Omega_a \lambda_{2a} = 0 \quad (24)$$

where

$$\tilde{\Phi} \equiv -\frac{e}{T} (\Phi_a - \Phi_c) > 0, \quad N_{a,c} = \int \frac{ds}{B} n_{e,a,c}. \quad (25)$$

Further, the phase space moment of Eq. (20b) yields,

$$\frac{4\pi i}{m_e^2} \int dE d\mu \tau_a F_e \bar{C}(g_a) = 2 \lambda_{1a} \omega_e^* \Omega_a N_a \frac{e \Delta \psi}{T} = \lambda_{1c} \omega_e^* \Omega_c N_c \frac{e \Delta \psi}{T}. \quad (26)$$

Substituting Eq. (26) into Eq. (17), then yields,

$$\begin{aligned} D_e(\omega) &= \frac{-8\pi e^2}{T m_e^2} \int_p dE d\mu F_e \Delta \psi^2 \frac{\Omega_a \Omega_c}{\Omega_a + \Omega_c} (\omega - \omega_T^*) \\ &+ \frac{2\lambda_{1a} e^2}{T} N_a \Omega_a \Delta \psi^2 \omega_e^* \\ &- 2m^2 \int_c \frac{ds}{B^2 r} \kappa \frac{\partial p_e}{\partial a} \psi_c^2 - 4m^2 \int_a \frac{ds}{B^2 r} \kappa \frac{\partial p_e}{\partial \alpha} \psi_a^2. \end{aligned} \quad (27)$$

We still need to evaluate the coefficient λ_{1a} , for which it is necessary to solve Eqs. (20a) and (20b) with the boundary condition given by Eq. (19). The solution for the central cell perturbed distribution is taken as a Maxwellian up to the separatrix, which is justified if Eq. (22) is satisfied. Then we define \tilde{g}_a from the relation

$$g_a = \frac{e \Delta \psi}{T} \omega_e^* \left[\lambda_{1c} - \frac{\omega - \omega_T^*}{\omega_e^*} + \lambda_{2c} \left(\frac{m_e v^2}{2T} - \frac{3}{2} - \tilde{\Phi} \right) + \tilde{g}_a \right].$$

where we have used that, $\frac{m_e v_a^2}{2} = \frac{m_e v_c^2}{2} - \tilde{\Phi} T$, where v_a and v_c are the electron speed in the anchor and central cell respectively. This relation follows from energy conservation of an orbit at the separatrix. By using Eqs. (19) and (21b), we find that on the separatrix,

$$\tilde{g}_a(\text{separatrix}) = 0. \quad (28)$$

From Eq. (20), we find that \tilde{g}_a satisfies the equation,

$$\bar{C}_\ell(\tilde{g}_a) = -S_{\text{wave}} \quad (29)$$

where

$$S_{\text{wave}} = i\Omega_a \left[\tilde{g}_a + A + B \left(\frac{m_e v^2}{2T} - \frac{3}{2} \right) \right] \quad (30)$$

with

$$\begin{aligned} A &= \lambda_{1c} - \lambda_{2c} \tilde{\Phi} - \frac{\omega}{\omega_e^*} + 1 - \eta \tilde{\Phi}_c - \eta \tilde{\Phi} \\ B &= \lambda_{2c} + \eta \\ \tilde{\Phi}_c &= -e\Phi_c/T. \end{aligned}$$

Here S_{wave} serves as an effective source of particles and energy. The solutions can be expressed in terms of the linearized Pastukhov problem which will be formulated in the following section.

III Solution in Terms of the Linearized Pastukhov Problem

We wish to relate the solution to Eq. (29) to the solution of the Pastukhov problem, which can be formulated as follows (for simplicity we limit ourselves to electron confining square-well potential of magnitude $|e\Phi|$, where one has an analytic solution for the Pastukhov problem).

Given a source $S_{e0}(v)$ of electrons, the particle loss rate ν_p and energy loss rate ν_{pE} is given by,

$$\begin{aligned} \nu_p &= \frac{\int d^3v S_{e0}(v)}{\int d^3v F_p(v)} \\ \nu_{pE} &= \frac{\int d^3v v^2 S_{e0}(v)}{\int d^3v v^2 F_p(v)} \end{aligned} \quad (31)$$

where $F_p(v)$ satisfies the Fokker-Planck equation,

$$C(F_p) = -S_{e0}(v) \quad (32)$$

with

$$C(F_p) = \int d^3v' \sum_j \frac{b_j}{m_e} \frac{\partial}{\partial \mathbf{v}} \cdot \frac{g^2 \mathbf{I} - \mathbf{g}\mathbf{g}}{g^3} \cdot \left(\frac{1}{m_e} \frac{\partial}{\partial \mathbf{v}} - \frac{1}{m_j} \frac{\partial}{\partial \mathbf{v}'} \right) F_p(\mathbf{v}) F_{pj}(\mathbf{v}') \quad (33)$$

and we have the boundary condition that $F_p(v)$ vanishes on the separatrix.

As long as $S_{e0}(\mathbf{v})$ is not too peaked near the separatrix, the values of ν_p and ν_E are insensitive to the precise form of $S_{e0}(\mathbf{v})$ if $|e\Phi|/T > 1$ ($|e\Phi|$ is the potential confining the electrons). The reason for this assertion is explained in Ref. 3. We further note that for the bulk of the distribution $F_p(v)$ is Maxwellian with a density n_0 and temperature T . It only deviates from a Maxwellian near the separatrix, which to an exponential small factor, does not change the moments of F_p . We write

$$F_p(v) = F_0(\mathbf{v})G(\mathbf{v}) \quad (34)$$

with

$$F_0 = \frac{n_0}{(2\pi T/m_e)^{3/2}} \exp\left(-\frac{m_e v^2}{2T}\right),$$

and then

$$\nu_p = \frac{1}{n_0} \int d^3v S_{e0}(v) ; \quad \nu_{pE} = \frac{m_e}{3n_0 T} \int d^3v v^2 S_{e0}(v). \quad (35)$$

If we substitute Eq. (34) into Eq. (33), we find,

$$C(F_p) \equiv \int d^3v' \sum_j \frac{b_j}{m_e} \frac{\partial}{\partial \mathbf{v}} F_0(v) F_{0j}(v') \frac{(\mathbf{I}g^2 - \mathbf{g}\mathbf{g})}{g^3} \frac{\partial}{\partial \mathbf{v}} G(\mathbf{v}) = -S_{e0}(\mathbf{v}) \quad (36)$$

where we have neglected the term $F_0(v') \partial G(\mathbf{v}') / \partial \mathbf{v}'$ as $\partial G(\mathbf{v}') / \partial \mathbf{v}'$ vanishes except near the separatrix. The contribution near the separatrix can be neglected as the Maxwellian weighting in the \mathbf{v}' integration produces an exponential small error. Equation (36) is the form of the collision operator used by Pastukhov⁶ and subsequent investigators,^{10,11} for solving for ν_p . It is found that

$$\begin{aligned} \nu_p &= 2 \left(\frac{1}{\pi}\right)^{1/2} \nu_e \frac{\left|\frac{e\Phi}{T}\right|^{1/2} (1+Z) \exp\left[-\frac{|e\Phi|}{T}\right]}{\ln[2R(1+Z)]}, \\ \nu_{pE} &= (1 + |e\Phi|/T) \nu_p, \\ \nu_e &= \frac{\sqrt{2\pi} n_0 e^4 \lambda_e}{m_e^{1/2} |e\Phi|^{3/2}}. \end{aligned} \quad (37)$$

where Z is the mean ion atomic number, R the mirror ratio in the anchor and Λ_e the Coulomb logarithm.

For the actual density and temperature we must solve the nonlinear algebraic problems posed by Eqs. (35) and (37) which can be rewritten as,

$$n_0 = \frac{\int S_{e0} d^3v}{\nu_p(n_0, T)} \quad (38)$$

$$\frac{3}{2} n_0 T = \frac{m_e \int S_{e0} v^2 d^3v}{2(|e\Phi|/T + 1)\nu_p(n_0, T)}. \quad (39)$$

These two nonlinear equations determine n_0 and T .

Now let us suppose that the source $S_{e0}(v)$ changes by a small amount δS and that the separatrix does not change. The density and temperature will change by δn and δT , and Eqs. (38) and (39) become,

$$n_0 + \delta n = \frac{\int S_{e0} d^3v + \int \delta S d^3v}{\nu_p(T, n_0) + \Delta T \frac{\partial \nu_p}{\partial T} + \delta n \frac{\partial \nu_p}{\partial n}} \quad (40)$$

$$\frac{3}{2} n_0 T + \frac{3}{2} \delta n T + \frac{3 n_0 \delta T}{2} = \frac{m_e \int d^3v v^2 (S_{e0} + \delta S)/2}{\nu_p E(T, n_0) + \delta T \frac{\partial \nu_{pE}}{\partial T} + \delta n \frac{\partial \nu_{pE}}{\partial n}}. \quad (41)$$

Now linearizing all the small quantities, and using the equilibrium relations, and redefining $-e\Phi/T \equiv \tilde{\Phi}$, we find,

$$2\nu_p \delta n + n_0 \delta T \frac{\partial \nu_p}{\partial T} = \int \delta S d^3v \quad (42)$$

$$2\nu_p \delta n + n_0 \delta T \frac{\partial \nu_p}{\partial T} + \nu_p \frac{\delta T n_0}{T(1 + \tilde{\Phi})} = \frac{m_e}{3T(1 + \tilde{\Phi})} \int \delta S v^2 d^3v. \quad (43)$$

Subtracting Eq. (42) from Eq. (43) yields,

$$\frac{\nu_p \delta T n_0}{T(1 + \tilde{\Phi})} = \frac{m_e}{3T(1 + \tilde{\Phi})} \int d^3v v^2 \delta S - \int d^3v \delta S. \quad (44)$$

Thus, δn and δT are determined from Eqs. (42) and (44) and the solution of this linearized Pastukhov problem is obtained. We still need to relate the linearized Fokker-Planck operator determining the linearized Pastukhov problem to the Fokker-Planck operator given in Eq. (29). We can then obtain a solution to our desired wave problem.

