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LOW FREQUENCY MODES IN A ROTATING PLASMA WITH FINITE AXIAL LENGTH

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Abstract

In a rotating plasma with finite axial length which simulates the central cell plasma in tandem mirrors, the dispersion relation for the low frequency electrostatic modes is derived using the full kinetic description. The analysis includes the effects of (1) the diamagnetic and $\mathbf{E}\times\mathbf{B}$ drifts, (2) the lowest order of the ion Larmor radius, (3) the bounce motion of particles along the magnetic field, and (4) the anisotropy of temperature. A new flute mode driven by the combination of the centrifugal force and $T_\perp > T_\parallel$ anisotropy is derived which may explain the oscillations observed in the TMX experiment.

I. Introduction

The low frequency ℓ=1 mode was observed in the central cell of the tandem mirror machines[1,2] and enhances the radial diffusion.[1] Here ℓ is the azimuthal mode number. Although the ℓ=1 flute mode[3,4] seems to be one of the important candidates for this mode, the identification of the mode is not clear.[1,2] Experimentally,[1] the bounce frequency of the ions between the mirrors is larger than the Doppler shifted frequency of the mode, $\tilde{\omega} = \omega - \omega_E$, where ω_E is the $\underset{\sim}{E}\times\underset{\sim}{B}$ rotation frequency and ω is the laboratory frequency. Hence, the effects of the finite length of the central cell or the bounce motion of particles along the magnetic field is studied theoretically in the present work. Since the drift kinetic description[5] misses the effect of the centrifugal force acting on ions[6] and the fluid description[4,7] misses the bounce motion of particles along the magnetic field, the full kinetic description[8] is used here. To simplify the problem, however, the model equilibrium magnetic field shown in Fig. 1 is adopted and the stability analysis is limited to electrostatic modes which have axial and radial structure. A general matrix dispersion relation is derived in Eqs. (31) and (35), that applies to a wide class of low frequency electrostatic drift modes.

Equations (35), (38), and (41) constitute the kinetic dispersion relation for the low frequency radially global drift modes. The dispersion relation includes the effects of (1) the diamagnetic drifts, (2) the $\underset{\sim}{E}\times\underset{\sim}{B}$ drift, (3) the lowest order of ion Larmor radius, (4) the bounce motion of particles along the magnetic field, and (5) the anisotropy of temperature. It is shown that the axially symmetric modes with low order axial mode numbers are significantly influenced by the bounce motion while the axially antisymmetric modes are not. As a simple application of the present theory, we reduce the dispersion relation for modes with $v_{zj}/L \gg |\omega - \omega_j^-|$, where v_{zj} is the thermal velocity parallel to the magnetic field of species j, L is a half length of the central cell, and ω_j^- is the lower characteristic frequency, Eq. (25), of a particle of species j in the equilibrium electric and magnetic fields. In this frequency regime, the parameter p measures the finite Larmor radius of ions, and the parameter q measures the anisotropy of temperature, which together determine the frequency and stability of the mode. The two parameters p, q are defined by Eqs. (51) and (59). For $|q| \ll 1$, the conventional flute modes given by Eq. (52) are the eigenmodes in the central cell. For $|q| \gg 1$, which may be more realistic since the anisotropy may easily exceed the ratio of the ion diamagnetic frequency to the ion cyclotron frequency, the new type of flute modes given by Eq. (61) are the eigenmodes of the central cell plasma. The new flute modes are driven unstable by the presence of the centrifugal force acting on ions and the temperature perpendicular to the magnetic field being higher than that parallel to the magnetic field. The frequency of the ℓ=1 new flute mode is equal to ω_E and is shown to be insensitive to the position of the conducting wall in contrast with the conventional ℓ=1 flute mode. We suggest that the present ℓ=1 flute mode may explain the ℓ=1 mode observed in the central cell of the TMX experiment.[1]

The equilibrium properties are briefly given in Sec. II. The stability properties for electrostatic modes are derived and investigated in Sec. III. The summary and discussions are given in Sec. IV.

