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BALLOONING STABILITY IN TOROIDAL DEVICES

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ABSTRACT

The marginal stability condition of ballooning instabilities for toroidal confinement devices is derived for low critical stability β ($\beta \leq 10\%$). The stability condition derived here should be applicable to EBT and multipoles as well as tokamaks and stellarators. For EBT and multipoles a more compact expression for the stability condition is possible and is given here in the appendix.

The MHD, finite β ballooning instability is expected to limit the maximum β in toroidal devices (tokamaks, stellerators, and EBT) as well as tandem mirrors. Thus it may behoove us to refine the critical β criterion which was first pointed out in the 1960's by a number of authors.[1-3] The usual way to determine the critical value is taking advantage of the fact that the growth rate, s, squared, is real in the MHD approximation. Hence the problem is reduced to solving for marginal stability condition,

$$v_A^2 \frac{d^2\phi}{d\ell^2} + w_o^2(\ell)\phi = 0 \tag{1}$$

where $w_o^2(\ell)$ is the local interchange growth rate, v_A is the Alfvén wave speed and ℓ is taken parallel to B. In actual experimental situations, this simplification is somewhat modified as the magnetic field strength, B, is usually a function of ℓ. But what will be described here presumably improves the usual assumption that $w_o^2(\ell)$ is expanded in Fourier series and retained to the first term, thus transforming Eq. (1) into Mathieu's equation.

We shall impose periodic boundary condition such that $\phi(\ell+L) = \phi(\ell)$. And introduce a new variable

$$\theta = 2\pi \frac{\ell}{L} . \tag{2}$$

Then Eq. (1) may be rewritten as

$$\frac{d^2\phi}{d\theta^2} - \gamma[h + f(\theta)]\phi = 0 \tag{3}$$

$$\gamma \equiv \frac{\overline{R}_c a}{(L/2\pi)^2} \beta \tag{4}$$

$$0 = \int_0^{2\pi} d\theta f(\theta) \tag{5}$$

$$\overline{R}_c^{\,2} \int \left(\frac{1}{R(\theta)} - h \frac{1}{\overline{R}_c} \right)^2 d\theta \equiv \int f^2(\theta) d\theta \tag{6}$$

$$\frac{h}{\overline{R}_c} \equiv \frac{1}{2\pi} \int \frac{d\theta}{R(\theta)} > 0 \tag{7}$$

Here R is defined as

$$\frac{1}{R} = \frac{d\ell nB^2}{2d\vec{n}} \cdot \frac{d\psi/d\vec{n}}{|d\psi/d\vec{n}|} \tag{8}$$

where $\vec{n}$ is the normal component to constant pressure surface and often (but not necessarily always) coincides with the (quasi) magnetic surface. The difficulty which arises in many ~~devices is that we cannot keep 1/R such that $d(\ell np)/d\vec{n} \cdot R$ is negative everywhere.~~

The function f is nondimensional and $\langle f^2 \rangle$ is taken usually between $0.5 \sim 2$. It is not too difficult to impose one particular value for $\langle f^2 \rangle$ but sometimes it is advantageous to define $\langle f^2 \rangle$ at our discretion.

(i) Method of low β approximation.

The equation (3) can be solved rather easily by numerical methods. But often it is convenient to have an analytical solution available. Here we try that. We order coefficients of Eq. (3) as $\gamma f(\theta) \sim O(\varepsilon)$, $\gamma h \sim O(\varepsilon^2)$. Usually h is $\sim 1/10$ whereas $f(\theta) \sim 1$, so this ordering is not inconsistent. Finally, $\gamma \sim \beta^2/h$ as we see later, so if $\beta \leqslant 0.1$, γ is considered to be small.

