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THEORY OF THE RENORMALIZED DIELECTRIC FOR ELECTROSTATIC DRIFT WAVE TURBULENCE IN TOKAMAKS

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ABSTRACT

A theory of collisionless electrostatic drift wave turbulence in a circular cylinder with shear is presented. A renormalized electron drift kinetic equation which is consistent with conservation of energy is derived. For low levels of turbulence, a perturbative solution indicates that the turbulence has a stabilizing effect and that total mode spectrum energy decreases. For strongly turbulent regimes, results quantitatively but not qualitatively different from those of Hirshman and Molvig are found. In particular, additional stabilizing terms lead to a lower saturation amplitude, but the basic picture of turbulent destabilization competing against linear and nonlinear shear damping persists.

I. INTRODUCTION

One of the central topics of current research in plasma theory is that of understanding the anomalous energy transport observed in tokamaks. Since it is thought that such transport is caused by low frequency microinstabilities of the drift wave variety,[1] it is obvious that the development of a theory of drift wave turbulence is a necessary prerequisite for a satisfactory theoretical explanation of empirical scaling laws. In this paper, a self-consistent theory of the renormalized dielectric for electrostatic universal mode turbulence in a cylindrical plasma with magnetic shear is presented.

Recent pioneering work of Hirshman and Molvig[2] has demonstrated the important effects that electron orbit stochasticity can have on drift waves. They showed that the presence of shear induces a coupling of stochastic radial $\underset{\sim}{E} \times \underset{\sim}{B}$ motion and poloidal free streaming which yields an aggregate electron orbit decorrelation rate ω_c, where $\omega_c = (k^2 v_t^2 D/3L_s^2)^{1/3}$ and D is the turbulent, test particle diffusion coefficient. Since at observed turbulence levels $\omega_c > \omega$, Hirshman and Molvig noted that the radial diffusion length $(D\omega_c^{-1})^{1/2}$, exceeds the electron linear Landau resonance point, x_e. Thus, an electron is diffused into resonance in less than one wave period and the fluid electron response characteristic of the region near the mode rational surface is eliminated. The contribution of the nonadiabatic electron response in the drift wave eigenmode equation is characterized by an inverse Landau resonance-like, turbulent

dissipation term, which destabilizes the mode. Saturation eventually occurs through a competition between electron destabilization and linear shear damping. The saturation criterion allows a self-consistent determination of D. It is worthwhile to note that the principal effect of the turbulence is to cause and regulate the input of the expansion free energy of the electron pressure gradient into the drift oscillation. It is also significant that the underlying modes persist in the presence of turbulence.

The analysis of Ref. 2 is based on the idea that in the presence of low frequency drift modes centered on closely packed rational surfaces, electron orbits will be stochastic. This stochasticity is incorporated into the theory by including a constituent $\delta x(\tau)$ in the electron orbit and then averaging the time history integrated electron response over a statistical distribution, usually $(2\pi D\tau)^{-1/2} \exp [-\delta x(\tau)^2 / D\tau]$, of that constituent. Of course, such a procedure is basically an application of resonance broadening theory[3] and is equivalent to replacing the $-(c/B_o)(\underset{\sim}{\nabla}\Phi \times \hat{n})\cdot\nabla f$ nonlinearity of the electron drift kinetic equation with the diffusion operator $-D\, \partial^2 f/\partial x^2$.

In view of the importance and significance of the results obtained by Hirshman and Molvig, a careful appraisal of the resonance broadening theory in the context of the application to the problem of drift wave turbulence in a sheared magnetic field is in order. As previously noted by Dupree and Tetreault[4] in their analysis of shearless drift wave turbulence, the resonance broadening procedure fails to preserve certain properties

