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Hamiltonian Four-Field Model for Nonlinear Tokamak Dynamics

R. D. Hazeltine, C. T. Hsu,
and *P. J. Morrison*

Institute for Fusion Studies
and
Department of Physics
The University of Texas at Austin
Austin, Texas 78712

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Institute for Fusion Studies and Department of Physics

University of Texas at Austin, Austin, Texas 78712

Abstract

The Hamiltonian four-field model is a simplified description of nonlinear tokamak dynamics that allows for finite ion Larmor radius physics as well as other effects related to compressibility and electron adiabaticity. Much simpler than a rigorous or even reduced description of the same physics, it still preserves essential features of the underlying exact dynamics. In particular, because it is a Hamiltonian dynamical system it conserves the appropriate Casimir invariants, as well as avoiding implicit, unphysical dissipation. Here the model is derived and interpreted, its Hamiltonian nature is demonstrated, and its constants of motion are extracted.

*Present address: MIT Plasma Fusion Center, Cambridge, MA 02139

I. Introduction

We present here a system of coupled fluid equations describing magnetized plasma motions in an axisymmetric confinement device, such as a tokamak. The system is intended to model such phenomena as sawtooth oscillation and tokamak disruption, especially in their nonlinear stages[1]. It is emphatically a simplified system, in which numerous geometrical and dynamical effects are neglected. On the other hand the equations attempt to represent non-ideal processes, including finite-ion-Larmor-radius (FLR) terms and electron adiabaticity, in a manner consistent with both simplicity and fundamental physical constraints. In particular, when explicit dissipation is omitted the model is shown to define a (generalized) Hamiltonian dynamical system. (See for example Ref 2).

The four-field model is so named because its essential distinction from reduced magnetohydrodynamics[3] (RMHD) is the need for four, rather than three, independent field variables. In this respect it most closely resembles two previous models, the approximate four-field model of Hazeltine, Kotschenreuther and Morrison[4] (hereafter referred to as HKM) and the asymptotic system of Hsu, Hazeltine, and Morrison[5] (HHM). More generally, however, the present model has much in common with numerous, earlier extensions of RMHD[6-8], especially in its motivation.

The usefulness of reduced fluid models is discussed elsewhere[3-8]. Here we only remark that the present four-field model is a generalization of RMHD that allows for slow evolution (frequencies comparable to the diamagnetic drift), long mean-free-path electron dynamics, and various effects of plasma compressibility, in a simple albeit non-rigorous way. Like its predecessors[4,5] it reproduces such features of kinetic and FLR physics as the "semi-collisional" conductivity; gyroviscosity-modified, nonlinear diamagnetic convection; curvature-modified drift-tearing instability; and diffusion in a stochastic magnetic field. Also like its predecessors it omits temperature gradients and kinetic effects of magnetic trapping. Finally, unlike the model of HKM, (but in common with the underlying physics it attempts to represent) its ideal version not only conserves energy but is a Hamiltonian dynamical system.

Three equivalent versions of the model are presented in Sec. II, which also includes interpretation of its most distinctive terms. The derivation is given in Sec. III, while Sec. IV is devoted to a discussion of the system's dynamical invariants.

The Hamilitonian property is an essential feature of the present model, which, in particular, played a major role in its derivation. It therefore seems appropriate to comment here upon the general significance of this property in such approximate field theories as RMHD and its extensions.

The phase-space conserving nature of Hamiltonian evolution depends upon rather delicate considerations, not always obvious from inspection of the system. Yet it has strong consequences, much stronger than, in particular, simple energy conservation (consider for example the energy conserving but non-Hamiltonian Boltzmann equation). In particular, Hamiltonian motion conserves not only phase-space volumes but numerous additional functionals of the field variables, such as Poincare invariants and the generalized helicities or Casimirs.

The simplest way to guarantee that some dynamical system is Hamiltonian is to demonstrate that it faithfully represents, at least in some asymptotic limit, the actual classical evolution of charged particles. Thus, for example, Vlasov theory and ideal magnetohydrodynamics can be shown to have the (generalized) Hamiltonian property[2]. However, not all systems of interest to plasma physics can be systematically derived from exact microscopic dynamics. Progress, especially in nonlinear regimes, frequently demands the use of simplified models in which the Hamiltonian property is problematic. A major concern in the application and interpretation of such models is the possibility of unphysical dissipation.

Physical dissipation enters exact formulations explicitly, through such mechanisms as collision operators or resistive terms. Its form (whether drag or diffusion, for example) is manifest in the equations, and its magnitude is arbitrarily adjustable through the size of certain coefficients (such as collision frequencies or resistivities). In the case of non-rigorously derived models, however, dissipation can enter implicitly and unintentionally, because of uncontrolled approximation. No resistivity or collisional term occurs in this case -- the system appears purely nondissipative -- yet phase-space conservation and other invariants may be lost. Significantly the magnitude and even the effective sign of this unphysical, fake dissipation is uncontrolled and typically difficult to determine.

It has been shown that RMHD is a Hamiltonian system[9]. Certain extensions of RMHD, discussed in Sec. III, similarly preserve the Hamiltonian property, and furthermore a Hamiltonian representation of two-dimensional FLR physics has been found[10]. Nonetheless it shall become clear that the Hamiltonian property of reduced fluid models must be considered extremely fragile. Amongst the myriad of physically plausible four-field models, each conserving energy and yielding correct, FLR-modified linear equations, only a tiny subset is Hamiltonian. One likely (although unproven) element of the Hamiltonian subset is the rigorously derived but complicated model of HHM. The system described in this work is shown to be Hamiltonian; we believe it is the subset's simplest member.

II. Description of the model.

A. Four-field equations

We present here the dissipationless version of the four-field model, noting that dissipative terms (resistivity, diffusion and viscosity) can be straightforwardly introduced *a posteriori*. The four normalized fields are W, ψ, p and v; they have the following physical significance:

W measures the scalar parallel *vorticity* ;

ψ measures the poloidal magnetic *flux* ;

p measures the electron *pressure* ;

v measures the ion *parallel velocity* .

In addition to the above normalized varaiables, the model involves three constant parameters: the electron beta, $\beta \equiv 8\pi n_c T_e/B_T^2$, where n_c is a constant measure of the plasma density and B_T is a constant measure of the toroidal magnetic field; $\delta \equiv c/(2\omega_{pi}a)$, the finite Larmor radius (FLR) parameter, where ω_{pi} is the ion plasma frequency and a is the plasma radius; and the temperature ratio, $\tau \equiv T_i/T_e$ (note that in previous work τ denoted a normalized time variable, for which we here use t).

We recall from HKM the following normalizations: $\psi = (\varepsilon B_T a)^{-1}A_\zeta$, where ε is the inverse aspect ratio and A_ζ is the toroidal component of the vector potential; $\varphi = c\Phi/(\varepsilon v_A B_T a)$, where Φ is the electrostatic potential and v_A is the Alfven speed; $v = V_{\|}/(\varepsilon v_A)^{-1}$, where $V_{\|}$ is the ion parallel velcocity; and $p = (\beta/\varepsilon)(n/n_c - 1)$, where n is the plasma density. We also introduce a velocity stream function, F, according to

$$(1 + \tau\beta\delta^2\nabla_\perp^2)F = \varphi + \delta\tau p, \qquad (1)$$

where $\nabla_\perp$ is the two-dimensional gradient operator in the plane transverse to the magnetic field. The function F, which differs somewhat from its counterpart in HKM, is a stream function in the sense that the normalized ion velocity tranverse to B is $\hat{\zeta}\times\nabla_\perp F$. The right-hand side of (1) evidently yields the expected combination of electric and diamagnetic drifts, while the $O(\delta^2)$ term involving $\nabla_\perp{}^2$ on the left-hand side gives an FLR correction. In terms of F, the normalized vorticity variable W is given by

$$W \equiv \nabla_\perp{}^2 F \ .$$

Similarly, the normalized parallel current density is related to ψ via

$$J \equiv \nabla_\perp{}^2 \psi \ .$$

Finally we define h, a normalized "horizontal" distance, by $h \equiv (R - R_0)/a$, where R is the major radius and R_0 the major radius of the magnetic axis. This quantity enters the equations only in the form $\nabla_\perp h$, which is the lowest-order field line curvature.

The four-field model can then be expressed as

$$(\partial/\partial t)W + [F,W] + \nabla_{\parallel} J + (1+\tau)(1+\tau\delta^2\beta\nabla_\perp{}^2)[h,p] =$$
$$\delta\tau\nabla_\perp\cdot[p+2\beta h,\nabla_\perp F] + (1/2)\tau^2\delta^3\beta\nabla_\perp{}^2[p+2\beta h,W]$$
$$- (1/2)\tau\delta\beta\nabla_\perp{}^2\nabla_{\parallel}(v+2\delta J), \qquad (2)$$

$$(\partial/\partial t)\psi + \nabla_{\parallel}\varphi - \delta\nabla_{\parallel} p = 0, \qquad (3)$$

$$(\partial/\partial t)p + [\varphi, p+2\beta h] = \beta\{2\delta[p,h] - \nabla_{\parallel}(v+2\delta J)\}, \qquad (4)$$

$$(\partial/\partial t)v + [\varphi,v] + (1/2)\nabla_{\parallel}[p + \tau(p-\delta\beta W)] = \delta^2\tau\beta[v,\nabla_\perp{}^2(F-\delta\tau p)]$$
$$+ 2\delta\tau\beta[v,h] \ . \qquad (5)$$

Here we use the conventional bracket symbol defined by

$$[f,g] \equiv \hat{\zeta}\cdot\nabla_\perp f\times\nabla_\perp g,$$

where $\hat{\zeta}$ is a unit vector in the toroidal direction. Also, the parallel gradient operator is defined by

$$\nabla_{\parallel} f \equiv \partial f/\partial\zeta + [f,\psi] .$$

Equations (3) and (4) express the generalized (collisionless) Ohm's law and the particle conservation law precisely as in HKM. Equation (2), the shear-Alfven law, differs from HKM in including several additional FLR and compressibility terms on the right hand side. Similarly the parallel acceleration law, Eq. (5), includes previously omitted physics. All the additional terms are numerically small, since δ and β are typically small in tokamak experiments. The significance of these correction terms is discussed in the Subsection C.

This system conserves the following energy (Hamiltonian) functional:

$$H \equiv (1/2)\langle |\nabla_{\perp} F|^2 + v^2 + |\nabla_{\perp}\psi|^2 + (1+\tau)p^2/(2\beta)\rangle, \qquad (6)$$

which differs from that of HKM. Here the angular brackets denote an integral over the system volume (effects of the volume boundary are ignored). This functional is easily understood to be the sum of the parallel and perpendicular fluid kinetic, poloidal magnetic field, and internal energies. In addition to the energy functional the four-field model conserves the following four Casimir (or "helicity" type) invariants:

$$C_1 = \langle A(\psi)\rangle$$

$$C_2 = \langle B(\psi)(p + 2\beta h)\rangle \qquad (7)$$

$$C_{3,4} = \langle C_{\pm}[2\delta\beta v + \beta\psi \pm (2\beta\tau)^{1/2}\delta(2\delta\beta W - \tau\delta^2\beta\nabla_{\perp}{}^2 p - p - 2\beta h)]\rangle .$$

These constants are associated with the magnetic helicity, density and generalizations of the cross helicity, respectively. When there are magnetic surfaces, such as in the case of axisymmetry or single helicity dynamics, the functions A, B and $C_\pm$ are arbitrary. For general three dimensional dynamics C_1 and C_2 remain conserved provided $A(\psi)=\psi$, $B(\psi)=$ constant and $C_\pm(x)=x$.