If we linearize the Fokker-Planck operator given in Eq. (36) about $F_p = F_0 G(\mathbf{v})$, we have,

$$F(v) = F_0(v)G(v) + \delta F$$

$$\delta F = F_0(v) h(v).$$

Then

$$C(F_p + \delta F) - C(F_p) \equiv C_{\ell p}(h(v)) = -\delta S. \quad (45)$$

With

$$\begin{aligned} C_{\ell p}(h(\mathbf{v})) &= \sum_j \int d^3 \mathbf{v}' \frac{b_j}{m_e} \frac{\partial}{\partial \mathbf{v}} \cdot \frac{(g^2 \mathbf{I} - \mathbf{g}\mathbf{g})}{g^3} \cdot \left(\frac{1}{m_e} \frac{\partial}{\partial \mathbf{v}} - \frac{1}{m_j} \frac{\partial}{\partial \mathbf{v}'} \right) \\ &\cdot [F_0(v)G(\mathbf{v})F_{0j}(v')h_j(\mathbf{v}') + F_{0j}(\mathbf{v}')G_j(\mathbf{v}')F_0(v)h(\mathbf{v})] \\ &= \sum_j \int d^3 \mathbf{v}' \frac{b_j}{m_e} \frac{\partial}{\partial \mathbf{v}} \cdot \left[F_0(v)F_{0j}(v') \frac{(I g^2 - \mathbf{g}\mathbf{g})}{g^3} \right] \\ &\cdot \left[\frac{h_j(\mathbf{v}')}{m_e} \frac{\partial}{\partial \mathbf{v}} G_e(\mathbf{v}) - \frac{G_e(\mathbf{v})}{m_j} \frac{\partial h_j(\mathbf{v}')}{\partial \mathbf{v}'} + \frac{G_j(\mathbf{v}')}{m_e} \frac{\partial h_e(\mathbf{v})}{\partial \mathbf{v}} - \frac{h_e(\mathbf{v})}{m_j} \frac{\partial}{\partial \mathbf{v}'} G_j(\mathbf{v}') \right]. \quad (46) \end{aligned}$$

We note that over most of phase space the solution for the electron distribution

$$C_{\ell p}(h_e(\mathbf{v})) = -\delta S$$

will be of the form

$$h_e(v) = \frac{\delta n_e}{n_0} + \frac{\delta T_e}{T} \left[\frac{m_e v^2}{2T} - \frac{3}{2} \right] \quad (47)$$

where $\delta n_e = \delta n_i = \delta n$ with δn defined in Eq. (40), and $\delta T_e = \delta T$ with δT defined in Eq. (40). We find that for the ion term we can have $h_i(\mathbf{v})$ as any arbitrary function since only the zeroth-order moment of $h_i(v)$ couples into the problem as the electron-ion energy exchange in the collision operator is negligible. This solution fails near the separatrix when $h_j(v)$ vanishes. Nonetheless, Eq. (47) is suitable for determining moments, as the deviation of the solution near the separatrix only introduces an exponentially small error if $\tilde{\Phi} \gg 1$. For similar reasons we can set $\partial G(\mathbf{v}')/\partial \mathbf{v}' = 0$ and $G(\mathbf{v}') = 1$, in the last two terms in the bracket of Eq. (46). The $G(\mathbf{v})$ in the second term of the bracket of Eq. (46) is one nearly everywhere except near the separatrix. As $G(\mathbf{v})$ is equal to zero on the separatrix this term does not directly produce particle flux. When $G(\mathbf{v})$ is replaced by unity, there is direct particle flux from this term, but it can be shown to be small by a factor $1/\tilde{\Phi}$ compared to the direct flux of the third term. Hence to order $1/\tilde{\Phi}$ we can replace $G(\mathbf{v})$ by unity in the second term. In the first term, we can replace $h(\mathbf{v}')$ by

$$h(\mathbf{v}') = \frac{\delta n}{n_0} + \frac{\delta T}{T} \left(\frac{1}{2} \frac{m_e v'^2}{T} - \frac{3}{2} \right). \quad (48)$$

The first term in the bracket of Eq. (46) then becomes (note that the δT term in the electron contribution vanishes to leading order, as $v^2 \sim \Phi/m_e \gg v'^2 \approx T/m_e$, $\mathbf{g} \doteq \mathbf{v}$),

$$\frac{\delta n}{n_0} \sum_j \int d^3 v' \frac{b_j}{m_j} \frac{\partial}{\partial \mathbf{v}} F_0(v) F_{0j}(v') \cdot \frac{\mathbf{I}g^2 - \mathbf{g}\mathbf{g}}{g^3} \cdot \frac{\partial G_e(\mathbf{v})}{m_e \partial \mathbf{v}} = \frac{-\delta n}{n_0} S_{e0}(\mathbf{v}).$$

where we used Eq. (36) to obtain the right-hand side.

We thereby find

$$C_{\ell p}(h) = C_{\ell}(h) - \frac{\delta n}{n_0} S_{e0}(v) = -\delta S \quad (49)$$

where $C_{\ell}(h)$ is given in Eq. (8) and is the linearized Fokker-Planck operator we are interested in for the wave problem. Comparing Eq. (49) with Eq. (29), shows that

$$\delta S = S_{\text{wave}} + \frac{\delta n}{n_0} S_{e0}(v). \quad (50)$$

Now observe that the equation and boundary conditions for h are identical to that for \tilde{g}_a if S_{wave} is given by Eq. (30); hence $h = \tilde{g}_a$.

Now substituting Eq. (50) into Eq. (42) and (44), and using $T \frac{\partial \nu_p}{\partial T} = \tilde{\Phi} \nu_p (1 + \mathcal{O}(1/\tilde{\Phi}))$, and Eq. (37), we find,

$$\begin{aligned} \nu_p \delta n + \frac{n_0 \delta T}{T} \tilde{\Phi} \nu_p &= \int S_{\text{wave}} d^3 v \\ \frac{\delta T}{T} \frac{\nu_p}{\tilde{\Phi}} n_0 &= \int d^3 v \left(\frac{m_e v^2}{3 \tilde{\Phi} T} S_{\text{wave}} - S_{\text{wave}} \right) + \mathcal{O}(1/\tilde{\Phi}). \end{aligned} \quad (51)$$

Now using Eq. (30) and (47) for $h(\mathbf{v})$, we find,

$$\nu_p \delta n + n_0 \frac{\delta T}{T} \tilde{\Phi} \nu_p = i \Omega_a (\delta n + n_0 A) \quad (52)$$

$$\frac{\delta T}{T} \frac{\nu_p}{\tilde{\Phi}} n_0 = i \Omega_a \left[-\delta n - n_0 A + \frac{n_0}{\tilde{\Phi}} \left(\frac{\delta T}{T} + B \right) \right] \left[1 + \mathcal{O} \left(\frac{1}{\tilde{\Phi}} \right) \right], \quad (53)$$

Now solving for δn and δT we find

$$\frac{\delta n}{n_0} = - \frac{-\Omega_a (\Omega_a + i \nu_p) - i \tilde{\Phi}^2 \nu_p \Omega_a + i \tilde{\Phi} B \nu_p \Omega_a}{(\Omega_a + i \nu_p)^2 + i \tilde{\Phi}^2 \nu_p \Omega_a} \quad (54)$$

$$\frac{\delta T}{T} = \frac{i \Omega_a [\nu_p A \tilde{\Phi} + i B (\Omega_a + i \nu_p)]}{(\Omega_a + i \nu_p)^2 + i \tilde{\Phi}^2 \nu_p \Omega_a}. \quad (55)$$

The distribution we are interested in, g_a , for the anchor response is

$$g_a = \omega_e^* \frac{e \Delta \psi}{T} \left[A + B \left(\frac{m_e v^2}{2T} - \frac{3}{2} \right) + \frac{\delta n}{n_0} + \frac{\delta T}{T} \left(\frac{m_e v^2}{2T} - \frac{3}{2} \right) \right]. \quad (56)$$

Then combining Eqs. (54)-(56) we have,

$$g_a = \frac{i\nu_p e \Delta \psi \omega_e^* \left\{ A(\Omega_a + i\nu_p) + \tilde{\Phi} B \Omega_a + \left(\frac{m_e v^2}{2T} - \frac{3}{2} \right) [B(\Omega_a + i\nu_p) + B \Omega_a \tilde{\Phi}^2 + A \Omega_a \tilde{\Phi}] \right\}}{T[(\Omega_a + i\nu_p)^2 + i\tilde{\Phi}^2 \nu_p \Omega_a]} \quad (57)$$

Comparing Eq. (57) with Eq. (21), then determines the two additional relations between the λ coefficients for our problem. Namely,

$$\lambda_{1a} = i\nu_p [A(\Omega_a + i\nu_p) + B\Omega_a \tilde{\Phi}] / R \quad (58)$$

$$\lambda_{2a} = i\nu_p [B(\Omega_a + i\nu_p + \Omega_a \tilde{\Phi}^2) + A\Omega_a \tilde{\Phi}] / R \quad (59)$$

with

$$R = (\Omega_a + i\nu_p)^2 + i\nu_p \Omega_a \tilde{\Phi}^2.$$

By substituting Eqs. (58) and (59) into Eqs. (23) and (24a), we find,

$$\begin{aligned} \lambda_{1c} U + \lambda_{2c} V &= -(a_1 Q + \eta b_1) \\ \lambda_{1c} W + \lambda_{2c} U &= -(a_2 Q + \eta b_2) \end{aligned} \quad (60)$$

with

$$\begin{aligned} U &= 2iN_a \Omega_a \nu_p (\Omega_a + i\nu_p) + N_c \Omega_c [(\Omega_a + i\nu_p)^2 + i\tilde{\Phi}^2 \nu_p \Omega_a] \\ V &= 2\nu_p^2 N_a \Omega_a \tilde{\Phi} \\ W &= \nu_p 2iN_a \Omega_a^2 \tilde{\Phi} + 2N_c \Omega_c \tilde{\Phi} [(\Omega_a + i\nu_p)^2 + i\tilde{\Phi}^2 \nu_p \Omega_a] / 3 \\ Q &= \omega / \omega_e^* - 1 + \eta \tilde{\Phi}_c \\ a_1 &= -2iN_a \Omega_a \nu_p (\Omega_a + i\nu_p) \\ a_2 &= -2iN_a \Omega_a^2 \nu_p \tilde{\Phi} \\ b_1 &= 2N_a \Omega_a \nu_p^2 \tilde{\Phi} \\ b_2 &= 2iN_a \Omega_a \nu_p (\Omega_a + i\nu_p). \end{aligned}$$

Now solving for λ_{1c} , and using $-N_c \Omega_c \lambda_{1c} = 2N_a \Omega_a \lambda_{1a}$, which is the quantity explicitly needed in the functional dispersional relation, we find,

$$2N_a \Omega_a \lambda_{1a} = -2i\nu_p N_a N_c \Omega_a \Omega_c R S / [U^2 - VW] \quad (61)$$

with

$$S = \left(\frac{\omega}{\omega_e^*} - 1 + \eta \tilde{\Phi}_c \right) [2i\nu_p N_a \Omega_a + N_c \Omega_c (\Omega_a + i\nu_p)] + i\eta \Omega_c \nu_p N_c \tilde{\Phi}.$$

The limiting forms of these equations are

$$-2N_a \Omega_a \lambda_{1a} \longrightarrow \begin{cases} 2i\nu_p N_a (\omega - \omega_e^* + \eta \tilde{\Phi}_c \omega_e^*) / \omega_e^*, & \Omega_a \gg \tilde{\Phi}^2 \nu_p \\ \frac{2N_c N_a \Omega_a \Omega_c [(\omega - \omega_e^* + \eta \tilde{\Phi}_c \omega_e^*)(\omega - \langle \omega_E \rangle) + \Omega_c \eta \omega_e^* \tilde{\Phi} N_c / N]}{N [(\omega - \langle \omega_E \rangle)^2 + 4\tilde{\Phi}^2 N_c N_a \Omega_a \Omega_c / (3N^2)] \omega_e^*}, & \nu_p \gg \tilde{\Phi}^2 \Omega_a \end{cases} \quad (62)$$

with $N = 2N_a + N_c$, $\langle \omega_E \rangle = (2\omega_{Ea} N_a + \omega_{Ec} N_c) / N$. The limiting result with $\Omega_a \gg \tilde{\Phi}^2 \nu_p$ was obtained in Ref. 3 (with $\eta = 0$ and $\omega_E = 0$). In Ref. 3 a low frequency boundary value regime is reported at very low collision frequency ($\omega \ll \nu_e$) which requires special boundary layer methods to obtain. We will not reproduce this regime in this work. The limiting result at very high collision frequency ($\nu_p \gg \tilde{\Phi}^2 \Omega_a$) was reported in Ref. 7. The results at intermediate results are new and our method bridges both limiting values.

