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Local Effect of Equilibrium Current on Tearing Mode Stability in the Semicollisional Regime (Part II)

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Abstract

The local effect of the equilibrium current on the linear stability of the $m \geq 2$ drift-tearing mode in the semicollisional regime is investigated analytically. The set of eigenmode equations describing the relevant mode dynamics inside the tearing layer, is solved variationally to obtain the mode dispersion relation. It is found that the stabilizing influence of the parallel thermal conduction on $m \geq 2$ drift-tearing mode is reduced by the inclusion of the local equilibrium current.

I. Introduction

In a recent paper,[1] the local effect of the local equilibrium current J_0 on the stability of low poloidal number tearing modes in the collisional regime[2,3] has been investigated.

The modification of the dispersion relation describing the $m \geq 2$ tearing mode has been found comparatively more significant than for the $m = 1$ mode, where the J_0 correction appears quite small. It was found that the FKR growth rate[2] is increased by the inclusion of the local J_0 terms, whereas the stabilization of the drift-tearing mode is enhanced; the classical $m = 1$ growth rate is slightly increased.

Also, it has been shown that, assuming the parallel equilibrium electric field $E_{\parallel}^{(0)}$ to be spatially constant, the presence of electron temperature gradients in the plasma is essential for the coupling of J_0 to the mode dynamics.

In deriving the relevant eigenmode equations inside the layer, the electron response has been taken to be strictly collisional, limiting the validity of the analysis to the outer, cooler regions of a tokamak discharge. The most interesting regime in which present day operates is, however, the semicollisional[2,3,4] where, in particular, the parallel thermal conduction has to be considered. The investigation of the local effect of J_0 in this regime appears to be especially interesting, since it has been surmised[5] that the presence of electron temperature gradients, coupled with the parallel thermal conduction, has a strong stabilizing effect on the $m \geq 2$ drift-tearing mode.

In the present paper, the electron response is derived from fluid theory including full semicollisional effects, and the formalism developed in Ref. 1 is used to derive the modification to the dispersion relation for the semicollisional $m \geq 2$ drift-tearing mode due to the local current.

The remainder of this article is organized as follows. In Sec. II, we derive the eigenmode equations, describing the plasma response in the semicollisional regime, including local J_0 effects. In Sec. III, the methodology developed in detail in Ref. 1 is used to obtain the variational functional for the semicollisional case. In Sec. IV, the appropriate limiting form of the variational functional is shown to recover the known result for $m \geq 2$ drift-tearing mode in the semicollisional regime. The effect of the local equilibrium current is then treated as a perturbation of this result, and the modified dispersion relation is obtained.

In Sec. V, we summarize our conclusions and remarks. In the Appendix, we present an alternative derivation of the semicollisional tearing mode dispersion relation in the absence of local J_0.

C.G.S. units with $c = 1$ are used throughout.

II. Eigenmode Equations

The electron dynamics of relevance in the present work is correctly represented by the following moment equations,[3,4] expressing particle conservation, momentum balance along the total magnetic field, and energy balance, respectively:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0, \tag{1}$$

$$\begin{aligned} \varepsilon'' \frac{\partial}{\partial t}\left(n m_e \hat{b}\cdot\mathbf{u}\right) = &- ne\hat{b}\cdot\mathbf{E} - \hat{b}\cdot\nabla p_e \\ &- (ne)^2\eta\hat{b}\cdot\mathbf{u}\left[1 + \frac{3}{2}\varepsilon''\left(\frac{m_e}{\eta n e^2}\right)\frac{\partial}{\partial t}\ell n T_e\right] - \varepsilon n\hat{b} \\ &\cdot\nabla T_e\left[1 - \frac{3\varepsilon'}{\nu}\frac{\partial}{\partial t}\ell n T_e - \frac{\varepsilon'}{\nu}\frac{\partial}{\partial t}\ell n\left(\hat{b}\cdot\nabla T_e\right)\right], \end{aligned} \tag{2}$$

$$\begin{aligned} &\frac{3}{2}n\left(\frac{\partial T_e}{\partial t} + \mathbf{u}\cdot\nabla T_e\right) + nT_e\nabla\cdot\mathbf{u} \\ &\quad + \nabla\cdot\left[\varepsilon n T_e\hat{b}\hat{b}\cdot\mathbf{u} - \chi\hat{b}\hat{b}\cdot\nabla T_e\right] - \nabla\cdot\left[\frac{5}{2}\frac{nT_e}{eB}\hat{b}\times\nabla T_e\right] \\ &\quad - \hat{b}\cdot\mathbf{u}\left[\varepsilon n\hat{b}\cdot\nabla T_e + (en)^2\eta\hat{b}\cdot\mathbf{u}\right] = 0, \end{aligned} \tag{3}$$

where

$$\mathbf{u} = \hat{b}\hat{b}\cdot\mathbf{u} + \frac{\mathbf{E}\times\hat{b}}{B} - \frac{\hat{b}\times\nabla p_e}{enB}. \tag{4}$$

These equations differ from the well-known transport equations derived by Braginskii,[6] to the required order,[4] by the inclusion of a time-dependent thermal force on the electrons and by the fact that the coefficient of the electron inertial term in the momentum equation, Eq. (2), is greater than unity. The physical origin of these terms, basically connected with the velocity dependence of the Coulomb cross-section (for a plasma, $\nu \sim v_e^{-3}$, where ν is the electron collision frequency and v_e is the electron thermal velocity), has been discussed in detail in Ref. 4. Here, we only remark that the inclusion of the time-dependent thermal force results in the thermoelectric growth rate[3,11] in the final dispersion relation and that this term does not couple with the local parallel equilibrium current J_0, and therefore does not play a crucial role in the present analysis. Also, the electron inertial term will be neglected in the subsequent derivation.

