IFS #241

Local Effect of Equilibrium Current on Tearing Mode Stability in the Collisional Regime (Part I)

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Abstract

The local effect of the equilibrium current on the linear stability of low poloidal number tearing modes in the collisional regime is investigated analytically. The set of eigenmode equations describing the relevant mode dynamics inside the tearing layer, including the local effect of the equilibrium current, is solved variationally to derive the dispersion relations for the $m \geq 2$ and $m = 1$ tearing mode. The local effect of the equilibrium current is found to increase the well known FKR growth rate for the classical tearing mode, to enhance stabilization for the $m \geq 2$ drift-tearing mode, and to slightly increase the growth rate for the $m = 1$ mode.

I. Introduction

In the linear stability analysis of low poloidal tearing modes,[1,2] the dispersion relation is obtained by matching the inner solution inside the tearing layer within the outer solution, valid in the ideal magnetohydrodynamic (MHD) region. In standard derivations of the relevant eigenmode equations in the inner region, it is customary[3,4,5] to neglect terms proportional to the local equilibrium current; because of this assumption, the resulting form of the equations allows the formulation of several elegant analytical techniques. The eigenmode problem becomes much more difficult whenever such terms are kept and most of the methods of solution are no longer applicable. Although these terms can be estimated to be small, the linear stability of tearing modes does critically depend on small, non-ideal effects, and it is therefore interesting to include the local equilibrium current in the analysis and to assess its effect on the stability properties of a number of modes. The modification of the basic set of eigenmode equations appears in an additional term in Ampere's law, due to resistivity perturbation in the generalized Ohm's law, and in the presence of the local current gradient, i.e., the so-called "kink term,"[6] in the equation expressing the quasi-neutrality condition.

The present analysis does not depart significantly from previous, "conventional" theories in treating the global effect of the equilibrium current. The current profile determines Δ', the well known[1,2,7] discontinuity in the derivative of the external solution; the problem in the ideal MHD region is presumed to have been solved and the Δ' is treated as a known parameter in the theory.

In this paper, only the collisional (hydrodynamic) regime[8] of the tearing mode is considered, limiting the validity of the present analysis to the outer region of some present day tokamak discharges. (It should be mentioned here that the neglect of compression ($\nabla \cdot \mathbf{u} = 0$) already implies that the analysis is valid only for low β tokamaks.) The extension of this study to the semicollisional[8] regime, in which for example the effect of the thermal conduction on the mode dynamics in higher temperature regions is considered, presented in a subsequent paper.

The remainder of this paper is organized in the following manner. In Sec. II, we derive the eigenmode equations; since the derivation is quite standard, apart from the inclusion of equilibrium current terms, only the final result is presented. In Sec. III, we formulate in detail the method of constructing the variational functional for the problem under present consideration. In Sec. IV, appropriate limiting forms of the variational functional are shown to recover the known results for the $m = 1$ and $m \geq 2$ tearing mode, respectively. The effect of the local equilibrium current is then treated as a perturbation of these results, and the modified dispersion relations are obtained. In Chapter V, we summarize our conclusions. In the Appendix, we give a derivation of the equation expressing the relation between the perturbed parallel current and perturbed electric field in the collisional regime, using the drift kinetic equation.

C.G.S. units with $c = 1$ are used throughout.

II. Eigenmode Equations

Let $\hat{b}_0 \equiv \mathbf{B}_0/B_0$ be a unit vector in the direction of the equilibrium magnetic field $\mathbf{B}_0$; $\hat{r}$ be a unit vector in the radial direction, normal to a flux surface; and $\hat{e} \equiv \hat{b}_0 \times \hat{r}$. In the slab model of the tokamak, these unit vectors, as well as the equilibrium quantities, depend only on the radius.

A conventional Fourier representation of any linearly perturbed quantity, $\tilde{g}$, is introduced, such that

$$\frac{\partial}{\partial t}\tilde{g} = -i\omega\tilde{g}, \tag{1a}$$

$$\hat{b}_0 \cdot \nabla\tilde{g} = ik\|(x)\tilde{g}, \tag{1b}$$

$$\hat{e} \cdot \nabla\tilde{g} = ik_\perp(x)\tilde{g}, \tag{1c}$$

$$\hat{r} \cdot \nabla\tilde{g} = \tilde{g}', \tag{1d}$$

where the radial variable $x = r - r_s$ denotes the distance from a mode rational surface, located at $r = r_s$. In the vicinity of such surface, where $k_{\|}(0) = 0$, we have

$$k_{\|}(x) \simeq k'_{\|} x. \tag{2}$$

It is convenient to formulate the eigenmode problem in terms of the perturbed electrostatic potential $\tilde{\phi}$ and the parallel component of the magnetic vector potential $\tilde{A}_{\|}$. For sufficiently small β (ratio between plasma pressure and magnetic pressure) it is consistent to assume[9]

$$\tilde{\mathbf{A}} = \hat{b}_0 \tilde{A}_{\|}, \tag{3}$$

implying that the radial magnetic perturbation becomes

$$\tilde{B}_r = i k_{\perp} \tilde{A}_{\|}, \tag{4}$$

and the components of the perturbed electric field are

$$\tilde{E}_{\|} \equiv \hat{b}_0 \cdot \tilde{\mathbf{E}} = -i k_{\|} \tilde{\phi} + i \omega \tilde{A}_{\|}, \tag{5a}$$

$$\tilde{E}_{\perp} \equiv \hat{e} \cdot \tilde{\mathbf{E}} = -i k_{\perp} \tilde{\phi}, \tag{5b}$$

$$\tilde{E}_r \equiv \hat{r} \cdot \tilde{\mathbf{E}} = -\tilde{\phi}'. \tag{5c}$$

The parallel component of Ampere's law takes the form

$$\tilde{J}_{\|} = \frac{1}{4\pi}(-\tilde{A}''_{\|} + k_{\perp}^2 \tilde{A}_{\|}) \simeq -\frac{\tilde{A}''_{\|}}{4\pi}, \tag{6}$$

where $\tilde{J}_{\|}$ is the perturbed parallel current. In Eq. (6) we introduced the approximation,[3] pertinent to low poloidal number modes and used throughout the present analysis, of assuming the mode radial wavelength to be much shorter than the azimuthal wavelength.