Then to the zeroth order we obtain

$$\phi_0 = 1 \, . \tag{9}$$

To the next order

$$\frac{d^2\phi_1}{d\theta^2} = \gamma f \phi_0 = \gamma f \, . \tag{10}$$

This is integrated once to yield

$$\frac{d\phi_1}{d\theta} = \gamma\int_0^\theta f(\theta)d\theta + C \equiv \gamma F(\theta) + C \tag{11}$$

$d\phi_1/d\theta$ satisfies the boundary condition in view of Eq. (5). Integrating once more

$$\phi_1 = \gamma \int_0^\theta d\theta F(\theta) + C\theta + C_1 \quad . \tag{12}$$

The constant C_1 may be absorbed by ϕ_o. The constant C must be chosen to satisfy the boundary condition that is

$$C = -\frac{\gamma}{2\pi} \int_0^{2\pi} d\theta\ F(\theta) \equiv -\frac{\gamma}{2\pi} G(2\pi) \quad . \tag{13}$$

We defined

$$F(\theta) = \int_0^\theta f(\theta)d\theta,\ F(2\pi) = 0 \tag{14}$$

$$G(\theta) = \int_0^\theta F(\theta)d\theta \tag{15}$$

Thus

$$\phi_1 = \gamma\left(G(\theta) - \frac{\theta}{2\pi}G(2\pi)\right) \tag{16}$$

To the next order

$$\frac{d^2\phi_2}{d\theta^2} = \gamma h + \gamma^2 f\phi_1 \ . \tag{17}$$

Integrating once

$$\frac{d\phi_2}{d\theta} = \gamma h\theta + \gamma^2\left[\int_0^\theta d\theta f G(\theta) - \frac{G(2\pi)}{2\pi}\int_0^\theta f(\theta)\theta d\theta\right] + C_2 . \tag{18}$$

C_2 is necessary to satisfy the boundary condition for ϕ_2, but the necessity of satisfying the boundary condition for Eq. (18) brings out the β_c. Since at $\theta = 0$, $d\phi_2/d\theta = C_2$, it follows that

$$2\pi h + \gamma\left[\int_0^{2\pi} d\theta f G(\theta) - \frac{G(2\pi)}{2\pi}\int_0^{2\pi} f(\theta)\theta d\theta\right] = 0. \tag{19}$$

The first term in the bracket gives [note $F(2\pi) = 0$]

$$\int_0^{2\pi} d\theta f G = FG\Big|_0^{2\pi} - \int G' F d\theta = -\int_0^{2\pi} F^2 d\theta \ . \tag{20}$$

Also

$$\int_0^{2\pi} f(\theta)\theta d\theta = F \cdot \theta\Big|_0^{2\pi} - \int_0^{2\pi} F d\theta = -G(2\pi) . \tag{21}$$

Hence

$$\gamma = \frac{2\pi h}{\int_0^{2\pi} F^2 d\theta - \frac{1}{2\pi}[G(2\pi)]^2} \quad . \tag{22}$$

Schwartz's inequality assures the positiveness of the divisor except for the trivial case ($f \equiv 0$).

Thus

$$\beta = \frac{2a\overline{R}_c}{(L/2\pi)^2} h \frac{\pi}{\int_0^{2\pi} F^2 d\theta - \frac{1}{2\pi}(G(2\pi))^2} \equiv \beta_{co} \cdot I \quad . \tag{23}$$

β_{co} is the critical β for $f = \cos\theta$ and I is the correction factor for other functional forms of $f(\theta)$.

(ii) Some Examples

Let us calculate some simple examples. Assuming h is small from Eq. (6) and letting $1/R(\theta) = (1/R_p)\cos\theta$ where R_p is the minimum field curvature radius, we get $\overline{R}_c = R_p$. Also $I = 1$ because $G(2\pi) = 0$. So we get

$$\beta = \frac{2a|R_p|}{(L/2\pi)^2} h = \beta_{co} \quad . \tag{24}$$

as is expected.

Take

$$f(\theta) = 1, \qquad 0 \leqslant \theta \leqslant \pi,$$

$$f(\theta) = -1, \qquad \pi \leqslant \theta \leqslant 2\pi. \tag{25}$$

Again, $\overline{R}_c = |R_p|$. The calcuation yields

$$\int_0^{2\pi} F^2 d\theta = \frac{2}{3}\pi^3 \tag{26}$$

$$G(2\pi) = \int_0^{\pi} d\theta \cdot \theta + \int_{\pi}^{2\pi} (2\pi - \theta)\, d\theta = \pi^2 . \tag{27}$$

Thus,

$$I = \frac{6}{\pi^2} \quad \text{or} \quad \beta = \frac{6}{\pi^2}\beta_{co}. \tag{28}$$