In Eqs. (1)–(4), n is the electron density, T_e is the electron temperature, $p_e = nT_e$ is the electron pressure, m_e is the electron mass, e is the absolute value of the electron charge; $\mathbf{E}$ and $\mathbf{B}$ are the total electric and magnetic fields, respectively, $\hat{b}$ is a unit vector in the direction of the total magnetic field $\mathbf{B}$; η is the Spitzer-Braginskii resistivity; χ is the thermal conductivity along the ambient magnetic field; ν is the electron collision frequency; ε, ε', ε'' are numerical transport coefficients tabulated in Ref. 4: in particular, ε is the familiar[6] coefficient of the thermal force and is equal to 0.71 for a singly-charged ion plasma.

The linearization of Eqs. (1)–(4) is carried out assuming that, for any linearly perturbed quantity $\tilde{g}$,

$$\frac{\partial}{\partial t}\tilde{g} = -i\omega\tilde{g}, \tag{5a}$$

$$\hat{b}_0 \cdot \nabla\tilde{g} = ik_{\parallel}(x)\tilde{g}, \tag{5b}$$

$$\hat{e} \cdot \nabla\tilde{g} = ik_{\perp}(x)\tilde{g}, \tag{5c}$$

$$\hat{r} \cdot \nabla\tilde{g} = \tilde{g}', \tag{5d}$$

where the radial variable $x = r - r_s$ denotes the distance from a mode rational surface, located at $r = r_s$. A locally orthogonal reference system is introduced, so that $\hat{b}_0 \equiv \mathbf{B}_0/B_0$ is a unit vector in the direction of the equilibrium magnetic field $\mathbf{B}_0$; $\hat{r}$ is a unit vector in the radial direction, normal to a flux surface; and $\hat{e} \equiv \hat{b}_0 \times \hat{r}$. In the slab model of the tokamak, these unit vectors, as well as the equilibrium quantities, depend only on the radius. Near a mode rational surface, where $k_{\parallel}(0) = 0$, we have

$$k_{\parallel}(x) \simeq k'_{\parallel}x. \tag{6}$$

For sufficiently small β (ratio between plasma pressure and magnetic pressure), it is consistent[7] to assume for the perturbed magnetic vector potential $\tilde{A}$,

$$\tilde{\mathbf{A}} = \hat{b}_0 \tilde{A}_{\parallel}, \tag{7}$$

implying that the radial magnetic perturbation becomes

$$\tilde{B}_r = ik_\perp \tilde{A}_{\parallel}, \tag{8}$$

and the components of the perturbed electric field are

$$\tilde{E}_{\parallel} \equiv \hat{b}_0 \cdot \tilde{\mathbf{E}} = -ik_{\parallel}\tilde{\phi} + i\omega \tilde{A}_{\parallel}, \tag{9a}$$

$$\tilde{E}_{\perp} \equiv \hat{e} \cdot \tilde{\mathbf{E}} = -ik_{\perp}\tilde{\phi}, \tag{9b}$$

$$\tilde{E}_r \equiv \hat{r} \cdot \tilde{\mathbf{E}} = -\tilde{\phi}', \tag{9c}$$

where $\tilde{\phi}$ is the perturbed electrostatic potential.

Making use of the above equations, we obtain, for the perturbed parallel current $\tilde{J}_{\parallel}$

$$\begin{aligned} &\eta \tilde{J}_{\parallel} \left[1 + \frac{isk_{\parallel}^2 D}{\omega} + \frac{isk_{\parallel}^2 D}{\omega} \frac{\frac{2}{3}(1+\varepsilon)\hat{\varepsilon}}{\left(1 + i\varepsilon''' s \frac{k_{\parallel}^2 D}{\omega}\right)} \right] \\ &= i\left(\omega \tilde{A}_{\parallel} - k_{\parallel}\tilde{\phi}\right) \left[1 - \frac{\omega_n^*}{\omega} - \frac{\omega_T^*}{\omega} \frac{\hat{\varepsilon}}{\left(1 + i\varepsilon''' s \frac{k_{\parallel}^2 D}{\omega}\right)} \right] \\ &+ \frac{3}{2}\eta J_0 \frac{e\omega_T^*}{T_e} \tilde{\phi} \left(\omega + i\varepsilon''' s k_{\parallel}^2 D\right)^{-1}, \end{aligned} \tag{10}$$

with the definitions

$$\omega_n^* \equiv -\frac{k_\perp}{eB} \frac{n'}{n} T_e, \tag{11a}$$

$$\omega_T^* \equiv -\frac{k_\perp}{eB} T_e', \tag{11b}$$

$$\hat{\varepsilon} \equiv 1 + \varepsilon + i\varepsilon\varepsilon' \frac{\omega}{\nu}, \tag{11c}$$

$$D \equiv \frac{T_e}{m_e \nu}, \tag{11d}$$

$$s \equiv \eta(Z=1)/0.51\eta\,(Z_{\text{eff}}), \tag{11e}$$

$$\frac{\chi}{n} \equiv \frac{3}{2}\varepsilon''' s D; \tag{11f}$$

ω_n^*, ω_T^* are the electron drift frequencies related to density gradients and electron temperature gradients, respectively; $\hat{\varepsilon}$ is defined in terms of the numerical transport coefficients ε, ε'; the quantity D is basically the diffusion coefficient of the electrons along the magnetic field B_0, when collisions inhibit free streaming at the thermal velocity; s is the numerical coefficient of the Spitzer-Braginskii conductivity and is equal to 1.96 for singly-charged ions; the numerical transport coefficient ε''' has been defined so that $(3/2)\varepsilon''' s$ is equal to the numerical coefficient of the parallel thermal conductivity, which is, $(3/2)\varepsilon''' s = 3.16$ for singly-charged ions. We note that apart from the local J_0 term, which is related to resistivity perturbations in Eq. (2), Eq. (10) has already been obtained in Ref. 3. The first eigenmode equation is obtained by combining Eq. (10) with the parallel component of Ampére's law in the form