II. Equilibrium

Since it is difficult to keep both the effects of the bounce motion of the particles along the magnetic field in the central cell and the centrifugal force on the ions due to $\underset{\sim}{E}\times\underset{\sim}{B}$ rotation simultaneously, it is necessary to adopt a simplified equilibrium for the problem. We introduce cylindrical polar coordinate system (r,θ,z) with $z=0$ at the midplane of the central cell. We assume that (a) the applied magnetic field is in the z direction and its magnitude is a finite constant for $|z| \leqslant L$ and infinite for $|z| > L$, where L is the half length of the central cell as defined in Fig. 1; (b) all of the particles are reflected at $z = \pm L$; (c) equilibrium properties are independent of θ and z; (d) the plasma column is rigidly rotating in the θ direction; and (e) the ratio of plasma pressure to magnetic pressure is sufficiently small that the axial diamagnetic field has a negligible effect on equilibrium and stability properties.

We consider here the rigid-rotor equilibrium distribution function given by

$$f_j^0(r,\underset{\sim}{v}) = N_{j0}\bar{F}_j(H_{\perp j} - \Omega_j p_{\theta j})G_j(v_z) \ , \tag{1}$$

where N_{j0} and Ω_j are constants, v_z is the axial velocity, $p_{\theta j}$ is the canonical angular momentum,

$$p_{\theta j} = m_j r(v_\theta + \frac{1}{2}\,\omega_{cj} r) \ , \tag{2}$$

and $H_{\perp j}$ is the perpendicular energy

$$H_{\perp j} = \frac{1}{2} m_j(v_r^2 + v_\theta^2) + e_j\phi_0(r) \ . \tag{3}$$

In Eqs. (2) and (3), e_j and m_j are the charge and mass of a particle of species j, $\phi_0(r)$ is the equilibrium electrostatic potential and $\omega_{cj} = e_jB/m_jc$ is the cyclotron frequency for the region of $|z| \leqslant L$. We further restrict the form of $\bar{F}_j$ and G_j as follows

$$\bar{F}_j(H_{\perp j} - \Omega_j p_{\theta j}) = \frac{m_j}{2\pi T_{\perp j}} \exp\left(-\frac{H_{\perp j}-\Omega_j p_{\theta j}}{T_{\perp j}}\right) \tag{4}$$

and

$$G_j(v_z) = \left(\frac{m_j}{2\pi T_{\parallel j}}\right)^{1/2} \exp\left(-\frac{m_j v_z^2}{2T_{\parallel j}}\right) , \tag{5}$$

where $T_\perp$ and $T_\parallel$ are the temperatures perpendicular and parallel to the magnetic field. We define the E×B rotation frequency as $\omega_E = \frac{c}{rB}\frac{d\phi_0}{dr}$ and assume ω_E is constant. Following Davidson,[8] Eq. (1) is then rewritten in terms of the quasi-neutrality condition, as

$$f_j^0 = N(r)F_j(V_r,V_\theta)G_j(v_z) , \tag{6}$$

where

$$N(r) = N_0 \exp\left(-\frac{r^2}{a^2}\right) \tag{7}$$

is the equilibrium density, and

$$F_j(V_r,V_\theta) = \frac{m_j}{2\pi T_{\perp j}} \exp\left(-\frac{m_j}{2T_{\perp j}} (V_r^2 + V_\theta^2)\right) . \qquad (8)$$

In Eqs. (6) to (8) a is the radius of the plasma column and

$$V_r = v_r \quad \text{and} \quad V_\theta = v_\theta - \Omega_j r . \qquad (9)$$

Note that ω_E and a are expressed as [8]

$$\omega_E = \frac{cT_{\perp e}T_{\perp i}}{eB(T_{\perp i}+T_{\perp e})} \left\{\frac{m_i(\Omega_i^2 + \Omega_i\omega_{ci})}{T_{\perp i}} - \frac{m_e(\Omega_e^2 + \Omega_e\omega_{ce})}{T_{\perp e}}\right\} \qquad (10)$$

and

$$a^2 = \frac{-2(T_{\perp e} + T_{\perp i})}{m_i(\Omega_i^2 + \Omega_i\omega_{ci}) + m_e(\Omega_e^2 + \Omega_e\omega_{ce})} , \qquad (11)$$

where the ions are assumed to be singly ionized. Defining the diamagnetic drift frequency for species j by

$$\omega_{*j} = \frac{cT_{\perp j}}{e_j B} \frac{1}{rN} \frac{dN}{dr} = -\frac{2cT_{\perp j}}{e_j Ba^2} , \qquad (12)$$

we have the relation

$$\Omega_j^2 + \Omega_j\omega_{cj} = \omega_{cj}(\omega_E + \omega_{*j}) . \qquad (13)$$