If

$$f = \alpha, \qquad 0 < \theta < \theta_1 , \tag{29}$$

$$f = -\delta , \qquad \theta_1 < \theta < 2\pi , \tag{30}$$

such that

$$\alpha\theta_1 = (2\pi - \theta_1)\delta . \tag{31}$$

Then after rather tedious, but straightforward calculations we get (by equating $\delta = 1$, thus $R_c = R_{bad}$)

$$I = \frac{6}{(2\pi - \theta_1)^2} , \tag{32}$$

$$\beta_c = \frac{2a|R_b|}{(L/2\pi)^2}\, h\, \frac{6}{(2\pi - \theta_1)^2} . \tag{33}$$

Thus, β_c could be smaller than the ordinary eigenvalue of Mathieu's equation.

If $f = f(\theta) = \cos N\theta$ where N is an integer, we get

$$F = \frac{1}{N} \sin\theta \ . \tag{34}$$

Thus, we arrive at $I = N^2$. In other words,

$$\beta_c = \frac{2a|R_p|h}{(L/2\pi N)^2} \ . \tag{35}$$

That is even in systems such as Octupole or EBT, the connection length is determined by one period of bad-good curvature.

(iii) Comparison with wave mechanical solution.

The usual method to solve eigenvalues could, of course, be used. Let us take the case of Eq. (25) slightly modified so that

$$f(\theta) = 1, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}, \quad \frac{3\pi}{2} \leqslant \theta \leqslant 2\pi$$
$$f(\theta) = -1, \qquad \frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2}. \tag{36}$$

The starting point is Eq. (32). We define

$$\gamma_1 = (1 + h)\gamma,$$
$$\gamma_2 = (1 - h)\gamma. \tag{37}$$

Then in the domain I where $f(\theta) = 1$, we have

$$\phi = C_1 \cos h\, \gamma_1^{1/2} \theta . \tag{38}$$

In domain II where $f(\theta) = -1$, we have

$$\phi = C_2 \cos \gamma_2^{1/2} (\pi - \theta). \tag{39}$$

The connection at $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ requires

$$\gamma_1^{1/2} \tan h\, \gamma_1^{1/2} \frac{\pi}{2} = \gamma_2^{1/2} \tan \gamma_2^{1/2} \frac{\pi}{2} . \tag{40}$$

The above equation can be solved for arbitrary γ numerically. In the small γ limit we obtain

$$\frac{\pi}{2}\gamma_1 - \frac{\pi^3}{24}\gamma_1^2 = \frac{\pi}{2}\gamma_2 + \frac{\pi^3}{24}\gamma_2^2 \tag{41}$$

or

$$h\pi = \gamma\frac{\pi^3}{12}(1 + h^2) \ . \tag{42}$$

Thus we get

$$\beta_c = \frac{a\bar{R}_c}{(L/2\pi)^2} \cdot 2h \cdot \frac{6}{\pi^2}\frac{1}{(1 + h^2)} \ . \tag{43}$$

This is to be compared with Eq. (28) ($h^2 << 1$).

We conclude that Eq. (23) is probably accurate enough for estimating critical β in normal situations.

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References

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APPENDIX

CLOSED FIELD LINE DEVICES

In the case of floating multipoles or EBT, the geometry is very simple, so that a more formal set of equations can be utilized. In these devices, closed field lines $\oint \frac{d\ell}{B} = \text{const}$ form the magnetic surfaces, ψ. And we can choose another orthogonal coordinate, χ as $\int B d\ell$ and the third coordinate $\vec{\nabla}\psi \times \vec{\nabla}\chi$ is an ignorable coordinate.