$$\tilde{J}_{\|} = \frac{1}{4\pi}\left(-\tilde{A}_{\|}'' + k_{\perp}^2 \tilde{A}_{\|}\right) \simeq -\frac{\tilde{A}_{\|}''}{4\pi}, \tag{12}$$

where we introduced the approximation, valid for low poloidal mode numbers and used throughout the present analysis, of assuming the mode radial wavelength to be much shorter than the azimuthal wavelength. Defining $\psi \equiv \omega \tilde{A}_{\|}/k_{\|}'$ and making use of Eq. (6), we get

$$\psi'' = \sigma_*\left(x^2\right)\left\{\psi - \left[x - \Lambda\left(x^2\right)\right]\phi\right\}, \tag{13}$$

where the generalized conductivity $\sigma_*\left(x^2\right)$ is given by[3,5]

$$\sigma_*\left(x^2\right) = \frac{-4\pi i}{\eta}\left\{\frac{\left(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*\right) + \left(\omega - \omega_n^*\right) i\varepsilon''' s \frac{k_{\|}^2 D}{\omega}}{\left(1 + i\varepsilon''' s \frac{k_{\|}^2 D}{\omega}\right)\left(1 + is\frac{k_{\|}^2 D}{\omega}\right) + \frac{2}{3}\hat{\varepsilon}(1+\varepsilon) is \frac{k_{\|}^2 D}{\omega}}\right\}, \tag{14}$$

and the equilibrium current appears through the factor

$$\Lambda\left(x^2\right) = \left\{\frac{-\frac{3}{2}\frac{\eta}{k_{\|}'}\frac{e\omega_T^*}{T_e} J_0}{-i\left(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*\right) + \varepsilon''' s\left(\omega - \omega_n^*\right)\frac{k_{\|}^2 D}{\omega}}\right\}. \tag{15}$$

Equation (13) is essentially a description of the electron dynamics; the other piece of needed information describes the ion dynamics and it is usually[8] obtained considering the plasma equation of motion in the form

$$m_i n_i \frac{\partial \mathbf{V}}{\partial t} = -\nabla P + \mathbf{J} \times \mathbf{B}, \tag{16}$$

where $\mathbf{V}$ is the ion (or bulk plasma) flow velocity, m_i is the ion mass, n_i is the ion density, $P = p_i + p_e$ is the total plasma pressure.

Solving Eq. (16) for the perpendicular component of the perturbed current, $\tilde{\mathbf{J}}_\perp$, and imposing quasineutrality in the form

$$\nabla \cdot \tilde{\mathbf{J}}_\parallel = -\nabla \cdot \tilde{\mathbf{J}}_\perp, \tag{17}$$

one finds[9]

$$\chi_A^2 \phi'' = x\psi'' - \frac{4\pi k_\perp}{B k_\parallel'} J_0' \psi, \tag{18}$$

where the Alfvén layer χ_A, defined as the distance from the rational surface at which $\omega \simeq k_\parallel v_A$ ($v_A = B_0 / (4\pi n_i m_i)^{1/2}$ is the Alfvén speed), is given by

$$\chi_A^2 \equiv \frac{\omega(\omega + \omega_i^*)}{\left(k_\parallel' v_A\right)^2}, \tag{19}$$

and the ion drift frequency is

$$\omega_i^* \equiv -\frac{k_\perp}{eB} \frac{p_i'}{n_i}. \tag{20}$$

In the derivation of Eqs. (13), (18) the following approximations have been used:

i) terms involving J_0 which are of order β have been neglected;

ii) terms involving J_0 which appear as Doppler shifts to the real part of the mode frequency have been neglected;

iii) contributions to the mode growth rate due to Ohmic heating terms are also neglected, since the corresponding thermal instability does not strongly couple to the tearing mode.

Neglecting terms of order $k_\parallel^2 D/\omega$ in Eqs. (14), (15), the eigenmode set constituted by Eqs. (13), (18) reduces exactly to the one studied in Ref. 1, where a derivation of Eq. (13) from kinetic theory rather than from fluid equations has been given, and also an extensive discussion on the eigenmode symmetry breaking properties of the J_0 terms has been presented.

In the following section, we outline the method of solution of Eqs. (13), (18) according to the formalism of Ref. 1.

III. Variational Principle

From the analysis of the dispersion relation of the collisional drift-tearing mode,[1,3] it turns out that the term $(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*)$ appearing in the numerator of σ_* tends to be much smaller than the factor $(\omega - \omega_n^*)$. Therefore, the following scaling is possible because of the presence of temperature gradients, and it is characterized by

$$\frac{k_\parallel^2 D}{\omega} < 1, \tag{21}$$

but, at the same time,

$$(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*) \sim (\omega - \omega_n^*)\,\frac{k_\parallel^2 D}{\omega}. \tag{22}$$

The existence of such a scaling, which includes in particular the effect of the thermal conduction, has been noted for example in Ref. 5 and is very interesting in the present context, since it is made possible by the temperature gradient, which also allows the coupling of the local equilibrium current to the eigenmode equations. We remark that, when $(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*) \ll (\omega - \omega_n^*)\, k_\parallel^2 D/\omega$, $\Lambda \to 0$ in Eq. (13), making the effect of the local current negligible. For this reason, since this study is intended to treat J_0 effects, we shall limit our discussion to the scaling expressed by Eqs. (21) and (22); we can now approximate σ_* by

$$\sigma_* \simeq \mu_1 + \mu_2 x^2, \tag{23}$$

where μ_1 is given by

$$\mu_1 \equiv \frac{-4\pi i}{\eta}\,(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*), \tag{24}$$

and μ_2 is defined as

$$\mu_2 \equiv \frac{4\pi\varepsilon''' s}{\eta}\,(\omega - \omega_n^*)\,\frac{{k'_\parallel}^2 D}{\omega}. \tag{25}$$