Equations (1)-(7) are the main results of this paper. We next rewrite the system in a form that makes manifest its Hamiltonian character.

B. Hamiltonian form

In order to display the Hamiltonian structure of the four-field model it is convient to introduce the following set of variables:

$$\begin{aligned}
\xi^1 &= \nabla_\perp^2(F - \delta\tau p/2),\\
\xi^2 &= \psi,\\
\xi^3 &= p + 2\beta h, \qquad\qquad (8)\\
\xi^4 &= v.
\end{aligned}$$

We shall refer to the ξ^i as "field variables" to distinguish them from the "physical variables" W, ψ, p and v.

When the total system energy is expressed in terms of the ξ^i, it becomes

$$\begin{aligned}
H[\xi] \equiv (1/2)\langle &\left| \nabla_\perp(\nabla_\perp^{-2}\xi^1) + (\delta\tau/2)\nabla_\perp(\xi^3 - 2\beta h)\right|^2\\
&+ \left|\nabla_\perp\xi^2\right|^2 + (1+\tau)(\xi^3 - 2\beta h)^2/(2\beta) + (\xi^4)^2\rangle, \qquad (9)
\end{aligned}$$

where $\nabla_\perp^{-2}$ represents the inverse Laplacian operator, whose occurrence in fluid Hamiltonians is conventional.

Now we can express the four field model for evolution of the ξ^i in the following form:

$$(\partial/\partial t)\xi^1 = [H_1, \xi^1] + \nabla_\parallel H_2 + [H_3, \xi^3] + [H_4, \xi^4], \qquad (10)$$

$$(\partial/\partial t)\xi^2 = \nabla_{\parallel}(H_1 + 2\delta\beta H_3), \tag{11}$$

$$(\partial/\partial t)\xi^3 = [H_1 + 2\delta\beta H_3, \xi^3] - \beta\nabla_{\parallel}(H_4 - 2\delta H_2), \tag{12}$$

$$(\partial/\partial t)\xi^4 = [H_1, \xi^4] - \beta\nabla_{\parallel}H_3 + \delta\tau[\xi^3 - 2\delta\beta\xi^1, H_4] . \tag{13}$$

Here functional derivatives of the Hamiltonian are indicated by subscripts, $H_i \equiv \delta H/\delta\xi^i$. They are given by

$$H_1 = -F,\ H_2 = -J,\ H_3 = [(1+\tau)/2\beta]p - (\delta\tau/2)W,\ H_4 = v, \tag{14}$$

and can easily be written in terms of the field variables by means of Eqs. (8). Note that Eqs. (10) - (13) are simpler in form than Eqs. (2) - (5), especially since the latter can only be used in conjunction with Eq. (1).

To express the four-field model in Hamiltonian form, first let F and G be arbitrary functionals of the fields ξ^i, with $F_i \equiv \delta F/\delta\xi^i$ as usual. Then, implicitly summing over paired indices, we define a Poisson bracket by

$$\{F,G\} = \langle c^{ij}{}_k\xi_k[F_i, G_j] + c^{ij}{}_2(F_i\partial G_j/\partial\zeta)\rangle, \tag{15}$$

where the coefficient matrix $c^{ij}{}_k$, which is symmetric with respect to its upper indices, has the following nonzero components:

$$\begin{aligned}
c_k{}^{1j} &= c_k{}^{j1} = \delta_{kj},\\
c_k{}^{23} &= c_k{}^{32} = 2\delta\beta\delta_{k2}\\
c_k{}^{33} &= 2\delta\beta\delta_{k3},\\
c_k{}^{34} &= c_k{}^{43} = -\beta\delta_{k2},\\
c_k{}^{44} &= -\delta\tau(\delta_{k3} - 2\delta\beta\delta_{k1}).
\end{aligned} \tag{16}$$

We remark that Eqs. (15) and (16) define a true Poisson bracket: it is bilinear, antisymmetric, it satisifies the Jacobi identity,

$$\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0, \tag{17}$$

and acts as a derivation; i.e.

$$\{F,GH\} = \{F,G\}H + G\{F,H\}.$$

We also remark that $C^{ij}{}_k$ is a rather simple matrix, at least in the sense of being sparse.

The Hamiltonian version of the Eqs. (2)-(5) is given by

$$(\partial/\partial t)\xi^i = \{\xi^i, H\}. \qquad (18)$$

The invariance of the "Casimirs" defined by Eqs. (7) then follows from the identities $\{C_i,F\}=0$ for $i=1-4$ and F arbitrary.

C. Discussion

Here we consider the significance of the new FLR and compressiblity terms appearing in the present model, basing our discussion on Eqs. (2)-(5) for convenience.

FLR corrections appear multiplied by $\tau\delta^2\beta$ or $\tau\delta\beta$, measuring the squared ion gyrodradius, ρ_i^2 (explicitly $2\tau\delta^2\beta = \rho_i^2/a^2$). Such terms occur in the ion dynamics described by Eqs. (2) and (5), in combination with the expected Laplacian factor, and have a well known interpretation in terms of averages over the Larmor orbit. The FLR terms manifest on the right-hand side of Eq. (2) describe, in particular, nonlinear diamagnetic convection and ion gyroviscosity. In linear theory (where the perturbation is assumed to vary more sharply than the equilbrium) these terms reproduce the ion drift-frequency corrections found in linearized gyrokinetic analysis[4-6].

Another type of FLR correction is most apparent in Eq. (5), although also present elsewhere: the $\delta\beta W$ correction to the ion pressure, $\tau p \rightarrow \tau(p - \delta\beta W)$. It can be identified with a well known residue from the "gyroviscous cancellation"; thus gyroviscosity is known[12] to modify the ion scalar pressure, p_i, in an FLR plasma according to

$$p_i \rightarrow p_i[1 - (2\Omega_i)^{-1} \mathbf{b} \cdot \nabla \times V_i], \qquad (19)$$

where Ω_i is the ion gyrofrequency, V_i is the ion fluid velocity and $\mathbf{b}$ is a unit vector in the direction of the magnetic field. When Eq. (19) is expressed in terms of the four-field normalized variables and reduced for large aspect ratio, it yields $p - \delta\beta W$.

All FLR terms in Eqs. (2) and (5) have been derived by systematic ordering procedures in previous work[5]; however the rigorous ordering also produces a host of additional corrections of similar form. Thus the present model, which is extremely simple compared to the rigorous version, contains a *selection* of gyroradius corrections. We presently discuss the grounds for this selectivity.

The remaining terms of interest involve the plasma compressibility, given by the right-hand side of Eq. (4). Equation (4) coincides with a previous conservation law and has been discussed in detail elsewhere[4]; we recall that the term involving h is the perpendicular compressibility, resulting from curvature of the magnetic field, while the term involving $\nabla_{\parallel}$ is the parallel compressibility of the electron flow, $V_{\parallel e} \propto v + 2\delta J$. The new feature here is the appearance of explicit compressibility terms in Eq. (2), as seen, for example, in its last term. We point out that the contribution of compressiblity to the shear-Alfven law, although rarely taken into account, is easily understood. First of all, the vorticity associated with diamagnetic acceleration, $\hat{\zeta} \cdot \nabla \times (d/dt)(\hat{\zeta} \times \nabla p)$, evidently involves $\nabla^2 (d/dt) p_i$ and therefore the

Laplacian of the compressibility, $p_i \nabla \cdot V_i$. Secondly, gyroviscosity can be shown[4,5] to contribute terms of the same form. Equation (2) displays the sum of these two contributions, which, together with the factor of (1/2), also occur in the rigorous version[5].

This comment helps explain the appearance of the modified vorticity, $\xi^1 = \nabla_\perp^2(F - \delta\tau p/2)$, as a basic field in the system. The second term correctly accounts for plasma compressibility in the shear-Alfven law. Perhaps fortuitously, it also contributes to a correct accounting of ion diamagnetic convection terms.

Thus the new terms are physically plausible, in the sense that rigorous ordering arguments yield correction terms of the same form. However, because the rigorous analysis also reveals numerous other FLR effects, the new terms do not make Eqs. (2)-(5) more "exact" in any formal sense. Why then do these particular corrections appear?

The correction terms in Eqs. (2)-(5) are best characterized as being the *minimal* additions to a cold-ion theory which preserve the following essential physical properties:

(i) Reasonable cold-ion ($\tau \to 0$) limit; specifically we require that the $\tau=0$ version agree with that of the previous four-field model, whose physical reasonableness was discussed in HKM.

(ii) Agreement in the linear regime with kinetic theory of ion diamagnetic effects; in particular we require that the ion diamagnetic frequency enter the linearized four-field model in the manner predicted by gyrokinetics[6].

(iii) Hamiltonian structure; we insist upon a dynamical law of the form of Eq. (18), where the bracket is anti-symmetric, satisifies Jacobi's identity, and acts as a derivation.

The four-field equations presented here satisfy these requirements, and they do so minimally, in the sense that the model obtained by omission of any term does not.

III. Derivation

Because we seek a drastically simplified description of FLR physics -- indeed, the simplest system that satisfies the requirements (i)-(iii) of Sec. II -- our derivation of the four-field model cannot rely on simple ordering procedures. Instead it is based on a mapping procedure that is motivated by asymptotically rigorous models.

A. The gyro map

A high-β version of RMHD that includes both electron and ion drift corrections, but excludes compressibility, is obtained by a rigorous ordering procedure in HHM. This three-field model is given by

$$(\partial/\partial t)\nabla_\perp^2\varphi + [\varphi, \nabla_\perp^2\varphi] + \nabla_\parallel J + (1+\tau)[h, p] + \delta\tau\nabla_\perp\cdot[p,\nabla_\perp\varphi] = 0, \quad (20)$$

$$(\partial/\partial t)\psi + \nabla_\parallel\varphi - \delta\nabla_\parallel p = 0, \quad (21)$$

$$(\partial/\partial t)p + [\varphi, p] = 0. \quad (22)$$

It conserves the following energy:

$$H = (1/2)\langle|\nabla_\perp\varphi|^2 + |\nabla_\perp\psi|^2 + 2\delta p\nabla_\perp^2\varphi - \tau\delta^2|\nabla_\perp p|^2 - 2(1+\tau)hp\rangle, \quad (23)$$

and is also a Hamiltonian system.

For reasons of clarity we now specialize to the axisymmetric case. The generalization to three dimensions is straightforward, involving nothing more than the replacement

$$[f,\psi] \rightarrow \nabla_\parallel f. \quad (24)$$

If this replacement is made in a Poisson bracket then it can be shown in general that the Jacobi identity is maintained.