IV Dispersion Relation

By using the results of the previous section for the response of collisional electrons, and the results of Refs. 2,3,8,9 for the response of collisionless ions (plus any other collisionless component such as a hot electron component in the anchor region), we find that the quadratic form $D(\omega)$, can be written as,

$$-D(\omega) \equiv R\psi_c^2 + S\psi_a^2 + W\Delta\psi^2 \quad (63)$$

$$\begin{aligned} R &= \int_0^{L_c} \frac{ds}{B} \left[\frac{n_i m_i c^2 k_\perp^2}{B^2} (\omega - \omega_{Ec}) (\omega - \omega_{Ec} - \omega_i^*) + m^2 \frac{\kappa c^2}{Br} \frac{\partial}{\partial \alpha} (P_\perp + P_\parallel) \right] \\ S &= 2 \int_{L_c}^\infty \frac{ds}{B} \left[\frac{n_i m_i c^2 k_\perp^2}{B^2} (\omega - \omega_{Ea}) (\omega - \omega_{Ea} - \omega_i^*) + m^2 \frac{\kappa c^2}{Br} \frac{\partial}{\partial \alpha} (P_\perp + P_\parallel) \right] \\ W &= \sum_j \frac{8\pi q_i^2}{T_j m_j^2} \int dE d\mu F_j \frac{\tau_a \tau_c (\omega - \omega_{Ea}) (\omega - \omega_{Ec}) (\omega - \omega_{Tj}^*)}{\omega \tau - \omega_{Ec} \tau_c - \omega_{Ea} \tau_a} \\ &\quad - 2\lambda_{1a} N_a \Omega_a \frac{\omega_e^* e^2}{T} \end{aligned} \quad (64)$$

where

$$P_\perp = \sum_j \frac{4\pi B^2}{m_j^2} \int \frac{dE d\mu \mu}{|v_\parallel|} F_j, \quad P_\parallel = \sum_j \frac{4\pi B}{m_j^2} \int dE d\mu |v_\parallel| F_j$$

$$\omega_i^* = \frac{m}{q_i n_{0i}} \frac{\partial}{\partial \alpha} P_{\perp i}.$$

In these expressions we have assumed that the passing ions as well as electrons can be taken as a Maxwellian distribution. The ion temperature in the central cell is T_i and the electron temperature in the central cell is $T_e = T$. We have neglected the collisional effects of ions, which is valid if $\nu_i/\omega \ll 1$ with ν_i the ion collisional frequency. We have also assumed for simplicity that the ions are in the high-bounce frequency limit ($\omega\tau_{bi} < 1$ with τ_{bi} the ion bounce time through the anchor region). For a more general response in other bounce frequency regimes the reader is referred to Ref. 2. The expression for λ_{1a} is given in Eq. (61).

By taking the variations in Eq. (63) with respect to ψ_a and ψ_c , we obtain the equations,

$$\begin{aligned} (R + W)\psi_c - W\psi_a &= 0 \\ -W\psi_c + (S + W)\psi_a &= 0. \end{aligned} \quad (65)$$

These equations lead the dispersion relation,

$$RS + W(R + S) = 0. \quad (66)$$

Using the dispersion relation we also find,

$$\frac{\psi_a}{\psi_c} = -\frac{R}{S}. \quad (67)$$

If $|W| \gg |R|, |S|$ or $\left| \frac{\partial W}{\partial \omega} \right| \gg |R/\omega|, |S/\omega|$, the lowest order dispersion relation is

$$W(R + S) = 0. \quad (68)$$

Thus, two types of modes are possible.

1. Flute modes, where

$$R + S = 0 \quad , \quad \psi_a/\psi_c = 1. \quad (69)$$

This mode is to lowest order independent of the collision properties and gives the usual MHD predictions for the flute interchange modes. Dissipative corrections only enter at first order where,

$$\delta\omega = \frac{RS}{W \frac{\partial}{\partial \omega}(R + S)}. \quad (70)$$

Here $\delta\omega$ is the correction to the lowest order frequency.

2. Trapped particle modes where to lowest order

$$W = 0, \quad (71)$$

while to next order

$$\delta\omega = -\frac{(R+S)}{RS} \frac{\partial W}{\partial \omega} \longrightarrow -\frac{1}{\frac{\partial W}{\partial \omega} R}, \quad \text{if } |R| \ll |S|. \quad (72)$$

If $|R|/|S| \ll 1$, then $|\psi_a/\psi_c| \ll 1$ and we have the usual polarization of the trapped particle mode. One should also note that if $S \gg W, R$, a trapped particle mode is also obtained, with $|\psi_a/\psi_c| \ll 1$, with the dispersion relation

$$R + W = 0. \quad (73)$$

This dispersion relation may give rapidly growing MHD-like trapped particle modes¹ if W is sufficiently small.

The dispersion relation has been investigated extensively in Ref. 2 in the collisionless case where $\lambda_{1a} = 0$. If $\Omega_a \gg \tilde{\Phi}^2 \nu_p$, λ_{1a} given in Eq. (62), introduces a dissipative contribution. We note that alternate forms for λ_{1a} are

$$2N_a \Omega_a \omega_e^* \lambda_{1a} \approx 2i\nu_p N_a (\omega - \omega_e^* + \eta \tilde{\Phi} \omega_e^*) = 2i\nu_p N_a (\omega - \omega_{enc}^* - \omega_{Ec}). \quad (74)$$

In the limit $\nu_p \gg \tilde{\Phi}^2 \Omega_a$, the response for λ_{1a} is reactive with considerable structure. We note that in this limit Eq. (62) can be written as

$$\frac{2N_a \Omega_a \omega_e^* \lambda_{1a} \frac{e^2}{T} =}{TN[\bar{\Omega}^2 + 4\tilde{\Phi} N_c N_a \Omega_a \Omega_c / N^2]} \left\{ \omega_{enc}^* \frac{N_c}{N} + 2\omega_{ena}^* \frac{N_a}{N} - \eta \tilde{\Phi} \omega_e^* (N_c - 2N_a) N \right\} + \frac{2N_a N_c \eta \omega_e^* \tilde{\Phi} (\omega_{Ea} - \omega_{Ec})}{N^2} \quad (75)$$

where $\bar{\Omega} = \omega - \langle \omega_E \rangle$, ω_{enc}^* and ω_{ena}^* refer to the central cell and anchor drift frequencies with gradients on the electron density. One may note that in this high collisional limit, the quantity W is much larger than R . Hence, the dispersion relation given in Eq. (71) is approximately $W = 0$, with roots

$$\omega = \omega_{Ea}, \quad \omega = \omega_{Ec} \quad (76)$$

and

$$\begin{aligned} \omega &= \langle \omega_E \rangle + \frac{1}{2} \left[\omega_{enc}^* \frac{N_a}{N} + 2\omega_{enc}^* \frac{N_a}{N} + \eta \tilde{\Phi} \omega_e^* \frac{(N_c - 2N_a)}{N} \right] \\ &\pm \frac{1}{2} \left[\left(\omega_{enc}^* \frac{N_c}{N} + 2\omega_{ena}^* \frac{N_a}{N} - \eta \tilde{\Phi} \omega_e^* \frac{(N_c - 2N_a)}{N_a} \right)^2 - \frac{8N_a N_c}{N^2} \eta \omega_e^* \tilde{\Phi} (\omega_{Ea} - \omega_{Ec}) \right]^{1/2} \end{aligned} \quad (77)$$

One observes that there is instability if

$$\frac{8N_a N_c}{N^2} \eta \omega_e^* \tilde{\Phi} (\omega_{Ea} - \omega_{Ec}) > \left(\omega_{enc}^* \frac{N_c}{N} + 2\omega_{ena}^* \frac{N_a}{N} - \eta \tilde{\Phi} \omega_e^* \frac{(N_c - 2N_a)}{N_a} \right)^2. \quad (78)$$

Thus, in the limit of very high electron collisional frequency one can obtain a fluid-like instability through the combination of temperature gradients, axial variation of the $\mathbf{E} \times \mathbf{B}$ drifts and the potential variation along the field line. These modes were reported in Ref. 7.

V Numerical Results

The previous sections give the detailed derivation of the dispersion relation of the dissipative trapped particle modes. In this section, a set of numerical results for the eigenfrequency (denoted as $\omega = \omega_R + i\gamma$) of Eq. (66) and discuss their relationship with analytic calculations. We also compare some results of the present theory with past theory³ that did not use energy conservation. A more complete application of our theory to concrete experimental situations is currently under investigation.

The parameters that are used for our numerical study, in units of ω_{ic}^* for frequency, are: $\omega_e^* = -1$, $\omega_{ia}^* = 2$, $N_c/N = 0.5$, $B_a/B_c = 2$, $N_{ip}/N_{ep} = 0.2$, $\hat{\gamma}_c^2 = 1$, $\hat{\gamma}_a^2 = -5$, $B_{\max}/B_c = 8$, $\tau_c/\tau = 0.9$ and $2L_a/L = .2$, where $2L_a$ is the total length of the two anchor regions, L the overall length of the system, B_{\max} the maximum of the magnetic field along the field line, the passing particle density in anchor region $n_{ps} = n_c(B_c/B_{\max})(T/|e\phi|\pi)^{1/2}$ or $N_{pc}/N_c = (B_c/B_{\max})(T/|e\phi|\pi)^{1/2} \cdot (L_p/L)$ and $\hat{\gamma}_{c,a}^2$ is defined as follows

$$\hat{\gamma}_{c,a}^2 = \frac{B_c^2}{\omega_e^{*2} m_i k_{\perp}^2} \int_0^{L_c, 2L_a} \frac{ds}{B_c^2} \frac{\kappa m^2}{r} \frac{\partial}{\partial \alpha} (P_{\perp} + P_{\parallel}).$$

One of the parameters that will be used in this study is λ^2 which is defined as

$$\lambda^2 = \frac{q^2 B_c^2}{T m_i c^2 k_{\perp}^2} \sim \frac{r^2}{m^2 \rho_L^2}$$

where ρ_L is the mean ion gyroradius using electron temperature, $\frac{m}{r}$ the perpendicular wavenumber of the perturbations.