We note that when σ_* is approximated as in Eq. (23) (unlike when one retains its full x^2 dependence as in Eq. (14)), the two J_0 terms in Eqs. (13) and (18) are essentially identical (a similar equality was shown in Ref. 1 to hold for the collisional case as well) and we have

$$(\mu_1 + \mu_2 x^2)\, \Lambda(x^2) = \frac{4\pi k_\perp J_0'}{k_\parallel' B}, \tag{26}$$

where we made use of the $T_e^{3/2}$ dependence of the Spitzer-Braginskii conductivity σ_{sp} in the equilibrium Ohm's law

$$J_0 = \sigma_{sp} E_{\|}^{(0)} \tag{27}$$

and assumed the parallel equilibrium electric field to be spatially constant to write

$$J_0' = \frac{3}{2}\frac{T_e'}{T_e} J_0. \tag{28}$$

We see that, even in the semicollisional regime, the presence of electron temperature gradients is essential for the coupling of the local J_0 to the mode dynamics and the local equilibrium current has to be treated on the same footing with its local gradient. Making use of Eq. (26), the eigenmode equations can evidently be cast in the following form

$$\psi'' = \sigma_* \left(x^2\right) \left\{\psi - \left[x - \Lambda\left(x^2\right)\right]\phi\right\}, \tag{29}$$

$$\phi'' + \frac{\lambda\mu_1}{\chi_A^2}\left[x - \Lambda\left(x^2\right)\right]\phi = \frac{\sigma_*\left(x^2\right)}{\chi_A^2}\left[x - \Lambda\left(x^2\right)\right]\left\{\psi - \left[x - \Lambda\left(x^2\right)\right]\phi\right\}, \tag{30}$$

where $\sigma_* \left(x^2\right)$ is given by its approximated form, Eq. (23).

Even though Eqs. (29), (30) closely resemble Eqs. of Ref. 1, the x^2 dependence of Λ makes the procedure of Ref. 1 no longer applicable in a straightforward way. The following approximation, however, will enable us to reduce the problem to the one solved before. Since the effect of the local equilibrium current shall be treated as a small perturbation of the known semicollisional result, a negligible error will be involved if we substitute x^2 by a suitable, average value over the radial extent of the mode; for the moment, let us denote this average simply as $\langle x^2 \rangle$; in Section IV, an explicit form for $\langle x^2 \rangle$ will be given. $\Lambda\left(x^2\right)$ can therefore be approximated by

$$\Lambda \simeq \frac{\rho\chi_A^2}{\left(\mu_1 + \mu_2 \langle x^2\rangle\right)}, \tag{31}$$

where

$$\rho \equiv \frac{4\pi k_\perp J_0'}{k_\|' B_0 \chi_A^2}. \tag{32}$$

Defining a new radial variable[1]

$$p \equiv x - \Lambda, \tag{33}$$

and a new field variable

$$Q(p) \equiv \sigma_*(p)(\psi - p\phi), \tag{34}$$

where

$$\sigma_*(p) \equiv \mu_1 + \mu_2(p^2 + 2\Lambda p + \Lambda^2), \tag{35}$$

we can write the eigenmode equations in the form

$$\psi'' = Q(p), \tag{36}$$

$$\phi'' + \rho p\phi = \frac{pQ(p)}{\chi_A^2}, \tag{37}$$

which is formally identical to the corresponding set of Ref. 1. We briefly outline the method of solution of Eqs. (36), (37), presented in detail in Ref. 1; treating the variable Q as a given inhomogeneous term at the right-hand side (r.h.s.) of Eqs. (36), (37), one formally solves these equations for ψ and ϕ in terms of Q; using the definition of Q as given by Eq. (34), one then constructs an appropriate combination of ψ and ϕ to obtain an integral equation for Q, which has the necessary structure to allow for a variational treatment. Appropriate boundary conditions are chosen by requiring the asymptotic behavior[1,8,10]

$$\psi \to \psi_0 \left(1 + \frac{\Delta'|x|}{2}\right) \tag{38}$$

for large x, where, we recall, ψ_0 is the value toward which the exterior ψ solution tends and Δ' is the logarithmic jump in the derivative of the external solution, defined by

$$\Delta' \equiv \frac{\psi'_+ - \psi'_-}{\psi_0}, \tag{39}$$

where ψ_+, ψ_- denote the external ψ solution on the right and on the left of the tearing layer, respectively. In the present context, no distinction is made between the radial variables x and p, because for $x \to \infty$, $p \to x$. The resulting integral equation for Q is[1]

$$\frac{Q(p)}{\sigma_*} = \frac{1}{\Delta'}\int_{-\infty}^{\infty} Q(p')dp' + \frac{1}{2}\int_{-\infty}^{\infty} g(p,p')Q(p')dp', \tag{40}$$

where

$$g(p,p') = g(p',p) \equiv |p-p'|\left[1-\frac{pp'}{\chi_A^2}K(p,p')\right], \tag{41}$$

and

$$\begin{aligned} K(p,p') = K(p',p) \equiv 1 &- \frac{\rho}{12}(p+p')(p-p')^2 \\ &+ \frac{\rho^2}{504}\left[p^6+p'^6-\frac{9}{5}pp'\left(p^4+p'^4\right)-\frac{9}{5}p^2p'^2\left(p^2+p'^2\right)+\frac{26}{5}p^3p'^3\right]. \end{aligned} \tag{42}$$

Noting that the kernel $g(p,p')$ is symmetric in p and p', the variational functional corresponding to the Euler-Lagrange equation Eq. (40) is given

$$\begin{aligned} S = \int_{-\infty}^{\infty} \sigma_*(p)E^2(p)dp &- \frac{1}{\Delta'}\left[\int_{-\infty}^{\infty}\sigma_*(p)E(p)dp\right]^2 \\ &- \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(p,p')\sigma_*(p)\sigma_*(p')E(p)E(p')dp\,dp'; \end{aligned} \tag{43}$$

we have expressed the functional S in terms of

$$E(p) \equiv \frac{Q(p)}{\sigma_*(p)}, \tag{44}$$

rather than $Q(p)$ itself, a form which is more useful for the subsequent calculations. By letting $\rho \to 0$, $E(p)$ becomes the ordinary perturbed parallel electric field; also, letting $\mu_2 \to 0$ in Eq. (43), cfr. Eq. (23), one recovers precisely the collisional form of S, a given in Ref. 1.