Starting with a two fluid theory, using the usual assumptions among which $|k_\perp \rho_i| << 1$ where $k_\perp$ is the wave vector perpendicular to $\vec{B}$, and ρ_i is the ion Larmor radius, we arrive at

$$\frac{d\psi}{dn_o} \frac{1}{\mu_o} \nabla_\parallel^2 n + \left(\vec{\nabla} \frac{1}{B^2}\right)_n (nkTB) - \frac{MS^2 n}{B\kappa} = 0 \qquad (1A)$$

where n_o is the unperturbed density, n is perturbed density, subscript n in $\vec{\nabla}\frac{1}{B^2}$ means $\vec{\nabla}\frac{1}{B^2} \cdot \vec{\nabla}\psi/|\vec{\nabla}\psi|$, $kT = kT_e + kT_i$ is assumed constant, B is the magnitude of $\vec{B}$, κ is the density gradient as defined $(\nabla \ell n n_o)_n$ with subscript n having the same meaning as $(\vec{\nabla}\frac{1}{B^2})_n$, M is the mass of ions and S^2 is the growth rate in the limit of $|k_\perp \rho_i| << 1$.

Since only S^2 appears, S^2 is real in Eq. (1A). Thus at marginal state the last term of Eq. (1A) can be made 0. Since B is a function of ℓ, ψ, where ℓ is the coordinate in the direction of B, Eq. (1A) will be written as

$$B \frac{\partial}{\partial \ell} \frac{1}{B} \frac{\partial n}{\partial \ell} + \frac{dP_o}{d\psi} \; B^2 \mu_o \frac{\partial}{\partial \psi} \; \frac{1}{B^2} \; = \; 0 \tag{2A}$$

or

$$\frac{\partial^2 n}{\partial \chi^2} + \frac{dP_o}{d\psi} \; \mu_o \left(\frac{\partial}{\partial \psi} \frac{1}{B^2} \right) n = 0. \tag{3A}$$

Since the field lines are closed, it follows that

$$\oint d\chi = \chi_o \propto \Sigma I_i \quad , \tag{4A}$$

where ΣI_i is the total net current (in coils) enclosed by the closed field lines.

We define $\left(\frac{\partial}{\partial \psi} \frac{1}{B^2} \right)_A$ as

$$\frac{\partial}{\partial \psi} \frac{1}{B^2} = \frac{d}{d\psi} \oint \frac{d\chi/B^2}{\chi_o} + \left(\frac{\partial}{\partial \psi} \frac{1}{B^2} \right)_A \quad . \tag{5A}$$

Since ψ and χ are orthogonal, the change of the order of operations involving χ and ψ is permitted. Then Eq. (5A) implies

$$\oint \left(\frac{\partial}{\partial \psi} \frac{1}{B^2} \right)_A d\chi = \left(\frac{\partial}{\partial \psi} \oint \frac{d\ell}{B} \right)_A = 0. \tag{6A}$$

Thus, Eq. (3A) is reduced to

$$\frac{\partial^2 n}{\partial \chi^2} + \frac{dP_o}{d\psi} \mu_o n \left(\frac{d}{d\psi} \oint \frac{d\chi}{B^2} \right) \Big/ \chi_o + \frac{dP_o}{d\psi} n \mu_o \left(\frac{\partial}{\partial \psi} \frac{1}{B^2} \right)_A = 0. \tag{7A}$$

Again, ordering three terms of Eq. (7A) as 1, ε^2, ε, we arrive at the critical $dP_o/d\psi$ as

$$\frac{dP_o}{d\psi} = \frac{\frac{1}{\mu_o}\frac{d}{d\psi}\oint\frac{d\ell}{B}}{\oint H^2(\chi)\,d\chi - \frac{J^2(\chi_o)}{\chi_o}} \equiv L(\psi) \tag{8A}$$

where

$$H(\chi) \equiv \left(\frac{d}{d\psi}\int_0^{\chi}\frac{d\chi}{B^2}\right)_A = \left(\frac{d}{d\psi}\int_0^{\chi}\frac{d\ell}{B}\right)_A \tag{9A}$$

$$J(\chi) = \int_0^{\chi} H(\chi)\,d\chi. \tag{10A}$$

If $|dP_o/d\chi|$ is larger than given in (8A), the plasma is unstable. Thus, in marginal stability cases

$$P = \int_{\psi_o}^{\psi} L(\psi)\,d\psi \tag{11A}$$

with $P(\psi_o) = 0$ at $\psi = \psi_o$. Note in a $\oint d\ell/B$ stable plasma

$$\frac{dP_o}{d\psi}\,\frac{d}{d\psi}\oint\frac{d\ell}{B} > 0. \tag{12A}$$