In Fig. 2(a) and Fig. 2(b), we present the eigenfrequencies vs. Pastukhov rate ν_p for different choices λ^2 (other parameters $\eta = \omega_{Ec} = \omega_{Ea} = 0$).

The figures show that when λ^2 and ν_p are relatively large, the eigenfrequency ω_R is approaching its asymptotic value ($\omega_e^* = -1$). The correction to the real frequency given by Eq. (70) agrees with the numerical results for large ν_p . The shift of the real frequency from ω_e^* at large ν_p differs appreciably from the theory of Ref. 2 for these parameters. The shift is a factor 2.5 larger than that in the previous theory. In Fig. 3, the comparison between growth rate of the previous theory and present theory is plotted. (In the previous theory, it was assumed that $\eta = \omega_{Ec} = \omega_{Ea} = 0$, hence we choose these parameters for our comparison).

The curves show that both theories give the same growth rate when ν_p is small, whereas, when ν_p becomes larger, the present theory predicts a higher growth rate. This result is expected from the structure of Eq. (62).

In Fig. 2(c), we plot the ratio ψ_a/ψ_c for the eigenmodes. We note that there is a substantial component of ψ_a with these parameters.

In Fig. 4(a), (b) and (c), we present the real frequency; growth rate and ψ_a/ψ_c ratio at small ν_p . We note that we have chosen parameters that correspond to an unstable trapped particle mode in the collisionless theory. The transition to trapped particle theory occurs when

$$\frac{\nu_p}{\omega_e^*} \lesssim \frac{N_{ps}}{N_c}.$$

In Fig. 5, we present growth rate γ vs. ν_p for different $\tilde{\Phi}$ (other parameters are the same as that in Fig. 2).

In fact, we can analytically obtain the growth rate in the limit $\tilde{\Phi}^2\omega/\nu_p \ll 1$. By substituting a formal solution $\omega = \omega_e^* + \delta\omega + i\gamma$ into expansion of Eqs. (64) and (66) in terms of ν_p^{-1} and doing tedious but straightforward manipulations, we find

$$\frac{\delta\omega}{\omega_e^*} \propto \tilde{\Phi}^2$$

and

$$\frac{\nu}{\omega_e^*} \propto \frac{\delta\omega}{\nu_p} \propto \tilde{\Phi}^2 \left| \frac{\omega_e^*}{\nu_p} \right|.$$

The numerical curve in Fig. 5 verifies the scaling (also see Fig. 6, which makes a direct comparison between the analytic approximation and numerical calculation). The analysis implies that the depth of electrostatic well increases the instability drive.

In Fig. 7(a) and 7(b), we choose parameters such that the temperature gradient and rotational drift enter the problem. When $\omega_{Ec}/|\omega_e^*| = 1$, $\omega_{Ea} = 0$ and $\eta = 1$, the eigenmodes has a much larger growth rate, which is consistent with the analytic prediction of Eq. (75).

The behavior of the modes in the high ν_p regime is substantially different from that the previous example and instability exists at large ν_p even without dissipation. For $\omega \gg \frac{N_{ps}}{N_a}$, we again recover collisionless trapped particle theory with an electric field present.^{12,13} Note that the parameters in this example produce a rapidly growing trapped particle instability with the growth rate plotted in Fig. 8.

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Figure Captions

1. Axial variation of magnetic field, ambipolar and electric field rotational frequency about midplane of a model tandem mirror configuration.
2. Dissipative trapped particle modes as a function of $\nu_p/|\omega_e^*|$ with $\eta = \omega_{Ec} = \omega_{Ea} = 0$ and $\tilde{\Phi} = 3$ and (a) Real part of the eigenfrequency; (b) Imaginary part (growth rate); (c) ψ_a/ψ_c ratio of the modes.
3. Growth rate predicted by the previous theory³ and present theory with $\lambda^2 = 200$, $\eta = \omega_{Ec} = \omega_{Ea} = 0$ and $\tilde{\Phi} = 3$.
4. Trapped particle modes for small dissipation with $\eta = \omega_{Ec} = \omega_{Ea} = 0$ and $\tilde{\Phi} = 3$ and (a) Real part (b) Growth rate (c) ψ_a/ψ_c ratio.
5. Growth rate as a function of $\nu_p/|\omega_e^*|$ for different ambipolar well depths at $\eta = \omega_{Ec} = \omega_{Ea} = 0$ and $\lambda^2 = 200$.
6. Comparison between the analytic approximation and numerical results at $\lambda^2 = 200$, $\omega_{Ec} = \omega_{Ea} = \eta = 0$ and $\tilde{\Phi} = 3$.
7. Trapped particle modes with $\eta = 1$, $\omega_{Ea} = 0$, $\omega_{Ec}/|\omega_e^*| = 1$ and $\tilde{\Phi} = 3$ and (a) Real part; (b) Growth rate.
8. Imaginary part of the eigenfrequency for the small $\nu_p/|\omega_e^*|$. (See caption of Fig. 7 for other parameters.)