of the exact, nonlinear drift kinetic equation. In particular, the orthogonality of $\underset{\sim}{E}_\perp$ and $\underset{\sim}{J}_\perp = ne(c/B_0)(\underset{\sim}{E}_\perp \times \hat{n})$ is violated when the $-(c/B_0)(\underset{\sim}{\nabla}\Phi \times \hat{n})\cdot\nabla f$ nonlinearity is replaced by the diffusion operator. This causes two difficulties: an erroneous renormalization of the adiabatic response and a failure to satisfy a constraint necessary for conservation of energy. Hirshman and Molvig circumvented the first difficulty by first integrating by parts, retaining the exact electron orbit, to extract the adiabatic response and then applying the resonance broadening procedure. Observe, however, that this is equivalent to writing $\tilde{f}_{k,\omega} = (|e|/T)\langle f\rangle\hat{\Phi}_{k,\omega} + \tilde{g}_{k,\omega}$, noting that $(\underset{\sim}{\nabla}\Phi \times \hat{n})\cdot\underset{\sim}{\nabla}\Phi = 0$, and then replacing $-(c/B_0)(\underset{\sim}{\nabla}\Phi \times \hat{n})\cdot\underset{\sim}{\nabla}\tilde{g}$ with $-D\partial^2\tilde{g}/\partial x^2$. Clearly, the operations of adiabatic response extraction and renormalization do not commute here. Hirshman and Molvig did not address the second difficulty in a substantive fashion.

Some questionable points in the resonance broadening procedure are also apparent from physically motivated observations. First, it is peculiar that the quadratically nonlinear term of the drift kinetic equation is replaced by a renormalization proportional only to $\tilde{f}$. Second, diffusion is characteristic of a Markov process. Thus, the resonance broadening theory has underlying it the presumption of the existence of a bath of fluctuations of short wavelength and high frequency, in comparison with typical mode scales, which cause the turbulent orbit scattering. This presumption is clearly inconsistent with the fact that the spatial and temporal scales of a typical test mode are comparable

to, not disparate from, the scales of the background modes. This is because other drift waves constitute the turbulent background medium. On the other hand, because of the strong resonant interaction between electrons and the mode, one might argue that application of the resonance broadening theory is a reasonable, approximate procedure for constructing a renormalized dielectric and that remedies for the defects discussed here would lead to quantitative, but not major qualitative, modifications to the Hirshman and Molvig theory. Clearly, further investigation is needed.

In this paper, we attempt to resolve some of the difficulties discussed by reconsidering the problem first studied by Hirshman and Molvig. The electron drift kinetic equation is renormalized using the procedure of Dupree and Tetreault.[4] The resulting renormalized equation contains an approximate representation of the $-(c/B_0)(\nabla\hat{\Phi} \times \hat{n})\cdot\nabla\tilde{f}$ nonlinearity which is composed of two types of terms, proportional to $\tilde{f}$ and $\hat{\Phi}$, and which retains non-Markovian effects. This renormalization permits an unambiguous extraction of the adiabatic response and is shown to satisfy constraints necessary for conservation of energy.

The renormalized drift kinetic equation contains terms involving complicated spectrum sums which have heretofore hindered actual calculations with this method. However, the two-dimensional nature of the wave vector specturm for cylindrical geometry with shear can be exploited by developing an approximate procedure, presented here, which greatly simplifies

the specturm summations. Thus, it is possible to solve the renormalized drift kinetic equation to obtain an analytic approximation to the turbulent electron response which, in turn, is used to construct the eigenmode equation (dielectric operator). The nonlinear stability probelm is analyzed in two regimes. For $\omega_c < \omega$, a perturbative analysis indicates that the turbulence has a stabilizing effect at small amplitude and total mode spectrum energy decreases. The rate of decrease is calculated. For $\omega_c > \omega$, moderate quantitative, but not qualitative, modifications to the basic picture of Hirshman and Molvig are found. In particular, additional stabilizing terms lead to a lower saturation amplitude, but the mechanism of turbulent destabilization competing against shear damping and turbulent mode broadening persists. This is consistent with the notion that in the presence of shear, resonant radial diffusion constitutes the dominant turbulent effect.

The paper is organized in the following manner. In Sec. II, the electron drift kinetic equation is renormalized. It is shown that the renormalized drift kinetic equation satisfies constraints necessary for conservation of energy and allows an unambiguous extraction of the adiabatic response. In Sec. III, the $\omega_c < \omega$ perturbative analysis is presented. The rate of decrease of total mode specturm energy is calculated. In Sec. IV, the $\omega_c > \omega$, strongly turbulent regime electron response is calculated and the eigenmode equation derived and solved. The physical motivation for procedures which simplify the calculation is given. In Sec. V, the results and their relation to those of Hirshman and Molvig are discussed.

