TWO-POINT THEORY OF CURRENT-DRIVEN ION-CYCLOTRON TURBULENCE

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Abstract

An analytical theory of current-driven ion-cyclotron turbulence which treats incoherent phase space density granulations (clumps) is presented. In contrast to previous investigations, we focus on the physically relevant regime of weak collective dissipation, where waves and clumps coexist. The threshold current for nonlinear instability is calculated, and is found to deviate from the linear threshold up to 7%. A necessary condition for the existence of stationary wave-clump turbulence is derived, and shown to be analogous to the test-particle model fluctuation-dissipation theorem result. The structure of three dimensional magnetized clumps is characterized. It is proposed that nonlinear instability is saturated by collective dissipation due to ion-wave scattering. For this wave-clump turbulence regime, it is found that the fluctuation level $(e\varphi/T_e)_{rms} \leqslant 0.1$, and that the modification of anomalous resistivity to levels predicted by conventional nonlinear wave theories is moderate. In marked contrast to the quasilinear prediction, it is also shown that ion heating significantly exceeds electron heating, and that a drag force on electrons is set up to counter-act the accelerating electric field along field lines and sustains a steady current.

I. Introduction

Strong current-driven ion-cyclotron waves have been identified in several experiments[1,2,3] and in satellite[4,5] data. Possible consequences, such as anomalous transport, ion heating, and anomalous resistivity have been the principal foci of attention. There is also increasing theoretical interest in anomalous transport[3,6] due to ion-cyclotron turbulence, which may be related to the formation of auroral arcs and ion-conic distributions along auroral field lines.

Traditionally, anomalous transport has been studied using quasilinear theory.[3] However, this theory is unsatisfactory for the following three reasons:

First, there remains, in the transport coefficients, a fluctuation spectrum $\langle E^2 \rangle_{\underset{\sim}{k}}$ which is frequently left undetermined or treated in an ad hoc way.

Second, its relevant time scale is that for evolution of the resonant part of the distribution function, which is usually longer than that for nonlinear interaction (this point will be discussed further in Sec. IV).

Third, the processes for average distribution function relaxation include only quasi-linear diffusion resulting from wave fluctuations. However, due to incoherent fluctuations, the processes for relaxation of the average distribution function also include a drag force, which counter-balances the D.C. electric field set up by the D.C. current. Hence, with the diffusion due to waves alone, the quasi-linear theory is inadequate for description of a steady state. Clearly, for time scales of interest, one needs a more complete nonlinear theory to describe plasma turbulence. In the past the conventional nonlinear theories[7,8] (which assume a broad fluctuation spectrum - this assumption is adopted here also) usually employed a stochastically-accelerated-particle-orbit calculation[9] to account for enhanced collective dissipation due to nonlinearity. These theories include only wave fluctuations and can only describe the initial phase of turbulence with linearly excited waves. As soon as these self-consistent waves are damped out by the nonlinear interactions, the theories fail. In particular, incoherent fluctuations were not taken into account at all, and thus the finite frequency line-width observed in turbulent plasmas cannot be explained with these theories.

Recently, a new plasma turbulence theory (clump theory) has been developed and applied to a one-dimensional model problem[11,12], which has also been studied using particle simulations.[13] This one-dimensional study focused on a physical situation dominated by incoherent ballistic fluctuations, rather than waves. The particle simulations that support the conclusions of the theoretical work used a low ion to electron mass ratio of $M_i/m_e = 4$. In this case, the electron and ion distribution functions overlap in velocity space, yielding very effective collisionless momentum exchange between particles and ballistic fluctuations. However, this situation is not at all universal. Indeed, plasma turbulence is usually associated with collective resonances, hence the most general treatment should contain both wave and non-wave fluctuations. In view of these considerations of relevance, we are motivated to study phase space turbulence in a three-dimensional, magnetized plasma which can support collective oscillations.

In this case, incoherent non-wave fluctuations (clumps) arise from the imperfect mixing of a Vlasov plasma. Particle-orbit stochasticity due to wave-particle interactions generates small-scale phase space density granulations that ballistically propagate at the resonant velocities, $v = \omega - n\omega_{ci}/k_{\|}$. By way of contrast, the wave fluctuations are generated from a coupling between the gradient of the average distribution function and the electric field, and hence are intrinsically coherent with the fluctuating electric field. By satisfying the requirement that the dielectric function vanishes ($\varepsilon_{\underset{\sim}{k},\omega} = 0$), wave-like phase space density fluctuations and the fluctuating electric field can self-consistently exist. Nonlinear modifications to the wave fluctuations do nothing more than introduce nonlinear collective dissipation into the dielectric function $\varepsilon_{\underset{\sim}{k},\omega}$, which modifies wave stability. Each type of fluctuation has a different character. The wave fluctuations of a given wavenumber k, satisfying the dispersion relation, appear to be periodic in time; the non-wave fluctuations, $\omega - n\omega_c = k_{\parallel} v_{\parallel}$, appear as macro-particles which propagate ballistically and are shielded by the short-range electric field.

The mechanism whereby these ballistic structures arise can be easily understood. The turbulent electric field rearranges the average particle distribution by scattering chunks of particles off their original locations in phase space, thus creating spatially-varying and complicated structures. At the same time, the tidal forces produced by the resulting turbulent electric field tend to tear the chunk of particles apart. However, if the size of a clump is small enough so that every particle in it feels the same tidal force, this clump can retain its structural integrity for a time long compared to the average correlation time of the system. Decay processes such as this will be offset by the continuous generation of clumps due to relaxation of the average particle distribution. The competing processes discussed above are described by an evolution equation of the two-point, one-time phase space density correlation function $\langle \delta f(1,t) \delta f(2,t) \rangle$, which has the form,

$$[\frac{\partial}{\partial t} + \underset{\sim}{v}_- \cdot \frac{\partial}{\partial \underset{\sim}{X}_-} - \frac{\partial}{\partial \underset{\sim}{q}_-} \cdot \underset{\approx}{D}(-) \cdot \frac{\partial}{\partial \underset{\sim}{q}_-}] <\delta f(1,t)\delta f(2,t)>$$

$$= S(1,2) \cong - \frac{\partial}{\partial t} \left(<f(1)><f(2)>\right) \quad (1)$$

where $\underset{\sim}{q}_-$ is the appropriate relative coordinate of phase space, and $\underset{\approx}{D}(-)$ is a relative diffusion coefficient, which vanishes at $\underset{\sim}{q}_- = 0$ where two points in phase space feel the same tidal forces. In the limit $\underset{\sim}{q}_- = 0$, the correlation function is secularly driven by the source, which evolves quasi-stationarily. At steady state, $t \to \infty$, a singular correlation function yields. The driving force, in Eq. (1), is related to relaxation of the average particle distribution due to turbulent scattering by both the wave and non-wave fluctuations.

It can be shown that clump fluctuations can be excited from thermal noise at a lower threshold free energy than that required for the wave fluctuations. This leads to the interesting paradox that in the limit of vanishing fluctuation amplitude, one would expect linear and nonlinear stability boundaries to coincide, yet in fact the nonlinear theory here yields a lower value of threshold free energy. This paradox can be resolved if one realizes that differnt time scales are appropriate to each case. The linear analysis of a nonlinear partial differential equation is valid for a finite time, after which transition to nonlinear evolution occurs. In a Vlasov plasma, the linear theory is valid up to a time τ, where $\tau < \tau_c$ and τ_c is the amplitude-dependent wave-particle decorrelation time. In a linearly stable plasma, nonlinear instability can occur if the free energy exceeds the threshold value. In this case, the thermal level of fluctuations will determine the size of τ_c, and the clump fluctuations then grow on this time scale.

Furthermore, when linear waves are weakly damped in plasmas, wave fluctuations can be emitted by the macro-particles (in analogy to those emitted by the discrete particles in the test-particle model) and can coexist with clump fluctuations in plasma turbulence. Because of this relationship between the wave and clump fluctuations, competition between formation and decay of the clump fluctuations can now be viewed as competition between emission and dissipation of waves. This can be understood as follows.

Schematically, Eq. (1) can be expressed as

$$(\gamma_\omega + \frac{1}{\tau_{c1}}) \langle \delta f \delta f \rangle_\omega = \langle S \rangle_\omega \tag{2}$$

where γ_ω and τ_{c1} are the growth rate and life-time of clump fluctuations, respectively. Note that the source is proportional to the fluctuation spectrum, hence we let $\langle S \rangle_\omega = (R_\omega/\tau_{c1})\langle \delta f \delta f \rangle_\omega$, where R_ω is an operator, which includes the effect whereby $\tilde{f}$ (incoherent fluctuation) is shielded by the plasma response function $\varepsilon_{\underset{\sim}{k},\omega}$. We can further express R_ω as $\sum_{\underset{\sim}{k}} \bar{R}_{\underset{\sim}{k},\omega}/|\varepsilon_{\underset{\sim}{k},\omega}|^2$, where $\bar{R}_{\underset{\sim}{k},\omega}$ is another operator proportional to the seed electric field fluctuations of macroparticles. Thus Eq. (2) yields

$$(\gamma_\omega \tau_{c1} + 1) \cong \sum_{\underset{\sim}{k}} \frac{\bar{R}_{\underset{\sim}{k},\omega}}{|\varepsilon_{\underset{\sim}{k},\omega}|^2} \tag{3}$$

$$\gamma_\omega \cong \frac{1}{\tau_{c1}} \left([\bar{R}_{\underset{\sim}{k},\omega}/(\mathrm{Im}\ \varepsilon_{\underset{\sim}{k},\omega})(\frac{\partial \mathrm{Re}\ \varepsilon_{\underset{\sim}{k},\omega}}{\partial \underset{\sim}{k}})]\Big|_{\underset{\sim}{k}=\underset{\sim}{k}_\omega} - 1 \right) \tag{4}$$

where the spectral k-sum is mainly determined by the pole contribution, $\mathrm{Re}\varepsilon_{\underset{\sim}{k},\omega}=0$, at the collective resonances. The steady state, $\gamma_\omega=0$, is determined by the collective dissipation $\mathrm{Im}\varepsilon$ and the generation of waves by clump emission $\bar{R}$. By contrast, in the previous analysis of the clump-dominant regime, the shielding factor $1/|\varepsilon|^2$ is not close to collective resonance, hence the process whereby nonlinear evolution was regulated by a finite collective dissipation ($\mathrm{Im}\varepsilon$) was not considered. Consequently, in stationary turbulence of the wave-clump regime, waves must be over-saturated; this finite collective dissipation can explain the finite frequency linewidth observed in stationary turbulence. Indeed, in general, waves should be viewed as broadened collective resonances. Progress[14,15] in this direction has also been made in other areas of plasma turbulence; in particular, in the clump theory[14] of drift-wave turbulence.

Turbulence in a magnetized plasma can exhibit different types of strongly correlated small-scale granulations in phase space. In the case of low-frequency turbulence in a magnetized plasma, such as drift-wave turbulence, perpendicular dynamics are due to the spatial wandering of guiding centers because of random $E\times B_o$ convection. Perpendicular scattering in velocity space is small, since that requires the time scale of the fluctuating electric field to be comparable to the cyclotron frequency. The perpendicular spatial extent of clumps is limited by the average length scale of the tidal force, which is the spectrum-averaged perpendicular wavelength $(\bar{k}_\perp)^{-1}$. In velocity space, the perpendicular scale of the clump can extend up to a thermal velocity v_t. In contrast, for high-frequency turbulence in a magnetized plasma, the gyro-motion of particles can resonate with the turbulent field and thus lead to perpendicular scattering in velocity space. Furthermore, the perpendicular length scale, $(\bar{k}_\perp)^{-1}$, of the turbulent field is smaller than the gyro-radius (usually $\bar{k}_\perp\rho \gtrsim 1$ in high frequency problems); the former then not only determines the perpendicular correlation length of the clump fluctuations, but also limits their sizes to be within $v_t(\bar{k}_\perp\rho)^{-1}$, (where $\bar{k}_\perp\rho > 1$) in the perpendicular velocity direction.

In ion-cyclotron turublence, the electric field is almost perpendicular to the magnetic field ($\bar{k}_\perp >> \bar{k}_\parallel$). The ion perpendicular diffusion dominates parallel diffusion, and the latter can thus be ignored for the time scale of interest. On the other hand, the electrons are strongly magnetized ($\bar{k}_\perp\rho_e << 1$) and are thus tied to the magnetic field lines. Their dynamics are essentially one-dimensional. These observations allow us to depict the clump structure more precisely. The electron clump appears in position space as a long, thin ($\bar{k}_\perp >> \bar{k}_\parallel$), cigar-shaped group of electrons travelling at a speed $v_\parallel \approx n\omega_{ci}/k_\parallel$; the ion clump, on the other hand, appears as a tether-rod with tether length ρ_i gyrating about the guiding center, if $\bar{k}_\perp\rho_i > 1$, and as a cigar if $k_\perp\rho_i < 1$, propagating at a ballistic velocity $v_\parallel \cong \omega - n\omega_{ci}/k_\parallel$. Both cigar and rod have radii $(\bar{k}_\perp)^{-1}$ and lengths $(\bar{k}_\parallel)^{-1}$ (Fig. 1). Quantitatively, the phase space density correlation functions of both species are described by the equations,

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_- \cdot \frac{\partial}{\partial \underset{\sim}{X}_-} - \frac{\partial}{\partial \underset{\sim}{g}_{\perp -}} \cdot \underset{\approx}{D}_\perp(-) \cdot \frac{\partial}{\partial \underset{\sim}{g}_{\perp -}}\right)^i \langle\delta f(1,t)\delta f(2,t)\rangle^i = \langle S\rangle^i \tag{5}$$

and

$$\left(\frac{\partial}{\partial t} + v_{\parallel -} \frac{\partial}{\partial Z_-} - \frac{\partial}{\partial v_{\parallel -}} \cdot D_\parallel(-) \cdot \frac{\partial}{\partial v_{\parallel -}}\right)^e \langle\delta f(1,t)\delta f(2,t)\rangle^e = \langle S\rangle^e, \tag{6}$$

for ion-cyclotron turbulence. At steady state, Eqs. (5) and (6) can be solved, yielding

$$\langle\delta f(1)\delta f(2)\rangle^i \cong \tau^i_{c1}(\underset{\sim}{X}_-,\underset{\sim}{v}_-)\langle S(1,1)\rangle^i \tag{7}$$

$$<\delta f(1)\delta f(2)>^e \cong \tau^e_{c1}(\underset{\sim}{X}_-, \underset{\sim}{v}_-)<S(1,1)>^e \qquad (8)$$

where

$$\tau^i_{c1} = -\tau_i \ \ell n \ \{\frac{3}{4}[\bar{k}^2_\perp X^2_{\perp-} + \frac{\bar{k}^2_\perp v^2_{\perp-}}{2\omega^2_{ci}} + \bar{k}^2_\parallel (Z^2_- + 2v_{\parallel-}Z_-\tau_i + 2v^2_{\parallel-}\tau^2_i)]\}^i \qquad (9)$$

$$\tau^e_{c1} = -\tau_e \ell n\{\frac{1}{3}[\frac{\bar{k}^2_\perp X^2_{\perp-}}{2} + \bar{k}^2_\parallel (Z^2_- + 2v_{\parallel-}Z_-\tau_e + 2v^2_{\parallel-}\tau^2_e)]\}^e \qquad (10)$$

and τ_i, τ_e are the particle-decorrelation times for ions and electrons, respectively. Eqs. (7), (8), (9) and (10) illustrate the strong correlation of the charged particles at small separations in phase space. It should be noted that, in Eq. (9), the term $\bar{k}^2_\perp v^2_{\perp-}/2\omega^2_{ci}$ can be expressed as $\bar{k}^2_\perp \rho^2_i (v_{\perp-}/v_{ti})^2/2$, indicating that, as discussed previously, the ion clump size in the perpendicular velocity space depends on the value of $\bar{k}_\perp \rho_i$ as compared with unity. On the other hand, in the parallel velocity space, both ion and electron clumps have short-range correlation. As a result, in velocity space the ion clump appears as a gyrating tether-disc perpendicular to the magnetic field, with the disc radius $v_{ti}(\bar{k}_\perp \rho_i)^{-1}$ and tether length v_{ti} if $\bar{k}_\perp \rho_i > 1$, and as a thin disc of radius v_{ti} if $\bar{k}_\perp \rho_i < 1$; the thickness of either disc is $(\bar{k}_\parallel \tau_i)^{-1}$. In contrast, the electron clump, in velocity space, appears as a thin disc of radius v_{te} and thickness $(\bar{k}_\parallel \tau_e)^{-1}$ (Fig. 1).

A relevant quantity worthy of examination is the velocity-integrated clump amplitude $<\delta n^2>$, which not only is important for determination of nonlinear instability, but also indicates the probability of observing the clumps experimentally. In ion-cyclotron turbulence, this global clump amplitude is much smaller than that for electron clumps. This is because ion finite-Larmor-radius effects sizably reduce the perpendicular velocity correlation length and thus the ion clump amplitude. As a result, electron clumps are more likely to be observed, and play a more significant role in the phase space dynamics.

When temperatures of both species are approximately equal ($T^i \cong T^e$), current-driven ion-cyclotron waves are weakly damped because the parallel phase velocity $\omega_{\underset{\sim}{k}}/k_{\parallel}$ can adjust so that there are long wavelength modes that avoid ion Landau-damping. Turbulence, therefore, consists of a mixture of waves and clumps. It is shown that because of small ion dissipation, the clump modifications to the threshold current for instability are not significant for an isotropic average ion distribution function. However, if the perpendicular ion temperation $T^i_{\perp}$ is greater than the electron temperature T^e, the clump effects will be more significant. This is because[16] when $T^i_{\perp} > T^e \cong T^i_{\parallel}$, the collective resonance frequency $\omega_{\underset{\sim}{k}}$ approaches ion cyclotron harmonic frequencies $n\omega_{ci}$ and enhances ion dissipation. In fact, this state can be reached from an initially isotropic ion distribution function by persistent wave-ion interaction and heating, so long as the current is sufficiently maintained. In the limit of very large perpendicular ion temperature, $T^i_{\perp} \gg T^e \sim T^i_{\parallel}$, waves are heavily damped, and thus the nonlocal, collective characters of turbulence is absent. Clumps, shielded by the short-range (Debye length) response, alone determine the evolution of the turbulence.

In this paper, we investigate the nonlinear theory of current-driven ion-cyclotron turbulence. In particular, the relevant regime of turbulence, where the clumps and waves coexist, is studied. Here we summarize the principal results of this paper:

(1) Renormalized one-point and two-point equations describing magnetized

Vlasov turbulence have been derived. Since particle motions evolve on two different time scales, the renormalized two-point equation describes the slow, relative motion; the renormalized one-point equation describes the rapid cyclotron motion and yields a renormalized dielectric function.

(2) Shapes of ion and electron clumps have been predicted. The electron clump appears cigar-shaped aligned along the magnetic field in position space, and appears as a thin disc perpendicular to the magnetic field in velocity space. The ion clump, when $\bar{k}_\perp \rho_i > 1$, appears as a gyrating tether-rod about the guiding center in position space and a gyrating tether-disc in velocity space; when $\bar{k}_\perp \rho_i < 1$, the ion clump appears cigar-shaped aligned along the magnetic field in position space, and appears as a thin disc perpendicular to the magnetic field in velocity space (Fig. 1).

(3) The clump amplitude of ions is much less than that of electrons because of ion finite-Larmor radius effects.

(4) When $T_\perp^i \geqslant T_\parallel^i \sim T^e$, ion-cyclotron turbulence is of the wave-clump type. When $T_\perp^i >> T_\parallel^i \sim T^e$, transition to the clump-dominant type of turbulence occurs. This transition takes place approximate when $(m_e/M_i)^{1/2} T_\perp^i / T^e \geqslant 0.1$, where waves are heavily damped.

(5) A necessary condition for the maintenance of stationary ion-cyclotron turbulence is given in Eq. (49), and is a Vlasov-theory analogue to the prediction of the fluctuation-dissipation theorem of the test-particle model. This results from the fact that the collective dissipation must balance

incoherent noise emission from macro-particles at steady state. This expression is valid for turbulence of the wave-clump type.

(6) The nonlinear growth rates in different regimes have been obtained, and are given in Eqs. (58) and (59). Nonlinear instabilities grow on the time scales of the particle decorrelation times.

(7) The clump modifications to the instability threshold in the wave-clump regime are small (reduction of threshold drift velocity is within 7%, Fig. 2). However, the modifications to the fluctuation level at saturation and the anomalous resistivity are more significant. Also while the quasi-linear theory predicts that electron parallel heating is larger than, or equal to, ion perpendicular heating for ion-cyclotron turbulence ($\bar{k}_\perp^2/\bar{k}_\parallel^2 \approx M_i/m_e$, $\bar{k}_\perp^2\rho_i^2 \gtrsim 1$), this theory predicts that ion perpendicular heating is the dominant mechanism for extraction of electron-beam energy.

The remainder of this paper is organized as follows: in Sec. II we present a method for the renormalization of the Vlasov equation for a magnetized plasma. The nonlinear terms are approximated by a non-Markovian diffusion process in velocity space and guiding-center space. Subsequently, this renormalized equation is used to obtain the nonlinear dielectric function. In Sec. III we renormalize the correlation equation. The triplet terms are approximated as diffusion in the relative coordinates of the corresponding phase space. In steady state, the correlation function can be calculated. In doing so, a steady-state condition for maintenance of stationary turbulence is derived. This is discussed in Sec. IV. In Sec. V, we investigate the clump instability. A mechanism of saturation by collective resonance damping due to nonlinear ion-wave interaction is proposed. The saturation level and anomalous resistivity are estimated in Sec. VI. Conclusions are discussed in Sec. VII.