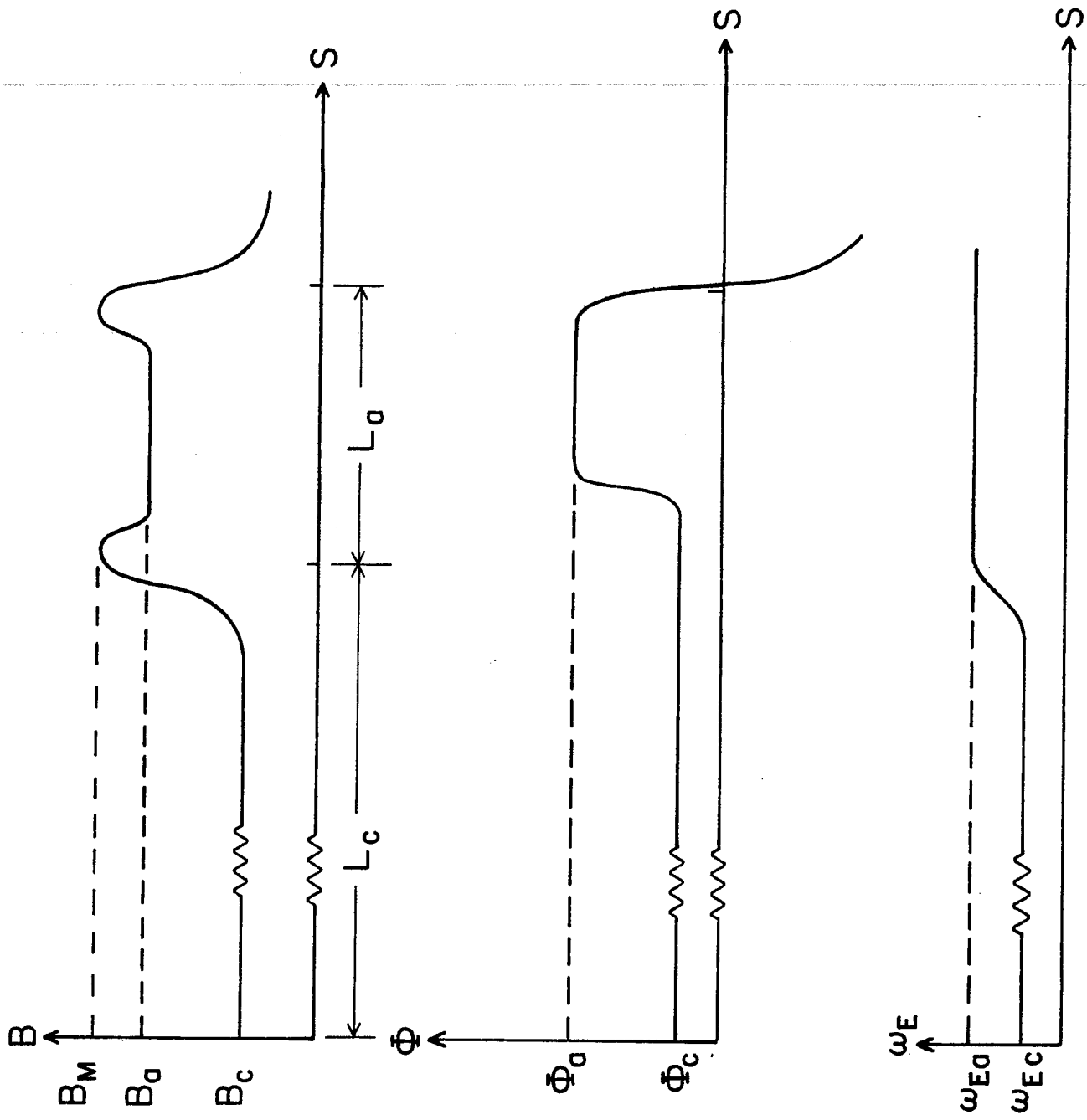


Fig. 1

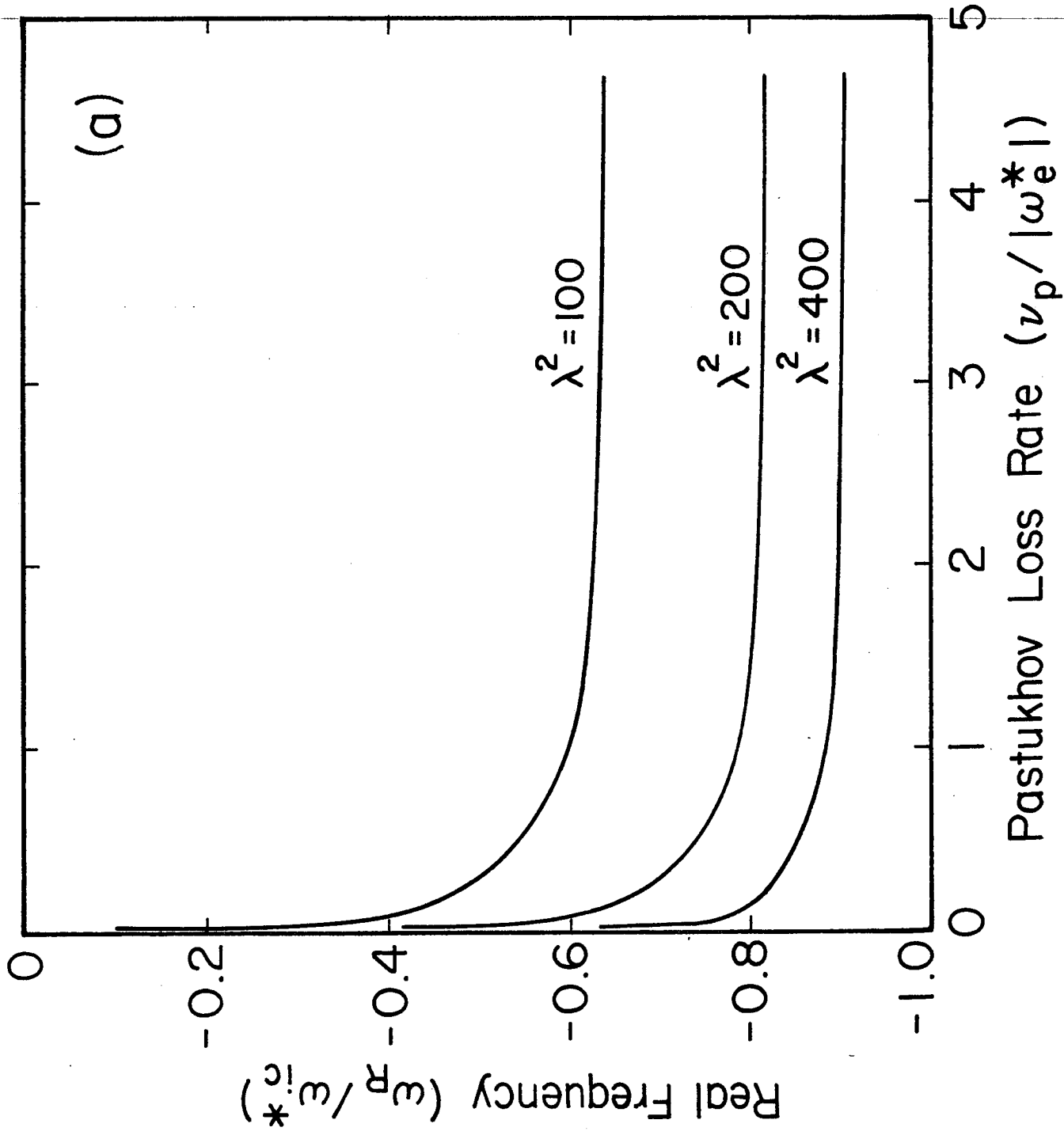


Fig. 2a

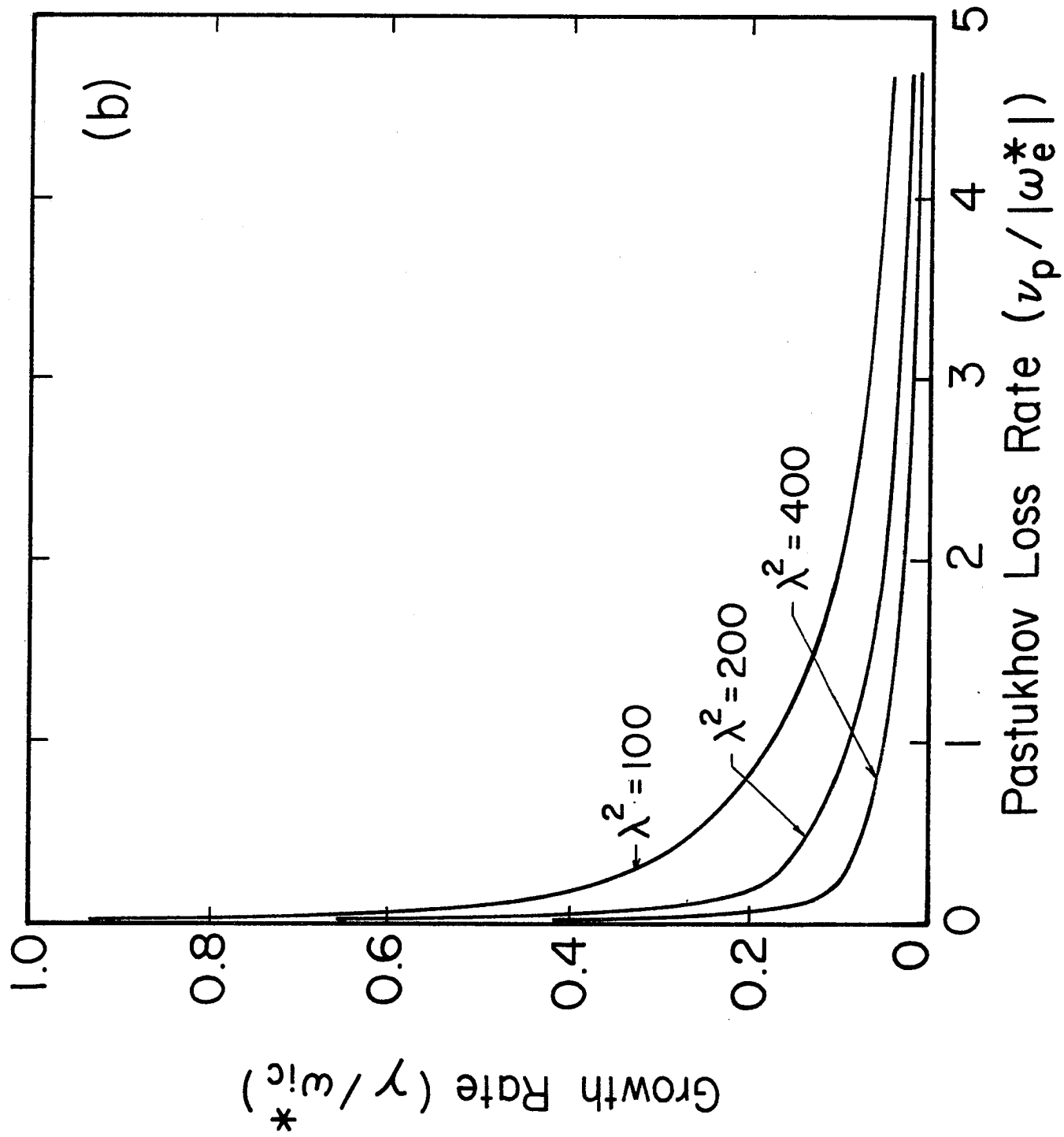


Fig. 2b