II. RENORMALIZED ELECTRON DRIFT KINETIC EQUATION

The renormalized drift kinetic equation for low frequency turbulence in a circular cylinder with shear is now derived. Neglecting the $E_{\|}\,\partial\langle f\rangle/\partial v_{\|}$ nonlinearity, which is unimportant because $E_{\|}$ is small, the nonlinear drift kinetic equation is given by

$$\frac{\partial \tilde{f}}{\partial t} + v_{\|}\hat{n}\cdot\nabla\tilde{f} + \nabla\cdot\left[\frac{c}{B_0}(\nabla\hat{\Phi}\times\hat{n})\tilde{f}\right] = \frac{c}{B_0}\frac{\partial\langle f\rangle}{\partial r}\frac{1}{r}\frac{\partial\hat{\Phi}}{\partial\theta}$$

$$- |e|\,\frac{\partial\langle f\rangle}{\partial\varepsilon}\,v_{\|}\hat{n}\cdot\nabla\hat{\Phi} \qquad (1)$$

The potential fluctuation $\hat{\Phi}$ can be decomposed as $\hat{\Phi} = \sum_{\underset{\sim}{m},\omega}\hat{\Phi}_{\underset{\sim}{m},\omega}(r)\exp[i(m\theta-n\phi-\omega t)]$ where $\underset{\sim}{m}$ refers to poloidal mode number m and toroidal mode number n and where r,θ,ϕ signify radius, poloidal, and toroidal angle, respectively. Associated with $\hat{\Phi}_{\underset{\sim}{m},\omega}(r)$ are the wave vector $\underset{\sim}{k} = [(\underset{\sim}{m}/r)\hat{\theta} - (n/R)\hat{z}]$, with poloidal component $k_\theta = m/r$, and the parallel wave number $k_{\|} = \underset{\sim}{k}\cdot\hat{n}\simeq k_\theta(r-r_0)/L_s$. Here, r_0 is the radius such that $\underset{\sim}{k}\cdot\underset{\sim}{B}(r_0) = 0$ and $L_s^{-1} = -rq'/Rq^2$ is the shear length. The electron diamagnetic drift frequency is $\omega_{*e} = -k_\theta\rho_s c_s/L_n$, where $c_s = (T_e/m_i)^{1/2}$ is the sound speed, $\rho_s = c_s/\Omega_i$, and $L_n^{-1} = (1/n_0)(\partial n_0/\partial r)$ is the density gradient scale length. Thus, $f_{\underset{\sim}{m},\omega}$ is determined from

$$-i\left(\omega-k_{\|}v_{\|}\right)\tilde{f}_{\underset{\sim}{m},\omega} + [N_\perp]_{\underset{\sim}{m},\omega} = i\,\frac{|e|}{T_e}\langle f\rangle\left(k_{\|}v_{\|} - \omega_e\right)\hat{\Phi}_{\underset{\sim}{m},\omega} \qquad (2a)$$

where the nonlinear term $[N_\perp]_{\underset{\sim}{m}}$ is

$$[N_\perp]_{\underset{\sim}{m},\omega} = -i\frac{c}{B_0}\sum_{\underset{\sim}{m}',\omega'}\left[\frac{1}{r}\frac{\partial}{\partial r}m'\,\hat{\Phi}_{\underset{\sim}{m}',\omega'} + \frac{m}{r}\left(\frac{\partial}{\partial r}\hat{\Phi}_{\underset{\sim}{m}',\omega'}\right)\right]\tilde{f}_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'} \qquad (2b)$$

Unless indicated by parentheses, radial derivatives act on all quantities to the right.