III. Stability for Electrostatic Modes

A. General dispersion relation

The linearized Vlasov equation for electrostatic perturbations can be expressed as

$$\left\{\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\nabla + \frac{e_j}{m_j}\left(-\nabla\phi_0 + \frac{1}{c}\,\underset{\sim}{v}\times\underset{\sim}{B}\right)\cdot\frac{\partial}{\partial\underset{\sim}{v}}\right\}\tilde{f}_j(t,\underset{\sim}{r},\underset{\sim}{v}) = \frac{e_j}{m_j}\,\nabla\tilde{\phi}(t,\underset{\sim}{r})\cdot\frac{\partial f_j^0}{\partial\underset{\sim}{v}}\,, \quad (14)$$

where $\tilde{f}_j$ is the perturbed distribution function for species j. Taking the form of $\tilde{\phi}$ and $\tilde{f}_j$ as

$$\tilde{\phi}(t,\underset{\sim}{r}) = \hat{\phi}(r,z)\;e^{i(\ell\theta-\omega t)}$$

$$\tilde{f}_j(t,\underset{\sim}{r},\underset{\sim}{v}) = \hat{f}_j(r,z,\underset{\sim}{v})\;e^{i(\ell\theta-\omega t)} \quad (15)$$

and using the relation

$$\frac{e_j}{m_j}\,\nabla\tilde{\phi}\cdot\frac{\partial f_j^0}{\partial\underset{\sim}{v}} = -\frac{e_j}{T_{\perp j}}\,f_j^0\left(\frac{d\tilde{\phi}}{dt} + i(\omega-\ell\Omega_j)\tilde{\phi} + \frac{T_{\perp j}-T_{\parallel j}}{T_{\parallel j}}\,v_z\,\frac{\partial\tilde{\phi}}{\partial z}\right)$$

in Eq. (14), we have

$$\hat{f}_j(r,z,\underset{\sim}{v}) = -\frac{e_j}{T_{\perp j}}\,\hat{\phi}(r,z)f_j^0 - \frac{e_j(\omega-\ell\Omega_j)}{T_{\perp j}}\,f_j^0 I_j - \frac{T_{\perp j}-T_{\parallel j}}{T_{\parallel j}}\,\frac{e_j}{T_{\perp j}}\,f_j^0 M_j\,, \quad (16)$$

where

$$I_j = i\int_{-\infty}^{t}dt'\,\tilde{\phi}(t'\underset{\sim}{r}')\;e^{-i(\ell\theta-\omega t)} = i\int_{-\infty}^{t}dt'\,\hat{\phi}(r',z')\;e^{i[\ell(\theta'-\theta)\;-\;\omega(t'-t)]} \quad (17)$$

and

$$M_j = \int_{-\infty}^{t} dt' v_z' \frac{\partial \tilde{\phi}(t', \underset{\sim}{r}')}{\partial z'} e^{-i(\ell\theta - \omega t)} = \int_{-\infty}^{t} dt' v_z' \frac{\partial \hat{\phi}}{\partial z'} (r', z') e^{i[\ell(\theta' - \theta) - \omega(t' - t)]} \ . \quad (18)$$

The quasi-neutrality condition, $\sum_{j=e,i} e_j \int d^3v\hat{f}_j = 0$, leads to the integral mode equation,

$$\sum_{j=e,i} \Big[\frac{T_{\perp i}}{T_{\perp j}} \hat{\phi}(r,z) + \frac{T_{\perp i}}{T_{\perp j}} (\omega - \ell\Omega_j) \int d^3v \ G_j F_j I_j + $$

$$+ \frac{T_{\perp i}}{T_{\perp j}} \frac{T_{\perp j} - T_{\parallel j}}{T_{\parallel j}} \int d^3v G_j F_j M_j\Big] = 0 \ , \quad (19)$$

for $\hat{\phi}(r,z)$ where G_j and F_j are given by Eqs. (5) and (8), respectively.