II. One-Point Equation Renormalization

Though the one-point theory cannot fully describe the nonlinear evolution of a plasma, renormalization provides a useful tool for determination of the nonlinear response of the waves. In this section, we consider renormalization for the case of high-frequency turbulence, and will derive the renormalized dielectric function $\varepsilon_{\underset{\sim}{k},\omega}$ and particle propagator G. In a uniformly magnetized and spatially homogeneous Vlasov plasma, the particle phase-space density of either species satisfies the equation,

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{X}} + \omega_c\, \underset{\sim}{v}\times\hat{z} \cdot \frac{\partial}{\partial \underset{\sim}{v}} - \frac{q}{M}\frac{\partial\varphi}{\partial \underset{\sim}{X}} \cdot \frac{\partial}{\partial \underset{\sim}{v}}\right)\delta f = \frac{q}{M}\frac{\partial\varphi}{\partial \underset{\sim}{X}} \cdot \frac{\partial\langle f\rangle}{\partial \underset{\sim}{v}} . \tag{11}$$

Introducing guiding-center coordinates, one can transform the above equation into

$$\left(\frac{\partial}{\partial t} + \omega_c \frac{\partial}{\partial\vartheta} + U\frac{\partial}{\partial Z} + \frac{c}{B}\left(\frac{\partial\varphi}{\partial \underset{\sim}{R}_\perp} \times \hat{Z}\right) \cdot \frac{\partial}{\partial \underset{\sim}{R}_\perp} - \frac{q}{M}\frac{\partial\varphi}{\partial \underset{\sim}{R}} \cdot \frac{\partial}{\partial \underset{\sim}{V}}\right)\delta f = \frac{q}{M}\frac{\partial\varphi}{\partial \underset{\sim}{R}} \cdot \frac{\partial\langle f\rangle}{\partial \underset{\sim}{V}} , \tag{12}$$

where $\underset{\sim}{R} = \underset{\sim}{X} - \dfrac{\underset{\sim}{v}\times\hat{z}}{\omega_c}$, $\underset{\sim}{V}=\underset{\sim}{v}$

$$\frac{\partial}{\partial \underset{\sim}{R}} = \frac{\partial}{\partial \underset{\sim}{X}} , \quad \frac{\partial}{\partial \underset{\sim}{V}} - \frac{1}{\omega_c}\hat{z} \times \frac{\partial}{\partial \underset{\sim}{R}} = \frac{\partial}{\partial \underset{\sim}{v}}$$

$$\vartheta = \cos^{-1}\left(\frac{\hat{x}\cdot \underset{\sim}{V}_\perp}{V}\right)$$

and $k \equiv |\underset{\sim}{k}_\perp|$, $\kappa \equiv k_\parallel$, $V \equiv |\underset{\sim}{v}_\perp|$, $U \equiv v_\parallel$, $R \equiv |\underset{\sim}{R}_\perp|$.

After a spatial-temporal Fourier transformation in guiding-center coordinates, we obtain

$$[\omega_c \frac{\partial}{\partial\vartheta} - i(\omega-\kappa U)]f_{\underset{\sim}{k},\omega} + \frac{q}{M}\sum_{\underset{\sim}{k}',\omega'} \Phi_{\underset{\sim}{k}',\omega'}\{\frac{kk'}{\omega_c}\sin(\psi_k-\psi_{k'})$$

$$- i\kappa'\frac{\partial}{\partial U} - ik'[\cos(\vartheta-\psi_{k'})\frac{\partial}{\partial V} - \frac{\sin(\vartheta-\psi_{k'})}{V}\frac{\partial}{\partial\vartheta}]\}f_{\underset{\sim}{k}'',\omega''}$$

$$= \frac{iq}{M}\Phi_{\underset{\sim}{k},\omega}[k\cos(\vartheta-\psi_k)\frac{\partial}{\partial V} + \kappa\frac{\partial}{\partial U}]\langle f\rangle \ , \qquad (13)$$

where $\psi_k \equiv \cos^{-1}(\frac{\hat{x}\cdot\underset{\sim}{k}_\perp}{k})$, $\underset{\sim}{k}'' = \underset{\sim}{k}-\underset{\sim}{k}'$, $\omega'' = \omega-\omega'$

and $f_{\underset{\sim}{k},\omega} = \int_0^\infty dt \int_{-\infty}^\infty d^3\underset{\sim}{x}\ \delta f(\underset{\sim}{x},\underset{\sim}{v},t)\exp[i\omega t-i\underset{\sim}{k}\cdot\underset{\sim}{x} + \frac{ikV}{\omega_c}\sin(\vartheta-\psi_k)]$

$$\Phi_{\underset{\sim}{k},\omega} = \int_0^\infty dt \int_{-\infty}^\infty d^3\underset{\sim}{x}\ \varphi(\underset{\sim}{x},t)\exp[i\omega t-i\underset{\sim}{k}\cdot\underset{\sim}{x} + \frac{ikV}{\omega_c}\sin(\vartheta-\psi_k)]$$

the Fourier components of fluctuations in guiding-center coordinates. In addition, we can define the Fourier components of fluctuations in real space coordinates,

$$\delta f_{\underset{\sim}{k},\omega} = f_{\underset{\sim}{k},\omega}\exp[-i\frac{kV}{\omega_c}\sin(\vartheta-\psi_k)],\ \varphi_{\underset{\sim}{k},\omega} = \Phi_{\underset{\sim}{k},\omega}\exp[-i\frac{kV}{\omega_c}\sin(\vartheta-\psi_k)].$$

To renormalize the nonlinearities, we assume that the fluctuating electric field has a broad spectrum, so that the electric field auto-correlation time is shorter than the particle decorrelation time. We then decompose $f_{\underset{\sim}{k},\omega}$ into

$$f_{\underset{\sim}{k},\omega} = f^{(1)}_{\underset{\sim}{k},\omega} + f^{(2)}_{\underset{\sim}{k},\omega} .$$

Here, $f^{(1)}_{\underset{\sim}{k},\omega}$ is the phase-incoherent component, which is not correlated with fluctuations of different k and ω, and $f^{(2)}_{\underset{\sim}{k},\omega}$ is the phase-coherent component, which is driven by a direct beating of $f_{\underset{\sim}{k}'',\omega''}$ and $\Phi_{\underset{\sim}{k}',\omega'}$. In addition, it is useful to notice that Eq. (13) can be written as

$$[\omega_c \frac{\partial}{\partial\vartheta} - i(\omega-\kappa U)]f_{\underset{\sim}{k},\omega} + \frac{q}{M}\sum_{\substack{\underset{\sim}{k}'\\ \omega'}}^{\not\xi} \Phi_{\underset{\sim}{k}',\omega'}\{\frac{kk'}{\omega_c}\sin(\psi_k-\psi_{k'})$$

$$- i\kappa'\frac{\partial}{\partial U} - ik'[\cos(\vartheta-\psi_{k'})\frac{\partial}{\partial V} - \frac{\sin(\vartheta-\psi_{k'})}{V}\frac{\partial}{\partial\vartheta}]\}f_{\underset{\sim}{k}'',\omega''}$$

$$= \frac{iq}{M}\Phi_{\underset{\sim}{k},\omega}[k\cos(\vartheta-\psi_k)\frac{\partial}{\partial V} + \kappa\frac{\partial}{\partial U}]\langle f\rangle$$

$$- \frac{q}{M}\Phi_{\underset{\sim}{k}_\xi,\omega_\xi}\{\frac{kk_\xi}{\omega_c}\sin(\psi_k-\psi_{k_\xi}) - i\kappa_\xi\frac{\partial}{\partial U} - ik_\xi[\cos(\vartheta-\psi_{k_\xi})\frac{\partial}{\partial V}$$

$$- \frac{\sin(\vartheta-\psi_{k_\xi})}{V}\frac{\partial}{\partial\vartheta}]\}f_{\underset{\sim}{k}-\underset{\sim}{k}_\xi,\omega-\omega_\xi} , \quad (14)$$

where the summation $\sum^{\not\xi}_{\underset{\sim}{k}',\omega'}$ does not include $\underset{\sim}{k}_\xi$ or ω_ξ, dummy variables.

Hence $f_{\underset{\sim}{k},\omega}$ is driven by the linear source and mode beating. Here $f^{(1)}_{\underset{\sim}{k},\omega}$ is driven by the former and $f^{(2)}_{\underset{\sim}{k},\omega}$ by the latter. Upon renormalization, the nonlinear terms on the left side yield the phase-coherent piece of the full nonlinearity. This phase-coherent piece can be written in a form

$$\int C_{\underset{\sim}{k},\omega}(\vartheta-\vartheta')f_{\underset{\sim}{k},\omega}(\vartheta')d\vartheta' + B_{\underset{\sim}{k},\omega}(\vartheta)\varphi_{\underset{\sim}{k},\omega} ,$$

where $C_{\underset{\sim}{k},\omega}$ and $B_{\underset{\sim}{k},\omega}$ are operators. In order to obtain $C_{\underset{\sim}{k},\omega}(\vartheta-\vartheta')$ and $B_{\underset{\sim}{k},\omega}(\vartheta)$, we will separate $f^{(2)}_{\underset{\sim}{k},\omega}$ from $f^{(1)}_{\underset{\sim}{k},\omega}$ in Eq. (14). Hence,

$$[\omega_c \frac{\partial}{\partial\vartheta} - i(\omega-\kappa U)]f^{(1)}_{\underset{\sim}{k},\omega} + \int C_{\underset{\sim}{k},\omega}(\vartheta-\vartheta')f^{(1)}_{\underset{\sim}{k},\omega}(\vartheta')d\vartheta'$$

$$= \frac{iq}{M}\Phi_{\underset{\sim}{k},\omega}[k\cos(\vartheta-\psi_k)\frac{\partial}{\partial V} + \kappa\frac{\partial}{\partial U}]\langle f\rangle - B_{\underset{\sim}{k},\omega}(\vartheta)\ \varphi_{\underset{\sim}{k},\omega} \quad (15)$$

$$[\omega_c \frac{\partial}{\partial\vartheta} - i(\omega-\kappa U)]f^{(2)}_{\underset{\sim}{k},\omega} + C_{\underset{\sim}{k},\omega}(\vartheta-\vartheta')f^{(2)}_{\underset{\sim}{k},\omega}(\vartheta')\ d\vartheta'$$

$$= -\frac{q}{M}\Phi_{\underset{\sim}{k}_\xi,\omega_\xi}\{\frac{kk_\xi}{\omega_c}\sin(\psi_k-\psi_{k_\xi}) - i\kappa_\xi\frac{\partial}{\partial U} - ik_\xi[\cos(\vartheta-\psi_{k_\xi})\frac{\partial}{\partial V}$$

$$-\frac{\sin(\vartheta-\psi_{k_\xi})}{V}\frac{\partial}{\partial\vartheta}]\}f_{\underset{\sim}{k}-\underset{\sim}{k}_\xi,\omega-\omega_\xi}. \quad (16)$$

Now, we can define a propagator $G_{\underset{\sim}{k},\omega}(\vartheta,\vartheta')$ so that

$$[\omega_c\frac{\partial}{\partial\vartheta} - i(\omega-\kappa U)]G_{\underset{\sim}{k},\omega}(\vartheta,\vartheta') + \int d\vartheta''C_{\underset{\sim}{k},\omega}(\vartheta-\vartheta'')G_{\underset{\sim}{k},\omega}(\vartheta'',\vartheta') = \omega_c\ \delta(\vartheta-\vartheta')\ . \quad (17)$$

Therefore $f^{(2)}_{\underset{\sim}{k},\omega}$ can be obtained in terms of $G_{\underset{\sim}{k},\omega}$,

$$f^{(2)}_{\underset{\sim}{k},\omega}(\vartheta) = \int d\vartheta' G_{\underset{\sim}{k},\omega}(\vartheta,\vartheta')(-\frac{q}{M})\Phi_{\underset{\sim}{k}_\xi,\omega_\xi}\{\frac{kk_\xi}{\omega_c}\sin(\psi_k-\psi_{k_\xi})$$

$$- i\kappa_\xi\frac{\partial}{\partial U} - ik_\xi[\cos(\vartheta'-\psi_{k_\xi})\frac{\partial}{\partial V} - \frac{\sin(\vartheta'-\psi_{k_\xi})}{V}\frac{\partial}{\partial\vartheta'}]\}f_{\underset{\sim}{k}-\underset{\sim}{k}_\xi,\omega-\omega_\xi}(\vartheta').$$

(18)

Substituting $f^{(2)}_{\underset{\sim}{k}'',\omega''}$ into $f^{(2)}_{\underset{\sim}{k}'',\omega''}$ on the left side of Eq. (13), we can obtain $C_{\underset{\sim}{k},\omega}(\vartheta-\vartheta')$ and $B_{\underset{\sim}{k},\omega}(\vartheta)$, the cyclotron harmonic component of which can be written as

$$C_{\underset{\sim}{k},\omega,n} = -\left(\frac{ik}{\omega_c}, \frac{\partial}{\partial U}, \frac{1}{V}\frac{\partial}{\partial V}, \frac{in}{V}\right) \sum_{\substack{k' \\ \omega' \\ n'}} \begin{bmatrix} D_{RR} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & D_{\underset{\sim}{vv}} & \\ 0 & & & \end{bmatrix} \begin{pmatrix} \frac{ik}{\omega_c} \\ \frac{\partial}{\partial U} \\ \frac{\partial}{\partial V} \\ \frac{in}{V} \end{pmatrix} \qquad (19)$$

$$B_{\underset{\sim}{k},\omega,n} = -\left(\frac{ik}{\omega_c}, \frac{\partial}{\partial U}, \frac{1}{V}\frac{\partial}{\partial V}, \frac{in}{V}\right) \sum_{\substack{k' \\ \omega' \\ n'}} \begin{bmatrix} B_{RR} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & B_{\underset{\sim}{vv}} & \\ 0 & & & \end{bmatrix} \begin{pmatrix} \frac{ik'}{\omega_c} \\ \frac{\partial}{\partial U} \\ \frac{\partial}{\partial V} \\ \frac{in'}{V} \end{pmatrix} \langle \varphi_{\underset{\sim}{k}',\omega'}\, \delta f^{*}_{\underset{\sim}{k}',\omega'} \rangle_{n'} \qquad (20)$$

where

$$D_{RR} = \frac{q^2}{M^2} |\varphi_{\underset{\sim}{k}',\omega'}|^2 G_{\underset{\sim}{k}'',\omega'',n''} \cdot \frac{k'^2}{2} J^2_{n'}(x') \ ,$$

$$D_{\underset{\sim}{vv}} = \frac{q^2}{M^2} |\varphi_{\underset{\sim}{k}',\omega'}|^2 G_{\underset{\sim}{k}'',\omega'',n''} \cdot$$

$$\begin{matrix} (\kappa' J_{n'}(x'))^2 & \frac{n'\omega_c\kappa'}{V} J^2_{n'}(x') & \frac{-k'\kappa'}{2}\frac{dJ^2_{n'}}{dx'} \\ \frac{n'\omega_c\kappa'}{V} J^2_{n'}(x') & \left(\frac{n'\omega_c}{V} J_{n'}(x')\right)^2 & \frac{-n'\omega_c k'}{2V}\frac{dJ^2_{n'}}{dx'} \\ \frac{-k'\kappa'}{2}\frac{dJ^2_{n'}}{dx'} & \frac{-n'\omega_c k'}{2V}\frac{dJ^2_{n'}}{dx'} & \left(k'\frac{dJ_{n'}}{dx'}\right)^2 \end{matrix}$$

$$B_{RR} = \frac{q^2}{M^2} G_{\underset{\sim}{k}'',\omega'',n''} \cdot \frac{kk'}{2} J_n^2(x) J_{n'}(x')$$

$$\beta_{\underset{\sim}{vv}} = \frac{q^2}{M^2} G_{\underset{\sim}{k}'',\omega'',n''} \cdot J_n(x) \cdot$$

$$\begin{matrix} \kappa'\kappa J_{n'}(x')J_n(x) & \frac{n\omega_c\kappa'}{V} J_{n'}(x')J_n(x) & -k\kappa' \frac{dJ_n}{dx} J_{n'}(x') \\ \frac{n'\omega_c\kappa}{V} J_{n'}(x')J_n(x) & \frac{n'n\omega_c}{V^2} J_{n'}(x')J_n(x) & \frac{-n'\omega_c k}{V}\frac{dJ_n}{dx} J_{n'}(x') \\ -k'\kappa \frac{dJ_{n'}}{dx'} J_n(x) & \frac{-n\omega_c k'}{V}\frac{dJ_{n'}}{dx'} J_n(x) & kk'\frac{dJ_n}{dx}\frac{dJ_{n'}}{dx'} \end{matrix}$$

and $x \equiv kV/\omega_c$, while n refers to the cyclotron harmonic number. In doing this, the ensemble average < > has included an average over the phase angles ψ_k and $\psi_{k'}$. That is,

$$\frac{q}{M} < \sum_{\underset{\sim}{k}',\omega'} \Phi_{\underset{\sim}{k}',\omega'} [\frac{kk'}{\omega_c} \sin(\psi_k - \psi_{k'}) - i\kappa' \frac{\partial}{\partial U} - ik'(\cos(\vartheta-\psi_{k'}) \frac{\partial}{\partial V}$$

$$- \frac{\sin(\vartheta-\psi_{k'})}{V} \frac{\partial}{\partial\vartheta}] \cdot f^{(2)}_{\underset{\sim}{k}'',\omega''}>$$

$$= \int d\vartheta' \sum_{\underset{\sim}{k}',\omega'} \frac{q^2}{M^2} |\varphi_{\underset{\sim}{k}',\omega'}|^2 <[\frac{kk'}{\omega_c} \sin(\psi_k-\psi_{k'}) - i\kappa' \frac{\partial}{\partial U}$$

$$- \frac{\partial}{\partial V}\left(ik'\cos(\vartheta-\psi_{k'})\right) + \frac{\partial}{\partial\vartheta}\left(\frac{ik'\sin(\vartheta-\psi_{k'})}{V}\right)] \cdot \exp[\frac{ik'V}{\omega_c} \sin(\vartheta-\psi_{k'})]$$

$$\cdot G_{\underset{\sim}{k}'',\omega''}(\vartheta,\vartheta')\exp[\frac{-ikV}{\omega_c} \sin(\vartheta'-\psi_k)] \cdot [\frac{kk'}{\omega_c} \sin(\psi_{k'}-\psi_k) - i\kappa' \frac{\partial}{\partial U}$$

$$+ ik'\cos(\vartheta-\psi_k) \frac{\partial}{\partial V} - \frac{ik'\sin(\vartheta'-\psi_k)}{V} \frac{\partial}{\partial\vartheta'}]> \cdot f_{\underset{\sim}{k},\omega}(\vartheta')$$

$$+ \int d\vartheta' \; \Phi_{\underset{\sim}{k},\omega}(\vartheta) \sum_{\substack{k' \\ \omega'}} \frac{q^2}{M^2} <[\frac{kk'}{\omega_c} \sin(\psi_k - \psi_{k'}) - \frac{\partial}{\partial U} (i\kappa')$$

$$- \frac{\partial}{\partial V} (ik'\cos(\vartheta - \psi_{k'})) + \frac{\partial}{\partial \vartheta} (\frac{ik'\sin(\vartheta - \psi_{k'})}{V})] \exp[\frac{ik'V}{\omega_c} \sin(\vartheta - \psi_{k'})]$$

$$\cdot \; G_{\underset{\sim}{k}'',\omega''}(\vartheta,\vartheta') \cdot \exp[-\frac{ikV}{\omega_c} \sin(\vartheta' - \psi_k)] \cdot [\frac{kk'}{\omega_c} \sin(\psi_k - \psi_{k'}) - i\kappa \frac{\partial}{\partial U}$$

$$- ik\cos(\vartheta' - \psi_k) \frac{\partial}{\partial V} - \frac{ik\sin(\vartheta' - \psi_k)}{V} \frac{\partial}{\partial \vartheta'}]> \cdot <\varphi_{\underset{\sim}{k}',\omega'} \delta f^*_{\underset{\sim}{k}',\omega'}> \quad ,$$

where averaged quantities $(<\cdots>)$ depends only on $\vartheta - \vartheta'$. The first and second $\int d\vartheta'$ integrals yield $\int C_{\underset{\sim}{k},\omega}(\vartheta,\vartheta') f_{\underset{\sim}{k},\omega}(\vartheta') d\vartheta'$ and $B_{\underset{\sim}{k},\omega}(\vartheta)\varphi_{\underset{\sim}{k},\omega}$, respectively.

We notice that the resulting renormalized terms are consistent with the non-Markovian structure of the Vlasov equation; in particular, all cyclotron harmonics are coupled. If one further makes a Markovian approximation, where $G_{\underset{\sim}{k}'',\omega'',n''}$ is replaced by $G_{\underset{\sim}{k}',\omega',n'}$, it follows that $C_{\underset{\sim}{k},\omega,n}$ is diagonalized. In this case $C_{\underset{\sim}{k},\omega,n}$ is be identical to that obtained by Dum and Dupree.[7] In that paper, the nonlinear dynamics are treated as a stochastic-acceleration problem for a turbulent-magnetized plasma.