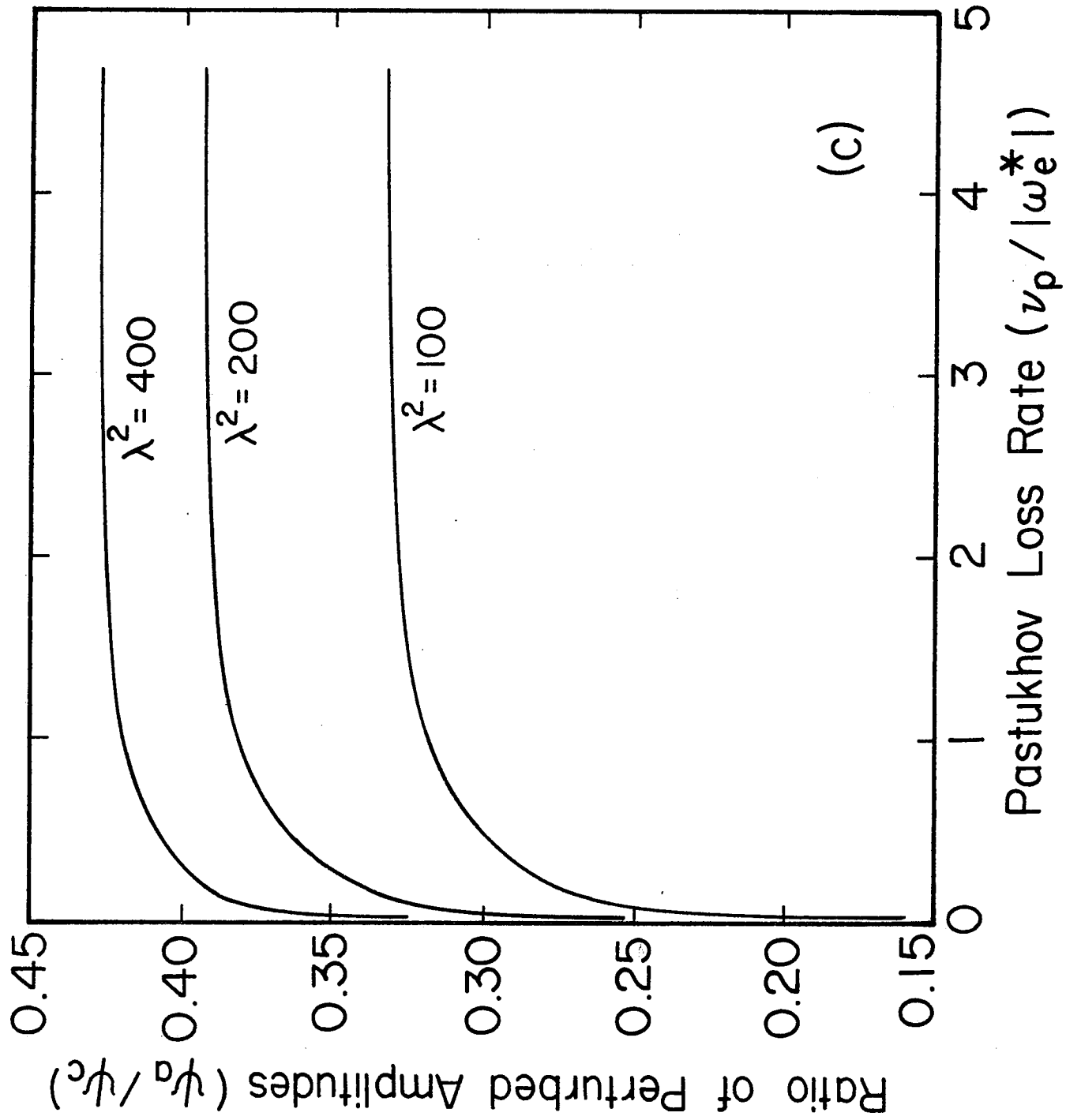


Fig. 2c

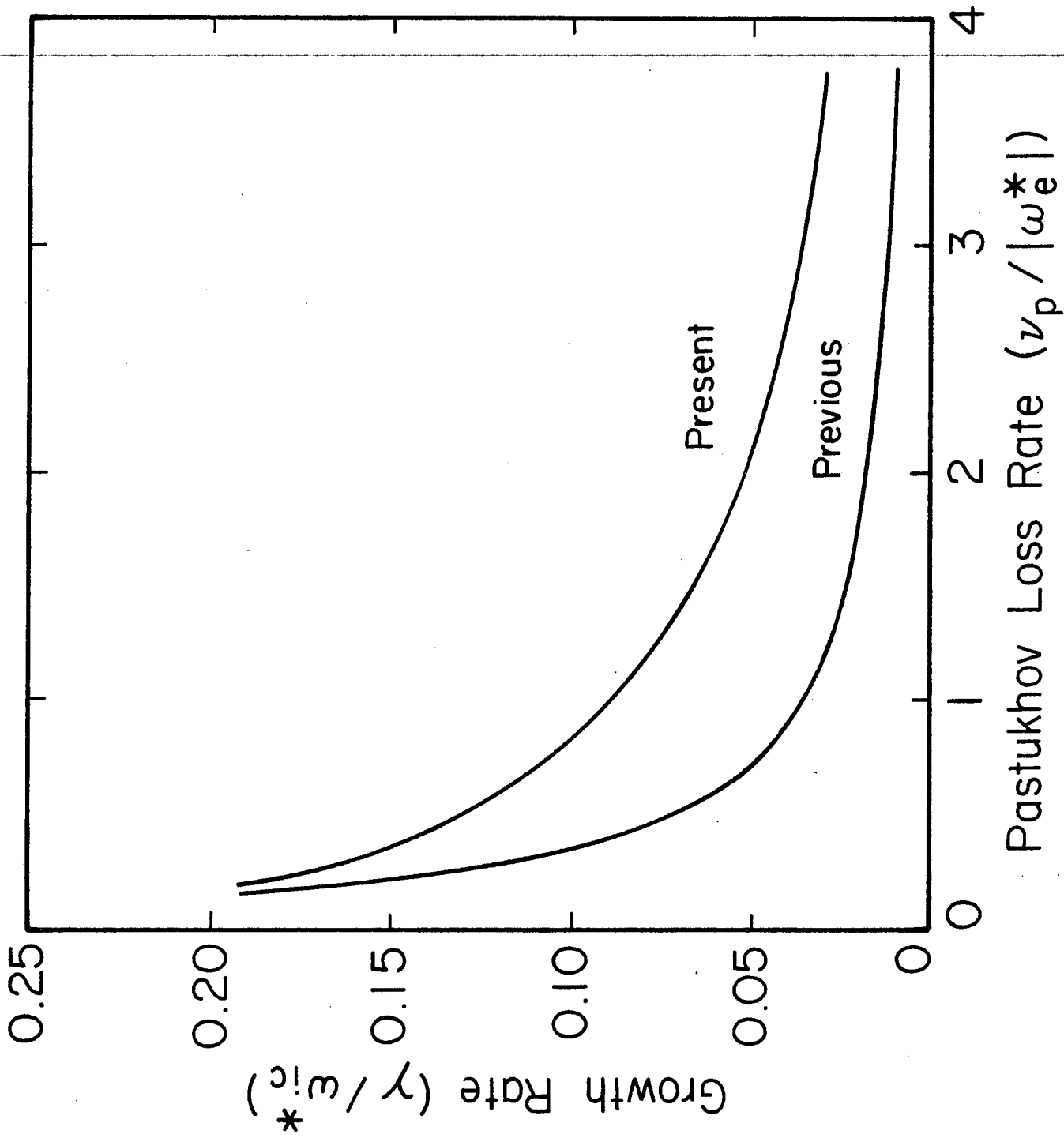


Fig. 3

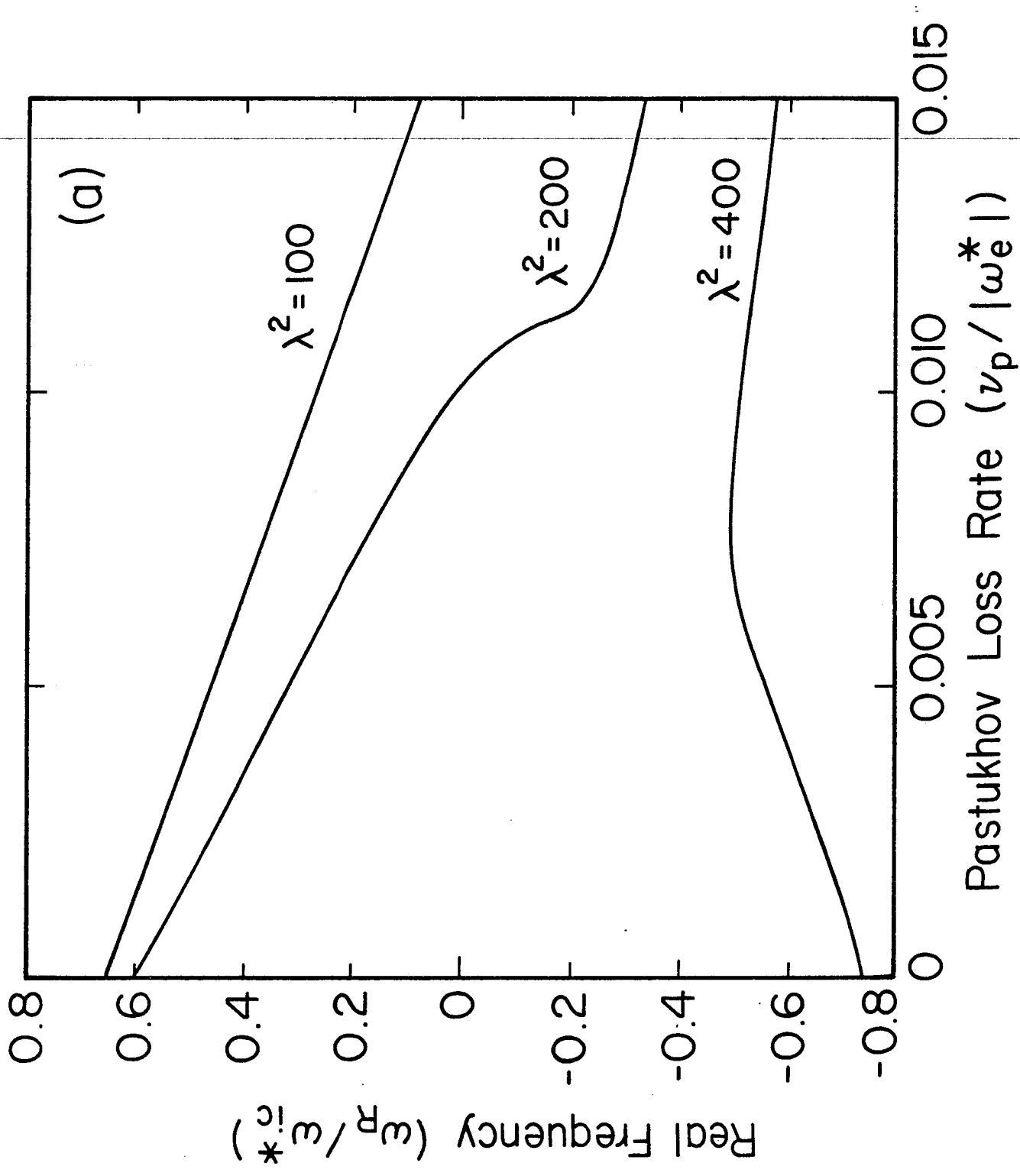


Fig. 4a

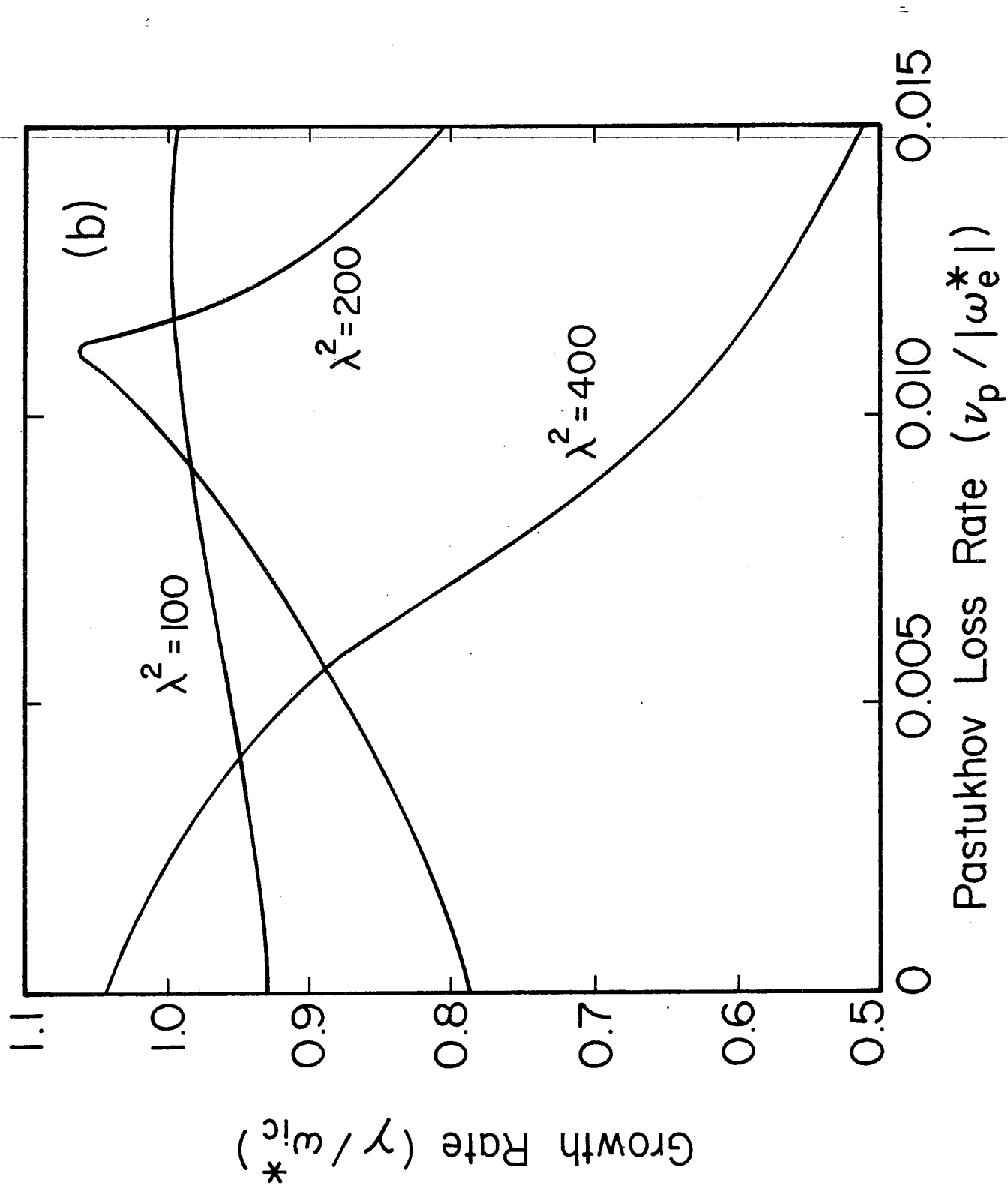