In order to evaluate the second and third terms of Eq. (19), we follow the method developed by Davidson[8] and extend it to the two-dimensional problem. We expand the perturbed electrostatic potential $\hat{\phi}$ as

$$\hat{\phi}(r,z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [\alpha_{mn} \cos(K_m z) + \beta_{mn} \sin(k_m z)]\phi_n(r) \ , \quad (20)$$

where $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ are expansion coefficients, $K_m = (m + \frac{1}{2})\pi/L$, $k_m = (m+1)\pi/L$ and the vacuum eigenfunction $\phi_n(r)$ is the solution of the radial eigenvalue equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\ell^2}{r^2}\right)\phi_n(r) = -\lambda_n^2 \phi_n(r)$$

subject to the boundary conditions $(rd\phi_n/dr)_{r=0} = 0$ and $\phi_n(r=b) = 0$. Here b is the radius of a conducting wall which surrounds the plasma column. The eigenfunction $\phi_n(r)$ is expressed as

$$\phi_n(r) = A_n J_\ell(\lambda_n r) \quad \text{and} \quad A_n = \sqrt{2}/bJ_{\ell+1}(\lambda_n b) \ , \tag{21}$$

where $J_\ell(x)$ is the Bessel function of the first kind of order ℓ and the eigenvalues $\{\lambda_n\}$ are determined from

$$J_\ell(\lambda_n b) = 0 \ . \tag{22}$$

Evidently $\tilde{\phi}(r=b,\theta,z,t) = \tilde{\phi}(r,\theta,z=\pm L,t) = 0$ are satisfied. In terms of Eqs. (17), (18) and (20), the two integrals of Eq. (19) are expressed as

$$\int d^3V G_j F_j \begin{Bmatrix} I_j \\ M_j \end{Bmatrix} = \int_0^\infty dV_\perp V_\perp F_j(V_\perp) \int_{-\infty}^\infty dv_z G_j(v_z) \times$$

$$\times \sum_{m,n=0}^{\infty} \Big[\alpha_{mn} \int_{-\infty}^0 d\tau \ e^{-i(\ell\theta+\omega\tau)} \begin{Bmatrix} i\ \cos(K_m z') \\ -K_m v_z' \ \sin(K_m z') \end{Bmatrix} +$$

$$+ \beta_{mn} \int_{-\infty}^0 d\tau \ e^{-i(\ell\theta+\omega\tau)} \begin{Bmatrix} i\ \sin(k_m z') \\ k_m v_z' \ \cos(k_m z') \end{Bmatrix}\Big] \int_0^{2\pi} d\zeta \phi_n(r') \ e^{i\ell\theta'}, \tag{23}$$

where $V_\perp = (V_r^2 + V_\theta^2)^{1/2}$ and ζ is the angle between V_r and V_θ. The last integral of Eq. (23) is written as

$$\int_0^{2\pi} d\zeta \phi_n(r') \ e^{i\ell\theta'} = 2\pi A_n \sum_{p,q=-\infty}^{\infty} J_p^2\Big(\frac{\lambda_n V_\perp}{\Delta_j}\Big) J_q\Big(\frac{\Omega_j - \omega_j^-}{\Delta_j} \lambda_n r\Big) J_{\ell-q}\Big(\frac{\omega_j^+ - \Omega_j}{\Delta_j} \lambda_n r\Big) \times$$

$$\times\ e^{i[\ell\theta+\{(p+q)\Delta_j+\ell\omega_j^-\}\tau]}\quad , \tag{24}$$

where Ω_j is the solution of Eq. (13), ω_j^+ and ω_j^- are the characteristic frequency of a particle of species j in the equilibrium field and are defined by

$$\omega_j^\pm = -\frac{1}{2}\,\omega_{cj}\{1 \pm \left(1+\frac{4\omega_E}{\omega_{cj}}\right)^{1/2}\}\ , \tag{25}$$

and $\Delta_j=\omega_j^+-\omega_j^-$. The derivation of Eq. (24) is given in Appendix I. Substitution of Eq. (24) into Eqs. (23) leads to