The total fluctuation $f_{\underset{\sim}{k},\omega,n}$ consists of a component $f^c_{\underset{\sim}{k},\omega,n}$ that is induced by the electric field, and another component $\tilde{f}_{\underset{\sim}{k},\omega,n}$ that is generated by the mode-coupling and is responsible for the localized structures. To obtain the renormalized dielectric function $\varepsilon_{\underset{\sim}{k},\omega}$, we need the coherent response $f^c_{\underset{\sim}{k},\omega,n}$ that satisfies

$$[-i(\omega - \kappa U - n\omega_c) + C_{\underset{\sim}{k},\omega,n}] f^c_{\underset{\sim}{k},\omega,n}$$

$$= \frac{iq}{M} \varphi_{\underset{\sim}{k},\omega} \{ [\exp[\frac{ikV}{\omega_c} \sin(\vartheta - \psi_k)] (k \cos(\vartheta - \psi_k) \frac{\partial}{\partial V} + \kappa \frac{\partial}{\partial U})]_n <f>$$

$$- (\frac{iM}{q})B_{\underset{\sim}{k},\omega,n})$$

$$\equiv \frac{iq}{M}\varphi_{\underset{\sim}{k},\omega}\{[\exp[\frac{ikV}{\omega_c}\sin(\vartheta-\psi_k)](k\cos(\vartheta-\psi_k)\frac{\partial}{\partial V} + \kappa\frac{\partial}{\partial U})]_n \langle f\rangle_M\} \quad . \tag{21}$$

The last equality indicates that the effect of $B_{\underset{\sim}{k},\omega,n}$ is to modify the average distribution function $\langle f\rangle$, yielding $\langle f\rangle_M$. In the conventional nonlinear theory of wave fluctuations, only the coherent fluctuation $f^c_{\underset{\sim}{k},\omega,n}$ is included. Hence, by substituting $f_{\underset{\sim}{k},\omega,n}$ with $f^c_{\underset{\sim}{k},\omega,n}$, $B_{\underset{\sim}{k},\omega,n}$ can be expressed as

$$B_{\underset{\sim}{k},\omega,n} = (\frac{ik}{\omega_c}, \frac{\partial}{\partial U}, \frac{1}{V}\frac{\partial}{\partial V}V, \frac{in'}{V}) \cdot \sum_{\substack{\underset{\sim}{k}' \\ \underset{\sim}{\omega}' \\ n'}} [\beta_{\underset{\sim}{vv}}] \cdot \begin{pmatrix} \frac{ik'}{\omega_c} \\ \frac{\partial}{\partial U} \\ \frac{\partial}{\partial V} \\ \frac{in'}{V} \end{pmatrix}$$

$$\cdot (\frac{iq}{M}) |\varphi_{\underset{\sim}{k}',\omega'}|^2 \sum_m J_{m-n'}(x')J_m(x')G^*_{\underset{\sim}{k}',\omega',m}[\kappa'\frac{\partial}{\partial U} + \frac{m\omega_c}{V}\frac{\partial}{\partial V}]\langle f\rangle \tag{22}$$

When the incoherent fluctuation $\tilde{f}_{\underset{\sim}{k},\omega,n}$ is included, using the Poisson equation

$$k^2\varphi_{\underset{\sim}{k},\omega} = 4\pi n_0[q\int \delta f^i_{\underset{\sim}{k},\omega}\, d\underset{\sim}{v}^i - e\int \delta f^e_{\underset{\sim}{k},\omega} d\underset{\sim}{v}^e] \quad ,$$

Eqs. (21) and (22) of both species yield

$$\varphi_{\underset{\sim}{k},\omega} \cdot \{1 + \frac{4\pi n_0}{\underset{\sim}{k}^2}\sum_n [\frac{q^2}{M^2}\int \frac{d\underset{\sim}{v}^i}{2\pi} J^2_n(x_i)\, G^i_{\underset{\sim}{k},\omega,n} \cdot (n\omega_{c_i} + \kappa U_i)\overline{\langle f_i\rangle}$$

$$+ \frac{e^2}{m^2} \int \frac{d\underset{\sim}{v}^e}{2\pi} J_n^2(x_e)\ G^e_{\underset{\sim}{k},\omega,n}(n\omega_{c_e} + \kappa U_e)\langle\overline{f_e}\rangle])$$

$$= \frac{4\pi n_0}{\underset{\sim}{k}^2} [q \int \delta\tilde{f}^i_{\underset{\sim}{k},\omega}\ d\underset{\sim}{v}^i - e \int \delta\tilde{f}^e_{\underset{\sim}{k},\omega}\ d\underset{\sim}{v}^e] \ .$$

Thus, the total potential fluctuation

$$\varphi_{\underset{\sim}{k},\omega} = \frac{4\pi n_0}{\underset{\sim}{k}^2} \frac{[q\int\delta\tilde{f}^i_{\underset{\sim}{k},\omega}\ d\underset{\sim}{v}^i - e\int\delta\tilde{f}^e_{\underset{\sim}{k},\omega} d\underset{\sim}{v}^e]}{\varepsilon_{\underset{\sim}{k},\omega}} \equiv \frac{\tilde{\varphi}^i_{\underset{\sim}{k},\omega} - \tilde{\varphi}^e_{\underset{\sim}{k},\omega}}{\varepsilon_{\underset{\sim}{k},\omega}} = \frac{\tilde{\varphi}_{\underset{\sim}{k},\omega}}{\varepsilon_{\underset{\sim}{k},\omega}} \qquad (23)$$

where $\varepsilon_{\underset{\sim}{k},\omega}$ is the dielectric function, containing contributions from the nonlinear wave-particle interactions. Equation (23) shows that the self-consistent field $\varphi_{\underset{\sim}{k},\omega}$ can be viewed as if it were produced by a source field $\tilde{\varphi}_{\underset{\sim}{k},\omega}$ caused by the incoherent fluctuations $\tilde{f}_{\underset{\sim}{k},\omega}$. Since $\tilde{f}_{\underset{\sim}{k},\omega}$ is incoherent with the fluctuating electric field, it satisfies the equation[10]

$$[-i(\omega - \kappa U - n\omega_c) + C_{\underset{\sim}{k},\omega,n}]\tilde{f}_{\underset{\sim}{k},\omega,n} = 0. \qquad (24)$$

That is, $\tilde{f}_{\underset{\sim}{k},\omega}$ propagates along the particle orbit, and behaves as a macro-particle. Here, it is apparent that $C_{\underset{\sim}{k},\omega,n}$ is responsible for the deviation from the unperturbed particle orbits.

III. Two-Point Correlation Equation

The incoherent fluctuation structure is elucidated by examination of the two-point, one-time correlation function of the phase-space density fluctuation, $<\delta f(1,t)\delta f(2,t)>$. This correlation evolves on a slow time scale, since the fast variation from the particle orbiting at cyclotron frequency averages out. Turbulent relative diffusion dominates the evolution of a plasma on this time scale. The necessarily weak relative diffusion at small separation in the presence of a positive-free energy source results in very strong correlation at small separation in phase space.

It is straightforward to obtain an equation of the one-time, two-point correlation function of either species particles from the Vlasov equation,

$$\left[\frac{\partial}{\partial t} + \underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{X}_1} + \underset{\sim}{v}_2 \cdot \frac{\partial}{\partial \underset{\sim}{X}_2} + \omega_c \underset{\sim}{v}_1 \times \hat{z} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} + \omega_c \underset{\sim}{v}_2 \times \hat{z} \cdot \frac{\partial}{\partial \underset{\sim}{v}_2}\right] <\delta f(1)\delta f(2)>$$

$$+ \frac{q}{M} < \left[\underset{\sim}{E}(1) \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} + \underset{\sim}{E}(2) \cdot \frac{\partial}{\partial \underset{\sim}{v}_2}\right] \delta f(1)\delta f(2)>$$

$$= -\frac{q}{M}\left[<\underset{\sim}{E}(1)\delta f(2)> \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} <f(1)> + <\underset{\sim}{E}(2)\delta f(1)> \cdot \frac{\partial}{\partial \underset{\sim}{v}_2} <f(2)>\right] . \qquad (25)$$

At this point, we could proceed with the renormalization of the triplet terms by treating each component of each point as shown previously. Then, arguing that variation of the correlation function in the relative coordinates is much stronger than that in the comoving coordinates, we could obtain an approximate renormalized equation for the correlation function. However, in cylindrical coordinates it is very tedious to perform the procedure of subtracting the renormalized equation of one point from that of the other point to obtain a renormalized equation in the relative coordinates (i.e., obtain $\partial/\partial \underset{\sim}{q}_- \cdot \underset{\approx}{D}(-) \cdot \partial/\partial \underset{\sim}{q}_-$ by subtracting $\partial/\partial \underset{\sim}{q}_1 \cdot \underset{\approx}{D}(1) \cdot \partial/\partial \underset{\sim}{q}_1$ from

$\partial/\partial q_2 \cdot \underset{\approx}{D}(2) \cdot \partial/\partial q_2)$. Hence the method described above is difficult and inefficient. Instead, we shall transform to the relative coordinates at the beginning of the calculation. Renormalization is then carried out in the relative-cylindrical coordinate. We note that this method can indeed preserve the property that decorrelation of nearby particles is caused by a relative force (Eq. (26)).

First, we introduce the relative coordinates and the comoving coordinates

$$\underset{\sim}{X}_- = \frac{1}{2}(\underset{\sim}{X}_1 - \underset{\sim}{X}_2) \quad , \quad \underset{\sim}{v}_- = \frac{1}{2}(\underset{\sim}{v}_1 - \underset{\sim}{v}_2) \; ,$$

$$\underset{\sim}{X}_+ = \frac{1}{2}(\underset{\sim}{X}_1 + \underset{\sim}{X}_2) \quad , \quad \underset{\sim}{v}_+ = \frac{1}{2}(\underset{\sim}{v}_+ + \underset{\sim}{v}_2) \; .$$

In addition, let

$$\underset{\sim}{E}(-) \equiv \frac{1}{2}(\underset{\sim}{E}(1) - \underset{\sim}{E}(2)) \quad , \quad \underset{\sim}{E}(+) \equiv \frac{1}{2}(\underset{\sim}{E}(1) + \underset{\sim}{E}(2)) \; .$$

We will assume that the plasma is homogeneous,

$$\frac{\partial}{\partial \underset{\sim}{X}_+} <\delta f(1)\delta f(2)> = 0 \; ,$$

and that variation of the correlation function is much stronger in the relative coordinates than in the comoving coordinates,

$$\left|\frac{\partial}{\partial \underset{\sim}{v}_-} <\delta f(1)\delta f(2)>\right| >> \left|\frac{\partial}{\partial \underset{\sim}{v}_+} <\delta f(1)\delta f(2)>\right| \; .$$

Equation (25) simplifies to

$$[\frac{\partial}{\partial t} + \underset{\sim}{v}_- \cdot \frac{\partial}{\partial \underset{\sim}{X}_-} + \frac{q}{M} (\underset{\sim}{v}_- \times B_0) \cdot \frac{\partial}{\partial \underset{\sim}{v}_-}] <\delta f(1)\delta f(2)>$$

$$+ \frac{q}{M} < \frac{\partial}{\partial \underset{\sim}{v}_-} \underset{\sim}{E}(-)\delta f(1)\delta f(2)> = - \frac{q}{M} [<\underset{\sim}{E}(1)\delta f(2)> \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} <f(1)>$$

$$+ <\underset{\sim}{E}(2)\delta f(1)> \cdot \frac{\partial}{\partial \underset{\sim}{v}_2} <f(2)>] \quad . \qquad (26)$$

To renormalize the triplet term $<\underset{\sim}{E}(-)\delta f(1)\delta f(2)>$, it is important to realize that the fast variation of $\delta f(1)\delta f(2)$ will beat with the fast variation of $\underset{\sim}{E}(-)$ yielding a net contribution to the ensemble-averaged triplet. Before ensemble averaging, we note that $\delta f(1)\delta f(2)$ satisfies

$$|\underset{\sim}{v}_+ \cdot \frac{\partial}{\partial \underset{\sim}{X}_+} \delta f(1)\delta f(2)| >> |\underset{\sim}{v}_- \cdot \frac{\partial}{\partial \underset{\sim}{X}_-} \delta f(1)\delta f(2)|$$

and $$|\underset{\sim}{v}_+ \times \underset{\sim}{B}_o \cdot \frac{\partial}{\partial \underset{\sim}{v}_+} \delta f(1)\delta f(2)| >> |\underset{\sim}{v}_- \times \underset{\sim}{B}_0 \cdot \frac{\partial}{\partial \underset{\sim}{v}_-} \delta f(1)\delta f(2)|$$

when $|\underset{\sim}{v}_-| << |\underset{\sim}{v}_+|$. Hence $\delta f(1)\delta f(2)$ evolves according to the equation,

$$[\frac{\partial}{\partial t} + \underset{\sim}{v}_+ \cdot \frac{\partial}{\partial \underset{\sim}{X}_+} + \frac{q}{M} \underset{\sim}{v}_+ \times \underset{\sim}{B}_0 \cdot \frac{\partial}{\partial \underset{\sim}{v}_+}] \delta f(1)\delta f(2)$$

$$+ \frac{q}{M} [\underset{\sim}{E}(+) \cdot \frac{\partial}{\partial \underset{\sim}{v}_+} \delta f(1)\delta f(2) + \underset{\sim}{E}(-) \cdot \frac{\partial}{\partial \underset{\sim}{v}_-} \delta f(1)\delta f(2)] = S(1,2)$$

where

$$S(1,2) = - \frac{q}{M} [\underset{\sim}{E}(1)\delta f(2) \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} <f(1)> + \underset{\sim}{E}(2)\delta f(1) \cdot \frac{\partial}{\partial \underset{\sim}{v}_2} <f(2)>] \quad .$$

Hence, $G^{(0)}$, the linear propagator of the one-point equation, is an approximate propagator for $\delta f(1)\delta f(2)$. We can then follow the procedure of one-point renormalization that the driven mode $(\delta f(1)\delta f(2))^{(2)}$ is substituted into $\langle \partial/\partial \underset{\sim}{v}_- \cdot \underline{E}(-)(\delta f(1)\delta f(2))\rangle$, to obtain

$$\frac{q}{M} \langle \frac{\partial}{\partial \underset{\sim}{v}_-} \underset{\sim}{E}(-)\delta f(1)\delta f(2)\rangle$$

$$\approx -\frac{q^2}{M^2} \sum_{\substack{\underset{\sim}{k}\\ \underset{\sim}{\omega}\\ n}} \left(\frac{1}{2\pi}\right)^4 \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}_-} G^{(0)}_{\underset{\sim}{k},\omega,n}\ J_n^2 |\varphi^{(-)}_{\underset{\sim}{k},\omega}|^2 \cdot \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}_-} \langle\delta f(1)\delta f(2)\rangle ,$$

where $\varphi_{\underset{\sim}{k},\omega}(-) = \varphi_{\underset{\sim}{k},\omega}(1-\exp[i\underset{\sim}{k}\cdot\underset{\sim}{X}_-])$. Finally, Eq. (26) becomes

$$\{\frac{\partial}{\partial t} + \underset{\sim}{v}_- \cdot \frac{\partial}{\partial \underset{\sim}{X}_-} + \frac{q}{M}\, \underset{\sim}{v}_-\times\underset{\sim}{B}_0 \cdot \frac{\partial}{\partial \underset{\sim}{v}_-} - \frac{q^2}{M^2} \sum_{\substack{\underset{\sim}{k}\\ \underset{\sim}{\omega}\\ n}} \left(\frac{1}{2\pi}\right)^4 \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}_-} [J_n^2 G^{(0)}_{\underset{\sim}{k},\omega,n} |\varphi_{\underset{\sim}{k},\omega}(-)|^2] \cdot$$

$$\cdot\ \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}_-}\}\langle\delta f(1)\delta f(2)\rangle \cong \langle S(1,2)\rangle \cong S(1,1), \qquad (27)$$

where the last equality is because $S(1,1)$ is a smooth function at small separation. The left-hand side renormalized terms have the form of Fokker-Plank operators acting on the correlation function. Also, the diffusion coefficient here is simpler than that obtained previously. This is because the separation of scales prevents the operator $\underset{\sim}{k} \cdot \partial/\partial \underset{\sim}{v}_-$ from mixing with the propagator of comoving coordinates.

The source terms of both species can be calculated explicitly in a way similar to that used in constructing the test-particle-model Lenard-Balescu equation, where we separate

$$\langle \underset{\sim}{E}\delta f\rangle_{\underset{\sim}{k},\omega} \quad \text{into} \quad \langle \underset{\sim}{E} f^c\rangle_{\underset{\sim}{k},\omega} \quad \text{and} \quad \langle \underset{\sim}{E}\tilde{f}\rangle_{\underset{\sim}{k},\omega} .$$

A somewhat more complicated procedure (details shown in Appendix B) using Eqs. (21), (24) yields

$$<S^e(1,1)> \cong [D_\parallel^{ee}+D_\parallel^{ei}](\frac{\partial<f_e>}{\partial U})^2 + [F_\parallel^{ee}+F_\parallel^{ei}]<f_e> \frac{\partial<f_e>}{\partial U} \tag{28}$$

$$<S^i(1,1)> \cong [\underset{\approx}{D}^{ii}+\underset{\approx}{D}^{ie}] : \frac{\partial<f_i>}{\partial \underset{\sim}{V}} \frac{\partial<f_i>}{\partial \underset{\sim}{V}} + [\underset{\sim}{F}^{ii}+\underset{\sim}{F}^{ie}] \cdot <f_i> \frac{\partial<f_i>}{\partial \underset{\sim}{V}} , \tag{29}$$

where

$$D_\parallel^{ee} = \frac{2e^2}{m^2} \sum_{\substack{\underset{\sim}{k} \\ \omega}} \frac{\kappa^2|\tilde{\varphi}^e_{\underset{\sim}{k},\omega}|^2}{(2\pi)^4|\varepsilon_{\underset{\sim}{k},\omega}|^2} G^{(e)}_{\underset{\sim}{k},\omega} , \quad D_\parallel^{ei} = \frac{2e^2}{m^2} \sum_{\substack{\underset{\sim}{k} \\ \omega}} \frac{\kappa^2|\tilde{\varphi}^i_{\underset{\sim}{k},\omega}|^2}{(2\pi)^4|\varepsilon_{\underset{\sim}{k},\omega}|^2} G^{(e)}_{\underset{\sim}{k},\omega}$$

$$F_\parallel^{ee} = \frac{4e}{m} \sum_{\substack{\underset{\sim}{k} \\ \omega}} \frac{\kappa<\delta\tilde{f}^e\tilde{\varphi}^e>_{\underset{\sim}{k}}}{(2\pi)^4|\varepsilon_{\underset{\sim}{k},\omega}|^2} G^{(e)}_{\underset{\sim}{k},\omega} \frac{\mathrm{Im}\chi_e}{<f_e>} , \quad F_\parallel^{ei} = \frac{4e}{m} \sum_{\substack{\underset{\sim}{k} \\ \omega}} \frac{<\delta\tilde{f}^e\tilde{\varphi}^e>_{\underset{\sim}{k}}}{(2\pi)^4|\varepsilon_{\underset{\sim}{k},\omega}|^2} G^{(e)}_{\underset{\sim}{k},\omega} \frac{\mathrm{Im}\chi_i}{<f_e>} \tag{30}$$

$$\underset{\approx}{D}^{ii} : \frac{\partial<f_i>}{\partial \underset{\sim}{V}} \frac{\partial<f_i>}{\partial \underset{\sim}{V}} = \frac{2q^2}{M^2} \sum_{\substack{\underset{\sim}{k} \\ \omega \\ n}} \frac{|\tilde{\varphi}^i_{\underset{\sim}{k},\omega}|^2}{(2\pi)^4|\varepsilon_{\underset{\sim}{k},\omega}|^2} J_n^2 G^{(i)}_{\underset{\sim}{k},\omega,n} [\frac{n\omega_c}{V}\frac{\partial<f_i>}{\partial V} + \kappa \frac{\partial<f_i>}{\partial U}]^2$$

$$\underset{\approx}{D}^{ie} : \frac{\partial<f_i>}{\partial \underset{\sim}{V}} \frac{\partial<f_i>}{\partial \underset{\sim}{V}} = \frac{2q^2}{M^2} \sum_{\substack{\underset{\sim}{k} \\ \omega \\ n}} \frac{|\tilde{\varphi}^e_{\underset{\sim}{k},\omega}|^2}{(2\pi)^4|\varepsilon_{\underset{\sim}{k},\omega}|^2} J_n^2 G^{(i)}_{\underset{\sim}{k},\omega,n} [\frac{n\omega_c}{V}\frac{\partial<f_i>}{\partial V} + \kappa \frac{\partial<f_i>}{\partial U}]^2$$

$$\underset{\sim}{F}^{ii} \cdot \frac{\partial<f_i>}{\partial \underset{\sim}{V}} = \frac{4q}{M} \sum_{\substack{\underset{\sim}{k} \\ \omega \\ n}} \frac{<\delta\tilde{f}^i\tilde{\varphi}^i>_{\underset{\sim}{k}}}{(2\pi)^4|\varepsilon_{\underset{\sim}{k},\omega}|^2} J_n^2 G^{(i)}_{\underset{\sim}{k},\omega,n} \frac{\mathrm{Im}\chi_i}{<f_i>} [\frac{n\omega_c}{V}\frac{\partial<f_i>}{\partial V} + \kappa \frac{\partial<f_i>}{\partial U}]$$

$$\underset{\sim}{F}^{ie} \cdot \frac{\partial<f_i>}{\partial \underset{\sim}{V}} = \frac{4q}{M} \sum_{\substack{\underset{\sim}{k} \\ \omega \\ n}} \frac{<\delta\tilde{f}^i\tilde{\varphi}^i>_{\underset{\sim}{k}}}{(2\pi)^4|\varepsilon_{\underset{\sim}{k},\omega}|^2} J_n^2 G^{(i)}_{\underset{\sim}{k},\omega,n} \frac{\mathrm{Im}\chi_e}{<f_i>} [\frac{n\omega_c}{V}\frac{\partial<f_i>}{\partial V} + \kappa \frac{\partial<f_i>}{\partial U}] . \tag{31}$$

and $\tilde{\varphi}^e_{\underset{\sim}{k}} = \frac{4\pi n_o e}{k^2} \int d^3\underset{\sim}{v}\delta\tilde{f}^e_{\underset{\sim}{k}}$, $\tilde{\varphi}^i_{\underset{\sim}{k}} = \frac{4\pi n_o q}{k^2} \int d^3\underset{\sim}{v}\delta\tilde{f}^i_{\underset{\sim}{k}}$, $\mathrm{Im}\chi_e$ and $\mathrm{Im}\chi_i$ are the imaginary parts of the electron and ion susceptibilities respectively. The sources are expressed in terms of Fokker-Planck operators. As explained earlier, these describe the relaxation of the average distribution functions. The diffusion arises from stochastic acceleration of charged particles by the turbulent-electric field. The drag force results from collisionless exchange of momenta between charged particles.

For the electron source terms we have a local cancellation between the diffusion and the drag,

$$[\frac{\partial}{\partial U} D^e_{\parallel} \frac{\partial}{\partial U} - \frac{\partial}{\partial U} F^{ee}_{\parallel}]\langle f_e\rangle = 0 .$$

This is due to the conservation of particle momenta of localized same-species interactions occurring in a one-dimensional system. An analogy to this is the head-on collisions of two identical particles, in which they appear as to exchange phase-space positions without interacting. Hence, $\langle f_e\rangle$ does not relax because the final state is the same as the initial state. Relaxation of $\langle f_e\rangle$ thus requires the interaction of the electrons with the ions. By contrast, the ion dynamics is three-dimensional, hence the constraint for the one-dimensional interactions is broken. The same-species interactions of ions can now contribute to relax the average distribution function $\langle f_i\rangle$.

