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ENERGY PRINCIPLE WITH GLOBAL INVARIANTS

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ABSTRACT

A variational principle is proposed for constructing equilibria with minimum energy in a toroidal plasma. The total energy is minimized subject to global invariants which act as constraints during relaxation of the plasma. These global integrals of motion are preserved exactly for all ideal motions and approximately for a wide class of resistive motions. We assume, specifically, that relaxation of the plasma is dominated by a tearing mode of single helicity. Equilibria with realistic current density and pressure profiles may be constructed in this theory, which is also used here to study current penetration in tokamaks. The second variation of the free energy functional is computed. It is shown that if the second variation of any equilibrium constructed in this theory is positive, the equilibrium satisfies the necessary and sufficient conditions for ideal stability.

I. Introduction

Variational principles have been known in magnetohydrodynamics (MHD) for some time. The formulation of a variational principle for the complete class of static equilibria of ideal toroidal plasmas is due to Kruskal and Kulsrud.[1] They characterized equilibria for ideal plasmas by a nondenumerable set of topological invariants derivable from the ideal hydromagnetic equations of motion. The linear stability of these equilibria was investigated by Bernstein, Frieman, Kruskal, and Kulsrud[2] in their celebrated paper on the Energy Principle. It is well known that the primary consequence of introducing a small amount of resistivity is relaxation in the topology of magnetic field lines, which makes the notion of topological invariants, exploited by Kruskal and Kulsrud, inapplicable. Taylor[3], however, has advanced the hypothesis that it is still possible to define a global invariant which is approximately conserved for small departures from perfect conductivity. Inspired by the conjecture due to Taylor, we have recently proposed a variational principle for constructing equilibria in toroidal plasmas in which relaxation is dominated by a tearing mode of single helicity[4]. In this paper, we discuss in detail the theoretical basis for our variational principle within the framework of Kruskal and Kulsrud. Equilibria with realistic current and pressure profiles may be constructed. We give criteria for stability of these special equilibria.

After this statement of purpose, we now give an overview of this

work. In Section II, we describe the variational principle due to Woltjer[5] which is chronologically the starting point for the developments leading to the present work. We explain in Section III how Taylor's conjecture[3] helps to overcome the basic limitation of Woltjer's work and attributes to it a special meaning for slightly nonideal plasmas. We then discuss briefly the successes of Taylor's simple theory, and the questions it naturally gives rise to. In the sections that follow, we attempt to answer partially some of these questions and formulate a variational principle with global invariants. In Section IV, we construct an infinite set of ideal invariants and select those that are approximately conserved by a rapidly growing tearing mode of single helicity. In Section V, we formulate the variational principle. In Section VII, for nonideal plasmas, we suggest a plausible variant of the thought experiment which Kruskal and Kulsrud originally gave for ideal plasmas[1]. On the basis of this thought experiment, which is supported by experimental observations, we derive an equilibrium equation. In Section VIII, we show that the generalized global invariants permitted by the quasi-ideal model of our earlier paper[4] are also conserved in the Double Tearing model and apply our theory to study current penetration in tokamaks. In Section XII, we show how to construct equilibria with finite pressure gradients in the framework of the present theory. In Section XIII, we compute the second variation of the free energy functional and show that equilibria for which the second variation is positive, satisfy the necessary and sufficient conditions for ideal stability. We conclude with a discussion of applications of this theory and work in progress.

II. Variational Principle of Woltjer

It has been suggested[6] that cosmic magnetic fields confining stellar plasmas of low density are force-free, which means that the magnetic field $\vec{B}$ obeys the equation

$$\nabla \times \vec{B} = \lambda \vec{B}, \tag{1}$$

where λ is any scalar function, arbitrary to the extent that

$$\vec{B} \cdot \nabla\lambda = 0, \tag{2}$$

a condition which follows by taking the divergence of Eq. (1). If the pressure of the plasma is neglected, Eq. (1) represents the most general solution of the magnetostatic equations.

There is no way to determine the function λ uniquely from the statements above. The case λ = constant, which lends itself to analytic solutions and is successful in explaining magnetic field patterns in the Crab Nebula[7], invites speculation as to why stellar plasmas should relax to force-free states independent of conditions at the moment of birth. Woltjer has argued[5] that an ideal plasma will relax into the state of lowest magnetic energy compatible with the invariant

$$K_o = \int_{V_o} d\tau \, \frac{\vec{A}\cdot\vec{B}}{2}, \tag{3}$$

where A is the vector potential defined by

$$\vec{B} = \nabla \times \vec{A}, \tag{4}$$

and V_o is the total volume of the system. By minimizing the total magnetic energy

$$W = \int_{V_o} d\tau \, \frac{B^2}{2} , \tag{5}$$

subject to the invariant K_o, Woltjer has derived the Euler-Lagrange equation

$$\nabla \times \vec{B} = \lambda_o \vec{B} , \tag{6}$$

where λ_o is a Lagrange multiplier.

Woltjer's variational principle is, however, subject to the following criticism. Ohm's law for an ideal plasma, given by

$$\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}) = 0, \tag{7}$$

implies that not only K_o, but an infinite number of constants of the motion, exist. As shown by Moffatt[8], and independently later by Taylor[3], Eq. (7) permits the stronger statement that

$$K = \frac{1}{2} \int_V d\tau \, \vec{A} \cdot \vec{B}, \tag{8}$$

is conserved for any arbitrary volume V bounded by a field line. If we adopt the heuristic interpretation due to Moffatt, K determines the degree of knottedness of the given field line. Given any initial state, the final state of the plasma is uniquely determined and characterized by the same values of the infinity of invariants K. By a suitable choice of the initial state, any solution of

the equilibrium Eq. (1) may be realized. However, this does not explain the important experimental observation that certain astrophysical plasmas (e.g., Crab Nebula[7]) or laboratory plasmas (e.g., Zeta[9]) should relax into the special force-free equilibria with constant $\lambda = \lambda_o$ independent of the initial preparation of the system. If the plasma is perfectly conducting, only an exceptional set of initial configurations may map into the special equilibria given by Eq. (6).

III. Taylor's Conjecture

For a small but finite departure from perfect conductivity, it is well known that topological constraints on field lines are relaxed and the plasma acquires accessibility to lower energy states which are inaccessible as long as the field lines remain frozen in the plasma. Field lines may now reconnect to give rise to topologically different configurations. K, which depends on the identity of individual field lines, is no longer a valid constant of the motion. Taylor[3] has advanced the remarkable hypothesis that for a slightly nonideal plasma the only one of the infinite set K that is approximately preserved is the global integral K_o given by Eq. (3) because it is the only one seemingly independent of any topological identification. The variational principle of Woltjer now assumes a special significance. The equilibrium Eq. (6), under Taylor's interpretation, now describes the unique final state a slightly nonideal plasma relaxes into, independent of the topological properties of the initial state.

Taylor's simple theory finds further support in experimental data from Zeta and HBTX. These triumphs have been discussed in detail by Taylor elsewhere[10,11]. We shall content ourselves here with a brief review. In cylindrical polar coordinates (r, θ, z), assuming that equilibrium quantities depend only on r, Eq. (6) has the solutions

$$B_r = 0, \quad B_\theta = B_o J_1(\lambda r), \quad B_z = B_o J_o(\lambda r), \tag{9}$$

where B_o is the amplitude of the magnetic field. The two constants

B_o and λ_o are determined by the values of K_o and the total toroidal flux which is conserved implicitly by virtue of the boundary conditions.

In experimental studies of self-reversal of the toroidal magnetic field in pinches, the parameter $F = B_z(a)/\langle B_z \rangle$, where a is the plasma radius, is measured as a function of time for increasing values of the pinching parameter $\theta = B_\theta(a)/\langle B_z \rangle$, which is a measure of the total toroidal current induced in the plasma. $\langle B_z \rangle$ represents the volume average of B_z. For the Bessel Function Model (BFM) given by Eq. (9), $\theta = (\lambda_o a/2)$. Reversal of B_z occurs when $\theta > 1.2$, which agrees satisfactorily with data from Zeta.

A simplifying feature of Eq. (6) is that it is a linear equation in $\vec{B}$. In simple geometries, if boundary conditions permit, particular solutions of the equation may be found by the method of separable variables and general solutions constructed by superposition. In the presence of helical symmetry, the general solutions for a periodic cylinder are known.[12] For particular values of K_o and the total toroidal flux, the axisymmetric solution (9) is only one of an infinity of solutions from which one has to choose the solution of lowest energy. Taylor has argued that the solution of lowest energy may be found by comparing the energies of the infinity of helically symmetric equilibria of which the BFM (9) is one. He has concluded that for $\theta \gtrsim 1.6$ the state of lowest energy is helically symmetric. He has also pointed out that the second transition at $\theta \simeq 1.6$ signifies the onset of current limitation for the toroidal pinch. It emerges naturally from his analysis that at this second transition, increasing the toroidal electric field at the wall increases only the

amplitude of the helical deformation without increasing the pinching parameter θ.