Following the method of Dupree and Tetreault, we renormalize Eq. (2) by constructing $C_{\underset{\sim}{m},\omega}\tilde{f}_{\underset{\sim}{m},\omega}$, the portion of $(N_\perp)_{\underset{\sim}{m},\omega}$ which is phase coherent with the fluctuation at $\underset{\sim}{m},\omega$. Because construction of the renormalized dielectric does not require an understanding of incoherent emission processes etc., the coherent approximation to $N_\perp$ is sufficient. Obviously, however, the phase coherency restriction eliminates many terms included in standard weak turbulence theory. Construction of the coherent approximation is implemented by iterative substitution of driven terms into $[N_\perp]_{\underset{\sim}{m},\omega}$, such that the overall phase is that of the fluctuation at $\underset{\sim}{m},\omega$. Hence, $C_{\underset{\sim}{m},\omega}\tilde{f}_{\underset{\sim}{m},\omega}$ is at least third order in fluctuating quantities. In addition, for simplicity and in order to make contact with the work of Hirshman and Molvig, we $C_{\underset{\sim}{m},\omega}\tilde{f}_{\underset{\sim}{m},\omega}$ include only terms corresponding to wave-particle interaction processes.

Proceeding, we now write $[N_\perp]_{\underset{\sim}{m},\omega}$ as:

$$[N_\perp]_{\underset{\sim}{m},\omega} \cong -i\,\frac{c}{B_o}\sum_{\underset{\sim}{m}',\omega'}\left[\frac{1}{r}\frac{\partial}{\partial r}\,m'\,\hat{\Phi}_{\underset{\sim}{m}',\omega'} + \frac{m}{r}\left(\frac{\partial\hat{\Phi}}{\partial r}_{\underset{\sim}{m}',\omega'}\right)\right]\tilde{f}^{(2)}_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'} \tag{3}$$

where the superscripts indicate order in fluctuating quantities. The corresponding terms of the form $f^{(1)}_{\underset{\sim}{m}',\omega'}\hat{\Phi}^{(2)}_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'}$ are part of the resonant three wave coupling processes and are thus dropped[4], since this analysis is limited to consideration of wave-particle interaction effects. It should be noted the consistency of the renormalized drift kinetic equation with conservation of energy is not affected by

the neglect of these terms. $\tilde{f}^{(2)}_{\underset{\sim}{m}-\underset{\sim}{m}',\ \omega-\omega'}$ is then determined by using Eq. (2), for the $\underset{\sim}{m}-\underset{\sim}{m}'$, $\omega-\omega'$ fluctuation, and noting that the phase coherency condition requires that $\tilde{f}^{(2)}_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'}$ be driven only by the direct interaction (beating) of independent test fluctuations at $\underset{\sim}{m},\omega$ and $\underset{\sim}{m}',\omega'$ in the presence of the remainder of the randomly phased turbulent background. Thus, $[N_\perp]_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'}$ is decomposed as

$$[N_\perp]_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'} \cong C_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'}\ \tilde{f}_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'} - i\frac{c}{B_o}\left[\frac{\underset{\sim}{m}'}{r}\hat{\Phi}^*_{\underset{\sim}{m}',\omega'}\frac{\partial\tilde{f}}{\partial r}_{\underset{\sim}{m},\omega} - \frac{\underset{\sim}{m}}{r}\left(\frac{\partial\tilde{f}^*}{\partial r}_{\underset{\sim}{m}',\omega'}\right)\hat{\Phi}_{\underset{\sim}{m},\omega} - \frac{\underset{\sim}{m}}{r}\left(\frac{\partial\hat{\Phi}^*}{\partial r}_{\underset{\sim}{m}',\omega'}\right)\tilde{f}_{\underset{\sim}{m},\omega} + \frac{\underset{\sim}{m}'}{r}\tilde{f}^*_{\underset{\sim}{m}',\omega'}\frac{\partial\hat{\Phi}}{\partial r}_{\underset{\sim}{m},\omega}\right] \quad (4)$$