$$\begin{aligned}\int d^3vG_jF_j\left\{\begin{matrix}I_j\\ M_j\end{matrix}\right\} &= \sum_{m,n=0}^{\infty}A_n\sum_{p,q=-\infty}^{\infty}J_q\left(\frac{\Omega_j-\omega_j^-}{\Delta_j}\lambda_n r\right)J_{\ell-q}\left(\frac{\omega_j^+-\Omega_j}{\Delta_j}\lambda_n r\right)\times\\ &\times 2\pi\int_0^\infty dV_\perp V_\perp F_j(V_\perp)J_p^2\left(\frac{\lambda_n V_\perp}{\Delta_j}\right)\times\\ &\times\left[\alpha_{mn}\int_{-\infty}^{\infty}dv_zG_j(v_z)\int_{-\infty}^{0}d\tau\, e^{-i\tilde{\omega}_j\tau}\left\{\begin{matrix}i\cos(K_m z')\\ -K_m v_z'\sin(K_m z')\end{matrix}\right\}+\right.\\ &\left.+\beta_{mn}\int_{-\infty}^{\infty}dv_zG_j(v_z)\int_{-\infty}^{0}d\tau\, e^{-i\tilde{\omega}_j\tau}\left\{\begin{matrix}i\sin(k_m z')\\ k_m v_z'\cos(k_m z')\end{matrix}\right\}\right]\ ,\end{aligned} \tag{26}$$

where $\tilde{\omega}_j=\omega-\ell\omega_j^--(p+q)\Delta_j$. The last integrals in Eqs. (26) are

$$\begin{aligned}\int_{-\infty}^{\infty}dv_zG_j(v_z)\int_{-\infty}^{0}d\tau\, e^{-i\tilde{\omega}_j\tau}\left\{\begin{matrix}i\cos(K_m z')\\ -K_m v_z'\sin(K_m z')\end{matrix}\right\} &= -\int_{-\infty}^{\infty}\frac{dv_zG_j(v_z)}{\tilde{\omega}_j-K_m v_z}\left\{\begin{matrix}1\\ K_m v_z\end{matrix}\right\}\times\\ \times\sum_{m'=0}^{\infty}\left[\delta_{m,m'}-(-1)^{m+m'}\frac{v_z}{L}\cot\left(\frac{\tilde{\omega}_j L}{v_z}\right)\left(\frac{1}{\tilde{\omega}_j+K_{m'}v_z}-\frac{1}{\tilde{\omega}_j-K_{m'}v_z}\right)\right]&\cos(K_{m'}z)\ ,\end{aligned} \tag{27}$$

and

$$\int_{-\infty}^{\infty} dv_z G_j(v_z) \int_{-\infty}^{0} d\tau\, e^{-i\tilde{\omega}_j \tau} \left\{ \begin{matrix} i \sin(k_m z') \\ k_m v_z' \cos(k_m z') \end{matrix} \right\} = - \int_{-\infty}^{\infty} \frac{dv_z G_j(v_z)}{\tilde{\omega}_j - k_m v_z} \left\{ \begin{matrix} 1 \\ k_m v_z \end{matrix} \right\} \times$$

$$\times \sum_{m'=0}^{\infty} \left[\delta_{m,m'} + (-1)^{m+m'} \frac{v_z}{L} \tan\left(\frac{\tilde{\omega}_j L}{v_z}\right) \left(\frac{1}{\tilde{\omega}_j + k_m' v_z} - \frac{1}{\tilde{\omega}_j - k_m' v_z} \right) \right] \sin(k_{m'} z) \ . \quad (28)$$

The derivations of Eqs. (27) and (28) are given in Appendix II.

Substituting Eqs. (26)-(28) into Eq. (19) and integrating over the plasma volume

$$\int_0^b drr A_{n'} J_\ell(\lambda_{n'} r) \frac{1}{L} \int_{-L}^{L} dz \left\{ \begin{matrix} \cos(K_{m''} z) \\ \sin(k_{m''} z) \end{matrix} \right\} \times \text{Eq. (19)} = 0 \ ,$$

we obtain the matrix representation of the integral mode equations

$$\sum_{m,n=0}^{\infty} \alpha_{m,n} \sum_{j=e,i} Q_j^s(m',n',m,n;\ell) = 0 \quad (29)$$

and

$$\sum_{m,n=0}^{\infty} \beta_{m,n} \sum_{j=e,i} Q_j^a(m',n',m,n;\ell) = 0 \ , \quad (30)$$