Experimental data from HBTX seems to corroborate Taylor's prediction of the second transition at $\theta \simeq 1.6$. But in the new generation of reversed field pinches like ZT - 40 at Los Alamos or ETA-BETA II at Padova, any evidence of this transition is yet to be found.[13] In these latter experiments, even self-reversal of the toroidal field seems to set in consistently at higher values of θ than is predicted by Taylor's theory.[13,14]

In spite of the successes of Taylor's theory, there are many aspects of the theory which are not satisfactory. There remains the important unanswered question of how and when a slightly nonideal plasma may preserve K_o approximately. Then, there is the unfortunate feature that for tokamak discharges, where the toroidal field is approximately constant across the plasma, Taylor's theory predicts flat current profiles which occur presumably only in the event of major disruptions when a large fraction of the magnetic energy of the plasma is suddenly released. But for a well-confined tokamak plasma, current density profiles are usually peaked at the center and vanish at the edge. Even in reversed field pinches, the toroidal current is small near the wall which in general violates the equilibrium equation $\vec{J} = \lambda_o \vec{B}$. In our earlier work[4], we have interpreted the preceding observations to imply that the replacement of the infinity of invariants for an ideal plasma by a single one was too drastic a step.

The important problem lies in determining what the additional constraints are. The infinite set K which is ideally preserved for every subdomain $V \subset V_o$ does not give us much of a clue. What we

really need is a set of global invariants which are integrals over the entire volume V_o and do not depend on the details of magnetic reconnection in any subdomain of V_o.

We shall see that it is possible to construct such a set for a plasma contained in a toroid within the framework of Kruskal and Kulsrud.

IV. Extended Class of Invariants for an Ideal Toroidal Plasma

It has been demonstrated by Kruskal and Kulsrud[1] that general solutions of the magnetostatic equations for plasmas bounded spatially have the property that the magnetic field lines lie on nested surfaces which are topologically toroids. Greene and Johnson[15] have shown that under these conditions the magnetic field $\vec{B}$ may be represented as

$$\vec{B} = \nabla\zeta \times \nabla\Psi + \nabla\Phi(\Psi) \times \nabla\theta, \tag{10}$$

in a magnetic coordinate system (Ψ, θ, ζ). For an arbitrary toroidal angle ζ which increases by 2π after each toroidal circuit, it is always possible to determine a θ which increases by 2π after each poloidal circuit. Ψ and Φ are respectively the poloidal and the toroidal flux functions. We note in Eq. (10) that Φ is specified as a function of Ψ and the functional relationship remains invariant under ideal perturbations which, as explained by Kruskal and Kulsrud, reflects the topological correspondence between the initial and the final states.

From Eqs. (4) and (10), the natural choice for the vector potential $\vec{A}$ is seen to be

$$\vec{A} = \Phi(\Psi)\nabla\theta - \Psi\nabla\zeta . \tag{11}$$

In what follows, we shall always assume that on the magnetic axis

$$\Phi = 0 \ , \tag{12}$$

which makes $2\pi\Phi(\Psi)$ the toroidal flux contained within the surface labeled by Ψ and assures the regularity of $\vec{A}$ on the axis despite the singularity in $\nabla\theta$.

For ideal motions of the plasmas, Ψ and $\Phi(\Psi)$ are invariants following a fluid element. We may, therefore, define the class $\mathcal{I}$ of ideal variations as those for which

$$\delta\Psi = -\vec{\xi}\cdot\nabla\Psi, \tag{13}$$

$$\delta\Phi = q(\Psi)\delta\Psi, \tag{14}$$

where $\vec{\xi}$ is a virtual displacement. We may now prove the following proposition:

For any volume V bounded by field lines (or, equivalently, between any two neighboring magnetic surfaces) the integral

$$K[\omega] = \frac{1}{2} \int_V d\tau \ \omega(\Psi,\Phi)\vec{A}\cdot\vec{B} \ , \tag{15}$$

for an arbitrary function $\omega(\Psi,\Phi)$, is conserved under the class $\mathcal{I}$ of ideal variations.

Proof: It follows from equations (10) and (11) that

$$\vec{A}\cdot\vec{B} = (\Phi\nabla\Psi - \Psi\nabla\Phi) \cdot \nabla\theta \times \nabla\zeta \ . \tag{16}$$

After some straightforward algebra, we find

$$\delta K[\omega] = \int_V d\tau \left[\left(\frac{\Psi}{2} \frac{\partial \omega}{\partial \Psi} + \frac{\Phi}{2} \frac{\partial \omega}{\partial \Phi} + \bar{\omega} \right) \nabla\Psi \cdot \nabla\theta \times \nabla\zeta (\delta\Phi - q\delta\Psi) \right.$$

$$+ \int_S ds \frac{\omega}{2} \hat{n} \cdot [(\Phi\delta\Psi - \Psi\delta\Phi)\nabla\theta \times \nabla\zeta - \delta\theta(\Phi\nabla\Psi - \Psi\nabla\Phi) \times \nabla\zeta], \quad (17)$$

where S is the surface bounding the volume V. The boundary condition $\hat{n} \cdot \vec{B} = 0$ on S implies

$$\hat{n} \times \nabla\Psi = \hat{n} \times \nabla\Phi = 0. \quad (18)$$

Since Ψ is a constant of the motion and uniquely identifies flux surfaces, we must have

$$\delta\Psi = \delta\Phi = 0 \ , \quad (19)$$

on S. Under variations I, subject to boundary conditions (18) and (19), we have shown thus

$$\delta K[\omega] = 0.$$

Q.E.D.

Since the space of $\omega(\Psi,\Phi)$ may be spanned by the space of step functions, the infinite set of K of Moffatt, defined by Eq. (8), is seen to be equivalent to the set $K[\omega]$.

It is easily seen that $K[\omega]$ is gauge invariant. For, under the gauge transformation $\vec{A} \to \vec{A} + \nabla\alpha$

$$K[\omega] \to K[\bar{\omega}] + \frac{1}{2} \int_V d\tau \ \omega(\Psi,\Phi) \ \nabla\alpha \cdot \vec{B}$$

$$= K[\omega] + \frac{1}{2} \int_V d\tau \, [\nabla\cdot(\alpha\omega\vec{B}) - \alpha\vec{B}\cdot\vec{\nabla}\omega]$$

$$= K[\omega],$$

using the single-valuedness of α, the boundary condition $\hat{n}\cdot\vec{B} = 0$ and the fact that Ψ and Φ are surface quantities.

We have remarked earlier that we do not yet have a detailed justification for K_o-conservation for a slightly nonideal plasma. In our earlier paper, we sought to remedy this situation somewhat by demonstrating that the quasi-ideal model of magnetic reconnection, invented by Kadomtsev[16] and independently by Monticello for the m=1, n=1 tearing mode in tokamaks, permits a generalized global invariant

$$G[\omega] = \frac{1}{2} \int_{V_o} d\tau \, \omega(\chi) \, \vec{A}\cdot\vec{B}, \tag{20}$$

where $\omega(\chi)$ is an arbitrary function of the helical flux function

$$\chi = q_s\Psi - \Phi, \tag{21}$$

preserved approximately by a tearing mode for which $q_s = m/n$. It is obvious that $G[\omega(\chi)]$ is a subset of the ideal set $K[\omega(\Psi,\Phi)]$ defined by Eq. (15).

V. Variational Principle

To keep things general, we minimize the magnetic energy W, defined by Eq. (5), subject to the infinity of global constraints

$$K[\omega_\alpha] = \frac{1}{2} \int_{V_o} d\tau \ \omega_\alpha(\Psi,\Phi)\vec{A}\cdot\vec{B} \ ,$$

where $\omega_\alpha(\Psi,\Phi)$ comprises a complete set of linearly independent basis functions. It is convenient to define the free energy functional

$$F = W - \sum_\alpha \lambda_\alpha K[\omega_\alpha], \tag{22}$$

where λ_α constitutes an infinite sequence of Lagrange multipliers, one corresponding to each constraint imposed on the system. A necessary and sufficient condition for extrema of W under variations $G \supset I$ is that the first variation of F vanish, i.e.,

$$\delta F = \delta W - \sum_\alpha \lambda_\alpha \delta K[\omega_\alpha] = 0 \ , \tag{23}$$

for unconstrained variations.

The first variation of W is

$$\delta W = \int_{V_o} d\tau \ [-\delta\Psi\nabla\zeta\cdot\vec{J} + \delta\Phi\nabla\theta\cdot\vec{J} - \delta\theta\nabla\Phi\cdot\vec{J}]$$

$$+ \int_{S_p} ds\, \hat{n}\cdot [\delta\Psi\, \vec{B}\times\vec{\nabla}\zeta - \delta\Phi\, \vec{B}\times\vec{\nabla}\theta + \delta\theta\, \vec{B}\times\vec{\nabla}\Phi]. \tag{24}$$

On S_p, the surface of the plasma in contact with the conducting wall, boundary conditions (18) and (19) hold; hence, Eq. (24) becomes

$$\delta W = \int_{V_o} d\tau\, [\delta\Phi\vec{J}\cdot\nabla\theta - \delta\Psi\vec{J}\cdot\nabla\zeta - \delta\theta\vec{J}\cdot\nabla\Phi]\ . \tag{25}$$

For the first variation of $K[\omega_\alpha]$ which is now an integral over the entire plasma volume V_o, we find similarly

$$\delta K[\omega_\alpha] = \int_{V_o} d\tau \left[\delta\Phi \left(\frac{\Psi}{2}\frac{\partial\omega_\alpha}{\partial\Psi} + \frac{\Phi}{2}\frac{\partial\omega_\alpha}{\partial\Phi} + \omega_\alpha\right) \nabla\Psi\cdot\nabla\theta\times\nabla\zeta\right.$$

$$\left. -\delta\Psi\ q\left(\frac{\Psi}{2}\frac{\partial\omega_\alpha}{\partial\Psi} + \frac{\Phi}{2}\frac{\partial\omega_\alpha}{\partial\Phi} + \omega_\alpha\right)\ \nabla\Psi\cdot\nabla\theta\times\nabla\zeta\right]. \tag{26}$$