The first eigenmode equation to be derived, describing essentially the electron dynamics inside the tearing layer, expresses the constitutive relationship between the perturbed parallel current, $\tilde{J}_{\|}$, given by Ampere's law, Eq. (6), and the perturbed parallel electric field, $\tilde{E}_{\|}$. The electron dynamics of relevance in the present work is correctly represented by the following moment equation,[10,11] expressing momentum balance along the total magnetic field:

$$\varepsilon'' \frac{\partial}{\partial t}(n m_e \hat{b} \cdot \mathbf{u}) = -ne\hat{b} \cdot \mathbf{E} - \hat{b} \cdot \nabla p_e$$
$$- (ne)^2 \eta \hat{b} \cdot \mathbf{u} \left[1 + \frac{3}{2}\varepsilon'' \left(\frac{m_e}{\eta n e^2} \right) \frac{\partial}{\partial t} \ell n \; T_e \right]$$
$$- \varepsilon \eta \hat{b} \cdot \nabla T_e \left[1 - \frac{3\varepsilon'}{\nu} \frac{\partial}{\partial t} \ell n \; T_e - \frac{\varepsilon'}{\nu} \frac{\partial}{\partial t} \ell n (\hat{b} \cdot \nabla T_e) \right], \qquad (7)$$

where

$$\mathbf{u} = \hat{b}\hat{b} \cdot \mathbf{u} + \frac{\mathbf{E} \times \hat{b}}{B} - \frac{\hat{b} \times \nabla p_e}{enB}. \qquad (8)$$

In Eqs. (7) and (8), n is the electron density, T_e is the electron temperature, $p_e = nT_e$ is the electron pressure, m_e is the electron mass, e is the absolute value of the electron charge, $\mathbf{E}$ and $\mathbf{B}$ are the total electric and magnetic fields, respectively, $\hat{b}$ is a unit vector in the direction of the total magnetic field $\mathbf{B}$; η is the Spitzer-Braginskii resistivity; ν is the electron collision frequency; ε', ε'' are numerical transport coefficients tabulated in Ref. 10; in particular, ε is the familiar coefficient of the thermal force[12] and is equal to 0.71 for a singly charged ion plasma. The above equation differs from the well known transport equation derived by Braginskii,[12] to the required order,[10] by the inclusion of a time dependent thermal force on the electrons and by the fact that the coefficient of the electron inertial term is greater than unity. The physical origin of these terms, basically connected with the velocity dependence of the Coulomb cross-section (for a plasma, $\nu \sim v_e^{-3}$, where v_e is the electron thermal velocity), has been discussed in detail in Ref. 10. Here, we only remark that the inclusion of the time dependent thermal force results in the thermoelectric growth rate[11,13] in the final dispersion relation and that this term does not couple with the local parallel equilibrium current J_0, and therefore does not play a crucial role in the present analysis. Also, the electron inertial term will be neglected in the subsequent derivation.

The linearization of Eq. (7) is quite straightforward, eliminating density and temperature perturbations through the following equations, expressing particle number and energy balance, respectively, in the collisional regime:

$$\omega \frac{\tilde{n}}{n} = \omega_n^* \frac{e\tilde{\phi}}{T_e}, \tag{9}$$

$$\omega \frac{\tilde{T}_e}{T_e} = \omega_T^* \frac{e\tilde{\phi}}{T_e}; \tag{10}$$

the result is

$$\eta \tilde{J}_\| = i(\omega \tilde{A}_\| - k_\| \tilde{\phi}) \left(1 - \frac{\omega_n^*}{\omega} - \frac{\hat{\varepsilon}\omega_T^*}{\omega}\right) + \frac{3}{2}\eta J_0 \frac{e\omega_T^*}{T_e}\frac{\tilde{\phi}}{\omega}, \tag{11}$$

with the definitions

$$\omega_n^* \equiv -\frac{k_\perp}{eB}\frac{n'}{n}T_e, \tag{12a}$$

$$\omega_T^* \equiv -\frac{k_\perp}{eB}T_e', \tag{12b}$$

$$\hat{\varepsilon} \equiv 1 + \varepsilon + i\varepsilon\varepsilon'\frac{\omega}{\nu}; \tag{12c}$$

ω_n^*, ω_T^* are the electron drift frequencies related to density gradients and electron temperature gradients, respectively; $\hat{\varepsilon}$ is defined in terms of the numerical transport coefficients ε, ε'.

We remark that the inclusion of J_0 in Eq. (11) destroys the diagonal character of the generalized "Ohm's law" of the form

$$\tilde{J}_\| = \sigma_* \tilde{E}_\|, \tag{13}$$

where the generalized "conductivity" does not have to be equal to the Spitzer-Braginskii conductivity and which, we stress, pertains only when the equilibrium electric field $E_\|^{(0)}$ is neglected inside the inner resistive layer. In the Appendix we present an alternative derivation of Eq. (11), using kinetic theory.

From Eqs. (2), (6) and (11) we obtain the first eigenmode equation in the form:

$$\psi'' = \mu_1 \left[\psi - (x - \lambda)\phi\right], \tag{14}$$

where

$$\psi \equiv \frac{\omega \tilde{A}_{\parallel}}{k'_{\parallel}}, \tag{15}$$

$$\mu_1 \equiv -\frac{4\pi i}{\eta}(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*), \tag{16}$$

and the equilibrium current appears through the factor

$$\lambda \equiv -i\frac{3}{2}\frac{\eta}{k'_{\parallel}}\frac{e\omega_T^*}{T_e}\frac{J_0}{(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*)}. \tag{17}$$

Equation (14), as remarked, is essentially a description of the electron dynamics; the other piece of needed information describes the ion dynamics, and it is usually[3] obtained considering the plasma equation motion in the form

$$m_i n_i \frac{\partial \mathbf{V}}{\partial t} = -\nabla P + \mathbf{J} \times \mathbf{B}, \tag{18}$$

where $\mathbf{V}$ is the ion (or bulk plasma) flow velocity, m_i is the ion mass, n_i is the ion density, $P = p_i + p_e$ is the total plasma pressure.

Solving Eq. (18) for the perpendicular component of the perturbed current, $\tilde{\mathbf{J}}_\perp$, and imposing quasineutrality in the form

$$\nabla \cdot \tilde{\mathbf{J}}_{\parallel} = -\nabla \cdot \tilde{\mathbf{J}}_\perp, \tag{19}$$

one finds[6]

$$x_A^2 \phi'' = x\psi'' - \frac{4\pi k_\perp}{B k'_{\parallel}} J'_0 \psi, \tag{20}$$

where the Alfvén layer x_A, defined as the distance from the rational surface at which $\omega \simeq k_{\parallel} v_A$ ($v_A = B_0/(4\pi n_i m_i)^{1/2}$ is the Alfvén speed), is given by

$$x_A^2 \equiv \frac{\omega(\omega + \omega_i^*)}{(k'_{\parallel} v_A)^2}, \tag{21}$$

and the ion drift frequency is

$$\omega_i^* \equiv -\frac{k_\perp}{eB}\frac{p'_i}{n_i}. \tag{22}$$

We remark that the J_0 term in Eq. (14), i.e., the λ factor, arising from perturbations of the Spitzer-Braginskii resistivity in the generalized Ohm's law, Eq. (11), gives rise to the "rippling mode" of Ref. 1, but has been usually neglected in the contest of modes with tearing symmetry, i.e., modes characterized by $\psi(0) \neq 0$. The J_0' term in Eq. (20) is the so-called "kink term" which has been similarly neglected in the inner region in most tearing mode analyses, although in the external MHD region it is precisely this term which determines the Δ'.