where we have changed the variable m" to m′ and define the matrix elements

$$Q_j^{\substack{a \\ s}}(m',n',m,n;\ell) = \frac{T_{\perp i}}{T_{\perp j}} \delta_{m',m}\, \delta_{n',n} -$$

$$- \sum_{p,q=-\infty}^{\infty} \frac{T_{\perp i}}{T_{\perp j}} A_{n'} A_n \int_0^b drr J_\ell(\lambda_{n'} r) J_q\left(\frac{\Omega_j - \omega_j^-}{\Delta_j} \lambda_n r\right) J_{\ell-q}\left(\frac{\omega_j^+ - \Omega_j}{\Delta_j} \lambda_n r\right) \times$$

$$\times 2\pi \int_0^\infty dV_\perp V_\perp F_j(V_\perp) J_p^2\left(\frac{\lambda_n V_\perp}{\Delta_j}\right) \times S_j^{\overset{a}{s}}(m',m,p+q,\ell) \; . \qquad (31)$$

In obtaining Eq. (31) we have used the orthogonality relation

$$A_{n'} A_n \int_0^b drr J_\ell(\lambda_{n'} r) J_\ell(\lambda_n r) = \delta_{n',n} \; . \qquad (32)$$

The finite length response functions S_j in Eq. (31) are defined by

$$S_j^s(m',m,p+q,\ell) = \int_{-\infty}^\infty \frac{dv_z G_j(v_z)}{\tilde{\omega}_j - K_m v_z} \left(\omega - \ell\Omega_j + \frac{T_{\perp j} - T_{\parallel j}}{T_{\parallel j}} K_m v_z\right) \times$$

$$\times \left[\delta_{m,m'} - (-1)^{m+m'} \frac{v_z}{L} \cot\left(\frac{\tilde{\omega}_j L}{v_z}\right)\left(\frac{1}{\tilde{\omega}_j + K_{m'} v_z} - \frac{1}{\tilde{\omega}_j - K_{m'} v_z}\right)\right] , \qquad (33)$$

and

$$S_j^a(m',m,p+q,\ell) = \int_{-\infty}^\infty \frac{dv_z G_j(v_z)}{\tilde{\omega}_j - k_m v_z} \left(\omega - \ell\Omega_j + \frac{T_{\perp j} - T_{\parallel j}}{T_{\parallel j}} k_m v_z\right) \times$$

$$\times \left[\delta_{m,m'} + (-1)^{m+m'} \frac{v_z}{L} \tan\left(\frac{\tilde{\omega}_j L}{v_z}\right)\left(\frac{1}{\tilde{\omega}_j + k_{m'} v_z} - \frac{1}{\tilde{\omega}_j - k_{m'} v_z}\right)\right] . \qquad (34)$$

Note that $\tilde{\omega}_j = \omega - \ell\omega_j^- - (p+q)\Delta_j$.

The dispersion relations for the eigenmodes of Eqs. (29) and (30) are given by

$$\det\Big[\sum_{j=e,i} Q_j^{\nu}(m',n',m,n;\ell)\Big] = 0 \quad \text{for } \nu=s \text{ or } a \,, \tag{35}$$

where m′ and m are the axial mode numbers, n′ and n the radial mode numbers and the superscript s or a denotes the axially symmetric or antisymmetric mode, respectively. The above matrix dispersion relation is valid for a wide class of electrostatic modes which satisfy the condition of quasi-neutrality. The terms with the $\cot(\tilde{\omega}_j L/v_z)$ and $\tan(\tilde{\omega}_j L/v_z)$ functions in Eqs. (33) and (34) describe the bounce motion of particles along the magnetic field in the central cell.