Since Eq. (23) must hold for arbitrary variations $\delta\Psi$, $\delta\Phi$, $\delta\theta$, terms multiplying them in the integrand of F must vanish independently. We are thus left with the following Euler-Lagrange equations for extrema of F:

$$\vec{J}\cdot\nabla\zeta - \frac{q}{J}\sum_\alpha \lambda_\alpha \left(\frac{\Psi}{2}\frac{\partial\omega_\alpha}{\partial\Psi} + \frac{\Phi}{2}\frac{\partial\omega_\alpha}{\partial\Phi} + \omega_\alpha\right) = 0, \tag{27}$$

$$\vec{J}\cdot\nabla\theta - \frac{1}{J}\sum_{\alpha}\lambda_{\alpha}\left(\frac{\Psi}{2}\frac{\partial\omega_{\alpha}}{\partial\Psi} + \frac{\Phi}{2}\frac{\partial\omega_{\alpha}}{\partial\Phi} + \omega_{\alpha}\right) = 0 \ , \tag{28}$$

$$\vec{J}\cdot\nabla\Phi = 0 \ , \tag{29}$$

where $J \equiv (\nabla\Psi\cdot\nabla\theta \times \nabla\zeta)^{-1}$ is the Jacobian. We may combine Eqs. (27), (28) and (29) into a more familiar form by casting $\vec{J}$ in the contravariant representation

$$\vec{J} = (J\vec{J}\cdot\nabla\zeta)\nabla\Psi\times\nabla\theta + (J\vec{J}\cdot\nabla\Psi)\nabla\theta\times\nabla\zeta + (J\vec{J}\cdot\nabla\theta)\nabla\zeta\times\nabla\Psi \ , \tag{30}$$

whence

$$\vec{J} = \sum_{\alpha}\lambda_{\alpha}\left(\frac{\Psi}{2}\frac{\partial\omega_{\alpha}}{\partial\Psi} + \frac{\Phi}{2}\frac{\partial\omega_{\alpha}}{\partial\Phi} + \omega_{\alpha}\right)\vec{B} \ . \tag{31}$$

In particular, for the generalized global invariant $G[\omega_{\alpha}(\chi)]$, Eq. (31) becomes

$$\vec{J} = \sum_{\alpha}\lambda_{\alpha}\left[\frac{\omega_{\alpha}'(\chi)}{2}\chi + \omega_{\alpha}(\chi)\right]\vec{B}. \tag{32}$$

The physically acceptable truncations of the infinite set of basis functions $\omega_{\alpha}(\chi)$ must satisfy two additional group properties. Since the solution space of the problem of extremizing W should not depend on the zero of χ, the truncated set of weight functions must span a function space which is invariant under addition of a constant to χ. That is,

$$\omega_{\alpha}(\chi + \chi_{o}) = \sum_{\beta=1}^{N} f_{\alpha\beta}(\chi_{o})\,\omega_{\beta}(\chi), \text{ for all } N, \tag{33}$$

where χ_o is a constant and $f_{\alpha\beta}(\chi_o)$, an arbitrary function of χ_o. Since the functions $\omega_\alpha(\chi)$ are assumed to be linearly independent

$$f_{\alpha\beta} \to \delta_{\alpha\beta} \quad \text{as} \quad \chi_o \to 0 \ . \tag{34}$$

Differentiating Eq. (33) with respect to χ_o, we get

$$\omega_\alpha'(\chi) = \sum_{\beta=1}^{N} f_{\alpha\beta}'(0)\ \omega_\beta(\chi). \tag{35}$$

Furthermore, solutions to the problem of extremizing W should be independent of the normalization of $\vec{B}$. Specifically, the problem should be unaltered under the transformation

$$\vec{B} \to C\vec{B}, \tag{36}$$

where C is a constant. Under conditions (34) through (36), the most natural choice for the class of admissible functions is

$$\omega_\alpha(\chi) = \chi^\alpha, \tag{37}$$

where α is a positive integer. With the choice (37), the equilibrium equation (31) becomes

$$\vec{J} = \sum_\alpha \frac{\lambda_\alpha}{2}\ (\alpha+2)\chi^\alpha \vec{B} \ . \tag{38}$$

We note that the global invariant K_o corresponds to $\alpha = 0$ of the hierarchy $K_\alpha = \int_{V_o} d\tau\ \chi^\alpha \frac{\vec{A}\cdot\vec{B}}{2}$. It may be plausibly argued

that since K_o is the only invariant that is independent of q_s, it is the sole member of the intersection of all sets, each of which contains an infinite number of constants of the motion for each assumed helicity. In the presence of at least two modes strongly coupled to one another, it is likely that K_o is the only surviving invariant. The final equilibrium state is then given by Eq. (6) which predicts flat current profiles in tokamak ordering. This is consistent with experimental observations (e.g., those performed on the Pulsator tokamak in Garching[17]), and results of numerical simulation[18] which demonstrate that nonlinear coupling of modes of different helicities result in major disruptions and flattening of the current profile.

VI. Comments on the Work of Kadomtsev

Kadomtsev[19] has also attempted to interpret on the basis of the quasi-ideal model discussed in our earlier paper[4], Taylor's conjecture on the global invariance of K_o. However, he concludes that K_o is not stricly an invariant in his model. In our last paper, we have traced the reason for this conclusion to the fact that he assumes $\Psi = 0$ on the magnetic axis, which is different from our choice $\Psi = 0$ on the wall. Our choice is motivated by the fact that $\oint \vec{A} \cdot d\vec{\ell}$ is a constant in time for any contour drawn in the poloidal or toroidal direction on a perfectly conducting wall. Since $\Psi(a)$, the value of Ψ on the wall, has to be a constant for all times by the above argument, and since $[\Psi(a) - \Psi(0)]$ is not conserved during reconnection with a shift of the axis, we believe it is inconsistent to assume $\Psi = 0$ on the axis.

Kadomtsev suggests correctly, however, that since in his model the helical flux χ and the differential toroidal flux $d\Phi$ is conserved, so is $\int d\Phi\ \chi$. He demonstrates then that though $\int d\Phi\ \chi$ is different from K_o, the problem of minimizing W subject to $\int d\Phi\ \chi$ still leads to Eq. (6) as the Euler-Lagrange equation.

We may carry Kadomtsev's argument even a step further. If χ and $d\Phi$ are conserved, so must be $\int d\Phi\ \chi^{\alpha}$. If we now seek minima of W subject to this infinite set of constraints, we obtain the Euler-Lagrange equation

$$\vec{J} = \sum_{\alpha} q_s \lambda_{\alpha}^{a} \frac{\alpha \chi^{\alpha-1}}{2} \vec{B} , \tag{39}$$

where λ_{α}^{a} is the Lagrange multiplier corresponding to the constraint $\int d\Phi\ \chi^{\alpha}$= constant. By a trivial redefinition of constants, Eq. (39) is seen to be the same as Eq. (38).

VII. Variant of the Thought Experiment of Kruskal and Kulsrud

We suggest now the following variant of the thought experiment of Kruskal and Kulsrud.[1] We imagine a slightly nonideal plasma contained in a toroidal vessel with perfectly conducting walls. The plasma is turbulent with tearing modes of all m and n. The existence of fine scale tearing destroys all invariants to some extent, except K_o. On a short time scale, however, the "moments" of $\vec{A}\cdot\vec{B}$ with respect to χ for the dominant mode, defined by K_α, will be better conserved than the "moments" corresponding to all other modes. Of all non-zero moments for the dominant mode, (the zeroth moment being K_o), we conjecture that the first moment $K_1 = \frac{1}{2}\int_{V_o} d\tau\ \chi\ \vec{A}\cdot\vec{B}$ will be the best conserved.

We seek, therefore, minima of W subject to the global invariants K_o and K_1. From Eq. (38), we find the equilibrium equation

$$\vec{J} = \left(\lambda_o + \frac{3\lambda_1}{2}\chi\right)\vec{B}. \tag{40}$$

The solutions of Eq. (40) are determined if we know the value of the normalization constant B_o for the magnetic field $\vec{B}$ and the two Lagrange multipliers λ_o, λ_1. These constants are to be determined from the three global invariants K_o, K_1 and $2\pi\Phi_p$, where Φ_p is the value of Φ on the wall.

For a physical system evolving in time subject to holonomic constraints, the Lagrange multipliers may be looked upon as "generalized forces" acting on the system due to the constraints. These forces are, of course, controlled by the experimentalist. One such

experimental condition is the vanishing of the current density at the edge. This condition is realized if we choose to examine those equilibria given by Eq. (40) for which

$$\frac{3\lambda_1}{2} = \frac{\lambda_o}{\Phi_p} \,. \tag{41}$$

Even though Eq. (40) does permit solutions for which the current density does not vanish at the edge, we do not consider them here because we are primarily interested in considering profiles which may be sustained on the transport time scale. With the choice (41) the equilibrium equation reduces to

$$\nabla\times\vec{B} = \lambda_o \vec{B} \left[1 + \frac{q_s \Psi - \Phi}{\Phi_p} \right] . \tag{42}$$

In our earlier paper[4], we have solved Equation (42) numerically in cylindrical geometry. A detailed discussion of the numerical results and the asymptotic solutions of Equation (42) will be published in a forthcoming paper[20] which will deal with further applications of this theory.