The structure of the eigenmode equations is significantly altered by the inclusion of these J_0 terms, to the extent that several elegant solution techniques,[3,4,5] applicable when such terms are neglected, are no longer possible.

In the following section, we present a method of solution which is suited to deal with Eqs. (14) and (20), and which, at the same time, provides continuous contact with previous formulations in the appropriate limit.

III. Variational Principle

We now consider the method of solution of the eigenmode equations; let us first show that the terms involving J_0 and J_0' in Eqs. (14) and (20) are identical if we assume the equilibrium parallel field $E_\parallel^{(0)}$ to be spatially constant (from Maxwell equations, this is true if the equilibrium magnetic field does not appreciably change in time on the typical time scale for the perturbation, a property which is assumed for all the quantities describing the equilibrium). The equilibrium Ohm's law is

$$J_0 = \sigma_{\text{sp}} E_\parallel^{(0)}, \tag{23}$$

where σ_{sp} is the Spitzer-Braginskii conductivity; from (23), since $\sigma_{\text{sp}} \sim T_e^{3/2}$ and we take $dE_\parallel^{(0)}/dx = 0$, we find

$$J_0' = \frac{3}{2}\frac{T_e'}{T_e} J_0, \tag{24}$$

which yields, making use of Eqs. (16) and (17),

$$\frac{4\pi k_\perp}{Bk'_\parallel}J'_0 = \lambda\mu_1. \tag{25}$$

Within the context of the approximations used in this work, the effect of the equilibrium current manifests itself only through its local gradient, which is, in turn, directly proportional to the local electron temperature gradient, and therefore it appears that the local equilibrium current and its local gradient are to be considered on the same footing in the present context.

The eigenmode equations can therefore be written as

$$x_A^2\phi'' = x\psi'' - \lambda\mu_1\psi, \tag{26}$$

$$\psi'' = \mu_1\left[\psi - (x-\lambda)\phi\right]; \tag{27}$$

eliminating the factor $\mu_1\psi$ in Eq. (26) by using Eq. (27), we obtain

$$\phi'' + \frac{\lambda\mu_1}{x_A^2}(x-\lambda)\phi = \frac{(x-\lambda)\mu_1}{x_A^2}[\psi - (x-\lambda)\phi], \tag{28}$$

which along with Eq. (27) form our basic set. Let us now define a new radial variable

$$p \equiv x - \lambda, \tag{29}$$

a new field variable

$$Q(p) \equiv \mu_1(\psi - p\phi), \tag{30}$$

and

$$\rho \equiv \frac{\lambda\mu_1}{x_A^2}, \tag{31}$$

so that we can write the eigenmode equations, in the form

$$\psi'' = Q(p), \tag{32}$$

$$\phi'' + \rho p\phi = \frac{pQ(p)}{x_A^2}. \tag{33}$$

By neglecting J_0 terms, i.e., letting $\rho \to 0$, $\lambda \to 0$, $p \to x$, Eqs. (32) and (33) are identical with the corresponding set of equations describing low poloidal mode number tearing modes in Ref. 5; in particular, the variable $Q(p)$ becomes the perturbed parallel current $\tilde{J}_{\|}$. We remark that, by defining the shifted radial variable p, the local equilibrium current effect appears explicitly in the eigenmode equations only through the term $\rho p \phi$ of Eq. (33). This term is responsible for the mixing of the even and the odd part of the fields due to J_0, in the p variable formulation.

We can formulate a variational principle for the variable Q suitably modifying the procedure of Ref. 5, which we briefly outline. Treating the variable Q as a given inhomogeneous term at the right-hand side (r.h.s.) of Eqs. (32) and (33), one formally solves these equations for ψ and ϕ in terms of Q; using the definition of Q as given by Eq. (30), one then constructs an appropriate combination of ψ and ϕ to obtain an integral equation for Q, which has the necessary structure to allow for a variational treatment. In the present analysis, the method of Ref. 5 is suitably extended to account for the presence of the extra J_0 term in Eq. (33). Equation (32) is easily solved, and its solution is

$$\psi(p) = \frac{1}{2}\int_{-\infty}^{\infty} |p - p'|Q(p')dp' + a_1 + b_1 p, \tag{34}$$

where a_1 and b_1 are arbitrary constants. The first term on the right-hand side of Eq. (34) is a particular integral, to which we have added a linear combination of the solutions of the corresponding homogeneous equation. To make contact with classical tearing mode theory, let us require the asymptotic behavior[3,4,5]

$$\psi \to \psi_0 \left(1 + \frac{\Delta'|x|}{2}\right), \tag{35}$$

for large x, where, we recall, ψ_0 is the value toward which the exterior ψ solution tends and Δ' is the logarithmic jump in the derivative of the external solution, defined by[1,2]

$$\Delta' \equiv \frac{\psi'_+ - \psi'_-}{\psi_0}, \tag{36}$$

where ψ_+, ψ_- denote the external ψ solution on the right and on the left of the tearing layer, respectively. In the present context, no distinction is made between the radial variables x and p, because for $x \to \infty$, $p \to x$. It then appears that we must choose

$$a_1 = \frac{1}{\Delta'} \int_{-\infty}^{\infty} Q(p')dp', \tag{37}$$

$$b_1 = 0, \tag{38}$$

so that the appropriate solution of Eq. (32) is

$$\psi = \frac{1}{2} \int_{-\infty}^{\infty} |p - p'| Q(p')dp' + \frac{1}{\Delta'} \int_{-\infty}^{\infty} Q(p')dp'. \tag{39}$$

After determining ψ as an integral of $Q(p)$, we must solve Eq. (33) to solve ϕ as a functional of $Q(p)$. The solution of the associated homogeneous equation can be expressed as a linear combination of the linearly independent set

$$\phi_1 = \sqrt{p} J_{1/3}(\zeta), \tag{40a}$$

$$\phi_2 = \sqrt{p} J_{-1/3}(\zeta), \tag{40b}$$

where $J_{\pm 1/3}(\zeta)$ are Bessel functions, and

$$\zeta = \frac{2}{3} \rho^{1/2} p^{3/2}. \tag{41}$$

The general solution of EQ. (33) is a combination of the homogeneous solution and the particular integral:

$$\begin{aligned}
\phi(p) &= a_2 \sqrt{p} J_{1/3}(\zeta) + b_2 \sqrt{p} J_{-1/3}(\zeta) + \frac{2\pi}{3\sqrt{3} x_A^2} \\
&\times \left\{ \frac{1}{2} \int_{-\infty}^{p} \sqrt{pp'} \left[J_{1/3}(\zeta) J_{-1/3}(\zeta') - J_{-1/3}(\zeta) J_{1/3}(\zeta') \right] p' Q(p') dp' \right. \\
&\left. - \frac{1}{2} \int_{p}^{\infty} \sqrt{pp'} \left[J_{1/3}(\zeta) J_{-1/3}(\zeta') - J_{-1/3}(\zeta) J_{1/3}(\zeta') \right] p' Q(p') dp' \right\},
\end{aligned} \tag{42}$$

where a_2, b_2 are arbitrary constants. When p is close to zero, the integrands in Eq. (42) contribute only for p' close to zero, since $Q(p')$ is assumed to be localized about $p' = 0$.