IV. Dispersion Relations

In order to evaluate explicitly the variational S, given by Eq. (43), it is necessary to choose a suitable trial function; in conventional theories, where J_0 terms are neglected, the eigenmode equations are invariant under space reflection and one can choose a trial function of definite parity; solutions with tearing symmetry have even parity (i.e., $\tilde{J}_{||}$, $\tilde{E}_{||}$, ψ even and ϕ odd) and the usual choice is[8,10]

$$E(x)_{\text{trial}} = \exp\left(-\alpha x^2/2\right), \tag{45}$$

where the variational parameter α measures the radial width ℓ_ω of the mode, i.e., $\ell_\omega \sim \alpha^{-1/2}$.

When J_0 terms are included, the space reflection symmetry of the eigenmode equations is broken, and one can no longer choose a trial function with definite parity; since we shall principally be treating the effect of the current as a correction to known results, we demand that the trial function reduces to Eq. (45) whenever the J_0 effect is neglected, to ensure continuous contact with previous analyses. The simplest choice for a mixed parity trial function with the required properties is[1]

$$E(p)_{\text{trial}} = (1 + \delta p) \exp\left(-\alpha p^2/2\right), \tag{46}$$

where the variational parameter δ, which is evidently proportional to ρ (cfr. Eq. (37)), measures the mixing of the even and odd part of the solution, caused by the inclusion of J_0. With the above choice for the trial function, S, as given by Eq. (43), becomes a polynomial in α and δ, and the extremum conditions on S, together with the condition that S vanishes for $E(p)_{\text{trial}} = E(p)$ (to lighten the notation, we use the same symbol to denote $E(p)$, the solution to Eqs. (40), (44), as to denote the trial function, containing variational parameters, which appears in Eq. (43)), are in this case[1,8]

$$\frac{\partial S}{\partial \delta} = 0, \tag{47}$$

$$\frac{\partial S}{\partial \alpha} = 0, \tag{48}$$

$$S = 0. \tag{49}$$

After the dispersion relation has been obtained, any mathematically consistent root must satisfy the following conditions, expressing the mode localization and the choice of the branch cut for $\alpha^{1/2}$ in evaluating the integrals appearing in Eq. (43), respectively:

$$\mathrm{Re}\,\alpha > 0, \tag{50}$$

$$\mathrm{Re}\,\alpha^{1/2} > 0; \tag{51}$$

no similar conditions have to be imposed on δ.

Substituting the trial function Eq. (46) in Eq. (43) we obtain, after a somewhat lengthy calculation,

$$\begin{aligned}
S =& \mu_1\left(\alpha^{-1/2}+\frac{\delta^2\alpha^{-3/2}}{2}\right)+\mu_2\left(\frac{\alpha^{-3/2}}{2}+\frac{3}{4}\delta^2\alpha^{-5/2}+2\Lambda\delta\alpha^{-3/2}+\Lambda^2\alpha^{-1/2}\right)\\
&-\frac{2\pi^{1/2}}{\Delta'}\alpha^{-1}\left[\mu_1+\mu_2\left(\alpha^{-1}+2\Lambda\delta\alpha^{-1}+\Lambda^2\right)\right]^2-\mu_1^2\left(2\alpha^{-3/2}+\frac{\alpha^{-5/2}}{\chi_A^2}\right)\\
&-\mu_1\mu_2\left(6\alpha^{-5/2}+\frac{5\alpha^{-7/2}}{\chi_A^2}\right)-\mu_2^2\left(\frac{7}{2}\alpha^{-7/2}+\frac{27}{4}\frac{\alpha^{-9/2}}{\chi_A^2}\right)\\
&+\delta^2\left[\mu_1^2\left(\alpha^{-5/2}+\frac{7}{2}\frac{\alpha^{-7/2}}{\chi_A^2}\right)+\mu_1\mu_2\left(5\alpha^{-7/2}+\frac{49}{2}\frac{\alpha^{-9/2}}{\chi_A^2}\right)\right.\\
&\left.+\mu_2^2\left(\frac{27}{4}\alpha^{-9/2}+\frac{321}{8}\frac{\alpha^{-11/2}}{\chi_A^2}\right)\right]\\
&+4\delta\Lambda\left[\mu_1\mu_2\left(-2\alpha^{-5/2}+\frac{\alpha^{-7/2}}{\chi_A^2}\right)+\mu_2^2\left(-\alpha^{-7/2}+\frac{11}{2}\frac{\alpha^{-9/2}}{\chi_A^2}\right)\right]\\
&+\Lambda^2\left[-2\mu_1\mu_2\left(2\alpha^{-3/2}+\frac{\alpha^{-5/2}}{\chi_A^2}\right)+\mu_2^2\left(-2\alpha^{-5/2}+\frac{9\alpha^{-7/2}}{\chi_A^2}\right)\right]\\
&+\frac{2}{3}\frac{\delta\rho}{\chi_A^2}\left(\mu_1^2\alpha^{-9/2}+9\mu_1\mu_2\alpha^{-11/2}+\frac{51}{4}\mu_2^2\alpha^{-13/2}\right)\\
&+\frac{4}{3}\frac{\Lambda\rho}{\chi_A^2}\left(\mu_1\mu_2\alpha^{-9/2}-\frac{15}{2}\mu_2^2\alpha^{-11/2}\right)\\
&+\frac{\rho^2}{\chi_A^2}\left(-\frac{2}{15}\mu_1^2\alpha^{-11/2}-\frac{22}{15}\mu_1\mu_2\alpha^{-13/2}+\frac{3}{10}\mu_2^2\alpha^{-15/2}\right).
\end{aligned}\tag{52}$$