The axisymmetric version of Eqs. (20)-(22) has the following Poisson bracket:

$$\{F,G\} = \langle U[F_U,G_U] + \psi([F_U,G_\psi] + [F_\psi,G_U]) + p([F_U,G_p] + [F_p,G_U]) + \delta\tau p[\nabla_\perp F_U;\nabla_\perp G_U]\rangle. \qquad (25)$$

Here, we have used $\delta F/\delta U \equiv F_U$, etc., and in the last term the "semicolon" notation is defined by

$$[A;B] = \textstyle\sum_i [A_i,B_i].$$

Because of the last term, the form of this bracket differs from previous brackets in that it involves more derivatives. Yet one can prove directly that Eq. (25) satisfies the Jacobi identity.

Now consider the zero ion-temperature limit. Setting τ equal to zero we obtain

$$(\partial/\partial t)\nabla_\perp{}^2\varphi + [\varphi, \nabla_\perp{}^2\varphi] + \nabla_\parallel J + [h, p] = 0, \qquad (26)$$

$$(\partial/\partial t)\psi + \nabla_\parallel\varphi - \delta\nabla_\parallel p = 0, \qquad (27)$$

$$(\partial/\partial t)p + [\varphi, p] = 0. \qquad (28)$$

Apart from removing the ion pressure from Eq, (26), the only effect of taking this limit has been to remove ion gyroviscosity physics. Observe that the term involving the parameter δ in Eq. (27), unlike the gyroviscous effect in Eq. (20), reflects electron physics; it is the Hall term.

At zero τ the Hamiltonian becomes

$$H = (1/2)\langle|\nabla_\perp\varphi|^2 + |\nabla_\perp\psi|^2 + 2\delta p\nabla_\perp{}^2\varphi - 2hp\rangle, \qquad (29)$$

and the Poisson bracket reduces to

$$\{F,G\} = \langle U[F_U,G_U] + \psi([F_U,G_\psi] + [F_\psi,G_U]) + p([F_U,G_p] + [F_p,G_U])\rangle, \qquad (30)$$

which differs from Eq. (25) only in that it lacks the gyro term.

Now comes the crucial observation: *Poisson brackets for systems without ion gyroviscosity physics can be mapped into those with ion gyroviscosity physics by a simple linear transformation.* The transformation amounts to changing to a frame moving at one-half the magnetization velocity. The magnetization velocity is defined by $v_M = (\nabla \times M)/ne$, where M is the magnetization. We call this transformation the *gyro map* .

The gyro map was first observed in Ref. 5 for a two-dimensional model with compressibilty. We will demonstrate it here for the brackets of Eqs. (20) - (22).

Technically the mapping we are refering to is a Lie algebra isomorphism; the brackets of Eqs. (25) and (30) are isomorphic. [In Sec. IV we use this algebraic fact to simply obtain the complicated constants of motion of Eqs. (7).] Physically the transformation amounts to defining a new variable U' by

$$U' = U + (\delta\tau/2)\nabla_\perp^2 p\,, \qquad (31)$$

which yields the following relation between the new and old stream functions:

$$\varphi' = \varphi + (\delta\tau/2)p\,. \qquad (32)$$

Here the second term evidently corresponds to the velocity of the moving frame. One can show that in reduced ordering, $(\delta\tau/2)\nabla_\perp^2 p = (\hat{\zeta}\cdot\nabla\times v_M)/2$, where $M = pB/B^2$.

By the chain rule for functional derivatives Eq. (31), the transformation on the field variables induces the following relations among the derivatives:

$$\delta/\delta U\big|_{U,p,\psi} = \delta/\delta U'\big|_{U',p,\psi}\ , \quad \delta/\delta\psi\big|_{U,p,\psi} = \delta/\delta\psi\big|_{U',p,\psi}$$

$$\delta/\delta p\big|_{U,p,\psi} = \delta/\delta p\big|_{U',p,\psi} + (\delta\tau/2)\nabla_\perp^2\delta/\delta U'\big|_{U',p,\psi}. \qquad (33)$$

Inserting $U = U' - (\delta\tau/2)\nabla_\perp^2 p$ and Eqs. (33) into Eq. (25) gives

$$\{F,G\} = \langle U'[F_{U'},G_{U'}] + \psi([F_{U'},G_\psi] + [F_\psi,G_{U'}]) + p([F_{U'},G_p] + [F_p,G_{U'}])\rangle. \qquad (34)$$

Equation (34) has precisely the same form as that of Eq. (30). Thus we see that the bracket for Eqs. (20) - (22) can be obtained from its $T_i = 0$ limit by reversing the transformation that we have just performed. We obtain the bracket for the four-field model in a similar way.

B. Four-Field Derivation

As noted our derivation of the new field equations begins with the cold-ion form of the previous four-field model[4]. This cold ion model is asymptotically correct and easily obtained by straightforward ordering arguments. Setting $\tau = 0$ in previous formulae (c.f. Sec. II A) we obtain

$$(\partial/\partial t)\nabla_\perp^2 F' + [F', \nabla_\perp^2 F'] + \nabla_\parallel J + [h, p] = 0, \qquad (35)$$

$$(\partial/\partial t)p + [F', p] + \beta\nabla_\parallel(v + 2\delta J) - 2\beta[h, F' - \delta p] = 0, \qquad (36)$$

$$(\partial/\partial t)\psi + \nabla_\parallel F' - \delta\nabla_\parallel p = 0, \qquad (37)$$

$$(\partial/\partial t)v + [F', v] + (1/2)\nabla_\parallel p = 0. \qquad (38)$$

Here F' is the velocity stream function, which in the $\tau = 0$ limit is equal to φ. The energy conserved by this system is

$$H = (1/2)\langle |\nabla_\perp F'|^2 + |\nabla_\perp\psi|^2 + v^2 + p^2/(2\beta)\rangle. \qquad (39)$$

We define the field variables by

$$(\xi^{1'}, \xi^{2'}, \xi^{3'}, \xi^{4'}) = (\nabla^2 F', \psi, p + 2\beta h, v). \qquad (40)$$

Hence, using the notation $H_i = \delta H/\delta\xi^{i'}$,

$$H_1 = -F', \; H_2 = -J, \; H_3 = p/2\beta, \; H_4 = v. \qquad (41)$$

The axisymmetric versions of Eqs. (35)-(38) can be written as

$$(\partial/\partial t)\xi^{j\prime} = [H_i, C'_k{}^{ij}\xi_k'], \qquad (42)$$

where the $C'_k{}^{ij}$ are given by the τ=0 limit of Eq. (16):

$$C'_k{}^{1j} = C'_k{}^{j1} = \delta_{kj},$$

$$C'_k{}^{23} = C'_k{}^{32} = 2\delta\beta\delta_{k2}$$

$$C'_k{}^{33} = 2\delta\beta\delta_{k3},$$

$$C'_k{}^{34} = C'_k{}^{43} = -\beta\delta_{k2}, \qquad (43)$$

and

$$C'_k{}^{44} = 0. \qquad (44)$$

Now the axisymmetric equations of motion can be expressed in Hamiltonian form,

$$(\partial/\partial t)\xi^{j\prime} = \{\xi^{j\prime}, H\}, \qquad (45)$$

where the bracket is defined by

$$\{F, G\} = \langle C'_k{}^{ij}\xi_k'[F_i, G_j]\rangle, \qquad (46)$$

for arbitrary functionals F and G. We omit the straightforward demonstration that this bracket, satisfying Jacobi's identity, is a proper Poisson bracket.

In other words the cold-ion limit of the previous four-field model is, like MHD, reduced MHD and many other models, a Hamiltonian system. One obvious result is that energy of Eq. (39) is conserved, since {H,H} = 0.

For finite T_i the Hamilitonian of Eq. (39) is altered, without rigorous justification, in two ways. Firstly, $F'=\varphi$ is replaced by F, the stream function of Eq. (1); this change is easily understood *a posteriori*, as shown below. Secondly, the internal energy is modified to include the ion contribution: $p^2/(2\beta) \rightarrow (1+\tau)p^2/(2\beta)$.

These unsurprising changes yield the Hamiltonian of Eq. (6), whose physical plausibility was discussed in Sec. II.

Less straightforward are the finite-τ modifications of the Poisson bracket. In this regard, it is convenient to treat the parallel and perpendicular dynamics separately.

Consider first the parallel dynamics. It is clear that our task is to justify the replacement of Eq. (44) by Eq. (16). We do this in an *ad hoc* manner, using three constraints to construct the coefficient C_k^{44}. First note that at finite τ the stream function F differs, to leading order in δ, from the potential φ by $\delta\tau p$, a term that gives rise to the ion diamagnetic drift. On the other hand, as first shown by Mikhailovskii[11], the parallel flow is advected only by the electrostatic drift, as indicated in Eq. (5). These two facts enforce the first term of Eq. (16). Finally one finds that the resulting bracket satisfies Jacobi's identity only if the remaining term of Eq. (16) is also appended.

Similar "brute-force" procedures -- inelegant but straightforward -- have been attempted in the construction of perpendicular dynamics at finite τ, but without success. The perpendicular dynamics, involving gyroviscosity and perpendicular compressibility, are much more complicated and the physical constraints less clear than in the parallel case. Notice in particular that each proposed finite-τ modification must be checked for consistency with the Jacobi identity; the unwieldy form of typical FLR corrections [cf., for example, Eq. (25)] makes such checks extremely tedious.

Fortunately the gyro-map permits a much simpler and more reliable implementation of FLR physics. To obtain the appropriate bracket for the above Hamiltonian we consider the reverse of the map defined by Eq. (31), setting

$$\xi^{1'} = \xi^{1''} + (\delta\tau/2)\nabla_\perp^2(\xi^{3''} - 2\beta h), \qquad (47)$$

$$\xi^{i'} = \xi^{i''} \quad , \quad i = 1,2,4;$$

where

$$(\xi^{1''}, \xi^{2''}, \xi^{3''}, \xi^{4''}) = (\nabla^2 F, \psi, p + 2\beta h, v).$$

The chain rule yields

$$\delta/\delta\xi^{i'} = \delta/\delta\xi^{i''} \quad , \quad i = 1,2,4;$$

$$\delta/\delta\xi^{3'} = \delta/\delta\xi^{3''} - (\delta\tau/2)\nabla_\perp^2 \delta/\delta\xi^{1''}. \qquad (48)$$

Inserting Eqs. (47) and (48) into the "parallel-corrected" $T_i = 0$ bracket, defined by Eqs. (43), (16) and (46), produces the correct four-field bracket, which together with the Hamilitonian of Eq. (6), produces Eqs. (2)-(5).

In Sec. II B we chose to write the Hamiltonian equations in terms of the variables ξ defined by Eq. (8). Thus the Hamiltonian of Eq. (6) becomes that of Eq. (9) and the bracket obtained above in terms of ξ'' becomes that given by Eqs. (15) and (16).

Notice that the electrostatic potential need not be defined for this closed system; the four field variables ξ^i are advanced in time without knowledge of φ. It is nonetheless of interest to identify φ in terms of the four fields. There are two arguments leading to the correct answer, as given by Eq. (1).

First we can demand agreement between Eq. (3), involving φ, and Eq. (11) for the ξ^i. The point here is that Eq. (3) is free of

FLR physics and easily derived from electron momentum conservation. Thus we use Eqs. (46)-(49) to find

$$\partial\xi^2/\partial t = -[\psi,\delta H/\delta\xi^{1\prime}] - 2\delta\beta[\psi,\delta H/\delta\xi^{3\prime}]$$
$$= -[\psi,\delta H/\delta\xi^1] - 2\delta\beta[\psi,\delta H/\delta\xi^3 - (\delta\tau/2)\nabla_\perp^2\delta H/\delta\xi^1]$$
$$= [\psi,(1+\delta^2\beta\tau\nabla_\perp^2)F - \delta\tau p] - \delta[\psi,p],$$

which agrees with Eq. (3) only if

$$\varphi = (1+\delta^2\beta\tau\nabla_\perp^2)F - \delta\tau p\ ,$$

as in Eq. (1).