From Eq. (4), and noting that $\hat{\Phi}^{(2)}_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'}$ terms have been excluded, $\tilde{f}^{(2)}_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'}$ is determined by

$$\left[-i(\omega-\omega')+i(k-k')_\parallel v_\parallel + \bar{C}_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'}\right]\tilde{f}^{(2)}_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'}$$

$$= i\frac{c}{B_o}\left[\frac{\underset{\sim}{m}'}{r}\hat{\Phi}^*_{\underset{\sim}{m}',\omega'}\frac{\partial\tilde{f}}{\partial r}_{\underset{\sim}{m},\omega} - \frac{\underset{\sim}{m}}{r}\frac{\partial\tilde{f}^*}{\partial r}_{\underset{\sim}{m}',\omega'}\hat{\Phi}_{\underset{\sim}{m},\omega} - \frac{\underset{\sim}{m}}{r}\left(\frac{\partial}{\partial r}\hat{\Phi}^*_{\underset{\sim}{m}',\omega'}\right)\tilde{f}_{\underset{\sim}{m},\omega} + \frac{\underset{\sim}{m}'}{r}\tilde{f}^*_{\underset{\sim}{m}',\omega'}\frac{\partial\hat{\Phi}}{\partial r}_{\underset{\sim}{m},\omega}\right] \quad (5)$$

and is thus given by

$$\tilde{f}^{(2)}_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'} = i\frac{c}{B_o}\bar{g}_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'}\left[\frac{\underset{\sim}{m}'}{r}\hat{\Phi}^*_{\underset{\sim}{m}',\omega'}\frac{\partial\tilde{f}}{\partial r}_{\underset{\sim}{m},\omega} - \frac{\underset{\sim}{m}}{r}\left(\frac{\partial\tilde{f}^*}{\partial r}_{\underset{\sim}{m}',\omega'}\right)\hat{\Phi}_{\underset{\sim}{m},\omega} - \left(\frac{\underset{\sim}{m}}{r}\frac{\partial\hat{\Phi}^*}{\partial r}_{\underset{\sim}{m}',\omega'}\right)\tilde{f}_{\underset{\sim}{m},\omega} + \frac{\underset{\sim}{m}'}{r}\tilde{f}^*_{\underset{\sim}{m}',\omega'}\frac{\partial\hat{\Phi}}{\partial r}_{\underset{\sim}{m},\omega}\right] \quad (6a)$$

$$\bar{g}_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'} = \left[-i(\omega-\omega') + i(\underset{\sim}{k}-\underset{\sim}{k}')_\parallel v_\parallel + \bar{C}_{\underset{\sim}{m}-\underset{\sim}{m}',\omega-\omega'}\right] \quad (6b)$$

Substituting $\tilde{f}^{(2)}_{\underset{\sim}{m-m'},\omega-\omega'}$ into Eq. (3) yields

$$[N_\perp]_{\underset{\sim}{m},\omega} = C_{\underset{\sim}{m},\omega} f_{\underset{\sim}{m},\omega} = -d_{\underset{\sim}{m},\omega}\, \tilde{f}_{\underset{\sim}{m},\omega} + b_{\underset{\sim}{m},\omega}\, \hat{\Phi}_{\underset{\sim}{m},\omega} \tag{7a}$$

where

$$d_{\underset{\sim}{m},\omega} = \sum_{\underset{\sim}{m}',\omega'} \frac{c^2}{B_o^2}\left[\frac{1}{r}\frac{\partial}{\partial r}\, \underset{\sim}{m}'\, \hat{\Phi}_{\underset{\sim}{m}',\omega'} + \frac{\underset{\sim}{m}}{r}\left(\frac{\partial \hat{\Phi}}{\partial r}_{\underset{\sim}{m}',\omega'}\right)\right]$$

$$\times\ \bar{g}_{\underset{\sim}{m-m}',\omega-\omega'}\left[\frac{\underset{\sim}{m}'}{r}\, \hat{\Phi}^*_{\underset{\sim}{m}',\omega'}\, \frac{\partial}{\partial r} - \frac{\underset{\sim}{m}}{r}\left(\frac{\partial \hat{\Phi}^*}{\partial r}_{\underset{\sim}{m}',\omega'}\right)\right] \tag{7b}$$