In Eq. (52), we recall, α and δ are variational parameters, μ_1 and μ_2 are given by Eqs. (24), (25), respectively, the local equilibrium current appears linearly in ρ and Λ, given by Eqs. (31) and (32), respectively. We notice that the corrections due to the local current appear in S as terms quadratic in ρ since δ must be of order ρ itself. Terms of higher order in ρ (e.g., terms of fourth order in J_0 such as $\delta^2\rho^2$ or $\delta^2\Lambda^2$) have been systematically ignored in the calculation.

We notice that, by letting $\mu_2 \to 0$ in Eq. (52), one recovers the collisional limit of S, cfr. Ref. 1, as it should be expected from the form of Eqs. (23)–(25). Also, from the previous derivation, it appears that we can choose for the average value of x^2, $\langle x^2 \rangle$, appearing in Λ its average over a Gaussian trial function, which is

$$\Lambda \simeq \frac{\rho\chi_A^2}{\mu_1 + (\mu_2/2\alpha)}. \tag{53}$$

We first proceed to show that S easily recovers the known result in the absence of the equilibrium current, in the appropriate limit. The "classical" form of S is obtained by neglecting terms proportional to J_0 in Eq. (52); also, since in the present section we shall only be interested in modes which are broader than the Alfvén layer, we consider the limit of S for $\left|\alpha^{1/2}\chi_A\right| < 1$:

$$\begin{aligned} S_{\rm cl} \simeq \mu_1\alpha^{-1/2} + \mu_2\frac{\alpha^{-3/2}}{2} - \frac{2\pi^{1/2}}{\Delta'}\alpha^{-1}\left(\mu_1 + \mu_2\alpha^{-1}\right)^2 \\ - \mu_1^2\frac{\alpha^{-5/2}}{\chi_A^2} - 5\mu_1\mu_2\frac{\alpha^{-7/2}}{\chi_A^2} - \frac{27}{4}\mu_2^2\frac{\alpha^{-9/2}}{\chi_A^2} = 0. \end{aligned} \tag{54}$$

The above form of S is still too complicated for a simple analytical solution, but it is possible to obtain a quite accurate answer by means of the following argument. In the Appendix, we present an exact variational calculation based on the method of Ref. 11, which allows a very simple treatment of the semicollisional conductivity given by Eq. (23) but it is not extendable to the case in which J_0 terms are included. We remark that the dispersion relations obtained by the two methods differ only by a few percent in the numerical factor in front of the damping rate.

Since the mode which we are going to discuss is characterized by a negligible value of Δ',[5] the corresponding term in S containing Δ' will be negligible, provided

$$\alpha = -\frac{\mu_2}{\mu_1}, \tag{55}$$

which, substituted in Eq. (54) and setting the resulting expression for S equal to zero, yields the dispersion relation

$$\mu_1^3 = \frac{2}{11}\mu_2^2\chi_A^2, \tag{56}$$

which can be written as

$$(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*)^3 = -i\frac{2}{11}\left(\frac{\eta}{4\pi}\right)\left[\varepsilon''' s\left(\omega - \omega_n^*\right) k_\parallel' D\right]^2 \frac{(\omega + \omega_i^*)}{\omega v_A^2}, \tag{57}$$

and which, for a drift tearing-type solution, $\omega^* \gg \gamma$, gives[5]

$$\omega \simeq \omega_0 + i\epsilon\varepsilon' \frac{\omega_0}{\nu}\omega_T^* + e^{7\pi i/6}\left(\frac{2}{11}\right)^{1/3}\left(\frac{\eta}{4\pi}\right)^{1/3}\left(\varepsilon''' s\hat{\varepsilon}\omega_T^* k_\parallel' D\right)^{2/3}\frac{\left(1+\frac{\omega_i^*}{\omega_0}\right)^{1/3}}{v_A^{2/3}}, \tag{58}$$

where the real mode frequency ω_0 is given by

$$\omega_0 \simeq \omega_n^* + (1+\epsilon)\omega_T^*. \tag{59}$$

The first imaginary term on the right-hand side of Eq. (58) is the so-called thermoelectric growth rate[12] (as in the collisional drift tearing result), while the last term provides a damping of the mode due to the stabilizing effect of the parallel thermal conduction. This dispersion relation corresponds to the root $(1)^{1/3} = e^{2\pi i/3}$ for μ_1, which is mathematically consistent, as it can be seen from Eq. (55). It is not necessary to impose the condition on $\alpha^{1/2}$ in the present case since, as it can be seen from Eqs. (54) and (55), the balance of the individual terms in S depends only on the sign of α and not of $\alpha^{1/2}$.