B. Dispersion relation for radially global modes

We will, henceforth, restrict our attention to equilibria in which $|\omega_E|$, $|\omega_{*j}|$, and $|\Omega_j|$ are much less than ω_{ci}. From Eq. (13) Ω_j is then

$$\Omega_j = -\frac{1}{2}\,\omega_{cj}\Big[1 - \Big\{1 + \frac{4(\omega_E+\omega_{*j})}{\omega_{cj}}\Big\}^{1/2}\Big] \,. \tag{36}$$

For electrons, we have from Eqs. (25) and (36)

$$\omega_e^+ = -\omega_{ce},\ \omega_e^- = \omega_E \quad \text{and} \quad \Omega_e = \omega_E+\omega_{*e} \,. \tag{37}$$

We neglect any effect of the finite Larmor radius of electrons and approximate $(\Omega_e-\omega_e^-)/\Delta_e \cong 0$ and $(\omega_e^+-\Omega_e)/\Delta_e \cong 1$. We then have from Eq. (31)

$$Q_e^\nu(m',n',m,n;\ell) = \frac{T_{\perp i}}{T_{\perp e}}[\delta_{m',m} - S_e^\nu(m',m,\ell)]\delta_{n',n} , \qquad (\nu\text{=s or a}) , \qquad (38)$$

where Eq. (32) has been used and $S_e^\nu(m',m,\ell)$ means $S_e^\nu(m',m,p+q=0,\ell)$. Hence $\tilde{\omega}_e$ in $S_e^\nu(m',m,\ell)$ is

$$\tilde{\omega}_e = \omega - \ell\bar{\omega}_e = \omega - \ell\omega_E . \qquad (39)$$

For ions, we will keep the lowest order effect of the finite Larmor radius of the ions. From Eq. (12),

$$\frac{\omega_{*i}}{\omega_{ci}} = -\frac{2\rho_{\perp i}^2}{a^2} = -\frac{2}{(\lambda_n a)^2}(\lambda_n\rho_{\perp i})^2 , \qquad (40)$$

where $\rho_{\perp i}^2 = T_{\perp i}/m_i\omega_{ci}^2$. For the radially global modes $\{\lambda_n a\}$ are not so large that we will keep the terms of order $(\omega_{*i}/\omega_{ci})$. A cumbersome, but straightforward calculation leads from Eq. (31) to

$$Q_i^\nu(m',n',m,n;\ell) = \delta_{m',m}\delta_{n',n} -$$

$$- [(1 - \lambda_n^2\rho_{\perp i}^2 - \frac{\omega_{*i}}{\omega_{ci}})\delta_{n',n} + \frac{\omega_{*i}}{\omega_{ci}} R(n',n;\ell)]S_i^\nu(m',m,\ell) , \qquad (41)$$

where $S_i^\nu(m',m,\ell)$ means $S_i^\nu(m',m,p+q=0,\ell)$, $\tilde{\omega}_i$ in $S_i^\nu(m',m,\ell)$ is

$$\tilde{\omega}_i = \omega - \ell\bar{\omega}_i \qquad (42)$$

and $R(n',n;\ell)$ is defined as

$$R(n',n;\ell) = \begin{cases} A_{n'}A_n \int_0^b drrJ_\ell(\lambda_{n'}r)\lambda_n rJ'_\ell(\lambda_n r) & \text{for } n' \neq n \\ 0 & \text{for } n'=n \ . \end{cases} \qquad (43)$$

where $J'_\ell(x) = dJ_\ell(x)/dx$.

Equations (35), (38), and (41) give the dispersion relation for low frequency electrostatic modes including the effects of (1) the diamagnetic drifts, (2) $\underset{\sim}{E}\times\underset{\sim}{B}$ drift, (3) the lowest order of ion Larmor radius, (4) the bounce motion of particles along the magnetic field, and (5) the difference between the parallel and perpendicular particle temperatures.

C. Influence of the bounce motion of bulk electrons

In the present subsection and the following two subsections we neglect the contributions from the resonant particles. In order to investigate the effect of the bounce motion of bulk electrons on Q_e, we will consider the three limiting cases below:

1. In the case of $L \to \infty$ but $|K_m v_{ze}|$, $|k_m v_{ze}| \gg |\tilde{\omega}_e|$, where v_{ze} is the thermal velocity of electrons parallel to the magnetic field, one has from Eqs. (33), (34), and (38)

$$Q_e^\nu(m',n',m,n;\ell) = \frac{T_{\perp i}}{T_{\parallel e}} \delta_{m',m}\ \delta_{n',n} \qquad (\nu = s \text{ or } a) \ . \qquad (44)$$