$$b_{\underset{\sim}{m},\omega} = \sum_{\underset{\sim}{m}',\omega'} \frac{c^2}{B_o^2}\left[\frac{1}{r}\frac{\partial}{\partial r}\, \underset{\sim}{m}'\, \hat{\Phi}_{\underset{\sim}{m}',\omega'} + \frac{\underset{\sim}{m}}{r}\left(\frac{\partial \hat{\Phi}}{\partial r}_{\underset{\sim}{m}',\omega'}\right)\right]$$

$$\times\ \bar{g}_{\underset{\sim}{m-m}',\omega-\omega'}\left[\frac{\underset{\sim}{m}}{r}\, \frac{\partial \tilde{f}^*}{\partial r}_{\underset{\sim}{m}',\omega'} + \frac{\underset{\sim}{m}'}{r}\, \tilde{f}^*_{\underset{\sim}{m}',\omega'}\, \frac{\partial}{\partial r}\right] \tag{7c}$$

Thus, the renormalized drift kinetic equation is

$$\left[-i\left(\omega - k_\parallel v_\parallel\right) + C_{\underset{\sim}{m},\omega}\right]\tilde{f}_{\underset{\sim}{m},\omega} = i\left|\frac{e}{T}\right| \langle f \rangle \left(k_\parallel v_\parallel - \omega_{*e}\right)\hat{\Phi}_{\underset{\sim}{m},\omega} \tag{8a}$$

$$C_{\underset{\sim}{m},\omega}\, \tilde{f}_{\underset{\sim}{m},\omega} = -d_{\underset{\sim}{m},\omega}\, \tilde{f}_{\underset{\sim}{m},\omega} + b_{\underset{\sim}{m},\omega}\, \hat{\Phi}_{\underset{\sim}{m},\omega} \tag{8b}$$

The distinction between $C_{\underset{\sim}{m-m}',\omega-\omega'}$ and $\bar{C}_{\underset{\sim}{m-m}',\omega-\omega'}$ is made because of the neglect of $\hat{\Phi}^{(2)}_{\underset{\sim}{m-m}',\omega-\omega'}$ terms in Eq. (5). Consistency with this approximation requires that $\bar{C}_{\underset{\sim}{m-m}',\omega-\omega'}\, \tilde{f}_{\underset{\sim}{m-m}',\omega-\omega'}$ equal $d_{\underset{\sim}{m-m}',\omega-\omega'} f_{\underset{\sim}{m-m}',\omega-\omega'}$ and that the $b_{\underset{\sim}{m-m}',\omega-\omega'} \hat{\Phi}_{\underset{\sim}{m-m}',\omega-\omega'}$ contribution be dropped.

The $d_{\underset{\sim}{m},\omega}$ and $b_{\underset{\sim}{m},\omega}$ operators involve sums over the turbulent spectrum. In view of the fact that the coherent approximation neglects emission and absorption by $\underset{\sim}{k},\omega$ components and that the phase coherency condition excludes non-factorizable driven fluctuations, the spectrum constituents are effectively limited to be normal modes only. Hereafter, that limitation is imposed and taken to be $\omega(m)$. Superfluous ω subscripts will be dropped. Note that such a limitation on the spectrum ensures treating a test mode and the background modes on an equal footing.

Having derived the renormalized drift kinetic equation, a brief discussion of its properties is useful at this point. First, a criterion (based on the exact drift kinetic equation) is established which the approximate $[N_\perp]_{\underset{\sim}{m}}$ given in Eq. (7) must satisfy in order that the renormalized drift kinetic equation be consistent with conservation of energy. Proceeding as in Ref. 4 by taking the $\frac{1}{2}mv_\parallel^2$ moment of Eq. (1), integrating over $v_\parallel$, and averaging yields

$$\frac{\partial}{\partial t}\left(\int dv_\parallel m \frac{v_\parallel^2}{2} \langle f\rangle\right) = -\underset{\sim}{\nabla}\cdot\underset{\sim}{q} + \langle E_\parallel J_\parallel\rangle \tag{9}$$

where $\underset{\sim}{q}$ is the heat flux due to $\underset{\sim}{E} \times \underset{\sim}{B}$ convection. In addition, the Poynting theorem gives

$$-\frac{\partial}{\partial t}\left(E^2/8\pi\right) = \langle E_\parallel J_{\parallel(e)}\rangle + \langle \underset{\sim}{E}_\perp \cdot \underset{\sim}{J}_{\perp(e)}\rangle + \langle \underset{\sim}{E}\cdot\underset{\sim}{J}_{(i)}\rangle \tag{10}$$