The second argument proceeds by directly ordering the Braginskii gyroviscosity tensor as in HHM. We express the ion velocity as

$$\mathrm{v} = \varepsilon v_A(\hat{\zeta}\times\nabla_\perp F + v\hat{\zeta}) + O(\varepsilon^2),$$

and compute the $O(\varepsilon)$ portion of the ion momentum balance equation; the result again is precisely Eq. (1).

.IV. Casimir Invariants

A. Derivation

Noncanonical field theories generally have a special class of constants of motion called Casimir invariants. These are entropy- or helicity-like constants, such as the magnetic and cross helicities of MHD. Since the four-field model, unlike ideal MHD, contains FLR physics and in addition is reduced, it is not obvious what these constants should be. [Direct calculation from Eqs. (2) - (5) leads to enormous and nearly hopeless labor.] We determine the Casimirs in this section using the bracket formalism.

By definition Casimir invariants are constants that commute with all functionals; i.e., C is a Casimir invariant if

$$\{C,F\} = 0 \quad \text{for all } F. \tag{49}$$

One can use Eq. (53) to obtain the constants. We begin with the two-dimensional, parallel corrected, cold-ion bracket of Eqs. (43), (16) and (46). Equation (49) can be manipulated, by partial integration, into the form

$$\{C,F\} = -\langle F_i [C_k{}^{ij} \xi'^{k}, C_j] \rangle = 0. \tag{50}$$

Here we have systematically set surface terms to zero. Independent of the boundary conditions necessary for the vanishing of these terms, the Casimirs so obtained will be constants of motion in the sense that their integrands will satisfy local conservation equations.

Now since Eq. (50) must be true for all functionals F, it follows that the coefficient of each F_i must vanish. This gives a system of four partial differential equations, which after some manipulation can be expressed as

$$[\xi^{2'},C_3] = 0, \tag{51}$$

$$[\xi^{2'},C_2] + [\xi^{3'},C_3] = 0, \tag{52}$$

$$[2\delta\beta\xi^{1'}-\xi^{3'},C_1] + [2\delta\beta\xi^{4'}+\beta\xi^{2'},C_4] = 0, \tag{53}$$

$$[2\delta\beta\xi^{4'}+\beta\xi^{2'},C^1] + [2\delta^2\beta\tau(2\delta\beta\xi^{1'}-\xi^{3'}),C_4] = 0 . \tag{54}$$

Equations (51) and (52) involve only the variables $\xi^{2'}$ and $\xi^{3'}$, while Eqs. (53) and (54) involve $\xi^{1'}$ and $\xi^{4'}$. Equations (51) and (52) respectively imply

$$C = \langle A(\xi^{2'})+\xi^{3'}B(\xi^{2'})\rangle$$

$$C = \langle k(\xi^{2'},\xi^{3'})\rangle , \tag{55}$$

where A, B and k are arbitrary functions of their arguments. Consistency between Eqs. (55) yields the following Casimir invariants:

$$C_1 = \langle A(\xi^{2'})\rangle$$

$$C_2 = \langle \xi^{3'}B(\xi^{2'})\rangle . \tag{56}$$

Similarly, Eqs. (53) and (54) imply

$$C_{3,4} = \langle C_{\pm}[2\delta\beta\xi^{4'}+\beta\xi^{2'}\pm(2\delta^2\beta\tau)^{1/2}(2\delta\beta\xi^{1'}-\xi^{3'})]\rangle , \tag{57}$$

where $C_{\pm}$ are arbitrary functions.

Now in order to obtain the Casimirs for the four-field model it is necessary to map from the primed to the physical variables. We know that the quantities thus obtained will be Casimirs, since the (parallel corrected) $T_i=0$ bracket is isomorphic to the four-field bracket written in terms of the physical variables; $W=\nabla_\perp{}^2F$, ψ, p and v. There is a one-to-one correspondence between Casimir invariants of isomorphic brackets. Thus we obtain the following Casimir invariants:

$$C_1 = \langle A(\psi)\rangle$$
$$C_2 = \langle (p + 2\beta h)B(\psi)\rangle \tag{58}$$
$$C_{3,4} = \langle C_\pm[2\delta\beta v + \beta\psi \pm (2\delta^2\beta\tau)^{1/2}(2\delta\beta W - \tau\delta^2\beta\nabla_\perp p - p - 2\beta h)]\rangle .$$

These quantities are constants for the axisymmetric version of Eqs. (2)-(5); i.e., where $\nabla_\parallel$ is replaced by $-[\psi,\cdot]$. For three dimensions the functions A, B and $C_\pm$ are restricted as mentioned in Sec. II A. This restriction, among other things, is discussed in the following subsection.

B. Discussion

The restriction of axisymmetry for constancy of the Casimir invariants, Eqs. (62), can be eased. In fact the existence of the above Casimirs for arbitary functions A, B and $C_\pm$ in three dimensions is tantomount to the existence of a solution $\tilde{\psi}$ to the following equation[2]:

$$\nabla_\parallel\tilde{\psi} = \partial\tilde{\psi}/\partial\zeta + [\tilde{\psi},\psi] = 0. \tag{59}$$

The question of the existence of a global $\tilde{\psi}$ is the same as that of the existence of a constant of motion for the one degree-of-freedom Hamiltonian system, for which the poloidal plane is the phase space, ζ is the time and ψ is the Hamiltonian. Said yet another way, the existence of $\tilde{\psi}$ is equivalent to the existence of magnetic surfaces. In the general case it is unlikely that $\tilde{\psi}$ exists (recall that ζ is a periodic variable).

Nevertheless, let us assume that $\tilde{\psi}$ exists and change variables; we will use the field $\tilde{\psi}$ instead of ψ. We wish to transform our three dimensional Poisson bracket, Eq. (15), into one written in terms of the variable $\tilde{\psi}$. To do this we relate ψ and $\tilde{\psi}$ variations of an arbitrary functional F. This yields

$$\nabla_{\parallel}(\delta F/\delta\psi) = [\delta F/\delta\tilde{\psi},\tilde{\psi}] \,. \qquad (60)$$

Upon inserting Eq. (60) into Eq. (15) we see that the transformation $\psi \rightarrow \tilde{\psi}$ takes the three dimensional four-field bracket into the axisymmetric bracket with $\tilde{\psi}$ replacing ψ. This bracket has the Casimir invariants of Eq. (58) for *arbitrary* functions A, B and $C_{\pm}$, but with $\tilde{\psi}$ replacing ψ.

Thus we have shown that the existence of the general Casimir invariants is tantomount to the existence of magnetic surfaces. It follows that the degree to which one believes magnetic surfaces exist in a tokamak discharge, should be the same as the degree to which one believes Casimir invariants with arbitrary functions A, B and $C_{\pm}$ exist.

One case in which solutions to Eq. (59) do exist is that of helical symmetry. Then one has $\psi(r,\tilde{\theta},t)$, where $\tilde{\theta} = \theta - \zeta/q_0$, and it can be shown by direct substitution that the following solves Eq. (59):

$$\tilde{\psi}(r,\tilde{\theta},t) = \psi(r,\tilde{\theta},t) + r^2/(2q_0) \,. \qquad (61)$$

Here $\tilde{\psi}$ is the helical flux function.

Let us next consider the meaning of the Casimir invariants. We have mentioned that these invariants are related to the magnetic and cross helicities. Specifically, they are the remnants of these ideal MHD quantities that survive our ordering procedure. The cross helicity also survives our inclusion of FLR physics, which is manifest in the fact that $v\cdot B$ has an additional term $v_M\cdot B$

arising from the gyro map. Since all four of our Casimir invariants have one of the two forms

$$\tilde{C}_1 = \langle f(\chi) \rangle \quad , \quad \tilde{C}_2 = \langle \Upsilon g(\chi) \rangle \quad , \tag{62}$$

where f and g are arbitrary functions, we will discuss their meaning in general terms for the fields χ and Υ. If we divide our physical domain up into cells, which we label by the value of χ at say the center, then the invariant $\tilde{C}_1$ determines the number of cells with a particular value of χ. This can be shown by picking f to be the characteristic function. The same procedure can be used to show that the invariant $\tilde{C}_2$ determines the sum of the values of Υ on those cells with a particular value of the field χ. Neither of these invariants determine spatial correlation, i.e. the placement of the cells with a given value.

To conclude we take limits of the Casimir invariants, Eqs. (58), and show that they reduce to previously obtained Casimirs invariants. To facilitate this we rewrite C_3 and C_4 as follows:

$$\tilde{C}_3 = \langle [C_-(D - a_i E) + C_+(D + a_i E)]/(4\delta\beta a_i) \rangle$$

$$\tilde{C}_4 = \langle [C_-(D - a_i E) - C_+(D + a_i E)]/(4\delta\beta) \rangle \, , \tag{63}$$

where $D \equiv \beta(\psi + 2\delta v)$, $E \equiv (1 + \delta^2\beta\tau\nabla_\perp^2)p - 2\delta\beta\nabla_\perp^2 F + 2\beta h$ and $a_i \equiv 2\delta^2\beta\tau$. In the cold ion limit $a_i, \tau \to 0$ and $F \to \varphi$, and the Casimirs invariants of Eq. (63) become

$$\tilde{C}_3 = \langle C_-(\psi + 2\delta v)/(2\delta) \rangle$$

$$\tilde{C}_4 = \langle [\nabla_\perp^2\varphi - (p + 2\beta h)/(2\delta\beta)] C_+(\psi + 2\delta v) \rangle \, . \tag{64}$$

We can further take the limit $\delta \to 0$ and obtain the invariants for compressible MHD (CRMHD):

$$\tilde{C}_3 = \langle v C_-'(\psi) \rangle$$

$$\tilde{C}_4 = \langle \nabla_\perp^2\varphi\, C_+(\psi) - (p + 2\beta h) v C_+'(\psi)/\beta \rangle \, . \tag{65}$$

This model was introduced in HKM.

V. Summary

The Hamiltonian four-field model is a simplified description of nonlinear tokamak dynamics that allows for finite ion Larmor radius physics as well as other effects related to compressibility and electron adiabaticity. Much simpler than a rigorous or even reduced description of the same physics, it still preserves essential features of the underlying exact dynamics.

The model is given by Eqs. (2) - (5), in terms of physical variables, and by Eqs. (10) - (13) in terms of the field variables, ξ_i. [The latter are defined by Eqs. (8).] A Hamiltonian expression of the model, in terms of a Hamiltonian functional and generalized Poisson bracket, is given by Eqs. (9) and (15) - (18).

Only the dissipationless form of the model is presented. In many applications such dissipative processes as resistivity and viscosity are appropriately included, in the conventional way -- for example, by appending ηJ to the right-hand side of Eq. (3). The Hamiltonian property is then lost, but it remains significant in that dissipation has been introduced in an explicit and physical way: as discussed in Section I, there is no fake dissipation.