VIII. On Current Penetration in Tokamaks

It has been known for some time that current penetration in the initial phase of a tokamak discharge proceeds anomalously fast, i.e., at a rate much faster than may be expected from the measured value of the electrical conductivity. It has been suggested by Stix[21] that this discrepancy may be accounted for by considering the redistribution of plasma current caused by the nonlinear evolution of the double tearing mode which has characteristics similar to the m=1, n=1 mode considered earlier. The following scenario emerges from the work of Stix and others.[22] In the initial stage of a tokamak discharge, after breakdown and ionization of the working gas, the plasma current increases, giving rise to a hollow current density profile which is known to be unstable to double tearing modes with m greater than the minimum value of q in the plasma (n=1). Since the mode for which m is the least is the fastest and the most dominant, as the current rises and q decreases, the m-number of the dominant mode decreases. Thus, the plasma experiences a sequence of double tearing instabilities during current rise. At each stage of the sequence, when the rational surfaces with the same value of q fulfill some criteria of proximity (see, for example, the numerical results of Carreras, Hicks and Waddell[23]), magnetic islands overlap after growing to large amplitudes, reconnect and cause rapid current diffusion.

Experimental observations on T-3[24] and T-4[25](I.V. Kurchatov Institute of Atomic Energy) and more recently on Alcator A

(MIT)[26] lend strong support to the above scenario.

In all these experiments, poloidal magnetic field fluctuations with m decreasing at least from 5 to 2 during current-rise have been identified. In particular, the Alcator A results[26] seem to indicate that these fluctuations are due to internal modes for which our model with a fixed plasma boundary suffices.

We suggest that during current rise, a tokamak plasma passes through a sequence of minimum-energy states. At every stage, relaxation is dominated by the fastest growing double tearing mode. The sequence proceeds from a high value of m, say m=5, to m=1 when the plasma settles down to the "internal sawtooth" regime and the current attains its maximum value. Thereafter, gross features of the discharge are determined by transport.

It may be shown easily that the quasi-ideal model of our earlier paper[4], which may be generalized to include multiple rational surfaces in the initial state, also permits the infinite set of global invariants G defined by Eq. (20). Fig. 1 shows the helical flux function $\chi(\Phi)$ before and after reconnection. We assume that in the initial state $\chi'(\Phi)$ vanishes at two points $\Phi = \Phi_{S1}$ and Φ_{S2} at which $\chi = \chi_{S1}$ and χ_{S2} respectively. There are two cases to be considered; $\chi_0 > \chi_{S2}$ [Fig. 1(a)] and $\chi_0 \leq \chi_{S2}$ [Fig. 1(b)]. In the initial state, $\Phi(\chi)$ has three branches:

$$\Phi_1 : [\chi_0, \chi_{S1}] \rightarrow [0, \Phi_{S1}] ,$$

$$\Phi_2 : [\chi_{S1}, \chi_{S2}] \rightarrow [\Phi_{S1}, \Phi_{S2}] ,$$

$$\Phi_3 : [\chi_{S2}, \chi_p] \rightarrow [\Phi_{S2}, \Phi_p] .$$

When any two surfaces of the same χ merge, they share a common separatrix. Thus, helical flux χ is conserved. Similarly, toroidal flux is also conserved. In the final state, the toroidal flux $\Phi_\infty(\chi)$ is monotonic and given in terms of the branches of the initial state by the following sets of equations:

Case(a), $\chi_0 > \chi_{S2}$:

$$= \Phi_1(\chi) \qquad \chi \in [\chi_0, \chi_{S2}]$$

$$\Phi_\infty(\chi) = \Phi_3(\chi) + \Phi_1(\chi) - \Phi_2(\chi) \qquad \chi \in [\chi_{S2}, \chi_{S1}]$$

$$= \Phi_3(\chi) \qquad \chi \in [\chi_{S1}, \chi_p] .$$

Case(b), $\chi_0 < \chi_{S2}$:

$$= \Phi_3(\chi) - \Phi_2(\chi) \qquad \chi \in [\chi_{S2}, \chi_0]$$

$$\Phi_\infty(\chi) = \Phi_3(\chi) + \Phi_1(\chi) - \Phi_2(\chi) \qquad \chi \in [\chi_0, \chi_{S1}]$$

$$= \Phi_3(\chi) \qquad \chi \in [\chi_{S1}, \chi_p] .$$

$[\Phi_{R1}, \Phi_{R2}]$ denotes the range over which reconnection takes

place. We may now use Eq. (20) to write

$$\frac{q_s}{(2\pi)^2} G_0 = \int_0^{\Phi_{R1}} d\Phi \frac{d\chi}{d\Phi} \omega[\Phi - \frac{1}{2}(\Phi\chi)'] + \int_{\Phi_{R1}}^{\Phi_{S1}} d\Phi \frac{d\chi}{d\Phi} \omega[\Phi - \frac{1}{2}(\Phi\chi)']$$

$$+ \int_{\Phi_{S1}}^{\Phi_{S2}} d\Phi \frac{d\chi}{d\Phi} \omega[\Phi - \frac{1}{2}(\Phi\chi)'] + \int_{\Phi_{S2}}^{\Phi_{R2}} \frac{d\chi}{d\Phi} \omega[\Phi - \frac{1}{2}(\Phi\chi)']$$

$$+ \int_{\Phi_{R2}}^{\Phi_P} d\Phi \frac{d\chi}{d\Phi} \omega[\Phi - \frac{1}{2}(\Phi\chi)'],$$

which, for the case (a), is

$$= \int_{\chi_0}^{\chi_{S2}} d\mu [\Phi_1 - \frac{1}{2}(\Phi_1\chi)'] + \int_{\chi_{S2}}^{\chi_{S1}} d\mu [\Phi_1 - \frac{1}{2}(\Phi_1\chi)']$$

$$+ \int_{\chi_{S1}}^{\chi_{S2}} d\mu [\Phi_2 - \frac{1}{2}(\Phi_2\chi)'] + \int_{\chi_{S2}}^{\chi_{S1}} d\mu [\Phi_3 - \frac{1}{2}(\Phi_3\chi)']$$

$$+ \int_{\chi_{S1}}^{\chi_P} d\mu [\Phi_3 - \frac{1}{2}(\Phi_3\chi)'],$$

where $d\mu[\chi] \equiv d\chi\ \omega(\chi)$ is an infinitesimal invariant measure convected by the plasma. Thus

$$\frac{q_s}{(2\pi)^2} G_0 = \int_{\chi_0}^{\chi_{S2}} d\mu [\Phi_1 - \frac{1}{2}(\Phi_1\chi)'] + \int_{\chi_{S2}}^{\chi_{S1}} d\mu [(\Phi_3 + \Phi_1 - \Phi_2) -$$

$$\frac{1}{2}\{(\Phi_3 + \Phi_1 - \Phi_2)\ \chi\}'] + \int_{\chi_{S1}}^{\chi_p} d\mu\ [\Phi_3 - \frac{1}{2}\ (\Phi_3\chi)']$$

$$= \int_{\chi_0}^{\chi_p} d\mu\ [\Phi_\infty - \frac{1}{2}\ (\Phi_\infty\chi)'] = \frac{q_s}{(2\pi)^2}\ G_\infty\ ,$$

and, for case (b), is similarly

$$= \int_{\chi_0}^{\chi_{S1}} d\mu\ [\Phi_1 - \frac{1}{2}\ (\Phi_1\chi)'] + \int_{\chi_{S1}}^{\chi_{S2}} d\mu\ [\Phi_2 - \frac{1}{2}\ (\Phi_2\chi)']$$

$$+ \int_{\chi_{S2}}^{\chi_{S1}} d\mu\ [\Phi_3 - \frac{1}{2}\ (\Phi_3\chi)'] + \int_{\chi_{S1}}^{\chi_p} d\mu\ [\Phi_3 - \frac{1}{2}\ (\Phi_3\chi)']$$

$$= \int_{\chi_{S2}}^{\chi_0} d\mu[(\Phi_3 - \Phi_2) - \frac{1}{2}\{(\Phi_3 - \Phi_2)\ \chi\}']$$

$$+ \int_{\chi_0}^{\chi_{S1}} d\mu[(\Phi_3 + \Phi_1 - \Phi_2) - \frac{1}{2}\{(\Phi_3 + \Phi_1 - \Phi_2)\ \chi\}']$$

$$+ \int_{\chi_{S1}}^{\chi_p} d\mu\ [\Phi_3 - \frac{1}{2}(\Phi_3\chi)']$$

$$= \int_{\chi_{S2}}^{\chi_p} d\mu\ [\Phi_\infty - \frac{1}{2}(\Phi_\infty\chi)'] = \frac{q_s}{(2\pi)^2}\ G_\infty\ ,$$

which is what we set out to prove.