In the present study, we are interested in modes which decay spatially when ζ, defined by Eq. (41), is still quite small. Expanding the Bessel functions to significant order in ζ, so to include terms up to ρ^2, we find after some straightforward algebra

$$\sqrt{pp'}\Big[J_{1/3}(\zeta)J_{-1/3}(\zeta') - J_{-1/3}(\zeta)J_{1/3}(\zeta')\Big] \simeq \frac{1}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)}(p-p')K(p,p'), \tag{43}$$

where

$$K(p,p') = K(p',p) \equiv 1 - \frac{\rho}{12}(p+p')(p-p')^2 + \frac{\rho^2}{504}\left[p^6 + p'^6 - \frac{9}{5}pp'(p^4+p'^4) - \frac{9}{5}p^2p'^2(p^2+p'^2) + \frac{26}{5}p^3p'^3\right]. \tag{44}$$

We can then write Eq. (42) as

$$\phi(p) = a_2\sqrt{p}J_{1/3}(\zeta) + b_2\sqrt{p}J_{-1/3}(\zeta) + \frac{1}{2x_A^2}\int_{-\infty}^{\infty}|p-p'|K(p,p')p'Q(p')dp'. \tag{45}$$

The arbitrary constants a_2 and b_2 are chosen by the requirement that the quantity $Q(p)$, given by Eq. (30), is localized, that is

$$\psi - p\phi \to 0, |p| \to \infty\ ,\ \zeta \ll 1. \tag{46}$$

We assume the existence of a matching region where, even if p is sufficiently large (i.e., $p \to x$) the current-dependent dimensional parameter ζ, given by Eq. (41), is still very small, as already noted. We remark that assumptions of this kind are peculiar to tearing mode theories, even when local J_0 terms are neglected.[5] The above boundary condition can be understood by the fact that, when $x \gg \lambda$, $p \to x$, the variable Q becomes the ordinary $\tilde{J}_{\|}$ which must indeed be localized, since in the exterior (MHD) region $\tilde{E}_{\|} \to 0$. According to Eq. (46), let us choose

$$a_2 = 0, \tag{47}$$

$$b_2 = 0, \tag{48}$$

so that the appropriate solution of Eq. (33) is

$$\phi(p) = \frac{1}{2x_A^2} \int_{-\infty}^{\infty} |p - p'| K(p,p') p' Q(p') dp'. \tag{49}$$

Combining Eqs. (30), (39) and (49), we obtain the integral equation

$$\frac{Q(p)}{\mu_1} = \frac{1}{\Delta'} \int_{-\infty}^{\infty} Q(p') dp' + \frac{1}{2} \int_{-\infty}^{\infty} g(p,p') Q(p') dp', \tag{50}$$

where

$$g(p,p') = g(p',p) \equiv |p - p'| \left[1 - \frac{pp'}{x_A^2} K(p,p') \right] \tag{51}$$

is a symmetric kernel. Following standard variational techniques,[3] it is straightforward to show that the functional

$$S[Q] = \int_{-\infty}^{\infty} \frac{Q^2(p)}{\mu_1} dp - \frac{1}{\Delta'} \left[\int_{-\infty}^{\infty} Q(p) dp \right]^2 - \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(p,p') Q(p) Q(p') dp dp' \tag{52}$$

is the variational quantity corresponding to the Euler-Lagrange Eq. (50) since, for an arbitrary variation δf,

$$\delta S = 0 \tag{53}$$

generates Eq. (50), and we have

$$S = 0 \tag{54}$$

for $f = Q$, where Q is the solution of Eq. (50). We remark that the symmetry of $g(p,p')$ is essential for a simple variational principle formulation in the above form.

IV. Dispersion Relations

In order to evaluate explicitly the variational functional S, given by Eqs. (52), it is necessary to choose a suitable trial function.

In conventional theories, where J_0 terms are neglected, the eigenmode equations are invariant under space reflection and one can choose a trial function of definite parity;

solutions with tearing symmetry have even parity (i.e., $\tilde{J}_{\|}$, $\tilde{E}_{\|}$, ψ even and ϕ odd) and the usual choice[3-5] is

$$Q(x) = \exp(-\alpha x^2/2), \tag{55}$$

where the variational parameter α measures the radial width of the mode ℓ_w, i.e., $\ell_w \sim \alpha^{-1/2}$.

When J_0 terms are included, the space reflection symmetry of the eigenmode equations is broken, and one can no longer choose a trial function with definite parity; since we shall be treating the effect of the current as a correction to known results, we demand that the trial function reduces to Eq. (55) whenever the J_0 effect is neglected, to ensure continuous contact with previous analyses. The simplest choice for a mixed trial function with the required properties is

$$Q(p) = (1 + \delta p) \exp(-\alpha p^2/2), \tag{56}$$

with variational parameters α and δ; α is still a measure of the radial mode width and δ, which is evidently proportional to ρ (cfr. Eq. (33)), measures the mixing of the even and odd part of the solution, caused by the inclusion of J_0. With the above choice for the trial function, S, as given by Eq. (52), becomes a polynomial in α and δ, and Eqs. (53) and (54) are in this case

$$\frac{\partial S}{\partial \delta} = 0, \tag{57}$$

$$\frac{\partial S}{\partial \alpha} = 0, \tag{58}$$

$$S = 0. \tag{59}$$

The mode localization condition and the choice of the branch cut in evaluating the integrals in Eq. (52), respectively, are ensured by demanding that any root of the dispersion relation satisfy independently

$$\mathrm{Re}\ \alpha > 0, \tag{60}$$

$$\mathrm{Re}\ \alpha^{1/2} > 0; \tag{61}$$

no similar restrictions have to be imposed on δ.