In large part because of its Hamiltonian property the present four field model conserves not only total energy but also four generalized helicities, or Casimir invariants. These constants of the motion, which are given by Eqs. (7), have considerable value in applications.

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Hamiltonian four-field model for nonlinear tokamak dynamics

R. D. Hazeltine, C. T. Hsu* and P. J. Morrison

Institute for Fusion Studies and Department of Physics

University of Texas at Austin, Austin, Texas 78712

Abstract

The Hamiltonian four-field model is a simplified description of nonlinear tokamak dynamics that allows for finite ion Larmor radius physics as well as other effects related to compressibility and electron adiabaticity. Much simpler than a rigorous or even reduced description of the same physics, it still preserves essential features of the underlying exact dynamics. In particular, because it is a Hamiltonian dynamical system it conserves the appropriate Casimir invariants, as well as avoiding implicit, unphysical dissipation. Here the model is derived and interpreted, its Hamiltonian nature is demonstrated, and its constants of motion are extracted.

*Present address: MIT Plasma Fusion Center, Cambridge, MA 02139

I. Introduction

We present here a system of coupled fluid equations describing magnetized plasma motions in an axisymmetric confinement device, such as a tokamak. The system is intended to model such phenomena as sawtooth oscillation and tokamak disruption, especially in their nonlinear stages[1]. It is emphatically a simplified system, in which numerous geometrical and dynamical effects are neglected. On the other hand the equations attempt to represent non-ideal processes, including finite-ion-Larmor-radius (FLR) terms and electron adiabaticity, in a manner consistent with both simplicity and fundamental physical constraints. In particular, when explicit dissipation is omitted the model is shown to define a (generalized) Hamiltonian dynamical system. (See for example Ref 2).

The four-field model is so named because its essential distinction from reduced magnetohydrodynamics[3] (RMHD) is the need for four, rather than three, independent field variables. In this respect it most closely resembles two previous models, the approximate four-field model of Hazeltine, Kotschenreuther and Morrison[4] (hereafter referred to as HKM) and the asymptotic system of Hsu, Hazeltine, and Morrison[5] (HHM). More generally, however, the present model has much in common with numerous, earlier extensions of RMHD[6-8], especially in its motivation.

The usefulness of reduced fluid models is discussed elsewhere[3-8]. Here we only remark that the present four-field model is a generalization of RMHD that allows for slow evolution (frequencies comparable to the diamagnetic drift), long mean-free-path electron dynamics, and various effects of plasma compressibility, in a simple albeit non-rigorous way. Like its predecessors[4,5] it reproduces such features of kinetic and FLR physics as the "semi-collisional" conductivity; gyroviscosity-modified, nonlinear diamagnetic convection; curvature-modified drift-tearing instability; and diffusion in a stochastic magnetic field. Also like its predecessors it omits temperature gradients and kinetic effects of magnetic trapping. Finally, unlike the model of HKM, (but in common with the underlying physics it attempts to represent) its ideal version not only conserves energy but is a Hamiltonian dynamical system.

Three equivalent versions of the model are presented in Sec. II, which also includes interpretation of its most distinctive terms. The derivation is given in Sec. III, while Sec. IV is devoted to a discussion of the system's dynamical invariants.

The Hamilitonian property is an essential feature of the present model, which, in particular, played a major role in its derivation. It therefore seems appropriate to comment here upon the general significance of this property in such approximate field theories as RMHD and its extensions.

The phase-space conserving nature of Hamiltonian evolution depends upon rather delicate considerations, not always obvious from inspection of the system. Yet it has strong consequences, much stronger than, in particular, simple energy conservation (consider for example the energy conserving but non-Hamiltonian Boltzmann equation). In particular, Hamiltonian motion conserves not only phase-space volumes but numerous additional functionals of the field variables, such as Poincare invariants and the generalized helicities or Casimirs.

The simplest way to guarantee that some dynamical system is Hamiltonian is to demonstrate that it faithfully represents, at least in some asymptotic limit, the actual classical evolution of charged particles. Thus, for example, Vlasov theory and ideal magnetohydrodynamics can be shown to have the (generalized) Hamiltonian property[2]. However, not all systems of interest to plasma physics can be systematically derived from exact microscopic dynamics. Progress, especially in nonlinear regimes, frequently demands the use of simplified models in which the Hamiltonian property is problematic. A major concern in the application and interpretation of such models is the possibility of unphysical dissipation.

Physical dissipation enters exact formulations explicitly, through such mechanisms as collision operators or resistive terms. Its form (whether drag or diffusion, for example) is manifest in the equations, and its magnitude is arbitrarily adjustable through the size of certain coefficients (such as collision frequencies or resistivities). In the case of non-rigorously derived models, however, dissipation can enter implicitly and unintentionally, because of uncontrolled approximation. No resistivity or collisional term occurs in this case -- the system appears purely nondissipative -- yet phase-space conservation and other invariants may be lost. Significantly the magnitude and even the effective sign of this unphysical, fake dissipation is uncontrolled and typically difficult to determine.

It has been shown that RMHD is a Hamiltonian system[9]. Certain extensions of RMHD, discussed in Sec. III, similarly preserve the Hamiltonian property, and furthermore a Hamiltonian representation of two-dimensional FLR physics has been found[10]. Nonetheless it shall become clear that the Hamiltonian property of reduced fluid models must be considered extremely fragile. Amongst the myriad of physically plausible four-field models, each conserving energy and yielding correct, FLR-modified linear equations, only a tiny subset is Hamiltonian. One likely (although unproven) element of the Hamiltonian subset is the rigorously derived but complicated model of HHM. The system described in this work is shown to be Hamiltonian; we believe it is the subset's simplest member.

II. Description of the model.

A. Four-field equations

We present here the dissipationless version of the four-field model, noting that dissipative terms (resistivity, diffusion and viscosity) can be straightforwardly introduced *a posteriori*. The four normalized fields are W, ψ, p and v; they have the following physical significance:

W measures the scalar parallel *vorticity* ;

ψ measures the poloidal magnetic *flux* ;

p measures the electron *pressure* ;

v measures the ion *parallel velocity* .

In addition to the above normalized varaiables, the model involves three constant parameters: the electron beta, $\beta \equiv 8\pi n_c T_e/B_T^2$, where n_c is a constant measure of the plasma density and B_T is a constant measure of the toroidal magnetic field; $\delta \equiv c/(2\omega_{pi}a)$, the finite Larmor radius (FLR) parameter, where ω_{pi} is the ion plasma frequency and a is the plasma radius; and the temperature ratio, $\tau \equiv T_i/T_e$ (note that in previous work τ denoted a normalized time variable, for which we here use t).

We recall from HKM the following normalizations: $\psi = (\varepsilon B_T a)^{-1} A_\zeta$, where ε is the inverse aspect ratio and A_ζ is the toroidal component of the vector potential; $\varphi = c\Phi/(\varepsilon v_A B_T a)$, where Φ is the electrostatic potential and v_A is the Alfven speed; $v = V_{\|}/(\varepsilon v_A)^{-1}$, where $V_{\|}$ is the ion parallel velcocity; and $p = (\beta/\varepsilon)(n/n_c - 1)$, where n is the plasma density. We also introduce a velocity stream function, F, according to

$$(1 + \tau\beta\delta^2\nabla_\perp^2)F = \varphi + \delta\tau p, \qquad (1)$$

where $\nabla_\perp$ is the two-dimensional gradient operator in the plane transverse to the magnetic field. The function F, which differs somewhat from its counterpart in HKM, is a stream function in the sense that the normalized ion velocity tranverse to B is $\hat{\zeta}\times\nabla_\perp F$. The right-hand side of (1) evidently yields the expected combination of electric and diamagnetic drifts, while the $O(\delta^2)$ term involving $\nabla_\perp^2$ on the left-hand side gives an FLR correction. In terms of F, the normalized vorticity variable W is given by

$$W \equiv \nabla_\perp^2 F \ .$$

Similarly, the normalized parallel current density is related to ψ via

$$J \equiv \nabla_\perp^2 \psi \ .$$

Finally we define h, a normalized "horizontal" distance, by $h \equiv (R - R_0)/a$, where R is the major radius and R_0 the major radius of the magnetic axis. This quantity enters the equations only in the form $\nabla_\perp h$, which is the lowest-order field line curvature.

The four-field model can then be expressed as

$$(\partial/\partial t)W + [F,W] + \nabla_\parallel J + (1+\tau)(1+\tau\delta^2\beta\nabla_\perp^2)[h,p] = \delta\tau\nabla_\perp\cdot[p+2\beta h,\nabla_\perp F] + (1/2)\tau^2\delta^3\beta\nabla_\perp^2[p+2\beta h,W] - (1/2)\tau\delta\beta\nabla_\perp^2\nabla_\parallel(v+2\delta J), \quad (2)$$

$$(\partial/\partial t)\psi + \nabla_\parallel\varphi - \delta\nabla_\parallel p = 0, \quad (3)$$

$$(\partial/\partial t)p + [\varphi, p+2\beta h] = \beta\{2\delta[p,h] - \nabla_\parallel(v+2\delta J)\}, \quad (4)$$

$$(\partial/\partial t)v + [\varphi,v] + (1/2)\nabla_\parallel[p + \tau(p-\delta\beta W)] = \delta^2\tau\beta[v,\nabla_\perp^2(F-\delta\tau p)] + 2\delta\tau\beta[v,h] \ . \quad (5)$$

Here we use the conventional bracket symbol defined by

$$[f,g] \equiv \hat{\zeta}\cdot\nabla_\perp f\times\nabla_\perp g,$$

where $\hat{\zeta}$ is a unit vector in the toroidal direction. Also, the parallel gradient operator is defined by

$$\nabla_{\parallel} f \equiv \partial f/\partial\zeta + [f,\psi] .$$

Equations (3) and (4) express the generalized (collisionless) Ohm's law and the particle conservation law precisely as in HKM. Equation (2), the shear-Alfven law, differs from HKM in including several additional FLR and compressibility terms on the right hand side. Similarly the parallel acceleration law, Eq. (5), includes previously omitted physics. All the additional terms are numerically small, since δ and β are typically small in tokamak experiments. The significance of these correction terms is discussed in the Subsection C.

This system conserves the following energy (Hamiltonian) functional:

$$H \equiv (1/2)\langle |\nabla_{\perp} F|^2 + v^2 + |\nabla_{\perp}\psi|^2 + (1+\tau)p^2/(2\beta)\rangle, \qquad (6)$$

which differs from that of HKM. Here the angular brackets denote an integral over the system volume (effects of the volume boundary are ignored). This functional is easily understood to be the sum of the parallel and perpendicular fluid kinetic, poloidal magnetic field, and internal energies. In addition to the energy functional the four-field model conserves the following four Casimir (or "helicity" type) invariants:

$$C_1 = \langle A(\psi)\rangle$$

$$C_2 = \langle B(\psi)(p + 2\beta h)\rangle \qquad (7)$$

$$C_{3,4} = \langle C_{\pm}[2\delta\beta v + \beta\psi \pm (2\beta\tau)^{1/2}\delta(2\delta\beta W - \tau\delta^2\beta\nabla_{\perp}{}^2 p - p - 2\beta h)]\rangle.$$

These constants are associated with the magnetic helicity, density and generalizations of the cross helicity, respectively. When there are magnetic surfaces, such as in the case of axisymmetry or single helicity dynamics, the functions A, B and $C_\pm$ are arbitrary. For general three dimensional dynamics C_1 and C_2 remain conserved provided $A(\psi)=\psi$, $B(\psi)=$constant and $C_\pm(x)=x$.