We now investigate the effect of J_0 on the result expressed by Eqs. (55), (56), by treating the effect of the local current as a perturbation on the result expressed by Eqs. (55), (56). The limit of S for $\left|\alpha^{1/2}\chi_A\right| < 1$ is given by

$$\begin{aligned}
S \simeq\; & \mu_1\left(\alpha^{-1/2} + \delta^2\frac{\alpha^{-3/2}}{2}\right) + \mu_2\left(\frac{\alpha^{-3/2}}{2} + \frac{3}{4}\delta^2\alpha^{-5/2} + 2\Lambda\delta\alpha^{-3/2} + \Lambda^2\alpha^{-1/2}\right)\\
& - \mu_1^2\frac{\alpha^{-5/2}}{\chi_A^2} - 5\mu_1\mu_2\frac{\alpha^{-7/2}}{\chi_A^2} - \frac{27}{4}\mu_2^2\frac{\alpha^{-9/2}}{\chi_A^2}\\
& + \frac{\delta^2}{\chi_A^2}\left(\frac{7}{2}\mu_1^2\alpha^{-7/2} + \frac{49}{2}\mu_1\mu_2\alpha^{-9/2} + \frac{321}{8}\mu_2^2\alpha^{-11/2}\right)\\
& + \frac{4\delta\Lambda}{\chi_A^2}\left(\mu_1\mu_2\alpha^{-7/2} + \frac{11}{2}\mu_2^2\alpha^{-9/2}\right) + \frac{\Lambda^2}{\chi_A^2}\left(-2\mu_1\mu_2\alpha^{-5/2} + 9\mu_2^2\alpha^{-7/2}\right)\\
& + \frac{2}{3}\frac{\delta\rho}{\chi_A^2}\left(\mu_1^2\alpha^{-9/2} + 9\mu_1\mu_2\alpha^{-11/2} + \frac{51}{4}\mu_2^2\alpha^{-13/2}\right)\\
& + \frac{4}{3}\frac{\Lambda\rho}{\chi_A^2}\left(\mu_1\mu_2\alpha^{-9/2} - \frac{15}{2}\mu_2^2\alpha^{-11/2}\right)\\
& + \frac{\rho^2}{\chi_A^2}\left(-\frac{2}{15}\mu_1^2\alpha^{-11/2} - \frac{22}{15}\mu_1\mu_2\alpha^{-13/2} + \frac{3}{10}\mu_2^2\alpha^{-15/2}\right).
\end{aligned} \tag{60}$$

The term involving Δ' has been neglected since, with α given by Eq. (5), it provides a correction of order ρ^4 (i.e., of fourth order in J_0) to S.

Since S is a variational quantity, the perturbative calculation is of the outmost simplicity; indeed, if we write Eq. (52) as

$$S = S_0 + \varepsilon^2 S_1, \tag{61}$$

where S_0 is the corresponding "classical" S, and $\varepsilon^2 \ll 1$ is proportional to ρ^2, and we assume

$$\alpha = \alpha_0 + \varepsilon^2 \alpha_1, \tag{62}$$

with α_0 given by Eq. (55), we have

$$S \simeq S_0(\alpha_0) + \left.\frac{\partial S_0}{\partial \alpha}\right|_{\alpha_0} \varepsilon^2 \alpha_1 + \varepsilon^2 S_1(\alpha_0) = S_0(\alpha_0) + \varepsilon^2 S_1(\alpha_0), \tag{63}$$

where only terms up to order ε^2 have been kept and we made use of the extremum condition on S_0.

The perturbative calculation for Eq. (60) is therefore carried out in the following way: after δ has been obtained by solving Eq. (47), we set S equal to zero, substituting α from Eq. (55) and making use of Eq. (56) in evaluating terms of order ρ^2. We then obtain, after some algebra,

$$\mu_1^3 = \frac{2}{11} \frac{\mu_2^2 \chi_A^2}{(1 - 164\rho^2\alpha^{-3})}, \tag{64}$$

or, making use again of Eqs. (55), (56), we obtain the dispersion relation

$$(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*)^3 = \frac{-i\frac{2}{11}\left(\frac{\eta}{4\pi}\right)\left[\varepsilon''' s(\omega - \omega_n^*) k_\parallel' D\right]^2 \frac{(\omega+\omega_i^*)}{\omega v_A^2}}{\left\{1 + \frac{30\left(\frac{\eta}{4\pi}\right)\left(\frac{4\pi k_\perp J_0'}{k_\parallel' B}\right)^2}{\left[\varepsilon''' s(\omega-\omega_n^*) D \frac{(\omega+\omega_i^*)}{v_A^2}\right]}\right\}}, \tag{65}$$

providing the modification to the result expressed by Eq. (57) due to the local J_0 contribution.

In the present case, the validity of the perturbative calculation is ensured by the condition

$$\left|\frac{4\pi k_\perp J_0'}{k_\parallel' B \mu_2^{1/2} \chi_A}\right| < 1; \tag{66}$$

for typical tokamak parameters, evaluated near the location of the $q = 2$ rational surface, q is the safety factor, the small quantity of Eq. (66) is of the order of $2 \cdot 10^{-2}$. Therefore, even though the numerical factors in front of the J_0 corrections in Eq. (65) is rather large, the equilibrium current does not modify significantly the result expressed by Eq. (57).

For a drift-type solution, $\omega^* \gg \gamma$, recalling that the mathematically consistent root of the dispersion relation is the one shown in Eq. (58), we conclude that the inclusion of J_0 results in decreasing the damping contribution of the parallel thermal conduction with respect to the result expressed by Eq. (57).

V. Conclusions

In the present paper, the local effect of the equilibrium parallel current J_0 on the stability of the tearing mode in the semicollisional regime has been investigated analytically, including, in particular, the effect of the parallel thermal conduction on the mode dynamics.

The equilibrium current enters the first eigenmode equation through its gradient, in the so-called "kink term," whereas J_0 appears in Ampére's law through resistivity perturbations in the "Ohm's law" provided by the electron momentum balance equation. Assuming the equilibrium parallel electric field to be spatially constant, both J_0 terms are identical for the form of the generalized conductivity corresponding to the scaling expressed by Eqs. (21), (22) and have the form of local current gradients, directly related to the local electron temperature gradient, because of the $T_e^{3/2}$ dependence of the Spitzer-Braginskii conductivity in the equilibrium Ohm's law. Analogously to the collisional case,[1] the presence of electron temperature gradients is essential for the local J_0 to couple to the mode dynamics.