Substituting the trial function given by Eq. (56) in Eq. (52) one obtains, after a somewhat lengthy but straightforward calculation:

$$S = \frac{\alpha^{-1/2}}{\mu_1} + \frac{\delta^2}{2}\frac{\alpha^{-3/2}}{\mu_1} - \frac{2\pi^{1/2}}{\Delta'}\alpha^{-1} - \frac{\alpha^{-5/2}}{x_A^2}\left(1 + 2\alpha x_A^2\right) + \frac{2}{3}\delta\rho\frac{\alpha^{-9/2}}{x_A^2}$$
$$+ \frac{7}{2}\delta^2\frac{\alpha^{-7/2}}{x_A^2}\left(1 + \frac{2}{7}\alpha x_A^2\right) - \frac{2}{15}\rho^2\frac{\alpha^{-11/2}}{x_A^2}. \tag{62}$$

Notice that it is important to keep all the terms quadratic in δ, ρ, because δ must scale as ρ. Effects due to to the local equilibrium current appear then in S as terms of order ρ^2, as it should be expected by the form of Eqs. (32), (33) and (56).

Before proceeding to study the effects of the local J_0 terms, we show that Eq. (62) easily recovers a number of previous known results derived in the absence of local current. By letting $\rho \to 0$, Eq. (57) implies $\delta \to 0$ (as, of course, it must be) and we obtain the "classical" form of S:

$$S_{\text{cl.}} = \frac{\alpha^{-1/2}}{\mu_1} - \frac{2\pi^{1/2}}{\Delta'}\alpha^{-1} - \frac{\alpha^{-5/2}}{x_A^2}\left(1 + 2\alpha x_A^2\right). \tag{63}$$

We recall that low poloidal member tearing modes can basically be grouped in two categories: the $m \geq 2$ modes, such as the $m = 2$ tearing mode, which are characterized by a finite discontinuity of ψ' across the tearing layer, whereas ψ is roughly constant inside it, and the $m = 1$ tearing mode, for which $\psi \to 0$ inside the layer and is therefore characterized by a very large value of Δ'. Although the nature of the magnetic perturbation ψ is very different for the two varieties of modes, the nature of $\tilde{E}_{\|}$ is essentially the same, enabling us to derive the dispersion relation for the two cases by taking appropriate limits of the same expression for S.

The $m \geq 2$ modes are characterized by a radial extent wider than the Alfvén layer x_A and by a finite value of Δ'; the condition $\ell_w > |x_A|$ is equivalent to $|\alpha^{1/2}x_A| < 1$ and therefore the appropriate limit of Eq. (63) to describe $m \geq 2$ modes is

$$S_{\text{cl.}} \simeq \frac{\alpha^{-1/2}}{\mu_1} - \frac{2\pi^{1/2}}{\Delta'}\alpha^{-1} - \frac{\alpha^{-5/2}}{x_A^2}. \tag{64}$$

Solving Eqs. (58) and (59) for S given by Eq. (52), we obtain for $\alpha^{1/2}$ and the dispersion relation, respectively,

$$\alpha^{1/2} = \frac{3}{2}\frac{\pi^{1/2}}{\Delta'}\mu_1, \tag{65}$$

and

$$\mu_1^3 x_A^2 = -\left(\frac{2\Delta'}{3^{3/4}\pi^{1/2}}\right)^4, \tag{66}$$

which can be written in more conventional notation as[3,4,11]

$$\omega(\omega+\omega_1^*)(\omega-\omega_n^*-\hat{\varepsilon}\omega_T^*)^3 = i\left(\frac{2\Delta'}{3^{3/4}\pi^{1/2}}\right)^4\left(\frac{\eta}{4\pi}\right)^3(k_\parallel' v_A)^2. \tag{67}$$

The dispersion relation for the $m=1$ tearing mode, which is characterized by a very large value of Δ', can be obtained from Eq. (63) by formally letting $\Delta'\to\infty$; S becomes

$$S_{\text{cl.}} \simeq \frac{\alpha^{-1/2}}{\mu_1} - \frac{\alpha^{-5/2}}{x_A^2} - 2\alpha^{-3/2}, \tag{68}$$

from which we find α and the dispersion relation to be given respectively by

$$\alpha x_A^2 = -1, \tag{69}$$

and

$$\mu_1 x_A^2 = -1, \tag{70}$$

which can be written as[3,4]

$$\omega(\omega+\omega_i^*)(\omega-\omega_n^*-\hat{\varepsilon}\omega_T^*) = -i\left(\frac{\eta}{4\pi}\right)(k_\parallel' v_A)^2. \tag{71}$$

We now investigate the effect of J_0 on the modes discussed; to describe the $m\geq 2$ tearing mode in the presence of local current, let us take the limit of S, Eq. (62), for $|\alpha^{1/2}x_A|<1$. This is given by

$$S \simeq \frac{\alpha^{-1/2}}{\mu_1} - \frac{\alpha^{-5/2}}{x_A^2} - \frac{2\pi^{1/2}}{\Delta'}\alpha^{-1} + \delta^2\left(\frac{\alpha^{-3/2}}{2\mu_1} + \frac{7}{2}\frac{\alpha^{-7/2}}{x_A^2}\right) + \frac{2}{3}\delta\rho\frac{\alpha^{-9/2}}{x_A^2} - \frac{2}{15}\rho^2\frac{\alpha^{-11/2}}{x_A^2}. \tag{72}$$

The variational parameter δ is determined by solving Eq. (57), which gives

$$\delta = -\frac{2}{3}\frac{\rho\mu_1}{(7\alpha\mu_1 + \alpha^3 x_A^2)}; \tag{73}$$

in principle, one should substitute this expression for δ in Eqs. (58) and (59) to obtain the dispersion relation, but we can derive the correction to the mode dynamics due to J_0 in a very simple way, if we treat the ρ^2 terms as a perturbation of the result derived in absence of current, i.e., Eqs. (65) and (66). This approach is consistent with the previous calculations, where we have already assumed the effect of the current to be small, expanding the Bessel functions in Eq. (43) for small argument.