Equations (1)-(7) are the main results of this paper. We next rewrite the system in a form that makes manifest its Hamiltonian character.

B. Hamiltonian form

In order to display the Hamiltonian structure of the four-field model it is convient to introduce the following set of variables:

$$\xi^1 = \nabla_\perp^2(F - \delta\tau p/2),$$
$$\xi^2 = \psi,$$
$$\xi^3 = p + 2\beta h, \tag{8}$$
$$\xi^4 = v.$$

We shall refer to the ξ^i as "field variables" to distinguish them from the "physical variables" W, ψ, p and v.

When the total system energy is expressed in terms of the ξ^i, it becomes

$$H[\xi] \equiv (1/2)\langle \left| \nabla_\perp(\nabla_\perp^{-2}\xi^1) + (\delta\tau/2)\nabla_\perp(\xi^3 - 2\beta h) \right|^2$$
$$+ \left|\nabla_\perp\xi^2\right|^2 + (1+\tau)(\xi^3 - 2\beta h)^2/(2\beta) + (\xi^4)^2\rangle, \tag{9}$$

where $\nabla_\perp^{-2}$ represents the inverse Laplacian operator, whose occurrence in fluid Hamiltonians is conventional.

Now we can express the four field model for evolution of the ξ^i in the following form:

$$(\partial/\partial t)\xi^1 = [H_1, \xi^1] + \nabla_{\|}H_2 + [H_3, \xi^3] + [H_4, \xi^4], \tag{10}$$

$$(\partial/\partial t)\xi^2 = \nabla_{\parallel}(H_1 + 2\delta\beta H_3), \qquad (11)$$

$$(\partial/\partial t)\xi^3 = [H_1 + 2\delta\beta H_3, \xi^3] - \beta\nabla_{\parallel}(H_4 - 2\delta H_2), \qquad (12)$$

$$(\partial/\partial t)\xi^4 = [H_1, \xi^4] - \beta\nabla_{\parallel}H_3 + \delta\tau[\xi^3 - 2\delta\beta\xi^1, H_4]\,. \qquad (13)$$

Here functional derivatives of the Hamiltonian are indicated by subscripts, $H_i \equiv \delta H/\delta\xi^i$. They are given by

$$H_1 = -F,\ H_2 = -J,\ H_3 = [(1+\tau)/2\beta]p - (\delta\tau/2)W,\ H_4 = v, \qquad (14)$$

and can easily be written in terms of the field variables by means of Eqs. (8). Note that Eqs. (10) - (13) are simpler in form than Eqs. (2) - (5), especially since the latter can only be used in conjunction with Eq. (1).

To express the four-field model in Hamiltonian form, first let F and G be arbitrary functionals of the fields ξ^i, with $F_i \equiv \delta F/\delta\xi^i$ as usual. Then, implicitly summing over paired indices, we define a Poisson bracket by

$$\{F,G\} = \langle c^{ij}{}_k\xi_k[F_i, G_j] + c^{ij}{}_2(F_i\partial G_j/\partial\zeta)\rangle, \qquad (15)$$

where the coefficient matrix $c^{ij}{}_k$, which is symmetric with respect to its upper indices, has the following nonzero components:

$$c_k{}^{1j} = c_k{}^{j1} = \delta_{kj},$$

$$c_k{}^{23} = c_k{}^{32} = 2\delta\beta\delta_{k2}$$

$$c_k{}^{33} = 2\delta\beta\delta_{k3},$$

$$c_k{}^{34} = c_k{}^{43} = -\beta\delta_{k2},$$

$$c_k{}^{44} = -\delta\tau(\delta_{k3} - 2\delta\beta\delta_{k1}). \qquad (16)$$

We remark that Eqs. (15) and (16) define a true Poisson bracket: it is bilinear, antisymmetric, it satisifies the Jacobi identity,

$$\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0, \qquad (17)$$

and acts as a derivation; i.e.

$$\{F,GH\} = \{F,G\}H + G\{F,H\}.$$

We also remark that $C^{ij}{}_k$ is a rather simple matrix, at least in the sense of being sparse.

The Hamiltonian version of the Eqs. (2)-(5) is given by

$$(\partial/\partial t)\xi^i = \{\xi^i, H\}. \tag{18}$$

The invariance of the "Casimirs" defined by Eqs. (7) then follows from the identities $\{C_i,F\}=0$ for $i=1-4$ and F arbitrary.

C. Discussion

Here we consider the significance of the new FLR and compressiblity terms appearing in the present model, basing our discussion on Eqs. (2)-(5) for convenience.

FLR corrections appear multiplied by $\tau\delta^2\beta$ or $\tau\delta\beta$, measuring the squared ion gyrodradius, ρ_i^2 (explicitly $2\tau\delta^2\beta = \rho_i^2/a^2$). Such terms occur in the ion dynamics described by Eqs. (2) and (5), in combination with the expected Laplacian factor, and have a well known interpretation in terms of averages over the Larmor orbit. The FLR terms manifest on the right-hand side of Eq. (2) describe, in particular, nonlinear diamagnetic convection and ion gyroviscosity. In linear theory (where the perturbation is assumed to vary more sharply than the equilbrium) these terms reproduce the ion drift-frequency corrections found in linearized gyrokinetic analysis[4-6].

Another type of FLR correction is most apparent in Eq. (5), although also present elsewhere: the $\delta\beta W$ correction to the ion pressure, $\tau p \rightarrow \tau(p - \delta\beta W)$. It can be identified with a well known residue from the "gyroviscous cancellation"; thus gyroviscosity is known[12] to modify the ion scalar pressure, p_i, in an FLR plasma according to

$$p_i \rightarrow p_i[1 - (2\Omega_i)^{-1}\mathbf{b}\cdot\nabla\times V_i], \qquad (19)$$

where Ω_i is the ion gyrofrequency, V_i is the ion fluid velocity and $\mathbf{b}$ is a unit vector in the direction of the magnetic field. When Eq. (19) is expressed in terms of the four-field normalized variables and reduced for large aspect ratio, it yields $p-\delta\beta W$.

All FLR terms in Eqs. (2) and (5) have been derived by systematic ordering procedures in previous work[5]; however the rigorous ordering also produces a host of additional corrections of similar form. Thus the present model, which is extremely simple compared to the rigorous version, contains a *selection* of gyroradius corrections. We presently discuss the grounds for this selectivity.

The remaining terms of interest involve the plasma compressibility, given by the right-hand side of Eq. (4). Equation (4) coincides with a previous conservation law and has been discussed in detail elsewhere[4]; we recall that the term involving h is the perpendicular compressibility, resulting from curvature of the magnetic field, while the term involving $\nabla_{\parallel}$ is the parallel compressibility of the electron flow, $V_{\parallel e} \propto v+2\delta J$. The new feature here is the appearance of explicit compressibility terms in Eq. (2), as seen, for example, in its last term. We point out that the contribution of compressiblity to the shear-Alfven law, although rarely taken into account, is easily understood. First of all, the vorticity associated with diamagnetic acceleration, $\zeta\cdot\nabla\times(d/dt)(\zeta\times\nabla p)$, evidently involves $\nabla^2(d/dt)p_i$ and therefore the

Laplacian of the compressibility, $p_i \nabla \cdot V_i$. Secondly, gyroviscosity can be shown[4,5] to contribute terms of the same form. Equation (2) displays the sum of these two contributions, which, together with the factor of (1/2), also occur in the rigorous version[5].

This comment helps explain the appearance of the modified vorticity, $\xi^1 = \nabla_\perp^2(F - \delta\tau p/2)$, as a basic field in the system. The second term correctly accounts for plasma compressibility in the shear-Alfven law. Perhaps fortuitously, it also contributes to a correct accounting of ion diamagnetic convection terms.

Thus the new terms are physically plausible, in the sense that rigorous ordering arguments yield correction terms of the same form. However, because the rigorous analysis also reveals numerous other FLR effects, the new terms do not make Eqs. (2)-(5) more "exact" in any formal sense. Why then do these particular corrections appear?

The correction terms in Eqs. (2)-(5) are best characterized as being the *minimal* additions to a cold-ion theory which preserve the following essential physical properties:

(i) Reasonable cold-ion ($\tau \to 0$) limit; specifically we require that the $\tau = 0$ version agree with that of the previous four-field model, whose physical reasonableness was discussed in HKM.

(ii) Agreement in the linear regime with kinetic theory of ion diamagnetic effects; in particular we require that the ion diamagnetic frequency enter the linearized four-field model in the manner predicted by gyrokinetics[6].

(iii) Hamiltonian structure; we insist upon a dynamical law of the form of Eq. (18), where the bracket is anti-symmetric, satisifies Jacobi's identity, and acts as a derivation.

The four-field equations presented here satisfy these requirements, and they do so minimally, in the sense that the model obtained by omission of any term does not.

III. Derivation

Because we seek a drastically simplified description of FLR physics -- indeed, the simplest system that satisfies the requirements (i)-(iii) of Sec. II -- our derivation of the four-field model cannot rely on simple ordering procedures. Instead it is based on a mapping procedure that is motivated by asymptotically rigorous models.

A. The gyro map

A high-β version of RMHD that includes both electron and ion drift corrections, but excludes compressibility, is obtained by a rigorous ordering procedure in HHM. This three-field model is given by

$$(\partial/\partial t)\nabla_\perp^2\varphi + [\varphi, \nabla_\perp^2\varphi] + \nabla_\parallel J + (1+\tau)[h, p] + \delta\tau\nabla_\perp\cdot[p,\nabla_\perp\varphi] = 0, \quad (20)$$

$$(\partial/\partial t)\psi + \nabla_\parallel\varphi - \delta\nabla_\parallel p = 0, \quad (21)$$

$$(\partial/\partial t)p + [\varphi, p] = 0. \quad (22)$$

It conserves the following energy:

$$H = (1/2)\langle |\nabla_\perp\varphi|^2 + |\nabla_\perp\psi|^2 + 2\delta p\nabla_\perp^2\varphi - \tau\delta^2|\nabla_\perp p|^2 - 2(1+\tau)hp\rangle, \quad (23)$$

and is also a Hamiltonian system.

For reasons of clarity we now specialize to the axisymmetric case. The generalization to three dimensions is straightforward, involving nothing more than the replacement

$$[f,\psi] \rightarrow \nabla_\parallel f. \quad (24)$$

If this replacement is made in a Poisson bracket then it can be shown in general that the Jacobi identity is maintained.

The axisymmetric version of Eqs. (20)-(22) has the following Poisson bracket:

$$\{F,G\} = \langle U[F_U,G_U] + \psi([F_U,G_\psi] + [F_\psi,G_U]) + p([F_U,G_p] + [F_p,G_U]) + \delta\tau p[\nabla_\perp F_U;\nabla_\perp G_U]\rangle. \qquad (25)$$

Here, we have used $\delta F/\delta U \equiv F_U$, etc., and in the last term the "semicolon" notation is defined by

$$[A;B] = \sum_i [A_i,B_i].$$

Because of the last term, the form of this bracket differs from previous brackets in that it involves more derivatives. Yet one can prove directly that Eq. (25) satisfies the Jacobi identity.