When $T^i_{\parallel} = T^i_{\perp}$, and $\langle f_i\rangle$ is Maxwellian, we can show (in Appendix B) that the same-species interactions cancel, i.e.,

$$[\frac{\partial}{\partial \underset{\sim}{V}} \cdot \underset{\approx}{D}^i \cdot \frac{\partial}{\partial \underset{\sim}{V}} - \frac{\partial}{\partial \underset{\sim}{V}} \cdot \underset{\sim}{F}^{ii}]\langle f\rangle^i = 0 .$$

The fact that $<f_i>$ is isotropic and Maxwellian is crucial for this local cancellation. This is because an isotropic Maxwellian distribution has no free energy by itself, thus the same-species interactions alone cannot relax $<f_i>$. In anisotropic collisionless plasmas, the same-species interactions are actually responsible for temperature isotropization. Here, in a current-driven plasma with $T^i_\parallel \neq T^i_\perp$, the isotropization terms are small in comparison to others that are responsible for relaxation of the current. This is because isotropization terms are proportional to the interaction of different harmonics n and m by a very weak coupling

$$(n-m) \frac{\omega_{ci}}{k_\parallel v_{ti}} \cdot \exp\left[-(n-m)^2 \frac{\omega_{ci}^2}{k_\parallel^2 v_{ti}^2}\right],$$

where the argument of the exponential has magnitude much larger than unity. Thus they are hereafter ignored.

Finally, we transform Eq. (27) to the guiding-center coordinates, and require that

$$\frac{\partial}{\partial \vartheta_-} < \delta f(1) \delta f(2)>^i = 0.$$

This is consistent with the fact that relative evolution, which determines $<\delta f(1)\delta f(2)>$, is slowly varying on the gyro-time scale. It then follows that

$$\{\frac{\partial}{\partial t} + U_- \frac{\partial}{\partial Z_-} - \frac{q^2}{M^2} \sum_{\substack{\underset{\sim}{k} \\ \underset{\sim}{\omega} \\ n}} \frac{J_n^2}{(2\pi)^4} |\varphi_{\underset{\sim}{k},\omega}|^2 G^{(i)}_{\underset{\sim}{k},\omega,n} \cdot \{[\frac{k^4 V_-^2}{4\omega_{ci}^2} + \frac{k^4 R_-^2}{8} + \frac{\kappa^2 k^2 Z_-^2}{2}] \frac{1}{\omega_{ci}^2} \frac{\partial}{\partial R_-^2}$$

$$+ \left[\frac{k^4 V_-^2}{8\omega_{ci}^2} + \frac{k^4 R_-^2}{4} + \frac{\kappa^2 k^2 Z_-^2}{2}\right] \frac{\partial^2}{\partial V_-^2}$$

$$+ \left[\frac{k^2 \kappa^2 V_-^2}{2\omega_{ci}^2} + \frac{k^2 \kappa^2 R_-^2}{2} + \kappa^4 Z_-^2\right] \frac{\partial^2}{\partial U_-^2}\}\Big)^i <\delta f(1)\delta f(2)> = <S>^i \quad . \tag{32}$$

This renormalized equation is only valid for high-frequency fluctuations. This is because in obtaining the simple expression for relative diffusion, we have assumed a separation of time scales between the comoving motion and relative motion, thus $|\frac{\partial}{\partial \vartheta_+} \delta f(1)\delta f(2)| >> |\frac{\partial}{\partial \vartheta_-} \delta f(1)\delta f(2)|$. Equation (32) then corresponds to the lowest order of the relative gyro-kinetic ordering, i.e. $\omega_c \frac{\partial}{\partial t} >> 1$.

It is straightforward to obtain the equation for the electron-correlation function,

$$\Big(\frac{\partial}{\partial t} + U_- \frac{\partial}{\partial Z_-} - \frac{e^2}{m^2} \sum_{\substack{k \\ \omega}} |\varphi_{k,\omega}|^2 G_{k,\omega}^{(e)} \kappa^4 Z_-^2 \frac{\partial^2}{\partial U_-^2}\Big)^e <\delta f(1)\delta f(2)>^e = <S>^e \quad . \tag{33}$$

Here, we have used $\bar{k}\rho_e << 1$ (highly magnetized electrons), hence the electron motion is one-dimensional.

IV. Steady State

In current-driven ion-cyclotron turbulence, the ion-perpendicular velocity is scattered, and the current-free energy is converted into perpendicular ion-thermal energy. During this process the average-distribution functions of electrons and ions vary slowly on a time scale much slower than that of nonlinear evolution. This can be shown as follows.

Nonlinear evolution occurs on a time scale

$$\tau_{NL} \approx \tau_e = \left(\frac{1}{\bar{\kappa}^2 D_{\parallel}^e}\right)^{1/3}$$

where $D_{\parallel}^e$ is the electron parallel diffusion coefficient. The quasi-linear flattening of the electron distribution function and the perpendicular heating of ions occur on the time scales of

$$\tau_{QL}^e \approx \frac{v_{te}^2}{D_{\parallel}^e}$$

and

$$\tau_{heat}^i \approx \frac{v_{ti}^2}{D_{\perp}^i} \approx \frac{v_{te}^2}{D_{\parallel}^e} \cdot \left[\left(\frac{1}{\bar{k}\rho_i}\right)\left(\frac{\bar{\kappa}^4 v_{te}^2 v_{ti}^2}{\omega_{ci}^4}\right)\right]$$

respectively, where v_{te}^2 and v_{ti}^2 are electron and ion thermal velocity respectively. For $|e\varphi/T_e| \ll 1$, it follows

$$\frac{\tau^e_{QL}}{\tau_{NL}} \gtrsim \omega^2_{ci}\tau^2_e >> [(\frac{1}{\bar{k}\rho_i})^4 (\frac{\bar{\kappa}^4 v^2_{te} v^2_{ti}}{\omega^4_{ci}})] \omega^2_{ci}\tau^2_e \approx \frac{\tau^i_{heat}}{\tau_{NL}} >> 1 .$$

Hence, we can regard the average-distribution functions as stationary over the time scale of interest. In this section, we will calculate the small-scale correlation functions for these quasi-stationary distribution functions.

Formally, one can invert the operators on the left side of Eqs. (32) and (33) (imposing the appropriate boundary conditions), and obtain

$$\langle \delta f(1) \delta f(2) \rangle^\sigma = \tau^\sigma_{cl}(\underset{\sim}{X}_-, \underset{\sim}{V}_-) \langle S \rangle^\sigma , \tag{34}$$

where τ^σ_{cl} is an operator of species σ. Physically, τ^σ_{cl} is the time for decorrelation of two nearby phase-space points, separated by $\underset{\sim}{X}_-, \underset{\sim}{V}_-$. This understanding enables one to estimate the eigenvalues of τ^σ_{cl} by solving for the relative trajectory of two initially-neighboring phase-space points.

In ion-cyclotron turbulence, electrons are strongly magnetized and their motion is restricted to the vicinity of the same field line, hence the perpendicular motion is negligible. On the other hand, since ion trajectories have moderate Larmor radii, the perpendicular electric field can scatter an ion off a field line. This perpendicular diffusion takes place sufficiently rapidly so that the parallel motion is simply free-streaming. (The relative importance of the perpendicular and the parallel diffusion of ions depends on the level of fluctuation and the ratio of the perpendicular and the parallel wavenumbers. Experiments[3] and computer simulations[6] both show strong perpendicular-ion heating, which indicates the dominance of the perpendicular-velocity diffusion.)

A set of coupled equations describing the relative trajectories can be obtained for each species,

$$\frac{d}{dt}\langle R_-^2\rangle = \frac{3}{4} Q \frac{\langle R_-^2\rangle}{\omega_{ci}^2} + \frac{1}{4}\frac{Q}{\omega_{ci}^4}\langle V_-^2\rangle + \frac{H}{2\omega_{ci}^2}\langle Z_-^2\rangle$$

$$\frac{d}{dt}\frac{\langle V_-^2\rangle}{\omega_{ci}^2} = \frac{1}{4} Q \frac{\langle R_-^2\rangle}{\omega_{ci}^2} + \frac{3}{4}\frac{Q}{\omega_{ci}^4}\langle V_-^2\rangle + \frac{H}{2\omega_{ci}^2}\langle Z_-^2\rangle \tag{35}$$

$$\frac{dZ_-}{dt} = U_-$$

for ions, and

$$\frac{d^3\langle Z_-^2\rangle}{dt^3} = F\langle Z_-^2\rangle$$

$$\frac{d\langle R_-^2\rangle}{dt} = 0 \tag{36}$$

$$\frac{d\langle V_-^2\rangle}{dt} = 0$$

for electrons, where

$$\binom{Q}{H} = \frac{q^2}{M_i^2}\left(\frac{1}{2\pi}\right)^4 \sum_{\underset{\sim}{k},\underset{\sim}{\omega},n} J_n^2 |\varphi_{\underset{\sim}{k},\omega}|^2 G^i_{\underset{\sim}{k},\omega,n} \binom{k^4}{k^2\kappa^2} \text{ , and } F = \frac{e^2}{m_e^2}\left(\frac{1}{2\pi}\right)^4 \sum_{\underset{\sim}{k},\underset{\sim}{\omega}} |\varphi_{\underset{\sim}{k},\omega}|^2 G^e_{\underset{\sim}{k},\omega}\kappa^4 .$$

The time-asymptotically dominant solutions are

$$<Z_-^2(t)>^i = <Z_-^2(0)>^i + 2<U_-(0)Z_-(0)>^i t + <U_-^2(0)>^i t^2$$

$$<X_{\perp -}^2(t)>^i = \frac{3}{4} \exp[t/\tau_i]\ \bar{k}^2 \{ <X_{\perp -}^2(0)>^i + \frac{<V_-^2(0)>^i}{2\omega_{ci}^2} + (\frac{\bar{\kappa}}{\bar{k}})^2 [<Z_-^2(0)>^i$$

$$+ 2<U_-(0)>^i \tau_i + 2<U_-^2(0)>^i \tau_i^2]\} \tag{37}$$

$$<Z_-^2(t)>^e = \frac{1}{3} \exp[t/\tau_e]\ \bar{\kappa}^2 \{ <Z_-^2(0)>^e + 2<U_-(0)Z_-(0)>^e \tau_e$$

$$+ 2<U_-^2(0)>^e \tau_e^2 + (\frac{\bar{k}}{\bar{\kappa}})^2 \frac{<X_\perp^2(0)>^e}{2}\} \tag{38}$$

$$<X_{\perp -}^2(t)>^e = <X_{\perp -}^2(0)>^e \ ,$$

where $\tau_i = \omega_{ci}^2/Q$, $\tau_e = (F)^{-1/3}$ and $<X_{\perp -}^2> \cong <R_-^2> + \frac{<V_-^2>}{2\omega_c}$. Notice that on the time scale of relevance, ions freely stream along the field line and experience turbulent scattering across the field line; while the electron motion is dominated by parallel diffusion. The exponential time-dependence of rapid decorrelation results from the nature of the relative-diffusion coefficients. When $\tau<\tau_i$ (or $t<\tau_e$), the relative motion is slow because of weak relative diffusion. When $t>\tau_i$ (or $t>\tau_e$) the particles have decorrelated, hence the relative motion becomes rapidly varying.

The clump life time $\tau_{cl}(\underset{\sim}{X}_-, \underset{\sim}{V}_-)$ is defined as the time that two nearby phase-space points require to achieve a separation of a distance $<|\underset{\sim}{k}|>^{-1}$, so that

$$\bar{k}^2 <X_{\perp -}^2(\tau_{cl})> + \bar{\kappa}^2 <Z_-^2(\tau_{cl})> = 1 \ . \tag{39}$$

to $1/\tau_i$. The reason is discussed here. The ion clump has basic correlation lengths $(\bar{\kappa})^{-1}$ and $(\bar{k})^{-1}$ in the parallel and perpendicular directions, respectively. Furthermore, in the perpendicular direction, the Lorentz force is much stronger than the relative turbulent electric field, thus correlation in the perpendicular velocity scale is determined by the relative gryro-motion and is $\omega_{ci}/\bar{k}$. In the parallel direction, on the other hand, the dynamics are essentially ballistic streaming. Hence, the parallel velocity correlation length does not directly result from parallel scattering, but from the fact that the parallel velocity dispersion leads to perpendicular particle-scattering by waves of different phase velocities, which causes decorrelation. Since the scattering occurs on the time scale τ_i, thus the parallel velocity correlation scales to τ_i. For electrons, the (relative) turbulent parallel electric field scatters relative parallel velocity, hence U_-^e scales to τ_e. This explains why in the expressions for clump life times the parallel velocities are associated with the decorrelation time, while the perpendicular velocities are not. It is instructive to note that in deriving the expression of ion-clump lifetime, we have used the fact that

$$|\omega_{ci} \frac{\partial}{\partial t}| >> 1 \qquad \text{or} \qquad \tau_i \omega_{ci} >> 1$$

in Sec. III. These are equivalent to the assumption that the ion Lorentz force is much stronger than the turbulent electric force field. In the limit of weak magnetic and strong fluctuating electric fields, where the straight-line-orbit limit can be recovered, the above analysis is invalid.

Following Boutros-Ghali, and Dupree[10], we can evaluate the clump component of the two-point correlation function,

$$\langle\tilde{f}\tilde{f}\rangle^{\sigma} \sim [\tau^{\sigma}_{c1}(X_-, V_-) - \tau_\sigma]\langle S\rangle^{\sigma} . \tag{42}$$

That is, the nonsingular part of the correlation function has an approximate magnitude $\tau_\sigma\langle S\rangle^\sigma$.

To investigate moments and extract results, one can integrate over velocities in Eq. (42). After spatial-temporal Fourier transformation and integration over V_1 and V_2, we have

$$\int dV_1 dV_2 \langle\delta\tilde{f}(1)\delta\tilde{f}(2)\rangle^e_k \cong - \int dX_- e^{ik\cdot X_-}$$

$$\cdot \int dV_- \tau_e \ln\left(\frac{\bar{k}^2}{8.15}\left(Z_-^2 + 2U_- Z_- \tau_e + 2U_-^2 \tau_e^2 + \frac{\bar{k}^2}{2\bar{\kappa}^2} X_{\perp -}^2\right)\right) \int dV_+ \langle S\rangle \tag{43}$$

and

$$\int dV_1 dV_2 \langle\delta\tilde{f}(1)\delta\tilde{f}(2)\rangle^i_k \cong -\int dX_- e^{ik\cdot X_-}$$

$$\cdot \int dV_- \tau_i \ln\left(\frac{\bar{k}^2}{2}\left(X_{\perp -}^2 + \frac{V_-^2}{2\omega_{ci}^2} + \frac{\bar{\kappa}^2}{\bar{k}^2}\left(Z_-^2 + 2U_- Z_- \tau_i + 2U_-^2 \tau_i^2\right)\right)\right) \int dV_+ \langle S\rangle . \tag{44}$$

Notice that the integrand has a width $1/\sqrt{\kappa}\tau_\sigma$ in U_-^σ, hence the integral $\int dU_-^\sigma$ introduces a factor $1/\tau_\sigma$ to cancel the coefficient τ_σ of the clump lifetime. Therefore, the original nonlinear equations of squared fluctuation amplitudes become linear after the velocity integration. Using Poisson's equation, we obtain

$$<\tilde{\varphi}^2>^i_{\underset{\sim}{k},\omega} = \frac{8A^i(\underset{\sim}{k})\Lambda_{n_\omega}(k\rho_i)}{\pi k^4} \sum_{\underset{\sim}{k}'} \frac{\left(\frac{\Lambda_{n'_\omega}(\sqrt{2}k'\rho_i)}{2\Lambda_{n'_\omega}(k'\rho_i)}\right)\kappa'(k')^4}{(2\pi)^3|\varepsilon_{\underset{\sim}{k}',\omega'}|^2} \cdot$$

$$\left\{<\tilde{\varphi}^2>^e_{\underset{\sim}{k}',\omega'}(\mathrm{Im}\chi^i_{\underset{\sim}{k}',\omega'})^2 + <\tilde{\varphi}^2>^i_{\underset{\sim}{k}',\omega'}\mathrm{Im}\chi^i_{\underset{\sim}{k}',\omega'}\mathrm{Im}\chi^e_{\underset{\sim}{k}',\omega'}\right\}\Bigg|_{\frac{\omega-n_\omega\omega_{ci}}{\kappa} = \frac{\omega'-n'_\omega\omega_{ci}}{\kappa'}} \tag{45}$$

and

$$<\tilde{\varphi}^2>^e_{\underset{\sim}{k},\omega} = \frac{8A^e(\underset{\sim}{k})}{\pi k^4} \sum_{\underset{\sim}{k}'} \frac{\kappa'(k')^4}{(2\pi)^3|\varepsilon_{\underset{\sim}{k}',\omega'}|^2} \cdot$$

$$\cdot \left\{<\tilde{\varphi}^2>^i_{\underset{\sim}{k}',\omega'}(\mathrm{Im}\chi^e_{\underset{\sim}{k}',\omega'})^2 + <\tilde{\varphi}^2>^e_{\underset{\sim}{k}',\omega'}\mathrm{Im}\chi^i_{\underset{\sim}{k}',\omega'}\mathrm{Im}\chi^e_{\underset{\sim}{k}',\omega'}\right\}\Bigg|_{\frac{\omega}{\kappa} = \frac{\omega'}{\kappa'}} \tag{46}$$

where terms proportional to $<\tilde{\varphi}^2>^e_{\underset{\sim}{k}',\omega'}(\mathrm{Im}\chi^i)^2$, $<\tilde{\varphi}^2>^i_{\underset{\sim}{k}',\omega'}\mathrm{Im}\chi^i\mathrm{Im}\chi^e$ in Eq. (45) and $<\tilde{\varphi}^2>^i_{\underset{\sim}{k}',\omega'}\mathrm{Im}\chi^i\mathrm{Im}\chi^e$, $<\tilde{\varphi}^2>^i_{\underset{\sim}{k}',\omega'}(\mathrm{Im}\chi^e)^2$ in Eq. (46) correspond to D^{ie}, F^{ie} and D^{ei}, F^{ei} respectively,

$$A^\sigma(\underset{\sim}{k}) = \int d\underset{\sim}{V}_- \int d\underset{\sim}{X}_- e^{i\underset{\sim}{k}\cdot\underset{\sim}{X}_-}\left[\tau^\sigma_{cl} - \tau_\sigma\right] ,$$

and

$$\Lambda_n(k\rho_i) = I_n(k^2\rho_i^2)e^{-k^2\rho_i^2} .$$

The summation over n has one dominant term $n=n_\omega$, for a given ω, because the ions can only interact with fluctuations with Doppler-shifted phase velocity $\omega-n\omega_c/\kappa$ near the ion thermal velocity. Since κ is small $(\kappa<<\omega_{ci}/v_{ti})$ in ion-cyclotron turbulence, only one harmonic is resonant.

To evaluate the $\underset{\sim}{k}'$ integral we note that $1/|\varepsilon_{\underset{\sim}{k}',\omega'}|^2$ can be expressed as $1/|\mathrm{Re}\varepsilon_{\underset{\sim}{k}',\omega'}|^2 + |\mathrm{Im}\varepsilon_{\underset{\sim}{k}',\omega'}|^2$, which is peaked where both $\mathrm{Re}\varepsilon_{\underset{\sim}{k}',\omega'}$ and $\mathrm{Im}\varepsilon_{\underset{\sim}{k}',\omega'}$ are minimal. Since the magnitude of $|\mathrm{Re}\varepsilon_{\underset{\sim}{k}',\omega'}|$ is generally larger than that of $|\mathrm{Im}\varepsilon_{\underset{\sim}{k}',\omega'}|$ (when $T^i_{\parallel} \lesssim T^i_{\perp}$), the minimum of $|\varepsilon_{\underset{\sim}{k}',\omega'}|^2$ is located near where $|\mathrm{Re}\varepsilon_{\underset{\sim}{k}',\omega'}| = 0$ and $|\mathrm{Im}\varepsilon_{\underset{\sim}{k}',\omega'}|$ is minimal. Hence the k' integral can be evaluated at the zero of $\mathrm{Re}\varepsilon_{\underset{\sim}{k}',\omega'}$ (for situations where the magnitude of $|\mathrm{Re}\varepsilon|$ is not neccessarily larger than that of $|\mathrm{Im}\varepsilon|$, see Appendix C).

As a result, ion-cyclotron clumps are associated with ion-cyclotron waves, for which $\mathrm{Re}\varepsilon_{\underset{\sim}{k},\omega} = 0$. In the limit $T^i_{\perp} \gg T^i_{\parallel}$, the magnitude of $|\mathrm{Im}\varepsilon_{\underset{\sim}{k},\omega}|$ is comparable to $|\mathrm{Re}\varepsilon_{\underset{\sim}{k},\omega}|$ because the normal mode frequencies are approaching cyclotron harmonics where the wave dissipation is the greatest. Hence, in the spectrum sum the approximation $\mathrm{Re}\varepsilon_{\underset{\sim}{k}',\omega'} = 0$ is not valid, and the ballistic frequencies do not coincide with the normal mode frequencies. In fact, the long-range character of waves disappears due to the highly dissipative media. Ion cyclotron turbulence then consists of only the clump fluctuations, analogous to ion-acoustic turbulence with equal ion and electron temperatures. In this regime, one must solve the integral equations numerically. Hereafter, we shall confine ourselves to the wave-clump regime.