Since S is a variational quantity, the perturbative calculation is of the outmost simplicity; indeed, if we write Eq. (72) as

$$S = S_0 + \varepsilon^2 S_1, \tag{74}$$

where S_0 is the corresponding "classical" S, given by Eq. (64), and $\varepsilon^2 \ll 1$ is proportional to ρ^2, and we assume

$$\alpha = \alpha_0 + \varepsilon^2 \alpha_1, \tag{75}$$

with α_0 given by Eq. (65), we have

$$S \simeq S_0(\alpha_0) + \left.\frac{\partial S_0}{\partial \alpha}\right|_{\alpha_0} \varepsilon^2 \alpha_1 + \varepsilon^2 S_1(\alpha_0) = S_0(\alpha_0) + \varepsilon^2 S_1(\alpha_0), \tag{76}$$

where only terms up to order ε^2 have been kept, and we made use of the extremum condition on S_0. Therefore, there is no need of determining α from Eq. (58), and the dispersion relation, correct to order ρ^2, is obtained by substituting Eq. (73) into Eq. (72), using Eqs. (65) and (66) in terms of order ρ^2, and setting the resulting expression, which is of the form of Eq. (76), equal to zero. We then obtain

$$\mu_1^3 x_A^2 = -\left(\frac{2\Delta'}{3^{3/4}\pi^{1/2}}\right)^4 \left(1 + \frac{17}{90}\rho^2\alpha^{-3}\right). \tag{77}$$

By recalling that the variational parameter α measures the radial mode width, $\ell_w \sim \alpha^{-1/2}$, we see that the J_0 term in Eq. (77) is of order ζ^4, where ζ, given by Eq. (41), has been treated as a small quantity in the derivation of S. Thus, the validity of our derivation in Sec. III ensures the validity of the present perturbative calculation, and using Eqs. (25), (31), (65) and (66), we can write Eq. (77) as

$$\omega(\omega+\omega_i^*)(\omega-\omega_n^*-\hat{\varepsilon}\omega_T^*)^3 = i\left(\frac{2\Delta'}{3^{3/4}\pi^{1/2}}\right)^4\left(\frac{\eta}{4\pi}\right)^3(k_\parallel' v_A)^2\left[1+\frac{17\pi}{360}\left(\frac{4\pi k_\perp J_0'}{k_\parallel' B\Delta'}\right)^2\right], \tag{78}$$

which summarizes explicitly the local J_0 effect on the $m \geq 2$ tearing mode in the collisional regime. As remarked, the validity of our derivation require the J_0' term to be a small correction, i.e.,

$$\left|\frac{4\pi k_\perp J_0'}{k_\parallel' B\Delta'}\right| < 1. \tag{79}$$

For typical present-day tokamak parameters at approximately two-thirds of the minor radius, where for example the $q = 2$ mode rational surface is located (q is the safety factor), the quantity $|4\pi k_\perp J_0' a/k_\parallel' B|$, where a is the plasma radius, is of the order of 1/3; since $|\Delta' a|$ is typically greater than one,[7] the above condition is usually well satisfied.

In the drift tearing limit[4,8] of Eq. (78), $\gamma \ll \omega^*$, we see that the local current gradient enhances stabilization of the $m \geq 2$ drift tearing mode, since the right-hand side of Eq. (67) provides a small damping to the overall growth rate.[14] In Ohmic equilibrium, the "pure" FKR result,[1] obtained by letting $\omega_* \to 0$ in Eq. (67), is not altered by the local current term which, for this case, is proportional to T_e' [Eq. 23]. If, however, we simply consider the limit $\gamma \gg \omega^*$ of Eq. (78), we see that the leading order effect of the J_0' term is to increase the known FKR growth rate.

To examine the effect of the local current gradient on the $m = 1$ tearing mode, let us take the limit of S for $\Delta' \to \infty$. We obtain

$$S \simeq \frac{\alpha^{-1/2}}{\mu_1} - \frac{\alpha^{-5/2}}{x_A^2}(1+2\alpha x_A^2) + \frac{\delta^2}{2}\frac{\alpha^{-3/2}}{\mu_1}$$

$$+\frac{7}{2}\delta^2\frac{\alpha^{-7/2}}{x_A^2}\left(1+\frac{2}{7}\alpha x_A^2\right)+\frac{2}{3}\delta\rho\frac{\alpha^{-9/2}}{x_A^2}+\frac{2}{15}\rho^2\frac{\alpha^{-11/2}}{x_A^2}. \tag{80}$$

Solving Eq. (57), we find that δ is given by

$$\delta=-\frac{\frac{2}{3}\frac{\rho}{\alpha x_A^2}}{\left(\frac{\alpha^2}{\mu_1}+\frac{7}{x_A^2}+2\alpha\right)}; \tag{81}$$

by proceeding as in the previous case, namely, treating the ρ^2 terms in Eq. (80) as a perturbation to the result expressed by Eqs. (69) and (70), we find

$$\mu_1 x_A^2=-\frac{1}{\left(1-\frac{17}{90}\rho^2\alpha^{-2}\right)}, \tag{82}$$

where, analogously to the $m\geq 2$ case, the local current effect turns to be of order ζ^4, similarly confirming the validity of the perturbative calculation. In this case, α is given by Eq. (69) and is, of course, different from the α appearing in the $m\geq 2$ result given by Eq. (77).

By substituting for α we find

$$\omega(\omega+\omega_i^*)(\omega-\omega_n^*-\hat{\varepsilon}\omega_T^*)=\frac{-i\left(\frac{\eta}{4\pi}\right)(k_\parallel' v_A)^2}{\left[1+\frac{17}{90}\left(\frac{4\pi k_\perp J_0'}{k_\parallel' B}\right)^2\frac{(\omega+\omega_i^*)\omega}{(k_\parallel' v_A)^2}\right]}; \tag{83}$$

which gives the local J_0 effect on the $m=1$ tearing mode. The condition for the validity of the derivation in this case can be written as

$$\left|\frac{4\pi k_\perp J_0' x_A}{k_\parallel' B}\right|<1. \tag{84}$$

For typical tokamak parameters estimated near the location of the $q=1$ rational surface, the small parameter in Eq. (84) is of the order of $5\cdot 10^{-3}$, making the effect of J_0 on the $m=1$ mode very small.

The modification to the classical $m = 1$ result due to J_0' is found by substituting $\omega \simeq i\gamma$ in Eq. (83), which then shows that the classical $m = 1$ growth rate is (very slightly) increased. Since, as it is known, the $m = 1$ tearing mode is inconsistent in the drift-tearing limit,[15] we cannot discuss the limit of Eq. (83) for $\omega^* \gg \gamma$.

In addition to studying the effect of the local current on the known $m \geq 2$ and $m = 1$ mode, we also investigated the possibility of entirely new modes by considering new dominant balances in S, due to the extra J_0 terms. To avoid misunderstandings, we do emphasize that the current terms are always "small" in our work, since all the forms of S have been derived by the method of Sec. III, assuming the small ζ expansion; we note however that the dispersion relation in many cases of interest follows by balancing terms in S which are indeed small (e.g., the $m \geq 2$ result follows from balancing $\Delta' x_A$ against $\alpha^{3/2} x_A^3$). In the present case we proceeded without treating the ρ^2 terms in Eq. (62) as a perturbation; rather we performed a full variational calculation by solving Eqs. (57), (58) and (59) independently.

Although a mathematically consistent result, in the sense of Eqs. (60) and (61), can be found, the corresponding region of parameter space does not describe low poloidal number tearing modes, and the result does not appear to be significant.

IV. Conclusions

In the present paper, the local effect of the equilibrium parallel current J_0 on the stability properties of low poloidal number tearing modes in the collisional regime has been investigated analytically.