Now consider the zero ion-temperature limit. Setting τ equal to zero we obtain

$$(\partial/\partial t)\nabla_\perp^2\varphi + [\varphi, \nabla_\perp^2\varphi] + \nabla_\parallel J + [h, p] = 0, \qquad (26)$$

$$(\partial/\partial t)\psi + \nabla_\parallel\varphi - \delta\nabla_\parallel p = 0, \qquad (27)$$

$$(\partial/\partial t)p + [\varphi, p] = 0. \qquad (28)$$

Apart from removing the ion pressure from Eq, (26), the only effect of taking this limit has been to remove ion gyroviscosity physics. Observe that the term involving the parameter δ in Eq. (27), unlike the gyroviscous effect in Eq. (20), reflects electron physics; it is the Hall term.

At zero τ the Hamiltonian becomes

$$H = (1/2)\langle|\nabla_\perp\varphi|^2 + |\nabla_\perp\psi|^2 + 2\delta p\nabla_\perp^2\varphi - 2hp\rangle, \qquad (29)$$

and the Poisson bracket reduces to

$$\{F,G\} = \langle U[F_U,G_U] + \psi([F_U,G_\psi] + [F_\psi,G_U]) + p([F_U,G_p] + [F_p,G_U])\rangle, \qquad (30)$$

which differs from Eq. (25) only in that it lacks the gyro term.

Now comes the crucial observation: *Poisson brackets for systems without ion gyroviscosity physics can be mapped into those with ion gyroviscosity physics by a simple linear transformation.* The transformation amounts to changing to a frame moving at one-half the magnetization velocity. The magnetization velocity is defined by $v_M = (\nabla \times M)/ne$, where M is the magnetization. We call this transformation the *gyro map* .

The gyro map was first observed in Ref. 5 for a two-dimensional model with compressibilty. We will demonstrate it here for the brackets of Eqs. (20) - (22).

Technically the mapping we are refering to is a Lie algebra isomorphism; the brackets of Eqs. (25) and (30) are isomorphic. [In Sec. IV we use this algebraic fact to simply obtain the complicated constants of motion of Eqs. (7).] Physically the transformation amounts to defining a new variable U' by

$$U' = U + (\delta\tau/2)\nabla_\perp^2 p\,, \qquad (31)$$

which yields the following relation between the new and old stream functions:

$$\varphi' = \varphi + (\delta\tau/2)p\,. \qquad (32)$$

Here the second term evidently corresponds to the velocity of the moving frame. One can show that in reduced ordering, $(\delta\tau/2)\nabla_\perp^2 p = (\hat{\zeta}\cdot\nabla\times v_M)/2$, where $M = pB/B^2$.

By the chain rule for functional derivatives Eq. (31), the transformation on the field variables induces the following relations among the derivatives:

$$\delta/\delta U\big|_{U,p,\psi} = \delta/\delta U'\big|_{U',p,\psi}\,, \quad \delta/\delta\psi\big|_{U,p,\psi} = \delta/\delta\psi\big|_{U',p,\psi}$$

$$\delta/\delta p\big|_{U,p,\psi} = \delta/\delta p\big|_{U',p,\psi} + (\delta\tau/2)\nabla_\perp^2\delta/\delta U'\big|_{U',p,\psi}. \qquad (33)$$

Inserting $U = U' - (\delta\tau/2)\nabla_\perp^2 p$ and Eqs. (33) into Eq. (25) gives

$$\{F,G\} = \langle U'[F_{U'},G_{U'}] + \psi([F_{U'},G_\psi] + [F_\psi,G_{U'}]) + p([F_{U'},G_p] + [F_p,G_{U'}])\rangle. \qquad (34)$$

Equation (34) has precisely the same form as that of Eq. (30). Thus we see that the bracket for Eqs. (20) - (22) can be obtained from its $T_i = 0$ limit by reversing the transformation that we have just performed. We obtain the bracket for the four-field model in a similar way.

B. Four-Field Derivation

As noted our derivation of the new field equations begins with the cold-ion form of the previous four-field model[4]. This cold ion model is asymptotically correct and easily obtained by straightforward ordering arguments. Setting $\tau = 0$ in previous formulae (c.f. Sec. II A) we obtain

$$(\partial/\partial t)\nabla_\perp^2 F' + [F', \nabla_\perp^2 F'] + \nabla_\parallel J + [h, p] = 0, \qquad (35)$$

$$(\partial/\partial t)p + [F', p] + \beta\nabla_\parallel(v + 2\delta J) - 2\beta[h, F' - \delta p] = 0, \qquad (36)$$

$$(\partial/\partial t)\psi + \nabla_\parallel F' - \delta\nabla_\parallel p = 0, \qquad (37)$$

$$(\partial/\partial t)v + [F', v] + (1/2)\nabla_\parallel p = 0. \qquad (38)$$

Here F' is the velocity stream function, which in the $\tau = 0$ limit is equal to φ. The energy conserved by this system is

$$H = (1/2)\langle|\nabla_\perp F'|^2 + |\nabla_\perp\psi|^2 + v^2 + p^2/(2\beta)\rangle. \qquad (39)$$

We define the field variables by

$$(\xi^{1'},\xi^{2'},\xi^{3'},\xi^{4'}) = (\nabla^2 F', \psi, p+2\beta h, v). \qquad (40)$$

Hence, using the notation $H_i = \delta H/\delta\xi^{i'}$,

$$H_1 = -F',\ H_2 = -J,\ H_3 = p/2\beta,\ H_4 = v. \qquad (41)$$

The axisymmetric versions of Eqs. (35)-(38) can be written as

$$(\partial/\partial t)\xi^{j\prime} = [H_i, C'_k{}^{ij}\xi_k'], \qquad (42)$$

where the $C'_k{}^{ij}$ are given by the τ=0 limit of Eq. (16):

$$C'_k{}^{1j} = C'_k{}^{j1} = \delta_{kj},$$

$$C'_k{}^{23} = C'_k{}^{32} = 2\delta\beta\delta_{k2}$$

$$C'_k{}^{33} = 2\delta\beta\delta_{k3},$$

$$C'_k{}^{34} = C'_k{}^{43} = -\beta\delta_{k2}, \qquad (43)$$

and

$$C'_k{}^{44} = 0. \qquad (44)$$

Now the axisymmetric equations of motion can be expressed in Hamiltonian form,

$$(\partial/\partial t)\xi^{j\prime} = \{\xi^{j\prime}, H\}, \qquad (45)$$

where the bracket is defined by

$$\{F, G\} = \langle C'_k{}^{ij}\xi_k'[F_i, G_j]\rangle, \qquad (46)$$

for arbitrary functionals F and G. We omit the straightforward demonstration that this bracket, satisfying Jacobi's identity, is a proper Poisson bracket.

In other words the cold-ion limit of the previous four-field model is, like MHD, reduced MHD and many other models, a Hamiltonian system. One obvious result is that energy of Eq. (39) is conserved, since {H,H} = 0.

For finite T_i the Hamiltonian of Eq. (39) is altered, without rigorous justification, in two ways. Firstly, $F'=\varphi$ is replaced by F, the stream function of Eq. (1); this change is easily understood *a posteriori*, as shown below. Secondly, the internal energy is modified to include the ion contribution: $p^2/(2\beta) \rightarrow (1+\tau)p^2/(2\beta)$.

These unsurprising changes yield the Hamiltonian of Eq. (6), whose physical plausibility was discussed in Sec. II.

Less straightforward are the finite-τ modifications of the Poisson bracket. In this regard, it is convenient to treat the parallel and perpendicular dynamics separately.

Consider first the parallel dynamics. It is clear that our task is to justify the replacement of Eq. (44) by Eq. (16). We do this in an *ad hoc* manner, using three constraints to construct the coefficient C_k^{44}. First note that at finite τ the stream function F differs, to leading order in δ, from the potential φ by $\delta\tau p$, a term that gives rise to the ion diamagnetic drift. On the other hand, as first shown by Mikhailovskii[11], the parallel flow is advected only by the electrostatic drift, as indicated in Eq. (5). These two facts enforce the first term of Eq. (16). Finally one finds that the resulting bracket satisfies Jacobi's identity only if the remaining term of Eq. (16) is also appended.

Similar "brute-force" procedures -- inelegant but straightforward -- have been attempted in the construction of perpendicular dynamics at finite τ, but without success. The perpendicular dynamics, involving gyroviscosity and perpendicular compressibility, are much more complicated and the physical constraints less clear than in the parallel case. Notice in particular that each proposed finite-τ modification must be checked for consistency with the Jacobi identity; the unwieldy form of typical FLR corrections [cf., for example, Eq. (25)] makes such checks extremely tedious.

Fortunately the gyro-map permits a much simpler and more reliable implementation of FLR physics. To obtain the appropriate bracket for the above Hamiltonian we consider the reverse of the map defined by Eq. (31), setting

$$\xi^{1'} = \xi^{1''} + (\delta\tau/2)\nabla_\perp^2(\xi^{3''} - 2\beta h), \qquad (47)$$

$$\xi^{i'} = \xi^{i''}, \quad i = 1,2,4;$$

where

$$(\xi^{1''}, \xi^{2''}, \xi^{3''}, \xi^{4''}) = (\nabla^2 F, \psi, p + 2\beta h, v).$$

The chain rule yields

$$\delta/\delta\xi^{i'} = \delta/\delta\xi^{i''}, \quad i = 1,2,4;$$

$$\delta/\delta\xi^{3'} = \delta/\delta\xi^{3''} - (\delta\tau/2)\nabla_\perp^2 \delta/\delta\xi^{1''}. \qquad (48)$$

Inserting Eqs. (47) and (48) into the "parallel-corrected" $T_i = 0$ bracket, defined by Eqs. (43), (16) and (46), produces the correct four-field bracket, which together with the Hamilitonian of Eq. (6), produces Eqs. (2)-(5).

In Sec. II B we chose to write the Hamiltonian equations in terms of the variables ξ defined by Eq. (8). Thus the Hamiltonian of Eq. (6) becomes that of Eq. (9) and the bracket obtained above in terms of ξ'' becomes that given by Eqs. (15) and (16).

Notice that the electrostatic potential need not be defined for this closed system; the four field variables ξ^i are advanced in time without knowledge of φ. It is nonetheless of interest to identify φ in terms of the four fields. There are two arguments leading to the correct answer, as given by Eq. (1).

First we can demand agreement between Eq. (3), involving φ, and Eq. (11) for the ξ^i. The point here is that Eq. (3) is free of

FLR physics and easily derived from electron momentum conservation. Thus we use Eqs. (46)-(49) to find

$$\partial\xi^2/\partial t = -[\psi,\delta H/\delta\xi^{1\prime}] - 2\delta\beta[\psi,\delta H/\delta\xi^{3\prime}]$$

$$= -[\psi,\delta H/\delta\xi^{1}] - 2\delta\beta[\psi,\delta H/\delta\xi^{3} - (\delta\tau/2)\nabla_\perp{}^2\delta H/\delta\xi^{1}]$$

$$= [\psi,(1+\delta^2\beta\tau\nabla_\perp{}^2)F - \delta\tau p] - \delta[\psi,p],$$

which agrees with Eq. (3) only if

$$\varphi = (1+\delta^2\beta\tau\nabla_\perp{}^2)F - \delta\tau p\ ,$$

as in Eq. (1).