$\underset{\sim}{J}_i$ includes ion $\underset{\sim}{E} \times \underset{\sim}{B}$, polarization, and parallel currents. Thus, shear damping appears in the energy balance as ion heating. Note that since $\underset{\sim}{J}_{(e)} = -n_o|e| \frac{c}{B_o}(\underset{\sim}{\nabla}\hat{\Phi} \times \hat{n})$, $\langle \underset{\sim}{E}_\perp \cdot \underset{\sim}{J}_{\perp(e)} \rangle = 0$, exactly. Thus, combining Eqs. (9) and (10) yields

$$\frac{\partial}{\partial t}\left(\int dv_\parallel \frac{mv_\parallel^2}{2}\langle f\rangle + \frac{E^2}{8\pi}\right) = -\underset{\sim}{\nabla}\cdot\underset{\sim}{q} - \langle \underset{\sim}{E}\cdot\underset{\sim}{J}_{(i)}\rangle \tag{11a}$$

$$\langle \underset{\sim}{E}\cdot\underset{\sim}{J}_{(e)}\rangle = -|e|\, n_o \int dv_\parallel \langle \hat{\Phi} N_\perp\rangle = 0 \tag{11b}$$

where Eq. (11a) is the energy balance relation and Eq. (11b) is the constraint. In effect, Eq. (11b) constitutes a test which approximate representations (renormalizations) of $N_\perp$ must pass so that constraints necessary, but by no means sufficient, for energy conservation are satisfied. Indeed, whether or not a theory actually conserves energy must be checked <u>aposteriori</u> using Eq. (11a).

That the renormalized $[N_\perp]_{\underset{\sim}{m}}$ given in Eq. (7) satisfies the constraint is verified by inserting that result into Eq. (11b). Thus

$$\int dv_\parallel \langle \hat{\Phi} N_\perp\rangle = \int dv_\parallel \int r\,dr \sum_{\underset{\sim}{m}} \hat{\Phi}^*_{\underset{\sim}{m}}\left(d_{\underset{\sim}{m}} \tilde{f}_{\underset{\sim}{m}} - b_{\underset{\sim}{m}}\hat{\Phi}_{\underset{\sim}{m}}\right) \tag{12a}$$

and integrating leading radial derivatives by parts gives

$$\langle \hat{\Phi} N_\perp\rangle = -\frac{c^2}{B_o^2}\int dr \sum_{\underset{\sim}{m},\underset{\sim}{m}'}\left\{\left\{\left(\frac{\partial \hat{\Phi}^*}{\partial r}_{\underset{\sim}{m}}\right) m' \hat{\Phi}_{\underset{\sim}{m}'} \bar{g}_{\underset{\sim}{m}-\underset{\sim}{m}'}\left[\frac{m'}{r}\hat{\Phi}^*_{\underset{\sim}{m}'}\frac{\partial \tilde{f}}{\partial r}_{\underset{\sim}{m}}\right.\right.\right.$$

$$\left.\left.\left. - \frac{m}{r}\left(\frac{\partial \tilde{f}^*}{\partial r}_{\underset{\sim}{m}'}\right)\hat{\Phi}_{\underset{\sim}{m}} - \frac{m}{r}\left(\frac{\partial \hat{\Phi}^*}{\partial r}_{\underset{\sim}{m}'}\right)\tilde{f}_{\underset{\sim}{m}} + \frac{m'}{r}\tilde{f}^*_{\underset{\sim}{m}'}\frac{\partial \hat{\Phi}}{\partial r}_{\underset{\sim}{m}}\right]\right\}\right.$$