IX. Finite Pressure in Ideal MHD

In the presence of finite pressure, the total energy W must include the contribution due to the internal energy of the plasma. From the equations of ideal MHD, the energy integral may be shown to be

$$W = \int_{V_o} d\tau \left[\frac{B^2}{2} + \frac{p}{\gamma - 1} \right] , \tag{43}$$

where p is the pressure and γ the specific heat of the plasma. The ideal MHD equations also conserve locally entropy and mass density. By integrating these equations in terms of the virtual displacement $\vec{\xi}(\vec{r},t)$, we obtain the Eulerian variations for the density and the pressure of the plasma, respectively:

$$\delta\rho = - \nabla\cdot(\rho\vec{\xi}) , \tag{44}$$

$$\delta p = - \gamma p \, \nabla\cdot\vec{\xi} - \vec{\xi}\cdot\nabla p . \tag{45}$$

Equations (44) and (45), combined with Eqs. (13) and (14), now comprise the class I of ideal variations.

The first variation of W is given by

$$\delta W = \int_{V_o} d\tau \, [\delta\Phi\vec{J}\cdot\nabla\theta - \delta\Psi\vec{J}\cdot\nabla\zeta - \delta\theta\vec{J}\cdot\nabla\Phi + \frac{\delta p}{\gamma - 1}]\cdot \tag{46}$$

Under variations I

$$\delta p = -\gamma \nabla \cdot (p\vec{\xi}) - (\gamma - 1)\, p'(\Psi)\, \delta\Psi \quad . \tag{47}$$

Since the plasma is bounded by a rigid wall

$$\hat{n} \cdot \vec{\xi} = o \tag{48}$$

on S_p. From (47) and (48), then

$$\delta W = \int_{V_o} d\tau [\delta\Psi \{\vec{J} \cdot (q\nabla\theta - \nabla\zeta) - p'(\Psi)\} - \delta\theta \vec{J} \cdot \nabla\Phi] \tag{49}$$

under I. For stationary equilibria δW should vanish for all admissible variations, which requires that

$$\vec{J} \cdot \nabla\Phi = o \ , \tag{50}$$

$$\vec{J} \cdot (q\nabla\theta - \nabla\zeta) = p'(\Psi) \quad . \tag{51}$$

Since

$$\vec{B} = \nabla\Psi \times (q\nabla\theta - \nabla\zeta) \quad , \tag{52}$$

we conclude, from Eqs. (50) and (51), that

$$\vec{J} = \lambda\vec{B} + p'(\Psi)\, \frac{B \times \nabla\Psi}{B^2} \quad , \tag{53}$$

where λ is an arbitrary scalar function. Therefore

$$\vec{J} \times \vec{B} = \vec{\nabla} p, \tag{54}$$

which proves that configurations for which W is stationary under variations of class I are general solutions of the magnetostatic equations. This is a somewhat simpler version of the original proof due to Kruskal and Kulsrud.[1]

X. Comments on the Work of Jukes

It has been suggested by Jukes[27] that the Energy Principle of ideal MHD may be generalized by replacing the assumption of perfect conductivity given by Eq. (7) with the equation

$$\delta K_o \equiv \int_{V_o} d\tau \nabla\Psi\cdot\nabla\theta\chi\nabla\zeta(\delta\Phi - q\delta\Psi) = o, \tag{55}$$

and retaining Eqs. (44) and (45) as constraints on variations of density and pressure. This procedure defines a class of variations G_s . Physically, Φ-surfaces are no longer tied to Ψ-surfaces everywhere, but density and entropy are still locally conserved between any two neighboring Ψ-surfaces. In a plasma turbulent with tearing modes, particles and entropy should be transported across magnetic surfaces. It is therefore unlikely that such stringent local constraints on density and entropy should hold. However, the problem of seeking stationary values of W subject to constraints (55), (44) and (45) is mathematically well-posed and instructive. The first variation of W is now

$$\delta W = \int_{V_o} d\tau \ [\delta\Phi\vec{J}\cdot\nabla\theta - \delta\Psi\{\vec{J}\cdot\nabla\zeta + p'(\Psi)\} - \delta\theta\vec{J}\cdot\nabla\Phi]. \tag{56}$$

From

$$\delta W - \lambda_o \delta K_o = o, \tag{57}$$

we obtain the Euler-Lagrange equations

$$\vec{J} \cdot \nabla\Phi = o, \tag{58}$$

$$\vec{J} \cdot \nabla\zeta = -p(\Psi) + \lambda_o \frac{q}{J} \tag{59}$$

$$\vec{J} \cdot \nabla\theta = \frac{\lambda_o}{J} . \tag{60}$$

Eqs. (58), (59) and (60), when substituted into the representation (30), combine to give

$$\vec{J} = \lambda_o \vec{B} - p'(\Psi)\nabla\Psi \times \nabla\theta. \tag{61}$$

It is easily seen that

$$\vec{J} \times \vec{B} = \nabla p. \tag{62}$$

In other words, Eq. (61) describes equilibria which are a subset of the complete class of Kruskal and Kulsrud. In the equilibrium equation (53), the function λ is, in particular

$$\lambda \equiv \frac{\vec{J}\cdot\vec{B}}{B^2}$$

$$= \lambda_o - \frac{p'}{B^2} \left[\{q|\nabla\Psi \times \nabla\theta|^2 - |\nabla\Psi|^2(\nabla\theta\cdot\nabla\zeta) + (\nabla\Psi\cdot\nabla\theta)(\nabla\Psi\cdot\nabla\zeta) \} \right] \tag{63}$$

However, Jukes derives the equilibrium equation

$$\vec{J} = \lambda_o \vec{B} + \frac{\vec{B}\times\vec{\nabla}p}{B^2} \tag{64}$$

$$p = \rho \sum_{\alpha} T_{\alpha} v_{\alpha} \ , \tag{86}$$

$$\rho \sum_{\alpha} \mu_{\alpha} u_{\alpha} = \frac{\gamma \rho}{\gamma - 1} - s \sum_{\alpha} T_{\alpha} v_{\alpha} \ . \tag{87}$$

In deriving Eqs. (83) and (84), we have used the relationship

$$\rho \sum_{\alpha} \mu_{\alpha} u_{\alpha f} + S \sum_{\alpha} T_{\alpha} v_{f} = p_{f} \ , \tag{88}$$

where $(\ )_f$ means partial differentiation either with respect to Ψ or Φ. Eq. (88) may be obtained by differentiating Eqs. (86) and (87). If we substitute Eqs. (83) - (85) in the contravariant representation (30) for $\vec{J}$, we get

$$\vec{J} = \sum_{\alpha} \lambda_{\alpha} \left[\frac{\Psi}{2} \frac{\partial w_{\alpha}}{\partial \Psi} + \frac{\Phi}{2} \frac{\partial w_{\alpha}}{\partial \Phi} + w_{\alpha} \right] \vec{B} - J \left[\frac{\partial p}{\partial \Psi} \nabla \Psi \times \nabla \theta - \frac{\partial p}{\partial \Phi} \nabla \zeta \times \nabla \Psi \right] . \tag{89}$$

It is easily seen that

$$\vec{J} \times \vec{B} = \vec{\nabla} p, \tag{90}$$

which means that Eq. (89) represents magnetostatic equilibria.

In the quasi-ideal model described in our earlier paper,

$$M\ [u_{\alpha}(\chi)] = \int_{V_o} d\tau\ u_{\alpha}(\chi)\ \rho \ , \tag{91}$$

$$S\ [v_{\alpha}(\chi)] = \int_{V_o} d\tau\ v_{\alpha}(\chi)\ s \ , \tag{92}$$

may be similarly argued to be approximately conserved on the time scale of the instability. We again make the choice

$$u_\alpha(\chi) = v_\alpha(\chi) \equiv \chi^\alpha \tag{93}$$

We now show that the basic conclusion of Taylor's theory for finite beta states is unaltered if we seek minima of W, subject not only to the constraint K_o, but also to the constraints M_o (the total mass) and S_o (the total entropy). From Eqs. (86), (87) and (89), the Euler-Lagrange equations are seen to be

$$p = \rho T_o \ , \tag{94}$$

$$\frac{T_o}{\gamma-1}\ \gamma - \ell n\ (p/\rho^\gamma) \quad = \rho\mu_o \ , \tag{95}$$

$$\vec{J} = \lambda_o \vec{B} \ . \tag{96}$$

Equations (94) and (95) give

$$\vec{\nabla} p = 0 \tag{97}$$

because μ_o and T_o are constants. Thus, in spite of the additional constraints M_o and S_o, the relaxed states are force-free.

The important question, as usual, lies in determining which members of the infinite hierarchies M_α and S_α are better

preserved in our variant of the thought experiment of Kruskal and Kulsrud. Since the energy of the system increases monotonically with the number of constraints, and since a resistive plasma seeks states of low energy it would be desirable to impose as few constraints as possible and yet reproduce gross features of a toroidal discharge. The optimum choice of constraints which give rise to equilibrium pressure profiles with nonvanishing spatial gradients, will be the subject of a future publication.