Fig. 4b

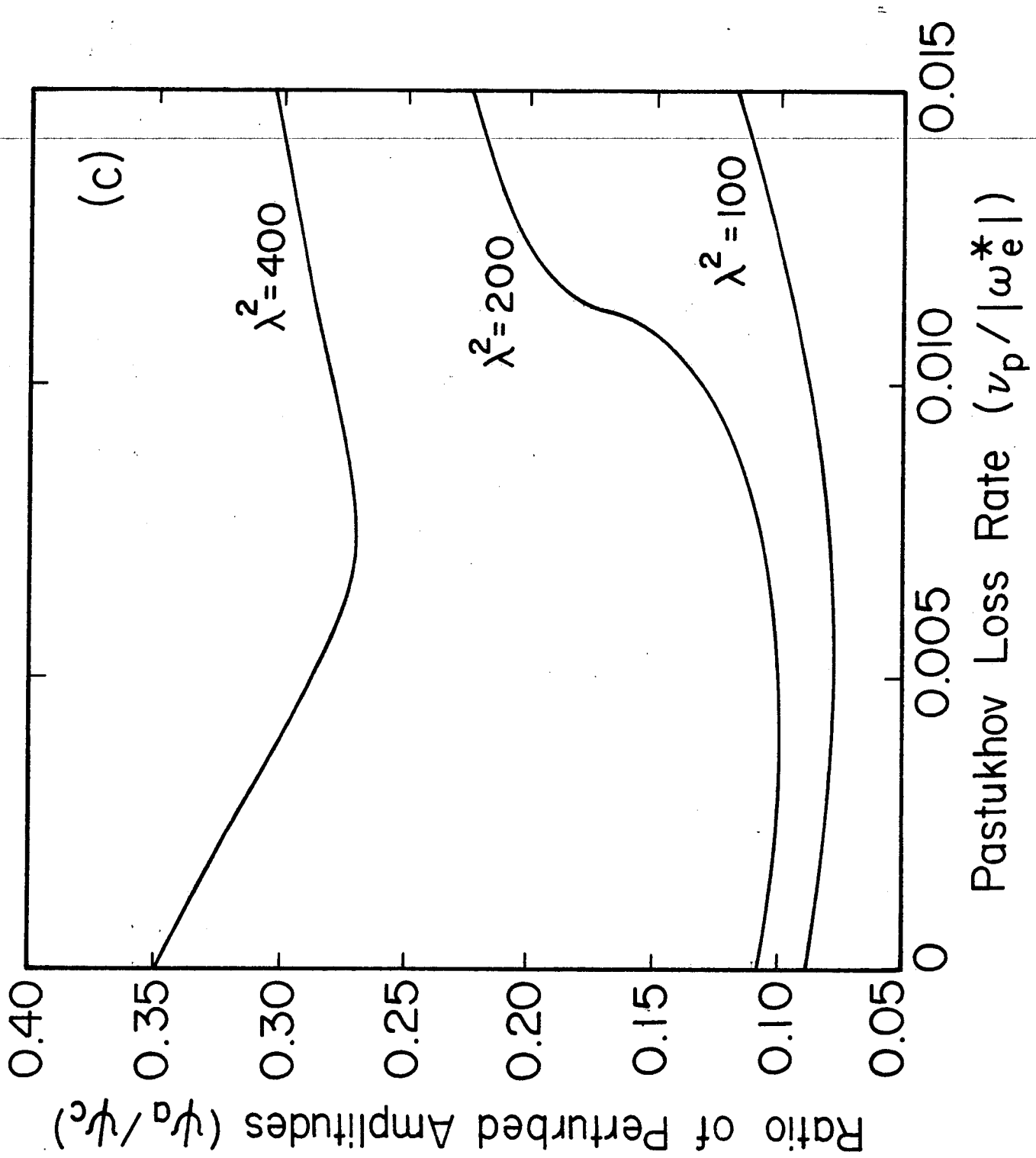


Fig. 4c

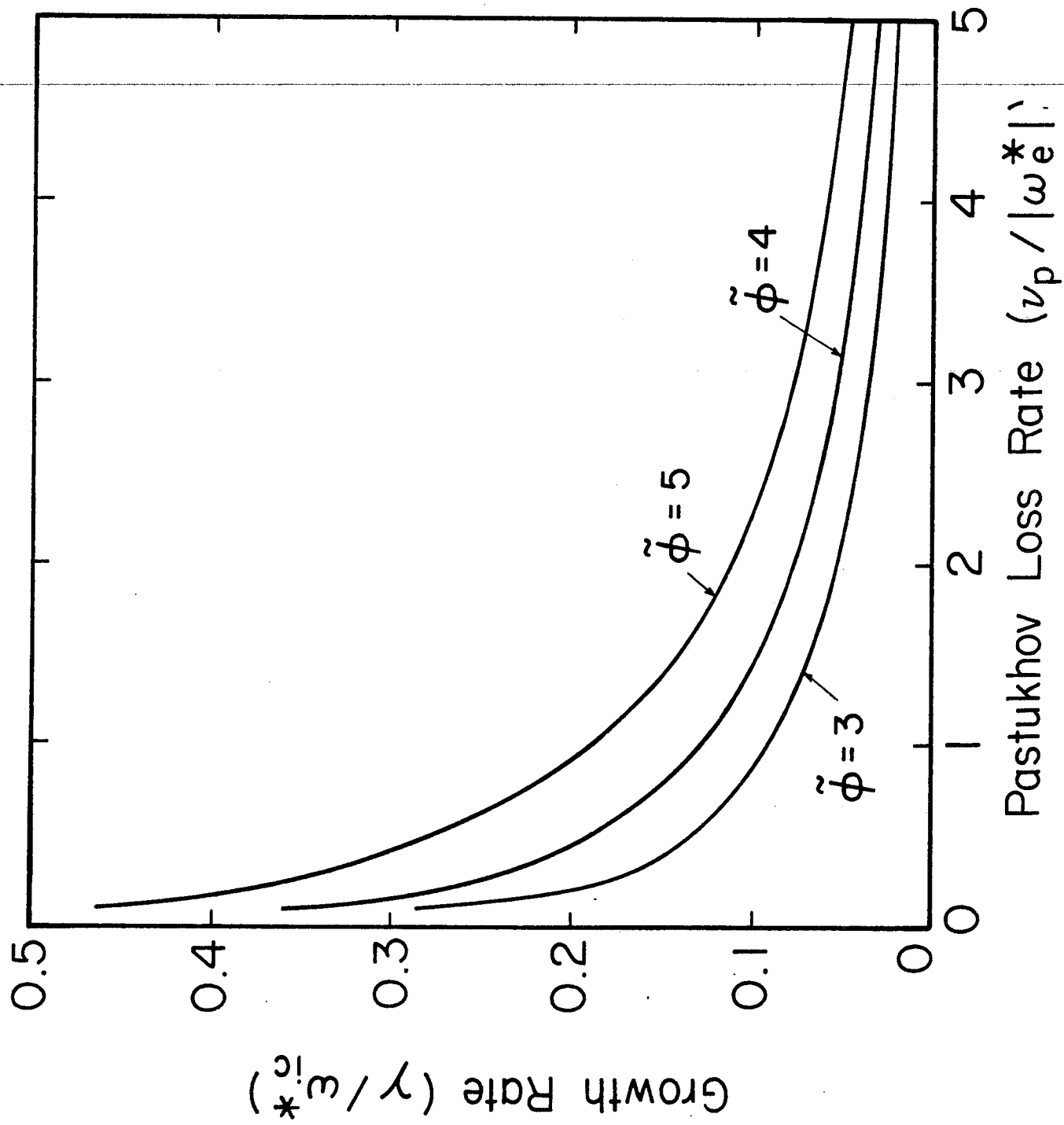


Fig. 5

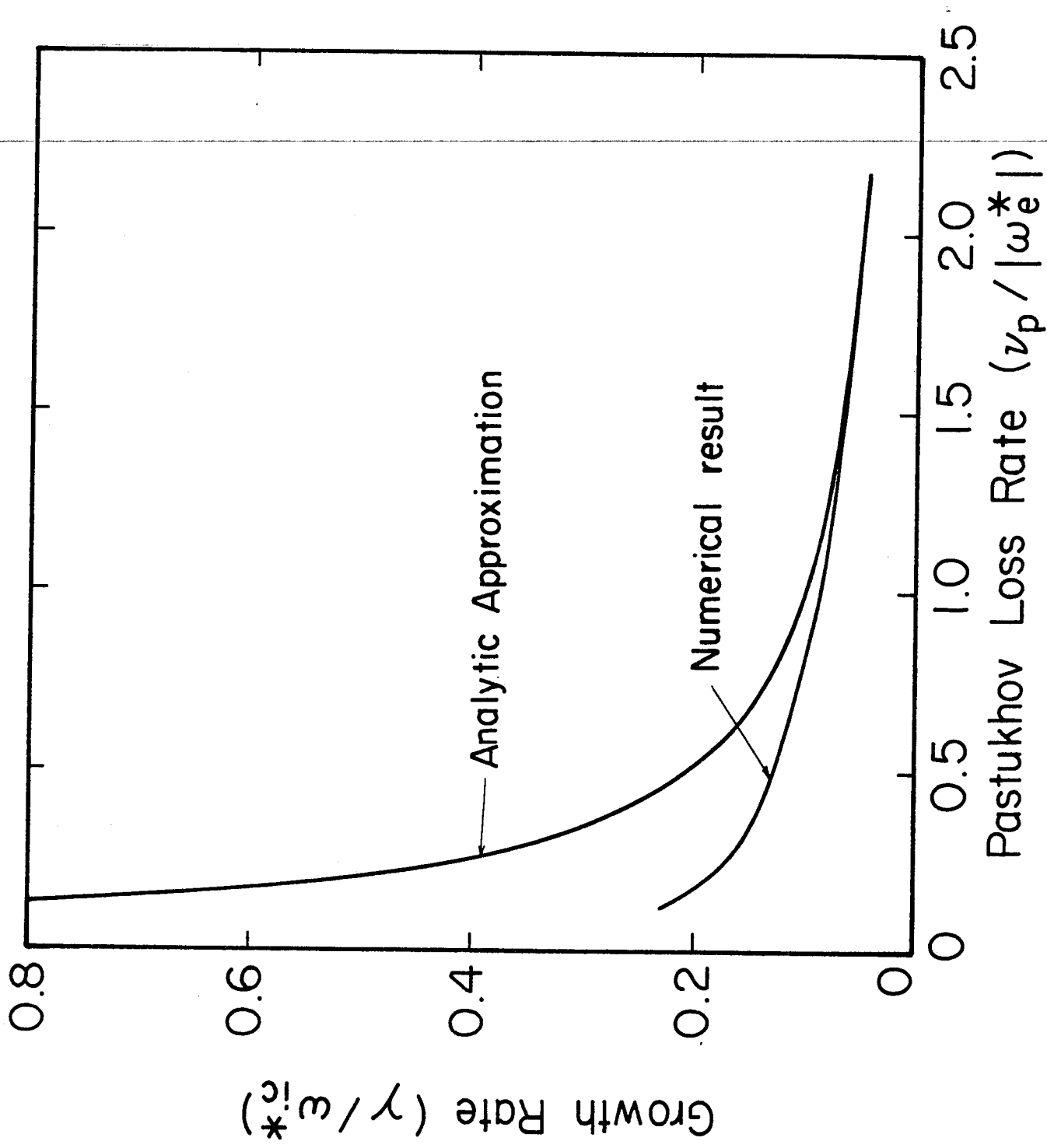