Thus, using the pole approximation, evaluating $\underset{\sim}{k}$ at $\underset{\sim}{k}_\omega$, and assuming that only one harmonic is dominant for a given ω, we obtain the expression

$$\langle\tilde{\varphi}^2\rangle^e_{\underset{\sim}{k},\omega}\Big|_{\underset{\sim}{k}=\underset{\sim}{k}_\omega} = \left\{\frac{a^e \underset{\sim}{k}^2\lambda_D^2}{|\mathrm{Im}\varepsilon_{\underset{\sim}{k},\omega}|}\left[(\mathrm{Im}\chi^e_{\underset{\sim}{k},\omega})^2 \langle\tilde{\varphi}^2\rangle^i_{\underset{\sim}{k},\omega} + \mathrm{Im}\chi^e_{\underset{\sim}{k},\omega}\,\mathrm{Im}\chi^i_{\underset{\sim}{k},\omega}\langle\tilde{\varphi}^2\rangle^e_{\underset{\sim}{k},\omega}\right]\right\}\Big|_{\underset{\sim}{k}=\underset{\sim}{k}_\omega} \quad (47)$$

$$\langle\tilde{\varphi}^2\rangle^i_{\underset{\sim}{k},\omega}\Big|_{k=k_\omega} = \left\{\frac{a^i \underset{\sim}{k}^2\lambda_D^2}{|\mathrm{Im}\varepsilon_{\underset{\sim}{k},\omega}|}\left[(\mathrm{Im}\chi^i_{\underset{\sim}{k},\omega})^2 \langle\tilde{\varphi}^2\rangle^e_{\underset{\sim}{k},\omega} + \mathrm{Im}\chi^e_{\underset{\sim}{k},\omega}\,\mathrm{Im}\chi^i_{\underset{\sim}{k},\omega}\langle\tilde{\varphi}^2\rangle^i_{\underset{\sim}{k},\omega}\right]\right\}\Big|_{\underset{\sim}{k}=\underset{\sim}{k}_\omega} . \quad (48)$$

Here, $a^e \sim 0.7$, $a^i \sim 0.15/\bar{k}^2\rho_i^2 \sim 0.1$, obtained from $A^e(\underset{\sim}{k}_\omega)$ and $A^i(\underset{\sim}{k}_\omega)$ respectively. A condition for stationary turbulence,

$$(a^e+a^i)\underset{\sim}{k}^2\lambda_D^2 \mathrm{Im}\chi^e_{\underset{\sim}{k},\omega}\mathrm{Im}\chi^i_{\underset{\sim}{k},\omega} = -|\mathrm{Im}\varepsilon_{\underset{\sim}{k},\omega}| = \mathrm{Im}\varepsilon_{\underset{\sim}{k},\omega} , \tag{49}$$

follows directly. The choice of sign is based on the physical motivation that the waves must be overdamped in the presence of a (nonlinear) noise source. Moreover, the structure of Eq. (49) persists in other plasma-turbulence problems, such as the wave-clump regime of ion-acoustic turbulence, drift-wave turbulence, etc.

The frequency linewidth due to the finite collective dissipation $\mathrm{Im}\varepsilon_{\underset{\sim}{k},\omega}$ can be expressed as

$$\Delta\omega_{\underset{\sim}{k}} \cong \left| \mathrm{Im}\varepsilon_{\underset{\sim}{k},\omega} \Big/ \left|\frac{\partial \mathrm{Re}\,\varepsilon_{\underset{\sim}{k},\omega}}{\partial\omega}\right| \right|_{\omega=\omega_{\underset{\sim}{k}}} . \tag{50}$$

For a given mass ratio, temperature ratio and drift velocity, $\Delta\omega_{\underset{\sim}{k}}$ may then be estimated numerically.

Physically, the above results may be summarized as follows. First, the multi-species plasma can relax its configuration to drive formation of small-scale fluctuations. This relaxation mechanism is similar to that in the test particle model, where only the interaction of different particle species contributes to this relaxation. Second, interactions that contribute to the driving source arise from the collective dissipation of wave-particle interactions and are proportional to $\mathrm{Im}\chi^i_{\underset{\sim}{k},\omega}$ and $\mathrm{Im}\chi^e_{\underset{\sim}{k},\omega}$. Third, waves that are emitted from clumps must be dissipated by the plasma in order to maintain stationary turbulence. Balancing the dissipation $\mathrm{Im}\varepsilon_{\underset{\sim}{k},\omega}$ with the driving source, Eq. (49) follows, i.e.

$$\mathrm{Im}\varepsilon_{\underset{\sim}{k},\omega} \propto \mathrm{Im}\chi^{e}_{\underset{\sim}{k},\omega}\mathrm{Im}\chi^{i}_{\underset{\sim}{k},\omega} .$$

Experimentally, one is concerned about the possibility of measuring the clump fluctuations in ion cyclotron turbulence. For the regime $T^i_\perp \approx T^i_\parallel \approx T^e$, wave fluctuations are weakly damped, the mixture of wave and clump fluctuations may make a clear identification of clumps difficult. However, if the fluctuation level of turbulence is sufficiently high so that the collective dissipation is large enough to damp the wave fluctuations, then the clump fluctuations will be more likely to be detected. For the regime $T^i_\perp >> T^i_\parallel \sim T^e$, turbulence is dominated by clump fluctuations and a direct measurement of clump fluctuations should be possible.

Next, one may be concerned about which species of clump fluctuation is more likely to be measured. The probability of measuring them is proportional to a statistical average of clump amplitudes. Therefore the relative size of $<\tilde{\varphi}^2>^e_{\underset{\sim}{k},\omega}$ to $<\tilde{\varphi}^2>^i_{\underset{\sim}{k},\omega}$ is a reasonable measure of likelihood of which species of clump fluctuation will be detected. From Eqs. (47) and (48), it is straightforward to obtain that

$$\frac{<\tilde{\varphi}^2>^e_{\underset{\sim}{k},\omega}}{<\tilde{\varphi}^2>^i_{\underset{\sim}{k},\omega}} = \frac{a^e|\mathrm{Im}\chi^e_{\underset{\sim}{k},\omega}|}{a^i|\mathrm{Im}\chi^i_{\underset{\sim}{k},\omega}|} \approx \frac{a^e}{a^i} = \frac{0.7}{\dfrac{0.15}{(\bar{k}\rho_i)^2}} \approx \quad 7. \tag{51}$$

The fact that $a^i << a^e$ is due to the effects of finite Larmor-radii of ions, which greatly weaken the driving source and also reduces the perpendicular velocity correlation length of ion clumps. As a consequence, the electron clump fluctuations are more likely to be observed than ion clump fluctuations.

V. Clump Instability

A state of stationary turbulence can be attained when damping of collective resonances balances emission. Both processes are related to the amplitude-dependent susceptibilities $\mathrm{Im}\chi^{e}_{\underset{\sim}{k},\omega}$ and $\mathrm{Im}\chi^{i}_{\underset{\sim}{k},\omega}$. When an imbalance occurs, the plasma will adjust its fluctuation level so as to satisfy the steady-state condition. Below the threshold-drift velocity, the free energy is insufficient for excitation of fluctuations, and the steady-state fluctuation level is that of thermal noise. Beyond the threshold-drift velocity, the free energy available can be used to excite the enhanced local phase-space fluctuations. This threshold-drift velocity is in general smaller than the threshold-drift velocity for linear instability because the clump instability can occur for $\mathrm{Im}\varepsilon<0$.

To obtain the threshold drift of the nonlinear instability, we can express Eq. (49) as

$$\mathrm{Im}\chi^{e}_{\underset{\sim}{k},\omega} = \frac{|\mathrm{Im}\chi^{i}_{\underset{\sim}{k},\omega}|}{1+(a^{e}+a^{i})\underset{\sim}{k}^{2}\lambda_{D}^{2}|\mathrm{Im}\chi^{i}_{\underset{\sim}{k},\omega}|} . \qquad (52)$$

where $\mathrm{Im}\chi^{e}_{\underset{\sim}{k},\omega}$ contains the free-energy source, the drift velocity v_{D}. When $T^{i}_{\parallel} \sim T^{i}_{\perp}$, the expressions for $\mathrm{Im}\chi^{e}_{\underset{\sim}{k},\omega}$ and $|\mathrm{Im}\chi^{i}_{\underset{\sim}{k},\omega}|$ are

$$\mathrm{Im}\chi^{e}_{\underset{\sim}{k},\omega} \approx \left[\frac{v_{D}}{v_{te}} - \frac{\omega}{\kappa v_{te}}\right] \left(\frac{1}{\underset{\sim}{k}^{2}\lambda_{D}^{2}}\right)$$

$$\mathrm{Im}\chi^{i}_{\underset{\sim}{k},\omega} \approx \frac{-\omega}{\kappa v_{t\parallel i}} \left(\frac{T^{e}}{T^{i}_{\perp}}\right) \Lambda_{n_{\omega}}(k^{2}\rho_{i}^{2}) \exp\left[\frac{-(\omega-n_{\omega}\omega_{ci})^{2}}{\kappa^{2}v^{2}_{t\parallel i}}\right]\left(\frac{1}{\underset{\sim}{k}^{2}\lambda_{D}^{2}}\right),$$

with ω and $\underset{\sim}{k}$ satisfying $\mathrm{Re}\,\varepsilon_{\underset{\sim}{k},\omega}$. A straightforward substitution of the above expressions into Eq. (52) yields

$$\frac{v^{thr}}{v_{te}} \approx \frac{\omega}{\kappa v_{te}} + \frac{\frac{\omega}{\kappa v_{t\parallel i}}\left(\frac{T^e}{T^i_\perp}\right)\Lambda_{n_\omega}\exp\left[-\frac{(\omega-n_\omega\omega_{ci})^2}{\kappa^2 v^2_{t\parallel i}}\right]}{1+\sqrt{\pi}\,(a^e+a^i)\,\frac{\omega}{\kappa v_{t\parallel i}}\left(\frac{T^e}{T^i_\perp}\right)\Lambda_{n_\omega}\exp\left[-\frac{(\omega-n_\omega\omega_{ci})^2}{\kappa^2 v^2_{t\parallel i}}\right]} . \tag{53}$$

While the linear threshold-drift velocity is

$$\left[\frac{v^{thr}}{v_{te}}\right]^L \approx \frac{\omega}{\kappa v_{te}} + \frac{\omega}{\kappa v_{t_{\parallel i}}}\left(\frac{T^e}{T^i_\perp}\right)\Lambda_{n_\omega}\exp\left[\frac{-(\omega-n_\omega\omega_{ci})^2}{\kappa^2 v^2_{t_{\parallel i}}}\right] . \tag{54}$$

The factor (a^i+a^e) in Eq. (53) accounts for the reduction of threshold-drift velocity.

The reduction of the threshold-drift velocity may be substantial if $(\underset{\sim}{k}^2\lambda_D^2)\mathrm{Im}\chi^i_{\underset{\sim}{k},\omega} \sim O(1)$, that is when $|\omega-n_\omega\omega_{ci}/\kappa v_{t_{\parallel i}}| < 1$, in which case the second term on the right of Eq. (53) can be sizably reduced. When $T^i_\perp > T^i_\parallel \sim T^e$, the wave fluctuations emitted by clumps tend to approach this regime, hence the clump effect may be enhanced. In Fig. 2 we plot both of the linear and nonlinar threshold-drift velocities versus $T^i_\perp/T^e\ \sqrt{m_e/M_i}$. As the latter increases the both threshold velocities increase, but the reduction Δv^{thr} does not increase as rapidly. When $T^i_\perp/T^e\ \sqrt{m_e/M_i} \sim 0.10$, there is a maximum Δv^{thr} with $\Delta v^{thr}/v^{thr} \sim 7\%$. This is because, upon minimizing the threshold velocity, the parallel-phase velocity $\omega_{\underset{\sim}{k}}/\kappa$ of the nonlinearly-excited fluctuations increases with $T^i_\perp/T_e\ \sqrt{m_e/M_i}$ more rapidly than that of the linear modes. According to Eq. (53), this increment in $\omega_{\underset{\sim}{k}}/\kappa$ tends to offset the reduction of

v^{thr} due to finite ion dissipation. This explains why the reduction Δv^{thr} saturates even in the case of strong ion dissipation.

By contrast, the ion-acoustic branch wave-clump turbulence is quite different. The ion-acoustic waves have an almost constant parallel-phase velocity. Hence, the first term of Eq. (53) is constant, and the nonlinear threshold-drift velocity is determined by the enhanced ion dissipation. Reduction of threshold drift can thus be very significant[11,17] (Appendix D).

If the drift velocity is above the nonlinear-threshold drift, the fluctuation will grow and hence the amplitude-dependent collective-resonance dissipation will increase till steady state results. The growing correlation function satisfies the equation,

$$[\gamma + \frac{1}{\tau_{c1}}] <\delta f_{(1)} \delta f_{(2)}>^{\sigma} \cong <S>^{\sigma} \tag{55}$$

$$<\delta f_{(1)} \delta f_{(2)}>^{\sigma} \cong \frac{\tau_{c1}^{\sigma}}{1+\gamma\tau_{c}^{\sigma}} <S>^{\sigma} . \tag{56}$$

These expressions are valid when the nonlinear growth rate γ is smaller than the shifted frequency $\omega-n\omega_c$. This has been assumed from the outset to guarantee a separation of time scales. It will be shown that the nonlinear growth rate γ scales with the particle trapping-time. Hence, so long as the fluctuation amplitude is small enough, the condition for separation of time scales can always be met. Following the same procedure as that in the previous section, one finds

$$\mathrm{Im}\varepsilon_{\underset{\sim}{k},\omega} = [\frac{a^{i}}{1+\gamma\tau_{i}} + \frac{a^{e}}{1+\gamma\tau_{e}}] \ \mathrm{Im}\chi^{i}_{\underset{\sim}{k},\omega} \mathrm{Im}\chi^{e}_{\underset{\sim}{k},\omega} \ (\underset{\sim}{k}^{2}\lambda_{D}^{2}) . \tag{57}$$

In the limit of weak growth, $\gamma\tau_i$, $\gamma\tau_e < 1$, so $1/1+\gamma\tau_i \sim 1-\gamma\tau_i$ and $1/1+\gamma\tau_e \sim 1-\gamma\tau_e$. This yields

$$\gamma(a^i\tau_i+a^e\tau_e) = \frac{(v_D-v_c)}{\mathrm{Im}\chi^e}\left|\frac{\partial}{\partial v_D}\,\mathrm{Im}\chi^e\right| , \tag{58}$$

where v_c is the critical drift velocity for maintaining the stationary turbulence at a certain fluctuation level, and hence is amplitude-dependent. The growth rate scales with the test-particle diffusion times, weighted by the relative sizes of the clumps of the corresponding species. When $\tau_e \sim \tau_i$, $a^e\tau_e >> a^i\tau_i$ the clump-growth rate is that of an electron-decorrelation time.

If the electron-drift velocity significantly exceeds the threshold, and $\gamma\tau_e$, $\gamma\tau_i \gtrsim 1$ then

$$\gamma \approx \left[\frac{a^i}{\tau_i} + \frac{a^e}{\tau_e}\right] \frac{\mathrm{Im}\chi^e\,\mathrm{Im}\chi^i\,\underset{\sim}{k}^2\lambda_D^2}{\mathrm{Im}\varepsilon} \sim \left[\frac{a^i}{\tau_i} + \frac{a^e}{\tau_e}\right] (a^i+a^e)^{-1} . \tag{59}$$

For ion-cyclotron turbulence, $a^i/\tau_i << a^e/\tau_e$, hence

$$\gamma \sim \frac{1}{\tau_e} . \tag{60}$$

Notice that Eq. (57) is quite general. In the case of ion-acoustic turbulence, $a^i = a^e$ but $\tau_e << \tau_i$, hence in the limit of small growth, the growth rate scales with the ion-decorrelation time τ_i. However, in the case of strong growth, the growth rate scales with the electron decorrelation time τ_e.

6. Saturation Level and Anomalous Resistivity

In ion-cyclotron turbulence, perpendicular nonlinear-ion scattering and heating is proposed as the dominant mechanism for saturation. Energetically, this nonlinear process transforms the electron current energy into the ion perpendicular thermal energy. Dynamically, this nonlinear effect leads to a broadened wave-ion resonance. The width of resonance, which is dependent upon the fluctuation level, yields a correction to the linear-ion dissipation. If the electron-drift velocity v_D does not significantly exceed the threshold velocity, the saturation level can thus be estimated with a perturbation expansion around the marginally stable state.

The saturation level (or the resonance width) can be estimated using Eq. (49), the steady-state condition and $\mathrm{Re}\,\varepsilon_{\underset{\sim}{k},\omega} = 0$ as a function of the drift velocity excess beyond the threshold. Since the anisotropic-ion distribution can lead to more effective wave-ion interactions as explained earlier, we shall confine ourselves to the case where $T^e_{\parallel} = T^i_{\parallel} = T^e_{\perp} < T^i_{\perp}$. Then Eq. (49) and $\mathrm{Re}\,\varepsilon_{\underset{\sim}{k},\omega} = 0$ can be expressed explicitly as

$$-Z'\left(\frac{\omega-\kappa v_D}{\kappa v_{te}}\right) + \frac{2T^e}{T^i_{\perp}} \Lambda_{n_\omega} \frac{(\omega + \frac{i}{\tau_i})}{\kappa v_{t_{\parallel i}}} Z\left(\frac{\omega - n_\omega \omega_{ci} + \frac{i}{\tau_i}}{\kappa v_{t_{\parallel i}}}\right)$$

$$-\Lambda_{n_\omega} Z'\left(\frac{\omega - n_\omega \omega_{ci} + \frac{i}{\tau_i}}{\kappa v_{t_{\parallel i}}}\right)$$

$$= i\left[\frac{(a^e + a^i)}{2}\right] \mathrm{Im}Z'\left(\frac{\omega-\kappa v_D}{\kappa v_{te}}\right) \Lambda_{n_\omega} \left[\mathrm{Im}Z'\left(\frac{\omega - n_\omega \omega_{ci} + \frac{i}{\tau_i}}{\kappa v_{t_{\parallel i}}}\right)\right.$$

$$+ \frac{T^e}{T^i_\perp} \, \mathrm{Im}[(\frac{\omega + \frac{i}{\tau_i}}{\kappa v_{t_{\parallel i}}}) \cdot Z(\frac{\omega - n_\omega \omega_{ci} + \frac{i}{\tau_i}}{\kappa v_{t_{\parallel i}}})] \;] \tag{61}$$

where the function Z is the plasma dispersion function. From the paragraph immediately following Eq. (38), ion decorrelation time τ_i is defined to be amplitude-dependent and can be estimated as

$$\frac{1}{\tau_i} \sim \frac{\pi c^2 \bar{k}^4}{2B_o^2 \bar{\kappa} v_{t_{\parallel i}}} \langle \varphi^2_{(x)} \rangle . \tag{62}$$

Since the ions are fluid-like, the dominant dependence on $1/\tau_i$ is in the Z functions. Let $1/\tau_i = 0$, we can obtain the threshold drift v^{thr} by solving the real and imaginary part of Eq. (61). In doing this, we have assumed that the fundamental harmonic is dominant and have evaluated $\bar{k}\rho_i$ at where Λ_o peaks. When $1/\tau_i$ is finite, we let $v_D = v^{thr} + \Delta v_D$, $\omega = \omega_o + \Delta\omega$, $\kappa = \kappa_o + \Delta\kappa$ where ω_o, κ_o are the frequency and parallel wavenumber at marginal stablility. Using a perturbation expansion around the marginally stable state, we obtain an approximately linear relation between $\Delta v_D / v_{te}$ and $\frac{1}{\tau_i}$.

$$\frac{\Delta v_D}{v_{te}} = Y \cdot \frac{1}{\kappa v_{t_{\parallel i}} \tau_i} \tag{63}$$

where Y is a function of $\sqrt{m_e/M_i}\; T^i_\perp / T^e$, and is shown in Fig. 3. Eq. (63) is recognizable as the expression $\gamma = 1/\tau_i$, where γ is the clump-enhanced growth rate. The saturation level can hence be estimated as a function of $\Delta v_D / v_{te}$ and $\sqrt{m_e/M_i}\; T^i_\perp / T^e$,

$$\frac{e^2\langle\varphi^2(x)\rangle}{T_e^2} = \frac{2}{\pi}\ \frac{m_e}{M_i}\left(\frac{T_\perp^i}{T^e}\right)^2\left(\frac{1}{\bar{k}\rho_i}\right)^4\left(\frac{\bar{\kappa}v_{te}}{\omega}\right)^2\left(\frac{\Delta v_D}{v_{te}}\right) Y^{-1}. \tag{64}$$

In Fig. 4, we show $e^2\langle\varphi^2(x)\rangle/T_e^2(\Delta v_D/v_D)$ as a function of $\sqrt{m_e/M_i}\ T_\perp^i/T^e$. The conventional nonlinear theory of wave fluctuations predicts that $e^2\langle\varphi^2(x)\rangle/T_e^2$ increases almost linearly with $\sqrt{m_e/M_i}\ T_\perp^i/T^e$. However, $e^2\langle\varphi^2(x)\rangle/T_e^2$ obtained using the clump theory increases more rapidly than that of conventional nonlinear theory, until $\sqrt{m_e/M_i}\ T_\perp^i/T^e \approx 0.045$. Then, after reaching a maximum value at $\sqrt{m_e/M_i}\ T_\perp^i/T^e \approx 0.05$, $e^2\langle\varphi^2(x)\rangle/T_e^2$ begins to decrease slightly. This is because $e\sqrt{\langle\varphi^2(x)\rangle}/T_e$ is inversely proportional to the parallel phase velocity $\omega/\kappa v_{te}$, Eq. (64), which increases more rapidly in the clump theory than that in the conventional nonlinear theory. (See paragraph immediately following Eq. (54)).

Among various anomalous transport coefficients, anomalous resistivity and heating are the most significant results of the clump theory. The anomalous resistivity results from a drag force, which is essential to maintain a steady current against a D.C. parallel electric field. In addition, this theory predicts a negligible turbulent diffusion of the electron distribution function, since this diffusion coefficient is proportional to the ion–clump amplitude, which is relatively small, as shown in the previous section. This indicates that there is relatively weak electron heating and that most of the electron–beam energy is converted into ion–perpendicular heating. By marked contrast, the conventional quasilinear theory prediction contains only the diffusion, and cannot treat a situation of a steady current. That theory also predicts that electron heating is greater than or equal to ion heating because of the fact that $\bar{k}^2\rho_i^2 \gtrsim 1$ and $(\bar{k}/\kappa)^2\ m_e/M_i \sim 1$ for ion–cyclotron turbulence, thus electron diffusion coefficient is larger than that of ions. Therefore, the wave and clump theories differ significantly in their prediction for the energy conversion of the electron current.