The equilibrium current enters the eigenmode equation expressing quasineutrality through its gradient, in the so-called "kink term", whereas it appears in Ampere's law through resistivity perturbations in the "Ohm's law" provided by the electron momentum balance equation. Assuming the equilibrium parallel electric field to be spatially constant, both J_0 terms are identical, and have the form of local current gradients, directly related to the local electron temperature gradient, because of the $T_e^{3/2}$ dependence of the Spitzer-Braginskii conductivity in the equilibrium Ohm's law. The presence of electron temperature gradients is therefore essential (within the approximation of a constant $E_{\|}^{(0)}$) for the local current to couple to the mode dynamics. Our method of solution, presented in Sec. III, suitably combines the two eigenmode equations to obtain an integral equation with a symmetric kernel for the perturbed parallel current $\tilde{J}_{\|}$ from which a variational principle can be formulated; this formulation is a substantial generalization of the method of Ref. 5, whose results are precisely reproduced when J_0 terms are correspondingly neglected in our equations.

The main approximation behind our approach is the expansion for small argument of the Bessel functions in the solution for ϕ, in Sec. III. We remark that the resulting variational principle formulation for $\tilde{J}_{\|}$ is reasonably tractable, it provides continuous contact with previous calculations where local J_0 terms were ignored and, perhaps most importantly, physical situations of interest (e.g., tokamak physics) do correspond to the relative expansion parameter being indeed small. However, we must acknowledge that the possibility of an arbitrary strong current is ruled out *a priori* in our formalism, even though, as noted, this is not restrictive for most cases of interest. Similarly, the potentially interesting case of $\Delta' \to 0$ for the $m \geq 2$ mode cannot be treated without a substantial enlarging of the present formalism. It is perhaps worth stressing that the above restrictions are not dictated by the perturbative calculation performed in Sec. IV, but rather by the expansion of the Bessel functions for small ξ, necessary to obtain a symmetric kernel in the integral equation for Q, Eq. (50).

Although the present paper treats only the collisional regime of the tearing mode (i.e., $k_{\|}^2 v_e^2/\nu\omega \to 0$), restricting our conclusions to rather low temperature plasmas, our formalism is powerful enough to deal with the semicollisional regime as well, including the effect of the thermal conduction along the magnetic field. The extension of the present study will be the subject of a forthcoming paper.

The main results of this paper are summarized in Eqs. (78) and (83), expressing the dispersion relations, modified by the local J_0 terms, for the $m \geq 2$ and the $m = 1$ tearing mode, respectively.

For $\gamma \gg \omega^*$, Eq. (78) describes the modification to the classical $m \geq 2$ tearing mode: the FKR growth rate is increased by the inclusion of J_0. For $\gamma_{\text{FKR}} \ll \omega^*$, Eq. (78) describes the effect of the local current on the drift-tearing mode; in this case the effect of J_0 is to enhance stability since, we recall,[14] the consistent root in this drift regime corresponds to the right-hand side of Eq. (67) providing damping.

Equation (83) describes the modification to the $m = 1$ tearing mode dispersion relation due to J_0; in the non-drift regime $\gamma \gg \omega^*$ (in which the mode is consistent) the effect of the local current is to slightly increase the known growth rate. The result that the effect of J_0 on the $m = 1$ mode is quite smaller than on the $m \geq 2$ mode is consistent with a dimensional analysis of Eq. (20).

We explicitly checked for the typical tokamak numbers that the current contributions were indeed small, thereby confirming the goodness of the approximation used.

We also considered the possibility that the local current is responsible for new modes, which do not exist when J_0 terms are neglected; although a mathematically consistent growing root can be found from the derived dispersion relation, the result does not appear to be relevant since it corresponds to an extremely narrow region of parameters space, which does not pertain to low poloidal number modes.

Current theoretical prediction on the linear stability of tearing modes are therefore confirmed, to leading order, by the present work, since the modifications due to the inclusion of the local current has been shown to be typically rather small.

As a concluding remark, we could tentatively predict that the discussed J_0 effects should be quite negligible in the weakly nonlinear regime. Even though our calculation is restricted to the linear case, it shows that the equilibrium current enters the analysis through the local electron temperature gradient if the equilibrium parallel electric field is spatially constant. If one considers a time scale over which the equilibrium magnetic field does not change appreciably, such an approximation should be satisfied. As a consequence of the formation of magnetic islands, and of the resulting large radial electron thermal conduction, the local flattening of the temperature profile will prevent the equilibrium current to effectively couple to the mode dynamics.

APPENDIX

Kinetic Theory Derivation of Eq. (11)

The most significant departure from conventional tearing mode theories in the present work is the inclusion of the parallel equilibrium electron field $E_{\parallel}^{(0)}$ in the derivation of the eigenmode equations, which alters, in particular, the character of Eq. (13) into the non-diagonal form given by Eq. (11); we remark that the constitutive relationship between $\tilde{J}_{\parallel}$ and $\tilde{E}_{\parallel}$ expressed by Eq. (13) is not a law of nature; rather, it only pertains when $E_{\parallel}^{(0)}$ is neglected.

In this appendix, we show that Eq. (11) can also be derived from kinetic theory, and is therefore quite general.

The drift kinetic equation[19] is

$$\frac{\partial f_1}{\partial t} + v_{\parallel}\hat{b}_0 \cdot \nabla f_1 - \frac{(\nabla\tilde{\phi} \times \mathbf{B})}{B^2} \cdot \nabla f_0 + iv_{\parallel}\frac{k_{\perp}\tilde{A}_{\parallel}}{B}\hat{r} \cdot \nabla f_0$$
$$+ \frac{e}{m}\left[E_{\parallel}^{(0)}\frac{\partial}{\partial v_{\parallel}}f_1 + \tilde{E}_{\parallel}\frac{\partial f_0}{\partial v_{\parallel}}\right] = C_{\ell}(f_0, f_1), \qquad (A.1)$$

where f_0 and f_1 are the equilibrium and perturbed electron distribution functions, respectively, $C_{\ell}(f_0, f_1)$ is the linearized collision operator, e is the electron charge, and m is the electron mass.

In the collisional regime, it is appropriate to choose

$$f_1 = \delta n\frac{\partial f_0}{\partial n} + \delta T\frac{\partial f_0}{\partial T} + \delta f, \qquad (A.2)$$

where δn, δT are density and temperature perturbations, respectively, and δf may be taken as a small (odd in $v_{\parallel}$) additional term.