XIII. Second Variations of the Free Energy

The energy function W, defined by equation (82) is stationary if and only if $\delta F = 0$ for unconstrained variations. It may be shown[28] that a sufficient condition for local minima of W under $\mathcal{G}$ is that the second variation $\delta^2 F > 0$ for unconstrained variations of the independent variables. Since equilibria that are minima of free energy are presumably stable, we now compute $\delta^2 F$, which may be written formally as

$$\delta^2 F \equiv \delta^2 W - \sum_{\alpha} \lambda_\alpha \, \delta^2 K[w_\alpha] - \sum_{\alpha} \mu_\alpha \, \delta^2 M[u_\alpha] - \sum_{\alpha} T_\alpha \, \delta^2 S[v_\alpha], \tag{98}$$

where α is a positive integer. In keeping with our earlier computation of δF, we vary Ψ, Φ, θ, p, and n independently of one another, defining the generalized class of variations $\mathcal{G}$. We keep the toroidal angle ζ fixed as before without any loss of generality. We now find

$$\delta^2 W = \int_{V_o} d\tau \left[\left| \delta \vec{B} \right|^2 + \vec{B} \cdot \delta^2 \vec{B} \right], \tag{99}$$

where

$$\delta \vec{B} = \nabla\zeta \times \nabla\delta\Psi + \nabla\delta\Phi \times \nabla\theta + \nabla\Phi \times \nabla\delta\theta, \tag{100}$$

and

$$\delta^2 \vec{B} = 2\nabla\delta\Phi \times \nabla\delta\theta \, . \tag{101}$$

Also,

$$\delta^2 K[w_\alpha] = \frac{1}{2} \int_{V_o} d\tau \left[\left(\delta\Psi^2 \frac{\partial^2 w_\alpha}{\partial \Psi^2} + 2\,\delta\Psi\delta\Phi \frac{\partial^2 w_\alpha}{\partial\Psi\partial\Phi} + \delta\Phi^2 \frac{\partial^2 w_\alpha}{\partial\Phi^2} \right) \right.$$

$$\times (\Phi\nabla\Psi - \Psi\nabla\Phi) \cdot \nabla\theta \times \nabla\zeta + 2 \left(\delta\Psi \frac{\partial w_\alpha}{\partial\Psi} + \delta\Phi \frac{\partial w_\alpha}{\partial\Phi} \right)$$

$$\times (\delta\Phi\nabla\Psi + \Phi\nabla\delta\Psi - \delta\Psi\nabla\Phi - \Psi\nabla\delta\Phi) \cdot \nabla\theta \times \nabla\zeta$$

$$+ 2 \left(\delta\Psi \frac{\partial w_\alpha}{\partial\Psi} + \delta\Phi \frac{\partial w_\alpha}{\partial\Phi} \right) (\Phi\nabla\Psi - \Psi\nabla\Phi) \cdot \nabla\delta\theta \times \nabla\zeta$$

$$+ 2 (\delta\Phi\nabla\Psi + \Phi\nabla\delta\Psi - \delta\Psi\nabla\Phi - \Psi\nabla\delta\Phi) \cdot \nabla\delta\theta \times \nabla\zeta\, w_\alpha$$

$$\left. + 2 (\delta\Phi\nabla\delta\Psi - \delta\Psi\nabla\delta\Phi) \cdot \nabla\theta \times \nabla\zeta\, w_\alpha \right], \qquad (102)$$

$$\delta^2 M [u_\alpha] = \int_{V_o} d\tau \left[\left(\delta\Psi^2 \frac{\partial^2 u_\alpha}{\partial\Psi^2} + 2\delta\Psi\delta\Phi \frac{\partial^2 u_\alpha}{\partial\Psi\partial\Phi} + \delta\Phi^2 \frac{\partial^2 u_\alpha}{\partial\Phi^2} \right) \rho \right.$$

$$\left. + 2\delta\rho \left(\delta\Psi \frac{\partial u_\alpha}{\partial\Psi} + \delta\Phi \frac{\partial u_\alpha}{\partial\Phi} \right) \right], \qquad (103)$$

and

$$\delta^2 S\,[v_\alpha] = \frac{1}{\gamma - 1}\int_{V_o} d\tau\,\Big[\Big(\delta\Psi^2\,\frac{\partial^2 v_\alpha}{\partial\Psi^2} + 2\delta\Psi\delta\Phi\,\frac{\partial^2 v_\alpha}{\partial\Psi\partial\Phi} + \delta\Phi^2\,\frac{\partial^2 v_\alpha}{\partial\Phi^2}\Big)\,\rho$$

$$\times \ln\,(p/\rho^\gamma) + 2\left(\delta\Psi\,\frac{\partial v_\alpha}{\partial\Psi} + \delta\Phi\,\frac{\partial v_\alpha}{\partial\Phi}\right)\delta\dot{\rho}\,\{\ln\,(p/\rho^\gamma) - \gamma\}$$

$$+ 2\left(\delta\Psi\,\frac{\partial v_\alpha}{\partial\Psi} + \delta\Phi\,\frac{\partial v_\alpha}{\partial\Phi}\right)\frac{\delta p}{p}\rho\rho + \delta\rho\,\frac{\delta p}{p}\,v_\alpha - \frac{\delta\rho^2}{\rho}\,\gamma v_\alpha - \frac{\delta p^2}{p^2}\,\rho v_\alpha\Big]. \tag{104}$$

We may differentiate identity (88) again to obtain

$$\rho\sum_\alpha \mu_\alpha u_{\alpha fg} + s\sum_\alpha T_\alpha\,v_{\alpha fg} = p_{fg} + \frac{1}{\gamma - 1}\left(\frac{p_f\rho_g}{\rho} + \frac{\rho_f p_g}{\rho}\right.$$

$$\left. + \frac{p_f p_g}{p} - \frac{\gamma p}{\rho^2}\,\rho_f\rho_g\right), \tag{105}$$

where $(\ )_g$ denotes partial differentiation either with respect to Ψ or Φ. Using identities (88) and (105), we can write

$$\sum_\alpha \mu_\alpha\delta^2 M\,[u_\alpha] + \sum_\alpha T_\alpha\delta^2 S\,[v_\alpha]$$

$$= \int_{V_o} d\tau\;\delta\Psi^2\left[p_{\Psi\Psi} + \frac{2p_\Psi\rho_\Psi}{(\gamma - 1)\rho} - \frac{p_\Psi^2}{(\gamma - 1)p} - \frac{\gamma}{\gamma - 1}\,\frac{\rho_\Psi^2}{\rho^2}\,p\right]$$

$$+ 2\delta\Psi\delta\Phi\left[p_{\Psi\Phi} + \frac{p_\Psi\rho_\Phi + \rho_\Psi p_\Phi}{(\gamma - 1)\rho} - \frac{1}{\gamma - 1}\,\frac{p_\Psi p_\Phi}{p} - \frac{\gamma}{\gamma - 1}\,\frac{\rho_\Psi\rho_\Phi}{\rho^2}\,p\right]$$

$$+ \delta\Phi^2\left[p_{\Phi\Phi} + \frac{2p_\Phi\rho_\Phi}{(\gamma - 1)\rho} - \frac{1}{\gamma - 1}\,\frac{p_\Phi^2}{p} - \frac{\gamma}{\gamma - 1}\,\frac{\rho_\Phi^2}{\rho^2}\,p\right]$$

$$+ \frac{2\delta\rho\delta\Psi}{\rho(\gamma - 1)}\left[\frac{\gamma p\rho_\Psi}{\rho} - p_\Psi\right] + \frac{2\delta\rho\delta\Phi}{(\gamma - 1)\rho}\left[\frac{\gamma p\rho_\Phi}{\rho} - p_\Phi\right]$$

$$+ \frac{2\delta p\delta\Psi}{(\gamma - 1)p}\left[p_\Psi - \frac{p}{\rho}\,\rho_\Psi\right] + \frac{2\delta p\delta\Phi}{(\gamma - 1)p}\left[p_\Phi - \frac{p}{\rho}\,\rho_\Phi\right]. \tag{106}$$

We now specialize to the case of a plasma contained in a periodic cylinder of radius a and length $2\pi R$ in which equilibrium quantities depend only on r. From considerations of symmetry, it is easily seen that all arbitrary variations on a given equilibrium state may be expressed in terms of real functions $\delta\Psi(r)$, $\delta\Phi(r)$, $-i\delta\theta(r)$ [$\equiv \delta\Gamma(r)$], $\delta\rho(r)$ and $\delta p(r)$ multiplied by [$\exp(im\theta + ikz)$], where k is determined by the quantum rule $k = n/R$; n is an integer. Assuming this form for the variations, $\delta^2 F$ may be reduced to the one-dimensional integral

$$\delta^2 F \equiv 2\pi^2 R \int_0^a dr\, r \Bigg[\left(\frac{m\delta\Psi + n\delta\Phi}{Rr} \right)^2 + \left(\frac{\delta\Psi' + n\Phi'\delta\Gamma}{R} \right)^2 + \left(\frac{\delta\Phi' - m\Phi'\delta\Gamma}{r} \right)^2$$

$$+ 2B_\theta \frac{n}{R}(\delta\Phi\delta\Gamma' + \delta\Phi'\delta\Gamma) - 2B_z \frac{m}{r}(\delta\Phi\delta\Gamma' + \delta\Phi'\delta\Gamma) - \sum_\alpha \frac{\lambda_\alpha}{rR}$$

$$\times \{(\delta\Phi\delta\Psi' - \delta\Psi\delta\Phi')\omega_\alpha + (\delta\Psi w_{\alpha\Psi} + \delta\Phi w_{\alpha\Phi})(\delta\Phi\Psi' - \delta\Psi\Phi' + \Phi\delta\Psi' - \Psi\delta\Phi')$$

$$- m(\delta\Psi w_{\alpha\Psi} + \delta\Phi w_{\alpha\Phi})\delta\Gamma(\Phi\Psi' - \Psi\Phi') - m(\delta\Phi\delta\Gamma\Psi' - \delta\Gamma\delta\Phi'\Psi - \delta\Gamma'\delta\Phi\Psi$$