Anomalous resistivity is obtained by taking the first moment of the electron-average distribution function. At steady state, where a D.C. electric field is neccessary, it follows,

$$\frac{eE_o}{m} \int U \frac{\partial\langle f_e\rangle}{\partial U} d\underset{\sim}{V}$$

$$= \frac{e}{m} \int d\underset{\sim}{V} \sum_{\underset{\sim}{k},\omega} \frac{U}{(2\pi)^4 |\varepsilon_{\underset{\sim}{k},\omega}|^2} \left(\frac{\omega_{pe}^2}{e\underset{\sim}{k}^2} \kappa^2 \frac{\partial}{\partial U} \left(G^e_{\underset{\sim}{k},\omega} \langle\tilde{\varphi}^2\rangle^i_{\underset{\sim}{k},\omega}\right) \frac{\partial\langle f_e\rangle}{\partial U}\right.$$

$$\left. -\mathrm{Im}\chi^i_{\underset{\sim}{k},\omega}\, \kappa \frac{\partial}{\partial U} \langle\delta\tilde{f}^e \tilde{\varphi}^e\rangle_{\underset{\sim}{k},\omega}\right) . \qquad (65)$$

The diffusion term is small because it is proportional to the amplitude of ion clumps. Thus, the drag force yields,

$$\frac{eE_o}{m} \approx \overline{\mathrm{Im}\chi^i_{\underset{\sim}{k},\omega}} \left(\frac{e^2\langle\varphi^2(x)\rangle}{T_e^2}\right) \bar{\kappa}\, v_{te}^2 , \qquad (66)$$

where $\overline{\mathrm{Im}\chi^i_{\underset{\sim}{k},\omega}}$ is the spectrum-averaged ion dissipation. With η_A defined by $E_o = \eta_A(n_o e v_D)$, we have the anomalous resistivity η_A,

$$\eta_A = \frac{4\pi\omega_{ci}}{\omega_{pe}^2} \left\{\frac{\frac{m_e}{M_i}\left(\frac{T_\perp^i}{T^e}\right)^2}{1.5(\bar{k}\rho_i)^4} \left(\frac{\overline{\kappa v}_{te}}{\omega}\right)^3 \left(\overline{\mathrm{Im}\chi^i_{\underset{\sim}{k},\omega}}\right) \left(\frac{\Delta v_D}{v_D}\right) Y^{-1}\right\} . \qquad (67)$$

In Fig. 5, we show the properly normalized anomalous resistivity

$$\frac{\eta_A}{\left(\frac{4\pi\omega_{ci}}{\omega_{pe}^2}\right)\frac{\Delta v_D}{v_D}} \quad \text{vs.} \quad \left(\frac{m_e}{M_i}\right)^{1/2}\frac{T_\perp^i}{T_e} .$$

A comparison between the anomalous resistivity of the clump theory and that of the conventional nonlinear theory of wave fluctuations is also shown in Fig. 5. The conventional nonlinear theory predicts an ever-increasing (but more slowly) resistivity as a function of $(m_e/M_i)^{1/2}\ T_\perp^i/T^e$, while the clump theory predicts a much higher value of resistivity at $(m_e/M_i)^{1/2}\ T_\perp^i/T^e \sim 0.03$. Then, the anomalous resistivity begins to decrease.

The saturation level and anomalous resistivity predicted in this theory are in good agreement with those obtained from the preliminary results of particle simulations.[17] It is also noticed that this theory predicts that the fluctuation level $e\varphi/T_e$ is quite moderate ($\lesssim 0.1$) for the wave-clump regime of ion-cyclotron turbulence. This conclusion suggests that a study of the clump regime of ion-cyclotron turbulence may be necessary to explain the anomalous transport associated with ion-cyclotron turbulence in the observations.

VII. Summaries and Conclusions

In this paper, we investigate the nonlinear theory of ion-cyclotron turbulence in the regime of strong wave-particle interaction. This type of turbulence contains two major constituents, the wave and non-wave (clump) fluctuations, in contrast to the case that is dominated by the non-wave fluctuations. Wave-wave interaction has been neglected because ion cyclotron waves are of the non-decay type that requires very high turbulence levels for wave-wave interaction, and also because we have focused our investigation on the small-scale structures of the phase space. The prinicipal results of this study are:

(1) Renormalized one-point and two-point equations describing magnetized Vlasov turbulence have been derived. Since particle motions evolve on two different time scales, the renormalized two-point equation describes the slow, relative motion; the renormalized one-point equation describes the rapid cyclotron motion and yields a renormalized dielectric function.

(2) Shapes of ion and electron clumps have been predicted. The electron clump appears cigar-shaped aligned along the magnetic field in position space, and appears as a thin disc perpendicular to the magnetic field in velocity space. The ion clump, when $\bar{k}\rho_i > 1$, appears as a gyrating tether-rod about the guiding center in position space and a gyrating tether-disc in velocity space; when $\bar{k}\rho_i < 1$, the ion clump appears cigar-shaped aligned along the magnetic field in position space, and appears as a thin disc perpendicular to the magnetic field in velocity space (Fig. 1).

(3) The clump amplitude of ions is much less than that of electrons because of ion finite-Larmor radius effects.

(4) When $T_{\perp}^{i} \geqslant T_{\parallel}^{i} \sim T^{e}$, ion-cyclotron turbulence is of the wave-clump type. When $T_{\perp}^{i} >> T_{\parallel}^{i} \sim T^{e}$, transition to the clump-dominant type of turbulence occurs. This transition takes place approximate when $(m_e/M_i)^{1/2} T_{\perp}^{i}/T^{e} \geqslant 0.1$, where waves are heavily damped.

(5) A necessary condition for the maintenance of stationary ion-cyclotron turbulence is given in Eq. (49), and is a Vlasov-theory analogue to the prediction of the fluctuation-dissipation theorem of the test-particle model. This results from the fact that the collective dissipation must balance incoherent noise emission from macro-particles at steady state. This expression is valid for turbulence of the wave-clump type.

(6) The nonlinear growth rates in different regimes have been obtained, and are given in Eqs. (58) and (59). Nonlinear instabilities grow on the time scales of the particle decorrelation times.

(7) The clump modifications to the instability threshold in the wave-clump regime are small (reduction of threshold drift velocity is within 7%, Fig. 2). However, the modifications to the fluctuation level at saturation and the anomalous resistivity are more significant. Also, while the quasi-linear theory predicts that electron parallel heating is larger than, or equal to, ion perpendicular heating for ion-cyclotron turbulence ($\bar{k}^2/\bar{\kappa}^2 \approx M_i/m_e$, $\bar{k}^2\rho_i^2 \geqslant 1$), this theory predicts that ion perpendicular heating is the dominant mechanism for extraction of electron-beam energy.

In deriving the global conditions for steady state turbulence, two interesting questions arise. First, supposing that the drift velocity v_D slightly exceeds the threshold velocity v^{thr}, then only the fluctuations $<\tilde{\varphi}^2>^{\sigma}_{\underset{\sim}{k}}$ within a narrow window of the wavenumber spectrum can be excited. However, the wavenumber spectra obtained from Eqs. (43) and (44) are broad and independent of v_D. Second, the spectrum of $<\varphi^2>_{\underset{\sim}{k},\omega}$ obtained from Eq. (34), by integrating the velocity and using Poisson's equation, are different from that obtained from the expression,

$$<\varphi^2>_{\underset{\sim}{k},\omega} = [<\tilde{\varphi}^2>^{i}_{\underset{\sim}{k},\omega} + <\tilde{\varphi}^2>^{e}_{\underset{\sim}{k},\omega}]/|\varepsilon_{\underset{\sim}{k},\omega}|^2 .$$

The latter expression has peaks near normal modes, $|\varepsilon_{\underset{\sim}{k},\omega}|^2 \approx 0$, while the former expression does not.

The answer to the first question is simply that as soon as fluctuations within the narrow window of the wavenumber spectrum are excited, they will spread out and cover the whole range of spectrum by mode coupling processes. The answer to the second question is as follows. The expression, Eq. (34), is only valid for the relative separation $|\underset{\sim}{X}_-| \lesssim 1/|\bar{\underset{\sim}{k}}|$. This limits validity of the long-wavelength $(|\underset{\sim}{k}| \leqslant |\bar{\underset{\sim}{k}}|)$ side of the spectrum resulting from Eq. (34). Since the poles of $|\varepsilon_{\underset{\sim}{k},\omega}|^2$ are generally located in this range of the spectrum, hence the expression of $<\varphi^2>_{\underset{\sim}{k},\omega}$ obtained from one method should not be compared with that from the other in this range.

It is natural to ask the question, what happens when both ion-cyclotron turbulence $(k/\kappa >> 1,\ k\rho_i \sim 1)$ and ion-acoustic turbulence $(k/\kappa \leqslant 1)$ exist at the same time? For a magnetized plasma with $T^i_\perp > T^e > T^i_\parallel$, the threshold-drift velocities of both are comparable. When one type of fluctuation is excited the other may also be excited. The presence of both types of fluctuations not only can change the structures of clumps, but their interplay leads to that the dissipation of one branch is determined by the other branch. Furthermore, ion-acoustic turbulence is intrinsically clump-dominated. When ion-cyclotron turbulence is coupled to ion-acoustic turbulence, its clump character can be greatly modified.

This kind of plasma turbulence with both ion-cyclotron and ion-acoustic fluctuations is a generalization of what has been investigated in this paper, and is now under investigation. The results will be published in the future.

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Appendixes

A. Linear Analysis

The linear dielectric function in a current-carrying, magnetized plasma with anisotropic-Maxwellian velocity distributions can be expressed as (Stix, 1962),

$$\varepsilon = 1 - \sum_{\sigma=i,e} \sum_{n=-\infty}^{\infty} \frac{\Lambda_n(k^2\rho_\sigma^2)}{|\underset{\sim}{k}|^2\lambda_{D_\sigma}^2} \left(\frac{T_\perp^\sigma}{T_\parallel^\sigma} + \left[\frac{(\omega-\kappa v_{D\sigma}-n\omega_{c\sigma})T_\perp^\sigma+n\omega_{c\sigma}T_\parallel^\sigma}{\kappa T_\parallel^\sigma v_{t\parallel}^\sigma}\right]\right.$$

$$\left.\cdot\, Z\left(\frac{\omega-\kappa v_{D\sigma}-n\omega_{c\sigma}}{\kappa v_{t\parallel\sigma}}\right)\right)$$

where the various quantities are defined as follows:

the perpendicular thermal speed

$$v_{t_{\perp\sigma}} = \left(\frac{2T_{\perp}^{\sigma}}{M_{\sigma}}\right)^{1/2} ,$$

the parallel thermal speed

$$v_{t_{\parallel\sigma}} = \left(\frac{2T_{\parallel}^{\sigma}}{M_{\sigma}}\right)^{1/2} ,$$

the perpendicular Debye length

$$\lambda_{D_{\sigma}} = \frac{v_{t\perp\sigma}}{\sqrt{2}\ \omega_{p_{\sigma}}} ,$$

the thermal gyroradius

$$\rho_{\sigma} = \frac{v_{t\perp\sigma}}{\sqrt{2}\ \omega_{c\sigma}} ,$$

$$\Lambda_n(k^2\rho_{\sigma}^2) = I_n(k^2\rho_{\sigma}^2)\ e^{-k^2\rho_{\sigma}^2} .$$

Z is the plasma dispersion function and σ stands for species.

For fluctuations in the ion-cyclotron regime, where $k\rho_i \sim 1$, $k \gg \kappa$ and $\omega - \kappa v_D/\kappa v_{te} \ll 1$, the dielectric can be simplified as

$$\varepsilon_{\underset{\sim}{k},\omega} \approx 1 + \frac{1}{2|\underset{\sim}{k}|^2\lambda_{De}^2} Z'\left(\frac{\omega-\kappa v_D}{\kappa v_{te}}\right) - \sum_n \frac{\Lambda_n(k^2\rho_i^2)}{(|\underset{\sim}{k}|^2\lambda_{Di}^2)}$$

$$\left\{\frac{T_\perp^i}{T_\parallel^i} + \frac{(\omega - n\omega_{ci})\left(\frac{T_\perp^i}{T_\parallel^i}\right) + n\omega_{ci}}{\kappa v_{t\parallel i}} \; Z\left(\frac{\omega - n\omega_{ci}}{\kappa v_{t\parallel_i}}\right)\right\} \tag{A-1}$$

or

$$\approx \frac{-1}{|\underset{\sim}{k}|^2\lambda_{De}^2}\left[1 + i\sqrt{\pi}\left(\frac{\omega-\kappa v_D}{\kappa v_{te}}\right)\right] - \sum_n \frac{\Lambda_n(k^2\rho_i^2)}{|\underset{\sim}{k}|^2\lambda_{Di}^2}$$

$$\cdot \left\{\frac{T_\perp^i}{T_\parallel^i} + \frac{(\omega - n\omega_{ci})\left(\frac{T_\perp^i}{T_\parallel^i}\right) + n\omega_{ci}}{\kappa v_{t\parallel i}} \; Z\left(\frac{\omega - n\omega_{ci}}{\kappa v_{t\parallel i}}\right)\right\} . \tag{A-2}$$

Here, we have assumed that the electrons have an isotropic-velocity distribution. Since the magnitude of $|\mathrm{Re}\varepsilon_{\underset{\sim}{k},\omega}|$ is larger than the magnitude of $|\mathrm{Im}\varepsilon_{\underset{\sim}{k},\omega}|$, except near $\mathrm{Re}\varepsilon_{\underset{\sim}{k},\omega}=0$, the zeros of $\varepsilon_{\underset{\sim}{k},\omega}$ will be located near $\mathrm{Re}\varepsilon_{\underset{\sim}{k},\omega} = 0$. Hence $\mathrm{Re}\varepsilon_{\underset{\sim}{k},\omega} = 0$ will determine the real part of the frequency ω, and $\mathrm{Im}\varepsilon$ will determine the imaginary part.

The analytical expression for the fundamental mode $\omega \approx \omega_{ci}$ is particularly simple, where $\mathrm{Re}\varepsilon_{\underset{\sim}{k},\omega} = 0$ can be expressed as

$$\frac{\omega}{\kappa v_{t\parallel i}} \mathrm{Re}Z\left(\frac{\omega-\omega_c}{\kappa v_{t\parallel i}}\right) \cong \frac{1 + \frac{T_\perp^i}{T_\parallel^e} - \frac{1-\Lambda_o(k^2\rho_i^2)}{k^2\rho_i^2}}{\Lambda_1(k^2\rho_i^2)} , \tag{A-3}$$

and the marginal stability, $\mathrm{Im}\varepsilon_{\underset{\sim}{k},\omega} = 0$, yields

$$\frac{v_D}{v_{t_{\parallel i}}} = -\left[\frac{1+\frac{T_\perp^i}{T_\parallel^e} - \frac{1-\Lambda_o}{k^2\rho_i^2}}{\Lambda_1}\right] \cdot \frac{1+\left(\frac{T_\parallel^i}{T_\perp^i}\right)\left(\frac{T_\parallel^e}{T_\parallel^i}\right)^{3/2}\left(\frac{M_i}{m_e}\right)^{1/2}\Lambda_1\, e^{\frac{-(\omega-\omega_{ci})^2}{\kappa^2 v_{t_{\parallel i}}^2}}}{\mathrm{Re}Z\left(\frac{\omega-\omega_{ci}}{\kappa v_{t_{\parallel i}}}\right)}, \qquad \text{(A-4)}$$

the threshold-drift velocity for exciting the mode $\varphi_{\underset{\sim}{k},\omega}$.

The minimal threshold velocity[16] to excite ion-cyclotron wave can be obtained by minimizing $v_D/v_{t_{\parallel i}}$ with respect to $\omega-\omega_c/\kappa v_{t_{\parallel i}}$ and $k^2\rho_i^2$. The latter takes a value $k^2\rho_i^2 \sim 2$ to yield the minimal threshold-drift velocity, and the value of $\omega-\omega_{ci}/\kappa v_{t_{\parallel i}}$ will be dependent on $T_\perp^i/T_\parallel^i$, $T_\parallel^i/T_\parallel^e$ and M_i/m_e. When $T_\parallel^e/T_\perp^i \sim 1$, Eqs. (A-3) and (A-4) can be simplied further,

$$\omega \approx \omega_{ci}\left[1+\frac{\Lambda_1}{1+\left(\frac{T_\perp^i}{T_\parallel^e}\right) - \Lambda_1 - \frac{1-\Lambda_o}{k^2\rho_i^2}}\right] \qquad \text{(A-5)}$$

$$\frac{v_D}{v_{t_{\parallel i}}} \approx \frac{\omega}{\kappa v_{t_{\parallel i}}}\left[1+\left(\frac{T_\parallel^i}{T_\perp^i}\right)\left(\frac{T_\parallel^e}{T_\parallel^i}\right)^{3/2}\left(\frac{M_i}{m_e}\right)^{1/2}\Lambda_1(k^2\rho_i^2)\, e^{\frac{-(\omega-\omega_c)^2}{\kappa^2 v_{t_{\parallel i}}^2}}\right], \qquad \text{(A-6)}$$

and the minimal threshold drift is,

$$\frac{v^{thr}}{v_{t_{\parallel i}}} = \left(\frac{1+\frac{T_\perp^i}{T_\parallel^e} - \frac{1-\Lambda_o}{k^2\rho_i^2}}{1+\frac{T_\perp^i}{T_\parallel^e} - \frac{1-\Lambda_o}{k^2\rho_i^2} - \Lambda_1}\right)\left\{\ln\left[2\left(\frac{T_\parallel^i}{T_\perp^i}\right)\left(\frac{T_\parallel^e}{T_\parallel^i}\right)^{3/2}\left(\frac{M_i}{m_e}\right)^{1/2}\Lambda_1\right]\right\}^{1/2}. \qquad \text{(A-7)}$$

B. Calculation of the Source Term

$$\langle S\rangle = -\frac{q}{M}\left[\langle \underset{\sim}{E}(1)\delta f(2)\rangle \cdot \frac{\partial\langle f\rangle}{\partial \underset{\sim}{v}_1} + \langle \underset{\sim}{E}(2)\delta f(1)\rangle \cdot \frac{\partial\langle f\rangle}{\partial \underset{\sim}{v}_2}\right]$$

$$= \frac{iq}{M}\sum_{\substack{\underset{\sim}{k}\\ \tilde{\omega}}} \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}}}{(2\pi)^4}\left\{\langle \Phi_{\underset{\sim}{k},\omega}(1)\quad f^*_{\underset{\sim}{k},\omega}(2)\left[\cos(\vartheta_1-\psi_k)k\,\frac{\partial\langle f\rangle}{\partial V_1} + \kappa\,\frac{\partial\langle f\rangle}{\partial U_1}\right]\right\rangle$$

$$+ \langle \Phi^*_{\underset{\sim}{k},\omega}(2)\ f_{\underset{\sim}{k},\omega}(1)\left[\cos(\vartheta_2-\psi_k)k\,\frac{\partial\langle f\rangle}{\partial V_2} + \kappa\,\frac{\partial\langle f\rangle}{\partial U_2}\right]\rangle\Big\} \ . \qquad \text{(B-1)}$$

The fluctuation $f_{\underset{\sim}{k},\omega}$ can be divided into a coherent part and an incoherent part, thus we have

$$\langle \Phi_{\underset{\sim}{k},\omega}(1) f^*_{\underset{\sim}{k},\omega}(2)\rangle = \frac{\varepsilon^*_{\underset{\sim}{k},\omega}}{|\varepsilon|^2}\left[\langle \tilde{\Phi}_{\underset{\sim}{k},\omega}(1)\tilde{f}^*_{\underset{\sim}{k},\omega}(2)\rangle + \langle \tilde{\Phi}_{\underset{\sim}{k},\omega} f^{c*}_{\underset{\sim}{k},\omega}(2)\rangle\right] \ . \qquad \text{(B-2)}$$

But

$$\tilde{\Phi}_{\underset{\sim}{k},\omega}(1) = \sum_n J_n(1)\ e^{in(\vartheta_1-\psi_k)}\tilde{\varphi}_{\underset{\sim}{k},\omega}$$

$$\tilde{\Phi}_{\underset{\sim}{k},\omega}\cos(\vartheta_1-\psi_k) = \sum_n \frac{nJ_n(1)}{x_1}\ e^{in(\vartheta_1-\psi_k)}\tilde{\varphi}_{\underset{\sim}{k},\omega}$$

$$f^c_{\underset{\sim}{k},\omega} = \frac{iq}{M}\sum_n e^{in(\vartheta-\psi_k)} G_{\underset{\sim}{k},\omega,n}\varphi_{\underset{\sim}{k},\omega} J_n\left[\frac{nk}{x}\,\frac{\partial\langle f\rangle}{\partial V} + \kappa\,\frac{\partial\langle f\rangle}{\partial U}\right] ,$$

where $x_1 \equiv kV_1/\omega_c$, $J_n(1) \equiv J_n(x_1)$ and $\cos\vartheta = \hat{x}\cdot\underset{\sim}{V}_\perp/V$. Therefore,

$$\langle \Phi_{\underset{\sim}{k},\omega}(1) f^{c*}_{\underset{\sim}{k},\omega}(2)\rangle = \frac{-iq}{M}\sum_n e^{in(\vartheta_1-\vartheta_2)} G^*_{\underset{\sim}{k},\omega,n} J_n(1) J_n(2)\langle|\varphi_{\underset{\sim}{k},\omega}|^2\rangle\left[\frac{nk}{x_2}\,\frac{\partial\langle f\rangle}{\partial V_2} + \kappa\,\frac{\partial\langle f\rangle}{\partial U_2}\right]$$

and

$$<\Phi_{\underset{\sim}{k},\omega}(1) f^{c*}_{\underset{\sim}{k},\omega}(2)\ \cos(\vartheta_1-\psi_k)> = -\frac{iq}{M}\sum_n e^{in(\vartheta_1-\vartheta_2)} G^*_{\underset{\sim}{k},\omega,n}\left(\frac{n}{x_1}\right) J_n(1) J_n(2) <|\varphi_{\underset{\sim}{k},\omega}|^2>$$

$$\cdot\ \left[\frac{nk}{x_2}\frac{\partial<f>}{\partial V_2} + \kappa\frac{\partial<f>}{\partial U_2}\right] .$$