The main result of this paper is given by Eq. (65), which provides the modification to the semicollisional mode of Ref. 5 due to the inclusion of the local equilibrium current. The local J_0 term is responsible for a decrease of the stabilization of the mode, due to the thermal conduction along the ambient magnetic field. The qualitative prediction of Ref. 5 is, however, not contradicted by the present result, since the local J_0 effect is obtained from perturbation theory and cannot become larger than the leading term.

We finally remark that the discussed J_0 effect appears to be relevant only to the linear analysis as it was already noted for the collisional regime. At least over a time scale over which the equilibrium magnetic field does not change appreciably, the equilibrium electric field can be assumed as uniform and therefore the local equilibrium current couple to the mode dynamics only through the local electron temperature gradient. As a consequence of the formation of magnetic islands, and of the resulting large radial electron thermal conduction, the local flattening of the temperature profile will prevent the equilibrium current to play a significant role.

Appendix: Alternative Derivation of the Semicollisional Tearing Mode Dispersion Relation in the Absence of Local Current

Because of the fairly complex expression for the semicollisional form of S, Eq. (54), even in the absence of J_0 terms, a somewhat approximate procedure has been employed to derive the mode dispersion relation given by Eq. (56). In this appendix, we present an alternative derivation of Eq. (56), based on the variational principle formulation of Ref. 11, which enables an exact and simple variational calculation in the present context, even though this method of solution cannot be extended to the case in which equilibrium current terms are included in the eigenmode equations.

If the J_0 terms are neglected, Eqs. (36), (37) can be combined into a single, integro-differential equation of the form[11]

$$\left(\frac{\chi_A^2 x^2 \mathcal{E}'}{x^2-\chi_A^2}\right)' + x^2\sigma_* \mathcal{E} = -\left(\Delta' + \frac{i\pi}{\chi_A}\right)^{-1} \frac{4x\chi_A^4}{(x^2-\chi_A^2)^2} \int_{-\infty}^{+\infty} \frac{x\mathcal{E}\,dx}{(x^2-\chi_A^2)^2}, \tag{A1}$$

where the field variable E is defined by

$$\mathcal{E}(x) \equiv \frac{\psi}{x} - \phi = \frac{\tilde{E}_{\|}}{ik_{\|}' x}. \tag{A2}$$

The variational functional reproducing the Euler-Lagrange equation, Eq. (A1), is evidently

$$S = (\Delta' + i\pi/\chi_A)(I_1 + I_2) + I_3^2, \tag{A3}$$

where

$$I_1 = \int_{-\infty}^{+\infty} \mathcal{E}\left(\frac{\mathcal{E}' x^2 \chi_A^2}{x^2-\chi_A^2}\right)' dx, \tag{A4}$$

$$I_2 = \int_{-\infty}^{+\infty} x^2 \sigma_* \mathcal{E}^2 dx, \tag{A5}$$

and

$$I_3 = 2\chi_A^2 \int_{-\infty}^{+\infty} \frac{x\mathcal{E}}{(x^2-\chi_A^2)^2} dx. \tag{A6}$$

By comparison with Eq. (45), it appears that the trial function for tearing-symmetry solutions is

$$\mathcal{E}(x)_{\text{trial}} = \frac{\exp(-\alpha x^2/2)}{x}; \tag{A7}$$

substituting the above trial function in Eqs. (A4)–(A6), using $\sigma_* = \mu_1 + \mu_2 x^2$, and simplifying the resulting expression for S making use of the fact that $|\alpha^{1/2}\chi_A| < 1$, $|\Delta'\chi_A| \ll 1$ for $m \geq 2$ tearing modes, we obtain

$$S = \mu_1 \alpha^{-1/2} + \frac{\mu_2}{2}\alpha^{-3/2} - \left(\frac{4\sqrt{2}-5}{4}\right)\alpha^{3/2}\chi_A^2 - \frac{\Delta'}{\pi^{1/2}}. \tag{A8}$$

The semicollisional mode discussed in Sec. IV corresponds to the

$$|\Delta'\chi_A| < \alpha^{3/2}\chi_A^3 \tag{A9}$$

limit of Eq. (A8), which gives

$$S \simeq \mu_1 \alpha^{-1/2} + \frac{\mu_2}{2}\alpha^{-3/2} - \left(\frac{4\sqrt{2}-5}{3}\right)\alpha^{3/2}\chi_A^2; \tag{A10}$$

the variational parameter α and the dispersion relation are obtained solving Eqs. (48), (49); they are respectively given by

$$\alpha = -\frac{3}{4}\frac{\mu_2}{\mu_1}, \tag{A11}$$

and (cfr. with Eq. (56))

$$\mu_1^3 = \left(\frac{3}{4}\right)^2 \left(4\sqrt{2}-5\right)\mu_2^2\chi_A^2. \tag{A12}$$

For a drift-type solution, Eqs. (A11) and (A12) yield

$$\omega \simeq \omega_0 + i\varepsilon\varepsilon'\frac{\omega_0}{\nu}\omega_T^* + e^{\frac{7\pi i}{6}}\left(\frac{3}{4}\right)^{2/3}\left(4\sqrt{2}-5\right)^{1/3}\left(\frac{\eta}{4\pi}\right)^{1/3}\left(\varepsilon''' s\hat{\varepsilon}\omega_T^* k_\parallel' D\right)^{2/3}\frac{\left(1+\frac{\omega_i^*}{\omega_0}\right)^{1/3}}{v_A^{2/3}}, \tag{A13}$$

with ω_0 given by Eq. (59). We remark that the above result differs from the one expressed by Eq. (58) only by a 21% difference in the numerical factor in front of the damping rate, and by a 15% difference in the numerical factor in front of the damping rate given by Eq. (39) of Ref. 5.

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