$$+ \delta\Psi\delta\Gamma'\Phi - \delta\Gamma\delta\Phi'\Psi - \delta\Gamma'\delta\Phi\Psi)w_\alpha + \frac{1}{2}(\delta\Psi^2 w_{\alpha\Psi\Psi} + 2\delta\Psi\delta\Phi w_{\alpha\Psi\Phi}$$

$$+ \delta\Phi^2 w_{\alpha\Phi\Phi})(\Phi\Psi' - \Psi\Phi')\} + \delta\Psi^2 \left[p_{\Psi\Psi} + \frac{1}{\gamma - 1}\left\{ \frac{2p_\Psi\rho_\Psi}{\rho} - \frac{p_\Psi^2}{p} - \gamma\frac{\rho_\Psi^2}{\rho^2}p \right\} \right]$$

$$+ 2\delta\Psi\delta\Phi \left[p_{\Psi\Phi} + \frac{1}{\gamma - 1}\left\{ \frac{p_\Psi\rho_\Phi + p_\Phi\rho_\Psi}{\rho} - \frac{p_\Psi p_\Phi}{p} - \frac{\gamma\rho_\Psi\rho_\Phi}{\rho^2}p \right\} \right]$$

$$+ \delta\Phi^2 \left[p_{\Phi\Phi} + \frac{1}{\gamma - 1}\left\{ \frac{2p_\Phi\rho_\Phi}{\rho} - \frac{p_\Phi^2}{p} - \frac{\gamma\rho_\Phi^2}{\rho^2}p \right\} \right] + \frac{2\delta\rho\delta\Psi}{(\gamma - 1)\rho}\left(\frac{\gamma p\rho_\Psi}{\rho} - p_\Psi \right)$$

$$+ \frac{2\delta\rho\delta\Phi}{(\gamma - 1)\rho}\left(\frac{\gamma p\rho_\Phi}{\rho} - p_\Phi \right) + \frac{2\delta p\delta\Psi}{(\gamma - 1)p}\left(p_\Psi - \frac{p}{\rho}\rho_\Psi \right) + \frac{2\delta p\delta\Phi}{(\gamma - 1)p}$$

$$\times \left(p_\Phi - \frac{p}{\rho}\rho_\Phi \right) \Bigg] . \qquad (107)$$

We now assert the following proposition. If we restrict variations to the ideal class I, $\delta^2 F$ is identically equal to the energy integral due to Bernstein, et al.[2] We prove this in the cylindrical case, for which $\delta^2 F$ reduces to the one-dimensional quadratic form due to Newcomb.[29]

Proof: In the periodic cylinder with equilibrium quantities depending only on r the ideal variations I are given by

$$\delta\Psi = -\xi_r \Psi' \tag{108}$$

$$\delta\Phi = -\xi_r \Phi' \tag{109}$$

$$\delta\theta = -\frac{\xi_\theta}{r} + \frac{\xi_z}{rq} \tag{110}$$

$$\delta\rho = -\xi_r \rho' - \rho\nabla \cdot \vec{\xi} \tag{111}$$

$$\delta p = -\xi_r p' - \gamma p \nabla \cdot \vec{\xi} \tag{112}$$

From the equilibrium equation

$$\vec{J} = \sum_\alpha \lambda_\alpha \left[\frac{\Psi}{2} w_{\alpha\Psi} + \frac{\Phi}{2} w_{\alpha\Phi} + w_\alpha \right] \vec{B} - Rp_\Psi \hat{z} + rp_\Phi \hat{\theta} \tag{113}$$

we can derive the relationship

$$B_\theta^2 + rB_\theta B_\theta' + rB_z B_z' + rRp_\Psi B_\theta + \hat{r}^2 p_\Phi B_z = 0 \quad . \tag{114}$$

Substituting Eqs. (108) through (112) into Eq. (107), and by repeated use of equations (113), (114), and the appropriate boundary conditions, we obtain after straightforward algebraic manipulations,

$$W = 2\pi^2 R \int_0^a dr\, r \left[\frac{1}{k^2r^2 + m^2} \left\{ \left(krB_z + mB_\theta \right) \xi_r' + \left(krB_z - mB_\theta \right) \frac{\xi_r}{r} \right\}^2 \right.$$

$$+ \left\{ \left(krB_z + mB_\theta \right)^2 - 2B_\theta (rB_\theta)' \right\} \frac{\xi_r^2}{r^2} + \gamma\rho (\nabla \cdot \vec{\xi})^2 + \frac{k^2r^2 + m^2}{r^2}$$

$$\left. \times \left\{ \Phi' \delta\Gamma - \frac{r}{k^2r^2 + m^2} \left[\left(krB_\theta - mB_z \right) \xi_r' - \left(krB_\theta + mB_z \right) \frac{\xi_r}{r} \right]^2 \right\} \right] \tag{115}$$

which is exactly the quadratic form due to Newcomb.

Q.E.D.

A corollary of the proposition above follows. If $\delta^2 F > 0$ for arbitrary variations of class G, then the equilibria must be linearly stable to all ideal perturbations. In other words, if an equilibrium given by the present variational principle is a local minimum of the free energy functional, it must satisfy the necessary and sufficient conditions of stability obtained from the Energy Principle of Bernstein, et. al.[2]

Conclusion

In conclusion, we indicate a few applications of this theory and report on work in progress.

We have assumed that relaxation is dominated at every stage of a toroidal discharge by a tearing mode of single helicity. For tokamaks in which only a few modes are typically observed, this is a more realistic scenario than for reversed field pinches in which virulent MHD activity often develops into turbulence. Clearly, it would be desirable to construct global invariants without reference to any particular dominant mode. A numerical simulation designed to examine the decay rates of energy and global constraints on the linear and nonlinear time scales may achieve this goal.

On the other hand, since experimental observations tend to support strongly the basic idea of imposing global constraints, our work raises an exciting possibility for the study of nonlinear evolution of tearing modes in which a crucial issue is resolution of the structure of the singular layer. If we can construct global invariants for tearing modes, we may be able to derive valuable information about their nonlinear evolution without resolving the structure of the singular layer.

The importance of the variational principle formulated in this dissertation is that it acts as a filter for states which have particularly robust stability properties. We have already shown from the second variation of the free energy that states which are local minima of free energy must be ideally stable.

Clearly, these states should be stable to a wider class of perturbations which include some or all of the resistive perturbations depending on the particular global constraints included in the theory. The precise windows of ideal and resistive stability for the equilibria considered here are under active study and will be reported in a forthcoming paper.

An interesting application of this theory, on which work is well under way[30], is the solution of the equilibrium equations in toroidal geometry. Recent discoveries of the "second region" of stability in tokamaks raises the question of accessibility from the low-β stable region to the high-β stable region without passing through the intermediate region of instability. The possibility that we may be able to generate a sequence of minimum-energy equilibria connecting the two regions of stability is indeed exciting.

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FIGURE CAPTIONS

Fig. 1 - Helical flux function $\chi(\Phi)$ before and after reconnection in the Double Tearing model.

Fig. 2 - q profiles during current rise for a sequence of double-tearing modes

Fig. 3 - $\overline{J}_z$ - profiles during current rise for a sequence of double-tearing modes.