Also, we have a term

$$\begin{aligned} <\tilde{\Phi}_{\underset{\sim}{k},\omega}(1)\tilde{f}^*_{\underset{\sim}{k},\omega}(1)> &= \sum_n e^{in(\vartheta_1-\vartheta_2)} <\tilde{\Phi}_{\underset{\sim}{k},\omega}(1)\tilde{f}^*_{\underset{\sim}{k},\omega}(2)>_n \\ &= \sum_n e^{in(\vartheta_1-\vartheta_2)} (2G^*_{\underset{\sim}{k},\omega,n}) <\tilde{\Phi}_{\underset{\sim}{k}}(1)\tilde{f}^*_{\underset{\sim}{k}}(2)>_n \\ &= 2\sum_n e^{in(\vartheta_1-\vartheta_2)} G^*_{\underset{\sim}{k},\omega,n} J_n(1) J_n(2)\ <\tilde{\varphi}_{\underset{\sim}{k}}\delta\tilde{f}^*_{\underset{\sim}{k}}> \end{aligned}$$

where the second equality assumes that the dominant time-dependence of $<\tilde{\Phi}_{\underset{\sim}{k}}(1,t_1)\tilde{f}^*_{\underset{\sim}{k}}(2,t_2)>_n$ is due to the Doppler-shifted ballistic propagation, i.e., $\omega - n\omega_c = \kappa v_{\parallel}$. Similarly, the term

$$<\tilde{\Phi}_{\underset{\sim}{k},\omega}(1)\tilde{f}^*_{\underset{\sim}{k},\omega}(2)\cos(\vartheta_1-\psi_k)> = 2\sum_n e^{in(\vartheta_1-\vartheta_2)}\frac{n}{x_2} J_n(2) J_n(1) G^*_{\underset{\sim}{k},\omega,n}(2) <\tilde{\varphi}_{\underset{\sim}{k}}\delta\tilde{f}^*_{\underset{\sim}{k}}> .$$

Hence, the source term of the ions becomes

$$\begin{aligned} <S(1,1)>^i = \frac{2q}{M}\sum_{\substack{\underset{\sim}{k}\\ \tilde{\omega}\\ n}} \frac{J_n^2 G^*_{\underset{\sim}{k},\omega,n}}{(2\pi)^2|\varepsilon_{\underset{\sim}{k},\omega}|^2} \Big\{\frac{q}{M}\left[|\tilde{\varphi}^e_{\underset{\sim}{k},\omega}|^2 + |\tilde{\varphi}^i_{\underset{\sim}{k},\omega}|^2\right]\left(\frac{nk}{x}\frac{\partial<f>}{\partial V} + \kappa\frac{\partial<f>}{\partial U}\right)^2 \\ - 2\left[\mathrm{Im}\chi^e_{\underset{\sim}{k},\omega} + \mathrm{Im}\chi^i_{\underset{\sim}{k},\omega}\right]<\delta\tilde{f}\tilde{\varphi}>^i_{\underset{\sim}{k}}\left(\frac{nk}{x}\frac{\partial<f>}{\partial V} + \kappa\frac{\partial<f>}{\partial U}\right)\Big\}^i \end{aligned} \qquad \text{(B-3)}$$

where we have let

$$|\varphi_{\underset{\sim}{k},\omega}|^2 = \frac{|\tilde{\varphi}_{\underset{\sim}{k},\omega}|^2}{|\varepsilon_{\underset{\sim}{k},\omega}|^2} = \frac{|\tilde{\varphi}^i_{\underset{\sim}{k},\omega}|^2 + |\tilde{\varphi}^e_{\underset{\sim}{k},\omega}|^2}{|\varepsilon_{\underset{\sim}{k},\omega}|^2} .$$

Furthermore, we will assume that

$$<\delta\tilde{f}\tilde{\varphi}>^\sigma_{\underset{\sim}{k}} = N_\sigma(\underset{\sim}{k})<f>^\sigma \frac{4\pi q^\sigma}{k^2} \qquad \text{(B-4)}$$

and $<f>^\sigma$ is a Maxwellian distribution. This yields

$$<|\tilde{\varphi}^i_{\underset{\sim}{k},\omega}|^2> = \frac{4\pi n_o q}{k^2} \sum_n \int d\underset{\sim}{V}(2G^i_{\underset{\sim}{k},\omega,n})<\delta\tilde{f}\tilde{\varphi}>^i_{\underset{\sim}{k}}$$

$$= \frac{(8\pi^2 n_o q)^2}{2\pi n_o k^4} N_i \sum_n \Lambda_n e^{-\frac{(\omega-n\omega_{ci})^2}{\kappa^2 v^2_{t\parallel i}}} \left(\frac{1}{\sqrt{\pi}\ \kappa v_{t\parallel i}}\right) . \qquad \text{(B-5)}$$

and

$$<|\tilde{\varphi}^e_{\underset{\sim}{k},\omega}|^2> = \frac{(8\pi^2 n_o e)^2}{2\pi n_o k^4} N_e\ e^{-\frac{(\omega-\kappa v_D)^2}{\kappa^2 v^2_{te}}} \left(\frac{1}{\sqrt{\pi}\kappa v_{te}}\right) . \qquad \text{(B-6)}$$

Hence, Eq. (B-3) can be reduced to

$$<S(1,1)>^i = \frac{2q}{M} \sum_{\substack{k\\ \omega\\ n}} \frac{J^2_n G^*_{\underset{\sim}{k},\omega,n}\left(\frac{4\pi e}{k^2}\right)}{(2\pi)^4|\varepsilon_{\underset{\sim}{k},\omega}|^2} \left(\frac{2\omega^2_{pi}}{k^2} \left[N_i(\underset{\sim}{k}) \sum_m \Lambda_m \frac{e^{-\frac{(\omega-m\omega_c)^2}{\kappa^2 v^2_{t\parallel i}}}}{v_{t\parallel i}}\right.\right.$$

$$+ \frac{N_e(\underset{\sim}{k})}{v_{te}} e^{-\frac{(\omega-\kappa v_D)^2}{\kappa^2 v_{te}^2}}] \; (\frac{\sqrt{\pi}}{\kappa}) \cdot (n\omega_{ci} + (\frac{T^i_\perp}{T^i_\parallel})\kappa U) \; <f_i>^2 \cdot (\frac{4}{v^4_{t_{\perp i}}})$$

$$+ 2[\mathrm{Im}\chi^e_{\underset{\sim}{k},\omega} + \mathrm{Im}\chi^i_{\underset{\sim}{k},\omega}] N_i(\underset{\sim}{k}) <f_i>^2 (n\omega_{ci} + (\frac{T^i_\perp}{T^i_\parallel})\kappa U)(\frac{2}{v^2_{t_{\perp i}}}) \; \}\Big|_{U = \frac{\omega - n\omega_c}{\kappa}} .$$

Rearranging

$$(n\omega_c + (\frac{T^i_\perp}{T^i_\parallel})\kappa U)\Big|_{u = \frac{\omega - n\omega_c}{\kappa}} = \omega + (\omega - m\omega_c)(\frac{T^i_\perp}{T^i_\parallel} - 1) + (m-n)\omega_c(\frac{T^i_\perp}{T^i_\parallel} - 1)$$

the ion source becomes

$$<S(1,1)>^i = \frac{2q}{M_i} \sum_{\substack{\underset{\sim}{k} \\ \underset{\sim}{\omega} \\ n}} \frac{J_n^2 G^*_{\underset{\sim}{k},\omega,n}}{(2\pi)^4 |\varepsilon_{\underset{\sim}{k},\omega}|^2} \frac{4\pi e}{k^2} \; \{ \; \{N_i(\underset{\sim}{k}) \; (\frac{2\omega^2_{pi}}{k^2 v^2_{t_{\perp i}}})$$

$$\sum_m [\frac{\omega}{\kappa} + \frac{(\omega - m\omega_c)}{\kappa} \; (\frac{T^i_\perp}{T^i_\parallel} - 1)] \Lambda_m \frac{e^{-\frac{(\omega - m\omega_c)^2}{\kappa^2 v^2_{t_{\parallel i}}}}}{v_{t_{\parallel i}}} \sqrt{\pi}$$

$$\cdot (n\omega_{ci} + (\frac{T^i_\perp}{T^i_\parallel})\kappa U)(\frac{2}{v^2_{t_{\perp i}}}) <f^i>^2 + \frac{4\omega^2_{pi}\sqrt{\pi}}{k^2 v^4_{t_{\perp i}} \kappa} N_e(\underset{\sim}{k}) \frac{e^{-\frac{(\omega-\kappa v_D)^2}{\kappa^2 v^2_{te}}}}{v_{te}}$$

$$(n\omega_{ci} + (\frac{T^i_\perp}{T^i_\parallel})\kappa U)^2 <f^i>^2) + \{2[\mathrm{Im}\chi^e_{\underset{\sim}{k},\omega} + \mathrm{Im}\chi^i_{\underset{\sim}{k},\omega}] N_i(\underset{\sim}{k})$$

$$\cdot\ (n\omega_{ci}+(\frac{T_{\perp}^{i}}{T_{\parallel}^{i}})\kappa U)(\frac{2}{v_{t_{\perp i}}^{2}})<f^{i}>^{2})\ )\Big|_{U=\frac{\omega-n\omega_c}{\kappa}}$$

$$+\frac{2q}{M_i}\sum_{\substack{k\\ \omega\\ n}}\frac{J_n^2 G^*_{k,\omega,n}}{(2\pi)^4|\varepsilon_{k,\omega}|^2}\frac{4\pi e}{k^2}\{N_i(k)(\frac{4\omega_{pi}^2}{k^2 v_{t_{\perp i}}^2})\sum_m (m-n)\omega_c(\frac{T_{\perp}^{i}}{T_{\parallel}^{i}}-1)\ \Lambda_m\frac{e^{-\frac{(\omega-m\omega_c)^2}{\kappa^2 v_{t_{\parallel i}}^2}}}{v_{t_{\parallel i}}}\sqrt{\pi}$$

$$\cdot(n\omega_{ci}+(\frac{T_{\perp}^{i}}{T_{\parallel}^{i}})\kappa U)(\frac{2}{v_{t_{\perp i}}^{2}})<f^{i}>^{2})\Big|_{U=\frac{\omega-n\omega_c}{\kappa}} \qquad \text{(B-7)}$$

where terms in the first spectral sum are the diffusion and drag, and those in the second spectral sum accounts for relaxation of anisotropy of the average-distribution function. The ion-ion diffusion will cancel the ion-ion drag. The remainder in the first bracket is the interactions between different species.

Finally, we can express source terms in terms of the electric field $<|\tilde{\varphi}^{\sigma}_{k,\omega}|^2>$, using Eqs. (B-5) and (B-6),

$$<S>^i \sim \frac{1}{(4\pi n_o q)^2}\sum_k \frac{J_{n_\omega}^2<f^i>_{\perp}^2}{(2\pi)^3|\varepsilon_{k,\omega}|^2}(\frac{2k^4\kappa^2}{\Lambda_{n_\omega}^2\pi^2})\{(\mathrm{Im}\chi^i_{k,\omega})^2<|\tilde{\varphi}^e_{k,\omega}|^2>$$

$$+\ \mathrm{Im}\chi^i_{k,\omega}\mathrm{Im}\chi^e_{k,\omega}<|\tilde{\varphi}^i_{k,\omega}|^2>\}\ \Big|_{\omega=n_\omega\omega_c+\kappa U_i}, \qquad \text{(B-8)}$$

$$<S>^e \sim \frac{1}{(4\pi n_o e)^2}\sum_k \frac{<f^e>_{\perp}^2}{(2\pi)^3|\varepsilon_{k,\omega}|^2}(\frac{2k^4\kappa^2}{\pi^2})\ \{(\mathrm{Im}\chi^e_{k,\omega})^2<|\tilde{\varphi}^i_{k,\omega}|^2>$$

$$+ \operatorname{Im}\chi^{i}_{\underset{\sim}{k},\omega} \operatorname{Im}\chi^{e}_{\underset{\sim}{k},\omega} \langle|\tilde{\varphi}^{e}_{\underset{\sim}{k},\omega}|^{2}\rangle \,)\Big|_{\omega=\kappa U_e} , \tag{B-9}$$

where we have assumed, in Eq. (B-8), that only one harmonic n_ω is dominant for a given frequency ω.

Appendix C

For situations where the magnitude of Reε is not neccessarily larger than that of Imε, we can evaluate the k' integral in Eqs. (45) and (46) in a slightly different way. First, we again assume that the k' integral is determined by the peak of $|\varepsilon_{\underset{\sim}{k}',\omega'}|^{-2}$. For a given value of ω', the integral $\int d\underset{\sim}{k}^3/|\varepsilon|^2$ can be evaluated by $\Delta\underset{\sim}{k}^3/|\varepsilon|^2$, where $\Delta\underset{\sim}{k}^3$ is the wavenumber "volume" contained in the peak of $|\varepsilon|^{-2}$. Since the functional dependence of ε on ω, $\underset{\sim}{k}$ and v_D is known, hence $\Delta\underset{\sim}{k}^3$ can be expressed by ω and v_D.

Second, a set of coupled equations similar to Eqs. (47) and (48) can immediately be obtained, and thus the neccessary condition for stationary turbulence can now be expressed as

$$\frac{|\varepsilon_{\underset{\sim}{k},\omega}|^2}{\Delta\underset{\sim}{k}^3} = \frac{(a^i+a^e)}{\kappa k^2} \operatorname{Im} \chi^{e}_{\underset{\sim}{k},\omega} \operatorname{Im}\chi^{i}_{\underset{\sim}{k},\omega} . \tag{C-1}$$

Notice that the terms on both sides of the equality are functions of ω, $\underset{\sim}{k}$ and v_D; however, these are not free parameters. They must satisfy a relation so as to minimize the drift velocity, yielding the threshold drift velocity v^{thr} for clump instability. When v_D exceeds v^{thr}, ω should be replaced by $\omega + i/\tau_c$, reflecting the fluctuation amplitude dependence of nonlinear saturation.

Appendix D

In the wave-clump regime of one-dimensional ion-acoustic turbulence, we can use Eq. (49) to determine the threshold drift velocity v^{thr} for clump instability. For this case, the reduction of threshold drift from that of the linear theory is substantial, in marked contrast to the ion-cyclotron case. This can be explained as follows.

The threshold drift v^{thr} is determined by

(a) $Re\varepsilon=0$,

(b) $Im\varepsilon=(a^e+a^i)\underset{\sim}{k}^2\lambda_D^2 Im\chi^e Im\chi^i$,

(c) Minimization of v_D with respect to ω and $\underset{\sim}{k}$.

For ion-acoustic fluctuations, the dielectric is

$$\varepsilon_{\underset{\sim}{k},\omega} = \frac{1}{\underset{\sim}{k}^2\lambda_D^2}\left[Z'\left(\frac{\omega}{\kappa v_{te}} - \frac{v_D}{v_{te}}\right) + \frac{T^e}{T^i} Z'\left(\frac{\omega}{\kappa v_{ti}}\right)\right] \qquad \text{(D-1)}$$

There are only two independent variables in $\varepsilon_{\underset{\sim}{k},\omega}$, i.e. ω/κ and v_D, hence relation (a) and (b) are sufficient to determine v^{thr}. Relation (c) is of no use for this case. However, in the case of the ion-cyclotron fluctuations, ω, k and v_D are independent variables, thus relation (c) has to be used. In fact, it is this minimization of v_D that results in the slight reduction for ion-cyclotron threshold drift.

A straightforward numerical method shows that the threshold reduction of ion-acoustic fluctuations is enhanced monotonically as T^i/T^e increases (Fig. D-1). When $T^i/T^e \gtrsim 0.5$, relations (a) and (b) yield unphysical complex values for v_D, indicating that the expansion around $Re\varepsilon=0$ fails. The ion-acoustic turbulence becomes of clump type, and the method described in Appendix C or numerical integration for Eqs. (45) and (46) should be used.

With mass ratio, $M_i/m_e = 100$ and temperature ratio, $T^i/T^e = 0.5$, we find that the threshold reduction $\Delta v^{thr}/v^{thr} \approx 40\%$. We expect that with $T^i/T^e = 1$, the threshold reduction should exceed 40%. This is consistent with that observed in a recent particle simulation by Berman et al.[18]

References

1. M. Yamada and H.W. Hendel, Phys. Fluids 21, 1555 (1978).
2. H. Bohmer and S. Fornaca, J. Geophys. Res. 84, 5234 (1979).
3. D.R. Dakin, T. Tajima, G. Benford and N. Rynn, J. Plasma Phys. 15, 175 (1976).
4. F.S. Mozer, C.W. Carlson, M.K. Hudson, R.B. Torbert, B. Parady, J. Yatteau and M.C. Kelly, Phys. Rev. Lett. 38, 292 (1977).
5. R.L. Lysak, H.K. Hudson and M. Temerin, J. Geophys. Res 85, 678 (1980).
6. H. Okuda, C.Z. Cheng and W.W. Lee, Phys. Fluids 24, 1060 (1981).
7. C.T. Dum and T.H. Dupree, Phys. Fluids 13, 2064 (1970).
8. S.P. Hirshman and K. Molvig, Phys. Rev. Lett. 42, 648 (1979).
9. T.H. Dupree, Phys. Fluids 9, 1733 (1966).
10. T. Boutros-Ghali and T.H. Dupree, Phys. Fluids 24, 1839 (1981).
11. T. Boutros-Ghali and T.H. Dupree, Phys. Fluids 25, 874 (1982).
12. D.J. Tetreault, Phys. Fluids 26, 3247 (1983).
13. R.H. Berman, D.J. Tetreault, and T.H. Dupree, Phys. Fluids 28, 155 (1985).
14. P.W. Terry and P.H. Diamond, in Statistical Physics and Chaos in Fusion Plasmas, W. Horton and L. Reichl, Eds., John Wiley & Sons, 1984, p. 335.
15. P.W. Terry and P.H. Diamond, Institute for Fusion Studies Report #114, 1984, to be published in Phys. Fluids.
16. K.F. Lee, J. Plasma Phys. 8, 379 (1972).
17. R.H. Berman, J.N. Leboeuf, T. Chiueh, and P.H. Diamond, Bull. Amer. Phys. Soc. 29, 1409 (1984).

18. R.H. Berman, T.H. Dupree, and D.J. Tetreault, Bull. Amer. Phys. Soc. 29, 1240 (1984).

Figure Captions

1. Illustrations for the electron clump and the ion clump with $\overline{k}_\perp \rho_i < 1$ (a), and the ion clump, with $\overline{k}_\perp \rho_i > 1$ and the perpendicular clump velocity of the order of the ion thermal velocity (b).

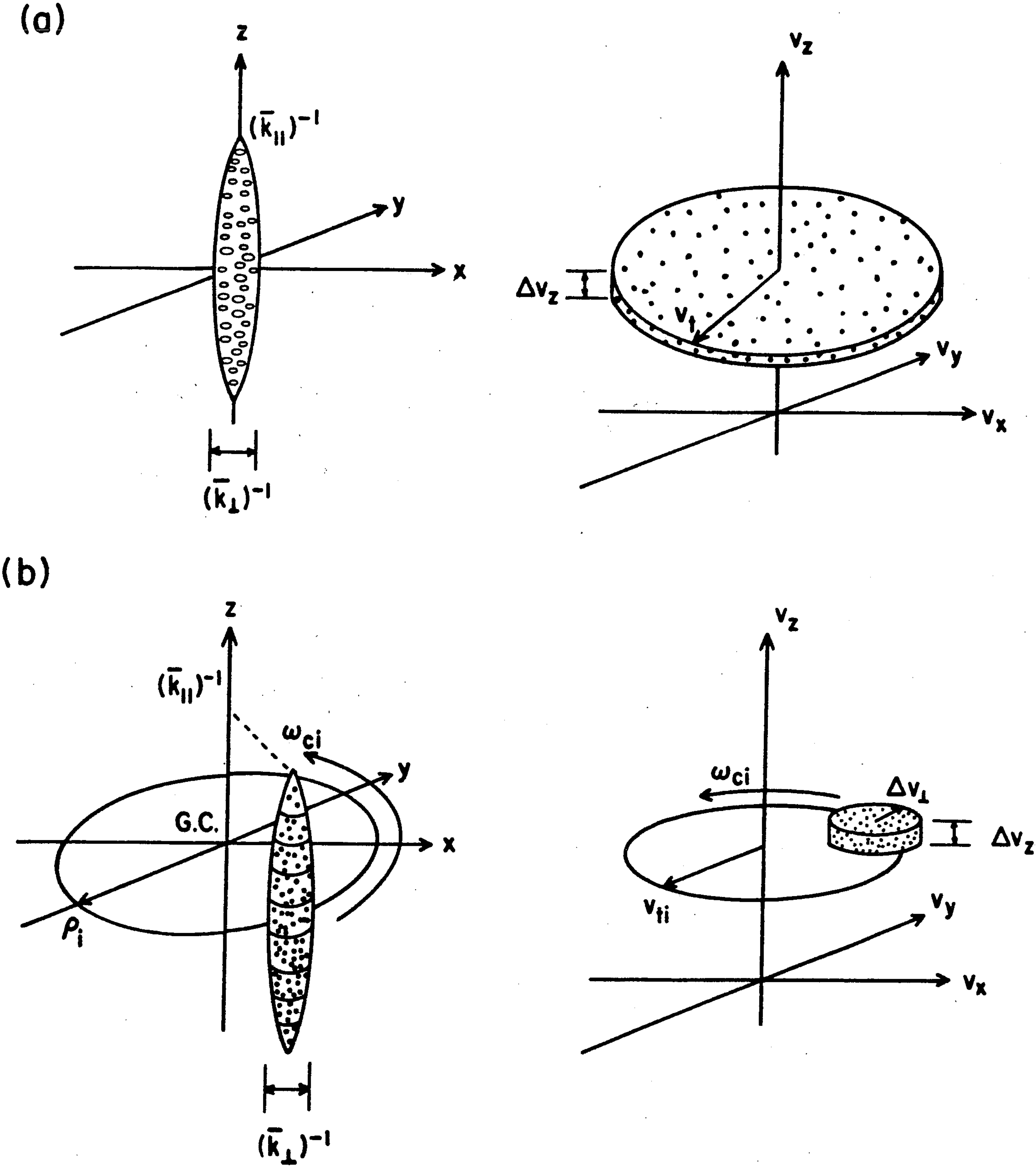
2. Threshold drift velocity normalized to the electron thermal velocity v^{thr}/v_{te} as a function of $(m_e/M_i)^{1/2}\, T_\perp^i/T^e$. Nonlinear and linear results are shown by the solid and dash lines respectively.