The equilibrium distribution function satisfies

$$\frac{e}{m}E_{\parallel}^{(0)}\frac{\partial f_0}{\partial v_{\parallel}} = C(f_0, f_0), \qquad (A.3)$$

for any density and temperature (we consider $E_{\parallel}^{(0)}$ spatially constant); f_0 can then be written as

$$f_0 = f_M[1 + v_{\parallel}h_{\rm sp}], \qquad (A.4)$$

where

$$f_M \equiv \frac{n}{\pi^{3/2} v_e^3} \exp(-v^2/v_e^2) \tag{A.5}$$

is the Maxwellian distribution function, $v_e = (2T/m)^{1/2}$ is the electron thermal speed, and the Spitzer function $h_{\rm sp}$ satisfies (cfr. Eq. (23))

$$J_0 \equiv e \int d^3v\, v_\| f_0 = e \int d^3v\, v_\|^2 f_M h_{\rm sp} = \sigma_{\rm sp} E_\|^{(0)}, \tag{A.6}$$

with the Spitzer-Braginskii conductivity given by

$$\sigma_{\rm sp} \equiv s \frac{ne^2}{m\nu}; \tag{A.7}$$

s is a numerical transport coefficient, equal to 1/0.51 for singly charged ions.

From Eq. (A.3), we have

$$\left[\delta T \frac{\partial}{\partial T} + \delta n \frac{\partial}{\partial n}\right] \frac{e}{m} \left[E_\|^{(0)} \frac{\partial f^{(0)}}{\partial v_\|}\right] = C_\ell \left(f_0, \delta n \frac{\partial f_0}{\partial n} + \delta T \frac{\partial f_0}{\partial T}\right), \tag{A.8}$$

so that the term $\frac{e}{m} E_\|^{(0)} \frac{\partial}{\partial v_\|} \left[\delta n \frac{\partial f_0}{\partial n} + \delta T \frac{\partial T_0}{\partial T}\right]$ in Eq. (A.1) evidently cancels with the corresponding term from $C_\ell(f_0, f_1)$.

We may next determine the perturbed δn and δT from the density and energy moments of Eq. (A.1), noting that the collision operator does not affect particle number and energy conservation. Actually, we shall use $\delta n_i = \delta n_e$ and $\nabla \cdot \tilde{\mathbf{J}} = 0$, so that we do not need the density moment. The energy moment of Eq. (A.1), neglecting Ohmic heating and small $k_\|^2 D/\omega$ corrections from parallel thermal conduction, yields

$$-i\omega \int d^3v \frac{mv^2}{2} \left[\delta n \frac{\partial f_0}{\partial n} + \delta T \frac{\partial f_0}{\partial T}\right] - ik_\perp \frac{\tilde{\phi}}{B} \int d^3v \frac{mv^2}{2} \left[\frac{dT}{dx} \frac{\partial f_0}{\partial T} + \frac{dn}{dx} \frac{\partial f_0}{\partial n}\right] = 0, \tag{A.9}$$

which reduces to

$$-i\omega \left[\delta n \frac{\partial}{\partial n} nT + \delta T \frac{\partial}{\partial T} nT\right] - ik_\perp \frac{\tilde{\phi}}{B} \left[\frac{dT}{dx} \frac{\partial}{\partial T} nT + \frac{dn}{dx} \frac{\partial}{\partial n} nT\right] = 0. \tag{A.10}$$

Hence,

$$n\delta T + T\delta n = -\frac{k_\perp \tilde{\phi}}{\omega B} \frac{d}{dx}(nT), \tag{A.11}$$

which, if we take

$$\frac{\delta n}{n} = \frac{e\tilde{\phi}}{T}\frac{\omega_n^*}{\omega}, \tag{A.12}$$

gives

$$\frac{\delta T}{T} = \frac{e\tilde{\phi}}{T}\frac{\omega_T^*}{\omega}, \tag{A.13}$$

with ω_n^*, ω_T^* given by Eqs. (12a) and (12b), respectively.

To obtain the current, it is useful to use the self adjoint properties of C_ℓ. We multiply Eq. (A.1) by $v_\| h_{\rm sp}$ and integrate over d^3v; taking note of the previous cancellation, we have

$$ik_\| \int d^3v\ v_\|^2 h_{\rm sp}\left[\delta n\frac{\partial f_0}{\partial n} + \delta T\frac{\partial f_0}{\partial T}\right] + i\frac{k_\perp \tilde{A}_\|}{B}\int d^3v\ v_\|^2 h_{\rm sp}\left[\frac{dn}{dx}\frac{\partial f_0}{\partial n} + \frac{dT}{dx}\frac{\partial f_0}{\partial T}\right]$$
$$- \frac{e}{T}\tilde{E}_\| \int d^3v\ v_\|^2 h_{\rm sp} f_0 = \int d^3v\ v_\| h_{\rm sp} C_\ell(f_0, \delta f). \tag{A.14}$$

Making use of the self-adjoint character of C_ℓ,

$$\int d^3v\ v_\| h_{\rm sp} C_\ell(f_0,\delta f) = \int d^3v\ \frac{\delta f}{f_0} C_\ell(f_0, v_\| h_{\rm sp} f_M) =$$
$$= \int d^3v\ \frac{\delta f}{f_0}\frac{e}{m}E_\|^{(0)}\frac{\partial f_0}{\partial v_\|} = -\frac{e}{T}E_\|^{(0)}\int d^3v\ v_\|\delta f. \tag{A.15}$$

Making use of Eqs. (A.6), (A.12), (A.13) and (A.15) in Eq. (A.14), and using Eq. (5a), we find

$$e\int d^3v\ v_\|\delta f = \sigma_{\rm sp}\tilde{E}_\|\left[1 - \frac{\omega_n^*}{\omega} - (1+\varepsilon)\frac{\omega_T^*}{\omega}\right], \tag{A.16}$$

where ε is the numerical transport coefficient defined in Section II.

With f_1 given by Eq. (A.2), the perturbed parallel current, defined by

$$\tilde{J}_\| \equiv e\int d^3v\ v_\| f_1, \tag{A.17}$$

is equal to

$$\tilde{J}_\| = \delta n\frac{\partial J_0}{\partial n} + \delta T\frac{\partial J_0}{\partial T} + e\int d^3v\ v_\|\delta f; \tag{A.18}$$

taking into account the $T^{3/2}$ dependence of J_0 through $\sigma_{\rm sp}$ and making use of Eqs. (A.13) and (A.16), we obtain

$$\tilde{J}_\parallel = \sigma_{\rm sp}\tilde{E}_\parallel \left[1 - \frac{\omega_n^*}{\omega} - (1+\varepsilon)\frac{\omega_T^*}{\omega}\right] + \frac{3}{2}\frac{e\tilde{\phi}}{T}\frac{\omega_T^*}{\omega}J_0, \qquad (A.19)$$

which is precisely Eq. (11).

Acknowledgments

One of the authors (F.C.) is pleased to thank Professor M.N. Rosenbluth for his valuable contribution to the present work. This work was supported by the U.S. Department of Energy, Contract No. DE-FG05-80-ET-53088.

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