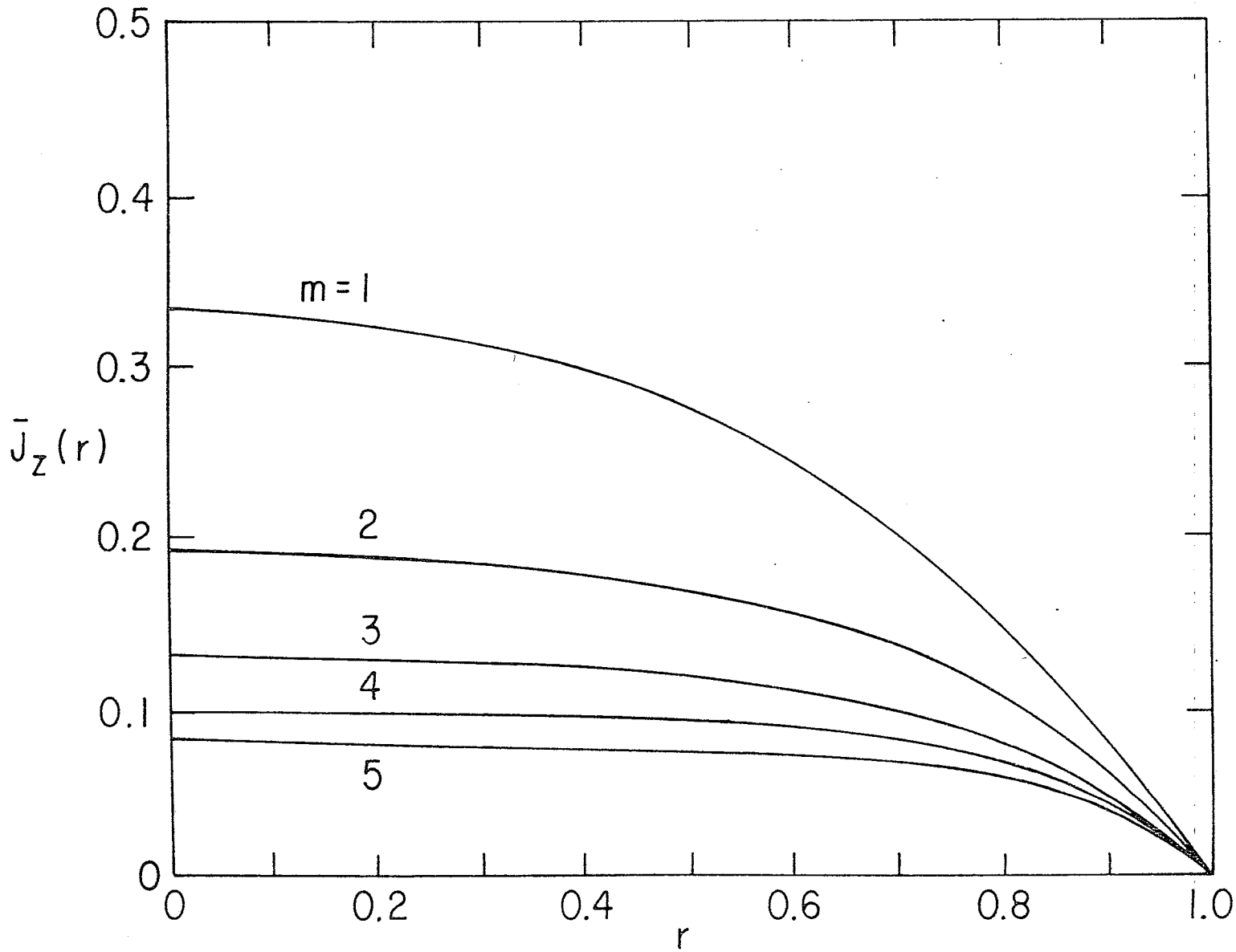

Fig. 3

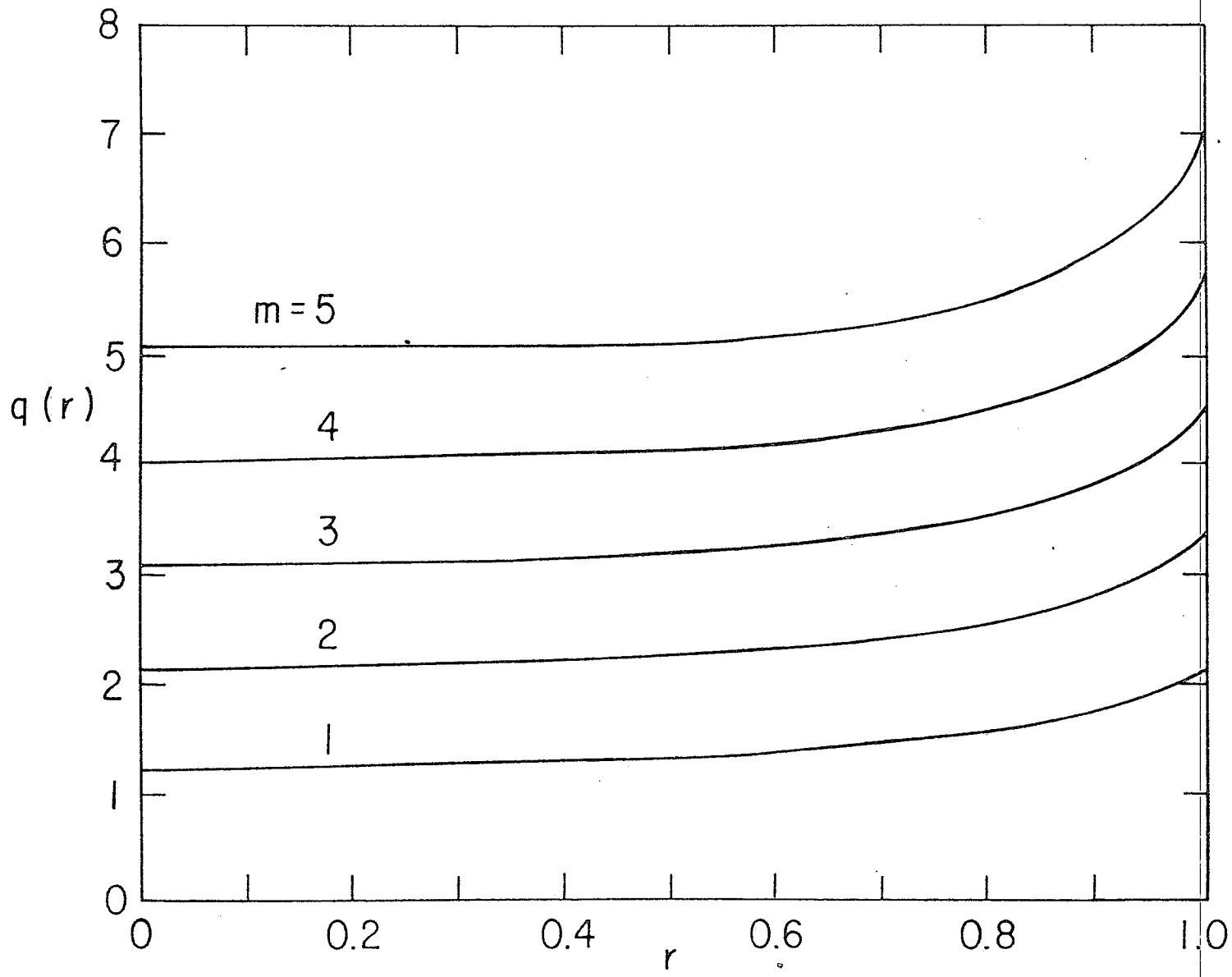

Fig. 2

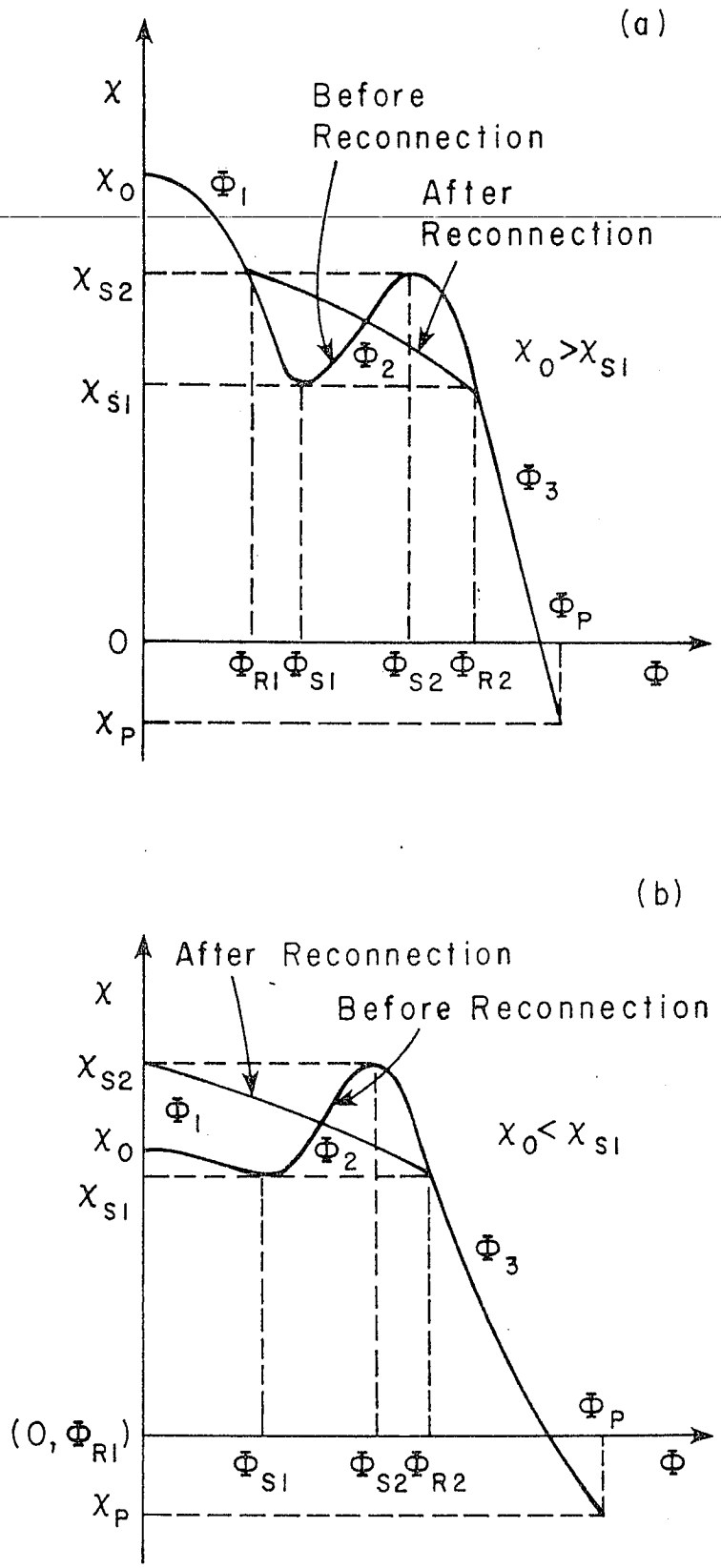
(a)
χ
Before
Reconnection
χ0
Φ1
After
Reconnection
χS2
χS1
Φ2
χ0>χS1
Φ3
ΦP
0
ΦR1 ΦS1
ΦS2 ΦR2
Φ
χP
(b)
After Reconnection
χ
Before Reconnection
χS2
Φ1
χ0
Φ2
χ0<χS1
χS1
Φ3
ΦP
(0, ΦR1)
ΦS1
ΦS2 ΦR2
Φ
χP

Fig. 1