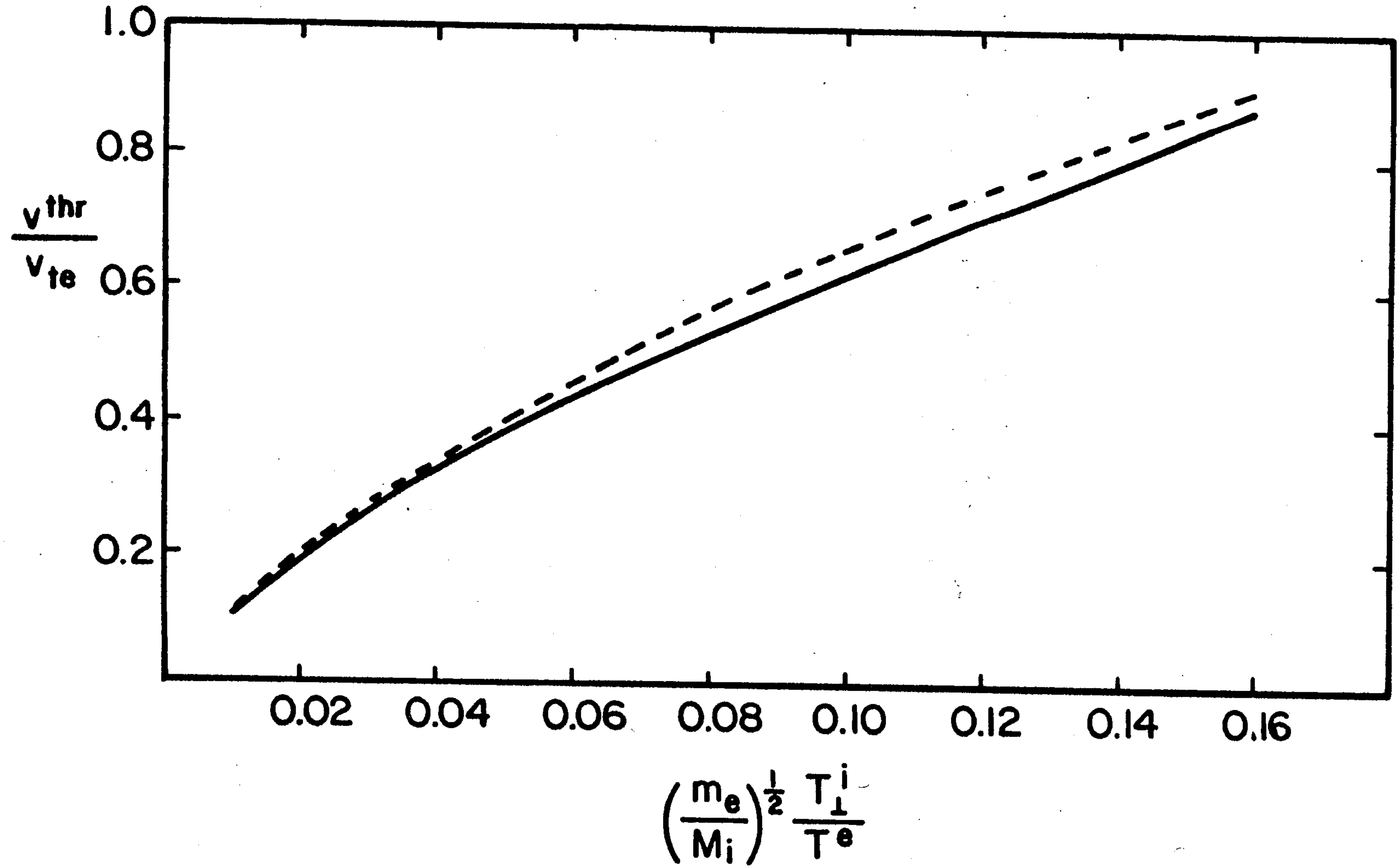
3. Ratio of the excess of drift velocity $\Delta v_D/v_{te}$ to the ion trapping rate $1/k_\prime v_{t_{\parallel i}} \tau_i$ as a function of $(m_e/M_i)^{1/2}\, T_\perp^i/T^e$. Results of the clump theory and the conventional nonlinear theory are shown by the solid and dash line, respectively.

4. Saturation level of the potential fluctuation $e^2\langle\varphi^2(x)\rangle/T_e^2$ normalized to $(\Delta v_D/v_D)$ as a function of $(m_e/M_i)^{1/2}\, T_\perp^i/T^e$. Results of the clump theory and the conventional nonlinear theory are shown by the solid and dash lines, respectively.

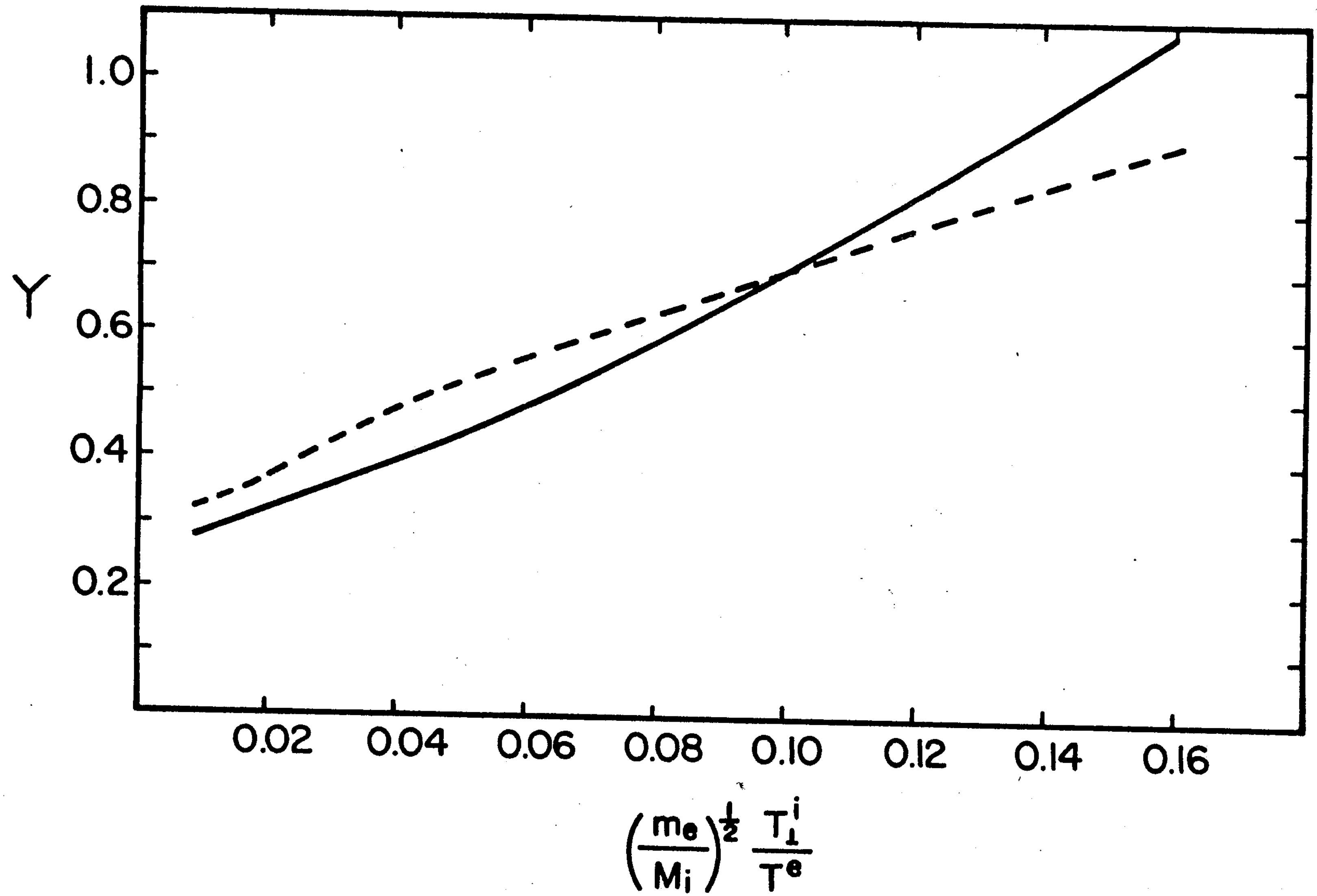
5. Anomalous resistivity η_A normalized to $(4\pi\omega_{ci}/\omega_{pe}^2)\ \Delta v_D/v_D$ as a function of $(m_e/M_i)^{1/2}\, T_\perp^i/T^e$. Results of the clump theory and the conventional nonlinear theory are shown by the solid and dash lines, respectively.

6. D-1. Threshold drift velocity for one-dimensional ion-acoustic turbulence of wave-clump type as a function T^i/T^e, with $M_i/m_e = 100$. Nonlinear and linear results are shown by the solid and dash lines respectively.

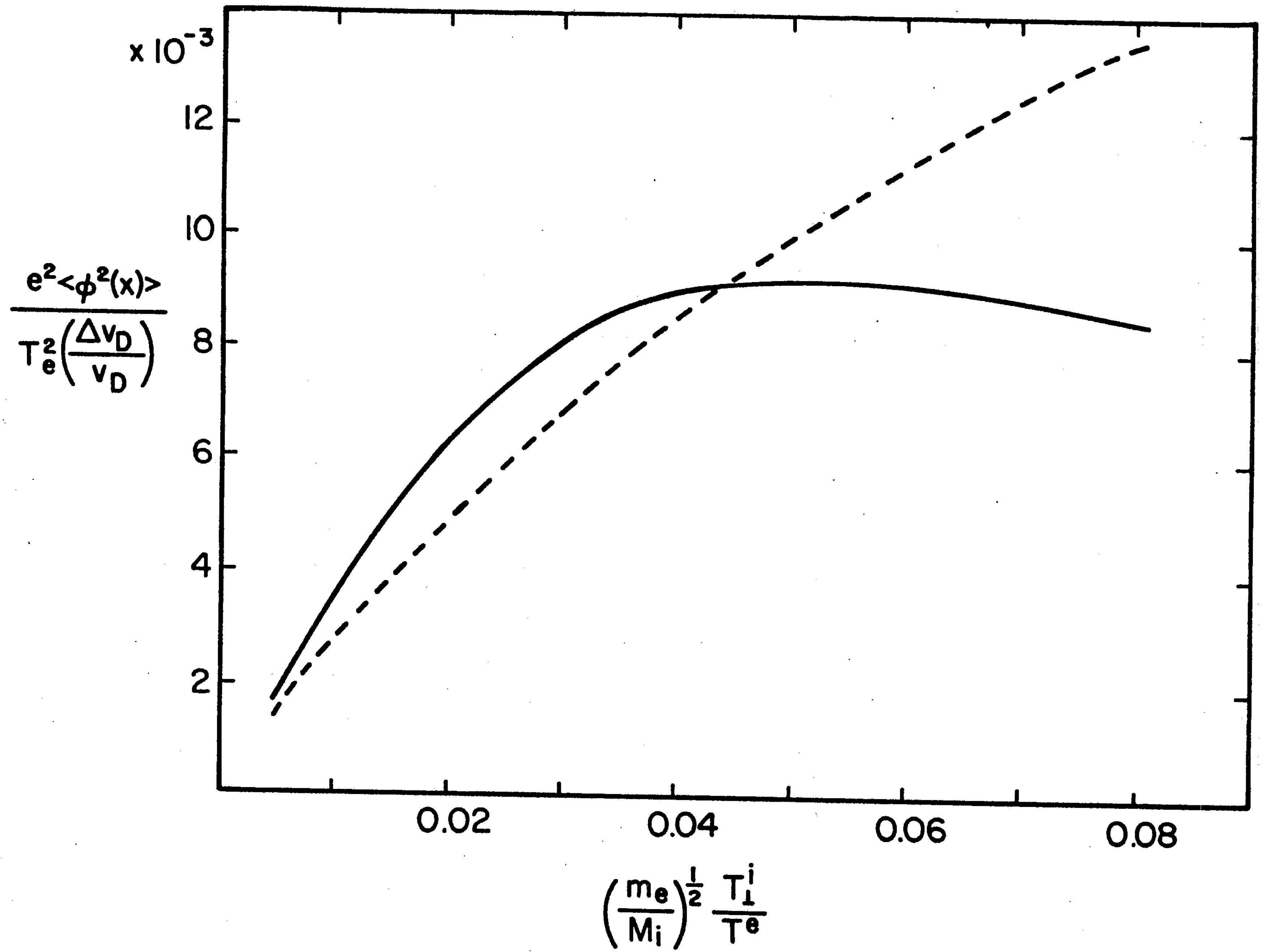

(a)
z
$(\bar{k}_{\parallel})^{-1}$
y
x
$(\bar{k}_{\perp})^{-1}$
v_z
Δv_z
v_t
v_y
v_x
(b)
z
$(\bar{k}_{\parallel})^{-1}$
ω_{ci}
y
G.C.
x
ρ_i
$(\bar{k}_{\perp})^{-1}$
v_z
ω_{ci}
$\Delta v_{\perp}$
Δv_z
v_{ti}
v_y
v_x



1.0
0.8
0.6
0.4
0.2
$\frac{v^{thr}}{v_{te}}$
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
$\left(\frac{m_e}{M_i}\right)^{\frac{1}{2}} \frac{T_\perp^i}{T^e}$



Y
1.0
0.8
0.6
0.4
0.2
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
$\left(\frac{m_e}{M_i}\right)^{\frac{1}{2}} \frac{T_\perp^i}{T^e}$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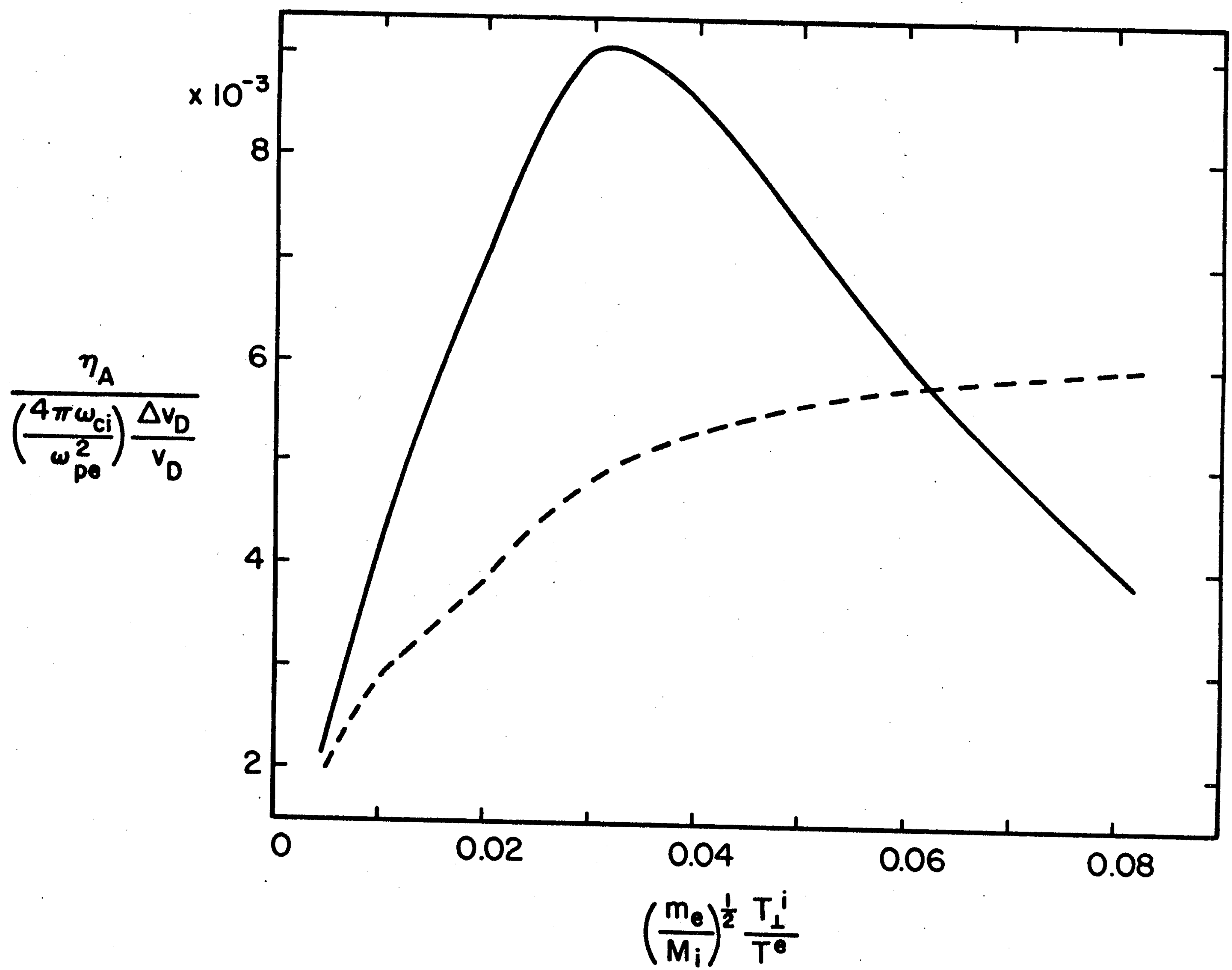
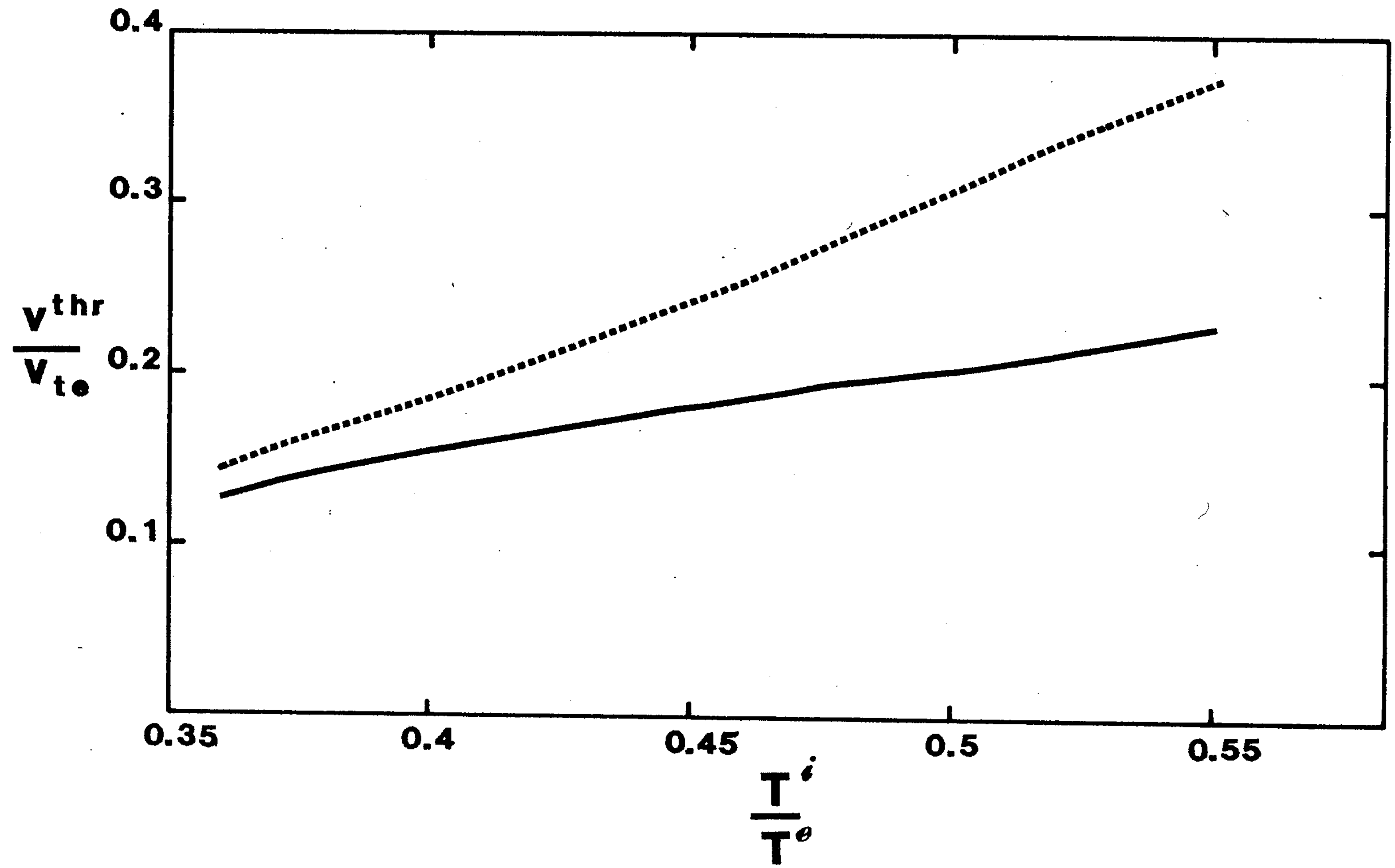
10
8
6
4
2
$\frac{e^2 <\phi^2(x)>}{T_e^2 \left(\frac{\Delta v_D}{v_D}\right)}$
0.02
0.04
0.06
0.08
$\left(\frac{m_e}{M_i}\right)^{\frac{1}{2}} \frac{T_\perp^i}{T^e}$



x 10^-3
8
6
4
2
0
0.02
0.04
0.06
0.08
$\frac{\eta_A}{\left(\frac{4\pi\omega_{ci}}{\omega_{pe}^2}\right)\frac{\Delta v_D}{v_D}}$
$\left(\frac{m_e}{M_i}\right)^{\frac{1}{2}}\frac{T_\perp^i}{T^e}$



0.4
0.3
0.2
0.1
$\frac{V^{thr}}{V_{te}}$
0.35
0.4
0.45
0.5
0.55
$\frac{T^{i}}{T^{e}}$