$$-\left\{\hat{\Phi}^{*}_{\underset{\sim}{m}}\, m\left(\frac{\partial\hat{\Phi}}{\partial r}\underset{\sim}{m'}\right)\bar{g}_{\underset{\sim}{m}-\underset{\sim}{m}'}\left[\frac{m'}{r}\hat{\Phi}^{*}_{\underset{\sim}{m}}\frac{\partial f}{\partial r}\underset{\sim}{m} - \frac{m}{r}\left(\frac{\partial f^{*}}{\partial r}\underset{\sim}{m'}\right)\hat{\Phi}_{\underset{\sim}{m}}\right.\right.$$
$$\left.\left. - \frac{m}{r}\left(\frac{\partial\hat{\Phi}^{*}}{\partial r}\underset{\sim}{m'}\right)\tilde{f}_{\underset{\sim}{m}} + \frac{m'}{r}\tilde{f}^{*}_{\underset{\sim}{m}'}\frac{\partial\hat{\Phi}}{\partial r}\underset{\sim}{m}\right]\right\}\Bigg) \qquad (12b)$$

Taking $\omega(-m) = -\omega$, which is true for drift waves, and interchanging $\underset{\sim}{m}$ with $\underset{\sim}{m}'$ gives $\langle\Phi N_{\perp}\rangle = -\langle\Phi N_{\perp}\rangle^{*}$. Similarly, interchanging $\underset{\sim}{m}$, $\underset{\sim}{m}'$ with $-\underset{\sim}{m}$, $-\underset{\sim}{m}'$ gives $\langle\Phi N_{\perp}\rangle = \langle\Phi N_{\perp}\rangle^{*}$. Thus, $\langle\hat{\Phi}N_{\perp}\rangle = 0$ and the renormalized drift kinetic equation is consistent with Eq. (11b). Note that detailed knowledge of $\bar{g}_{\underset{\sim}{m}-\underset{\sim}{m}'}$ was not necessary for construction of this proof.

Regarding extraction of the adiabatic response, it is easily verified that in the limit where $\tilde{f}_{\underset{\sim}{m}} = \frac{|e|}{T_e}\hat{\Phi}_{\underset{\sim}{m}}\langle\tilde{f}\rangle$ and similarly for all $\tilde{f}_{\underset{\sim}{m}'}$, $C_{\underset{\sim}{m}}\tilde{f}_{\underset{\sim}{m}} = 0$. Thus, the adiabatic response is annihilated and can be unambigously extracted. The renormalized drift kinetic equation for the nonadiabatic response $g_{\underset{\sim}{m}}$ is

$$-i\left(\omega - k_{\parallel}v_{\parallel}\right)\tilde{g}_{\underset{\sim}{m}} - d_{\underset{\sim}{m}}\tilde{g}_{\underset{\sim}{m}} + \bar{b}_{\underset{\sim}{m}}\hat{\Phi}_{\underset{\sim}{m}} = i\frac{|e|}{T}\left(\omega - \omega_{*e}\right)\hat{\Phi}_{\underset{\sim}{m}}\langle\tilde{f}\rangle \qquad (13)$$

where $\bar{b}_{\underset{\sim}{m}}$ contains the nonadiabatic responses, $\tilde{g}_{\underset{\sim}{m}}$, of the background modes.

In general terms, note that in the renormalized drift kinetic equation the nonlinear term of the drift kinetic equation is replaced by <u>two</u> quasilinear-like terms. The importance of the $b_{\underset{\sim}{m}}$ term is obvious and is discussed in Ref. 4. Note, however, that since $\hat{\Phi}^{(2)}$ terms have been dropped, Poisson's equation never appears in the derivation of the renormalized drift kinetic equation. Thus, to attribute to $b_{\underset{\sim}{m}}$ the role of

representing a "back reaction of the particles on the field"[5] is incomplete. Also of significance is the non-Markovian character ($\underset{\sim}{m}$ dependency) of $d_{\underset{\sim}{m}}$ and $b_{\underset{\sim}{m}}$. This feature, which is necessary for consistency with conservation of energy, indicates that the renormalized drift kinetic equation is spatially and temporally non-local, unlike the diffusion equation used by Hirshman and Molvig.