The second argument proceeds by directly ordering the Braginskii gyroviscosity tensor as in HHM. We express the ion velocity as

$$\mathrm{v} = \varepsilon v_A(\hat{\zeta}\times\nabla_\perp F + v\hat{\zeta}) + O(\varepsilon^2),$$

and compute the $O(\varepsilon)$ portion of the ion momentum balance equation; the result again is precisely Eq. (1).

IV. Casimir Invariants

A. Derivation

Noncanonical field theories generally have a special class of constants of motion called Casimir invariants. These are entropy- or helicity-like constants, such as the magnetic and cross helicities of MHD. Since the four-field model, unlike ideal MHD, contains FLR physics and in addition is reduced, it is not obvious what these constants should be. [Direct calculation from Eqs. (2) - (5) leads to enormous and nearly hopeless labor.] We determine the Casimirs in this section using the bracket formalism.

By definition Casimir invariants are constants that commute with all functionals; i.e., C is a Casimir invariant if

$$\{C,F\} = 0 \quad \text{for all } F. \tag{49}$$

One can use Eq. (53) to obtain the constants. We begin with the two-dimensional, parallel corrected, cold-ion bracket of Eqs. (43), (16) and (46). Equation (49) can be manipulated, by partial integration, into the form

$$\{C,F\} = -\langle F_i [C_k{}^{ij} \xi'^k, C_j] \rangle = 0. \tag{50}$$

Here we have systematically set surface terms to zero. Independent of the boundary conditions necessary for the vanishing of these terms, the Casimirs so obtained will be constants of motion in the sense that their integrands will satisfy local conservation equations.

Now since Eq. (50) must be true for all functionals F, it follows that the coefficient of each F_i must vanish. This gives a system of four partial differential equations, which after some manipulation can be expressed as

$$[\xi^{2'},C_3] = 0, \tag{51}$$

$$[\xi^{2'},C_2] + [\xi^{3'},C_3] = 0, \tag{52}$$

$$[2\delta\beta\xi^{1'}-\xi^{3'},C_1] + [2\delta\beta\xi^{4'}+\beta\xi^{2'},C_4] = 0, \tag{53}$$

$$[2\delta\beta\xi^{4'}+\beta\xi^{2'},C^1] + [2\delta^2\beta\tau(2\delta\beta\xi^{1'}-\xi^{3'}),C_4] = 0. \tag{54}$$

Equations (51) and (52) involve only the variables $\xi^{2'}$ and $\xi^{3'}$, while Eqs. (53) and (54) involve $\xi^{1'}$ and $\xi^{4'}$. Equations (51) and (52) respectively imply

$$C = \langle A(\xi^{2'}) + \xi^{3'}B(\xi^{2'})\rangle$$

$$C = \langle k(\xi^{2'},\xi^{3'})\rangle, \tag{55}$$

where A, B and k are arbitrary functions of their arguments. Consistency between Eqs. (55) yields the following Casimir invariants:

$$C_1 = \langle A(\xi^{2'})\rangle$$

$$C_2 = \langle \xi^{3'}B(\xi^{2'})\rangle. \tag{56}$$

Similarly, Eqs. (53) and (54) imply

$$C_{3,4} = \langle C_\pm[2\delta\beta\xi^{4'}+\beta\xi^{2'}\pm(2\delta^2\beta\tau)^{1/2}(2\delta\beta\xi^{1'}-\xi^{3'})]\rangle, \tag{57}$$

where $C_\pm$ are arbitrary functions.

Now in order to obtain the Casimirs for the four-field model it is necessary to map from the primed to the physical variables. We know that the quantities thus obtained will be Casimirs, since the (parallel corrected) $T_i = 0$ bracket is isomorphic to the four-field bracket written in terms of the physical variables; $W = \nabla_\perp^2 F$, ψ, p and v. There is a one-to-one correspondence between Casimir invariants of isomorphic brackets. Thus we obtain the following Casimir invariants:

$$C_1 = \langle A(\psi)\rangle$$
$$C_2 = \langle (p + 2\beta h)B(\psi)\rangle \qquad (58)$$
$$C_{3,4} = \langle C_{\pm}[2\delta\beta v + \beta\psi \pm (2\delta^2\beta\tau)^{1/2}(2\delta\beta W - \tau\delta^2\beta\nabla_{\perp}p - p - 2\beta h)]\rangle .$$

These quantities are constants for the axisymmetric version of Eqs. (2)-(5); i.e., where $\nabla_{\parallel}$ is replaced by $-[\psi, \cdot\,]$. For three dimensions the functions A, B and $C_{\pm}$ are restricted as mentioned in Sec. II A. This restriction, among other things, is discussed in the following subsection.

B. Discussion

The restriction of axisymmetry for constancy of the Casimir invariants, Eqs. (62), can be eased. In fact the existence of the above Casimirs for arbitary functions A, B and $C_{\pm}$ in three dimensions is tantomount to the existence of a solution $\tilde{\psi}$ to the following equation[2]:

$$\nabla_{\parallel}\tilde{\psi} = \partial\tilde{\psi}/\partial\zeta + [\tilde{\psi},\psi] = 0. \qquad (59)$$

The question of the existence of a global $\tilde{\psi}$ is the same as that of the existence of a constant of motion for the one degree-of-freedom Hamiltonian system, for which the poloidal plane is the phase space, ζ is the time and ψ is the Hamiltonian. Said yet another way, the existence of $\tilde{\psi}$ is equivalent to the existence of magnetic surfaces. In the general case it is unlikely that $\tilde{\psi}$ exists (recall that ζ is a periodic variable).

Nevertheless, let us assume that $\tilde{\psi}$ exists and change variables; we will use the field $\tilde{\psi}$ instead of ψ. We wish to transform our three dimensional Poisson bracket, Eq. (15), into one written in terms of the variable $\tilde{\psi}$. To do this we relate ψ and $\tilde{\psi}$ variations of an arbitrary functional F. This yields

$$\nabla_{\parallel}(\delta F/\delta\psi) = [\delta F/\delta\tilde{\psi},\tilde{\psi}] . \qquad (60)$$

Upon inserting Eq. (60) into Eq. (15) we see that the transformation $\psi \to \tilde{\psi}$ takes the three dimensional four-field bracket into the axisymmetric bracket with $\tilde{\psi}$ replacing ψ. This bracket has the Casimir invariants of Eq. (58) for *arbitrary* functions A, B and $C_{\pm}$, but with $\tilde{\psi}$ replacing ψ.

Thus we have shown that the existence of the general Casimir invariants is tantomount to the existence of magnetic surfaces. It follows that the degree to which one believes magnetic surfaces exist in a tokamak discharge, should be the same as the degree to which one believes Casimir invariants with arbitrary functions A, B and $C_{\pm}$ exist.

One case in which solutions to Eq. (59) do exist is that of helical symmetry. Then one has $\psi(r,\tilde{\theta},t)$, where $\tilde{\theta} = \theta - \zeta/q_0$, and it can be shown by direct substitution that the following solves Eq. (59):

$$\tilde{\psi}(r,\tilde{\theta},t) = \psi(r,\tilde{\theta},t) + r^2/(2q_0) . \qquad (61)$$

Here $\tilde{\psi}$ is the helical flux function.

Let us next consider the meaning of the Casimir invariants. We have mentioned that these invariants are related to the magnetic and cross helicities. Specifically, they are the remnants of these ideal MHD quantities that survive our ordering procedure. The cross helicity also survives our inclusion of FLR physics, which is manifest in the fact that $v \cdot B$ has an additional term $v_M \cdot B$

arising from the gyro map. Since all four of our Casimir invariants have one of the two forms

$$\tilde{C}_1 = \langle f(\chi) \rangle \quad , \quad \tilde{C}_2 = \langle \Upsilon g(\chi) \rangle \quad , \tag{62}$$

where f and g are arbitrary functions, we will discuss their meaning in general terms for the fields χ and Υ. If we divide our physical domain up into cells, which we label by the value of χ at say the center, then the invariant $\tilde{C}_1$ determines the number of cells with a particular value of χ. This can be shown by picking f to be the characteristic function. The same procedure can be used to show that the invariant $\tilde{C}_2$ determines the sum of the values of Υ on those cells with a particular value of the field χ. Neither of these invariants determine spatial correlation, i.e. the placement of the cells with a given value.

To conclude we take limits of the Casimir invariants, Eqs. (58), and show that they reduce to previously obtained Casimirs invariants. To facilitate this we rewrite C_3 and C_4 as follows:

$$\tilde{C}_3 = \langle [C_-(D - a_i E) + C_+(D + a_i E)]/(4\delta\beta a_i) \rangle$$

$$\tilde{C}_4 = \langle [C_-(D - a_i E) - C_+(D + a_i E)]/(4\delta\beta) \rangle \ , \tag{63}$$

where $D \equiv \beta(\psi + 2\delta v)$, $E \equiv (1 + \delta^2\beta\tau\nabla_\perp{}^2)p - 2\delta\beta\nabla_\perp{}^2 F + 2\beta h$ and $a_i \equiv 2\delta^2\beta\tau$. In the cold ion limit $a_i, \tau \to 0$ and $F \to \varphi$, and the Casimirs invariants of Eq. (63) become

$$\tilde{C}_3 = \langle C_-(\psi + 2\delta v)/(2\delta) \rangle$$

$$\tilde{C}_4 = \langle [\nabla_\perp{}^2\varphi - (p + 2\beta h)/(2\delta\beta)] C_+(\psi + 2\delta v) \rangle \ . \tag{64}$$

We can further take the limit $\delta \to 0$ and obtain the invariants for compressible MHD (CRMHD):

$$\tilde{C}_3 = \langle v C_-'(\psi) \rangle$$

$$\tilde{C}_4 = \langle \nabla_\perp{}^2\varphi\, C_+(\psi) - (p + 2\beta h) v C_+'(\psi)/\beta \rangle \ . \tag{65}$$

This model was introduced in HKM.

V. Summary

The Hamiltonian four-field model is a simplified description of nonlinear tokamak dynamics that allows for finite ion Larmor radius physics as well as other effects related to compressibility and electron adiabaticity. Much simpler than a rigorous or even reduced description of the same physics, it still preserves essential features of the underlying exact dynamics.

The model is given by Eqs. (2) - (5), in terms of physical variables, and by Eqs. (10) - (13) in terms of the field variables, ξ_i. [The latter are defined by Eqs. (8).] A Hamiltonian expression of the model, in terms of a Hamiltonian functional and generalized Poisson bracket, is given by Eqs. (9) and (15) - (18).

Only the dissipationless form of the model is presented. In many applications such dissipative processes as resistivity and viscosity are appropriately included, in the conventional way -- for example, by appending ηJ to the right-hand side of Eq. (3). The Hamiltonian property is then lost, but it remains significant in that dissipation has been introduced in an explicit and physical way: as discussed in Section I, there is no fake dissipation.

In large part because of its Hamiltonian property the present four field model conserves not only total energy but also four generalized helicities, or Casimir invariants. These constants of the motion, which are given by Eqs. (7), have considerable value in applications.

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