Fig. 6

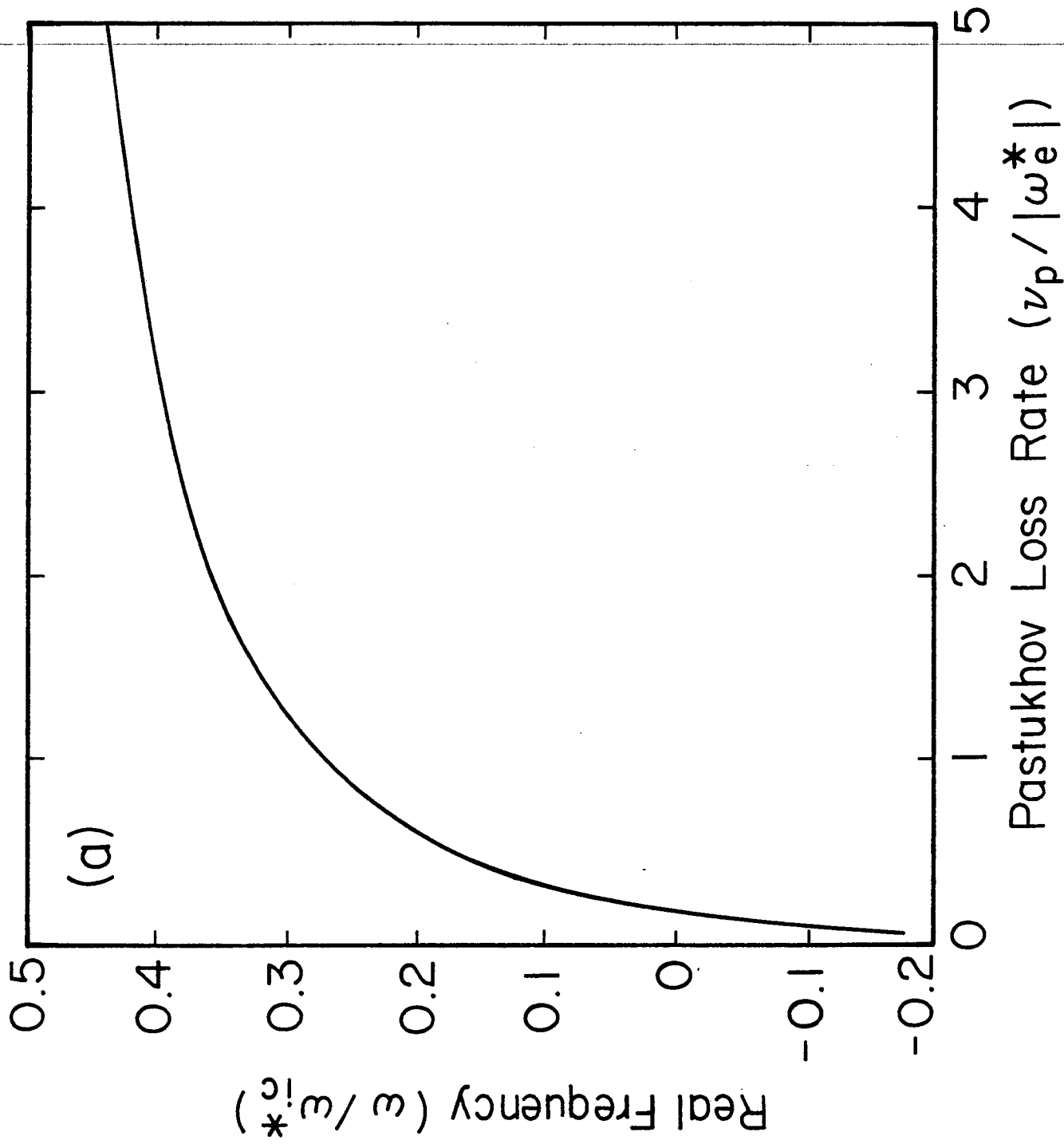


Fig. 7a

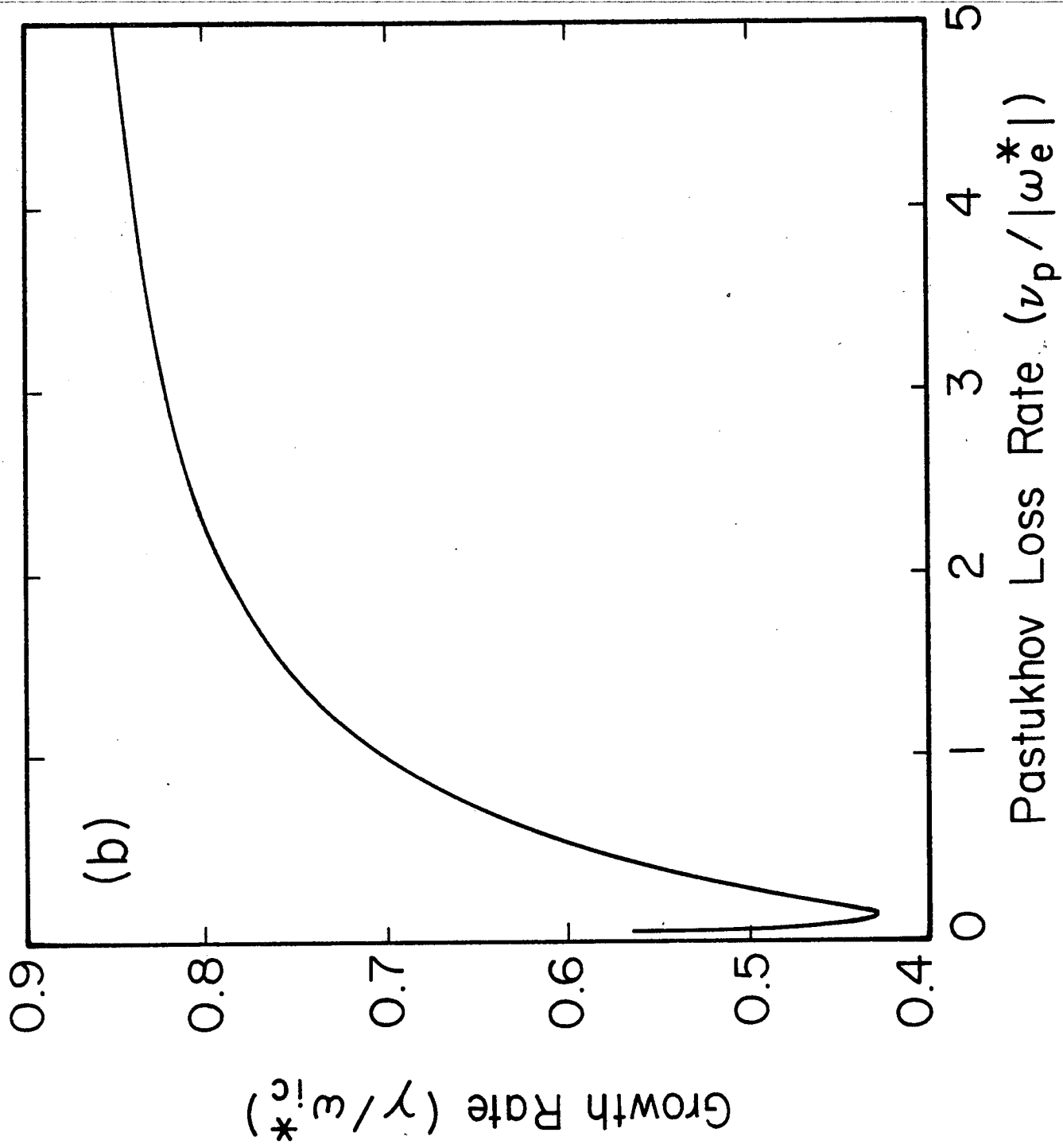


Fig. 7b

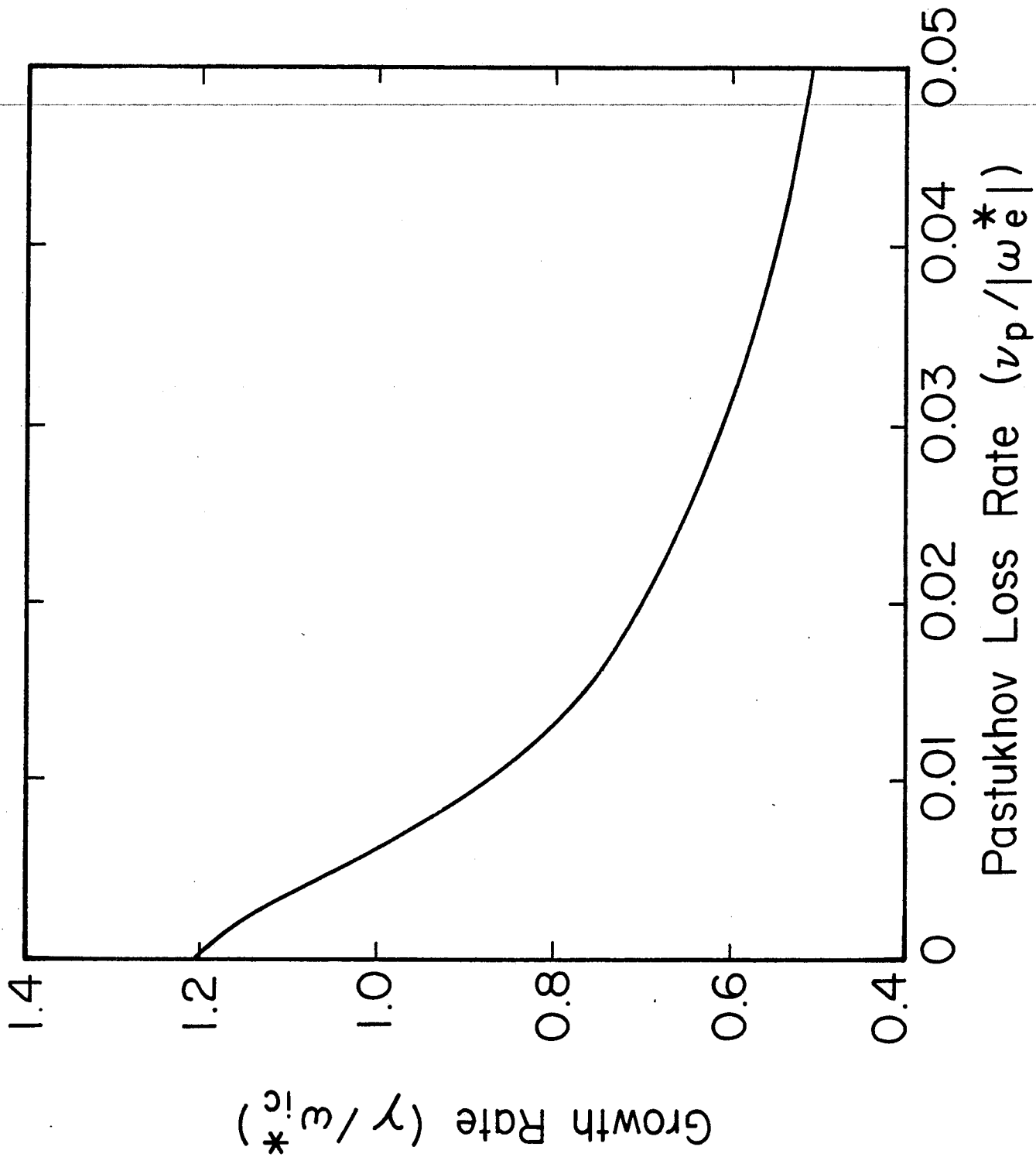


Fig. 8