2. Letting $K_m = 0$, one has Q_e^s of the conventional flute modes,

$$Q_e^s(m',n',m,n;\ell) = \frac{T_{\perp i}}{T_{\perp e}}\left(1 - \frac{\omega-\ell\Omega_e}{\omega-\ell\bar{\omega}_e}\right)\delta_{m',m}\ \delta_{n',n}\ . \tag{45}$$

3. In the case of $v_{ze}/L \gg |\tilde{\omega}_e|$, one has

$$Q_e^s(m',n',m,n;\ell) = \frac{T_{\perp i}}{T_{\perp e}}\left[\frac{T_{\perp e}}{T_{\parallel e}}\ \delta_{m',m} - \frac{2(-1)^{m'+m}}{K_m K_m' L^2}\ \frac{\omega-\ell\Omega_e}{\omega-\ell\bar{\omega}_e}\right]\delta_{n',n} \tag{46}$$

and

$$Q_e^a(m',n',m,n;\ell) = \frac{T_{\perp i}}{T_{\parallel e}}\ \delta_{m',m}\ \delta_{n',n}\ . \tag{47}$$

Equations (44) and (47) and the first terms of Eq. (46) are the same and are derived from the usual Boltzman relation $\tilde{n} = Ne\tilde{\phi}/T_{\parallel e}$. The second term of Eq. (46) is, on the other hand, due to the bounce motion along the magnetic field. Equation (46) shows that the symmetric modes with low m' and m are significantly affected by the bounce motion. Since it is assumed that any K_m is zero in Eq. (45), the conventional flute modes miss the effect of the bounce motion. Equation (47) shows that even in the case of $v_{ze}/L \gg |\omega-\ell\omega_E|$, the antisymmetric modes in the z direction are not influenced by the bounce motion.

D. Conventional flute modes

For comparison we will derive the dispersion relation of the conventional flute modes from our theory. Letting $K_m = 0$ in Eq. (33) for ions and substituting Eqs. (41) and (45) in Eq. (35), one has

$$\det\Big[\frac{T_{\perp i}}{T_{\perp e}}\Big(1 - \frac{\omega-\ell\Omega_e}{\omega-\ell\omega_e^-}\Big)\delta_{n',n} + \delta_{n',n}$$

$$- \Big\{\Big(1 - \lambda_n^2\rho_{\perp i}^2 - \frac{\omega_{*i}}{\omega_{ci}}\Big)\delta_{n',n} + \frac{\omega_{*i}}{\omega_{ci}} R(n',n;\ell)\Big\} \frac{\omega-\ell\Omega_i}{\omega-\ell\omega_i^-}\Big] = 0 . \qquad (48)$$

From Eqs. (25) and (36) the approximate expressions for the characteristic ion frequencies are

$$\omega_i^- = \omega_E\Big(1 - \frac{\omega_E}{\omega_{ci}}\Big) \quad \text{and} \quad \Omega_i = (\omega_E + \omega_{*i})\Big(1 - \frac{\omega_E+\omega_{*i}}{\omega_{ci}}\Big) . \qquad (49)$$

Using Eqs. (37) and (49) in Eq. (48), the dispersion relation reduces to

$$\det\Big[\Big\{p - 1 + \frac{\omega^2}{(\omega-\ell\omega_E)(\omega-\ell\omega_E-\ell\omega_{*i})}\Big\}\delta_{n',n} + R(n',n;\ell)\Big] = 0 , \qquad (50)$$

where

$$p = -\frac{\omega_{ci}}{\omega_{*i}}\Big(\lambda_n^2\rho_{\perp i}^2 + \frac{\omega_{*i}}{\omega_{ci}}\Big) = \frac{1}{2}\lambda_n^2 a^2 - 1 . \qquad (51)$$

In order to obtain the approximate solution of Eq. (50), we neglect the off-diagonal terms in $R(n',n;\ell)$ and derive

$$\omega = \ell\left(\omega_E + \frac{1}{2}\,\omega_{*i}\right)\left(1 - \frac{1}{p}\right)\left[1 \pm i\left\{\frac{(\omega_E + \frac{1}{2}\,\omega_{*i})^2 - p\omega_{*i}^2/4}{(\omega_E + \frac{1}{2}\,\omega_{*i})^2 (p-1)}\right\}^{1/2}\right] . \qquad (52)$$

We have shown that Eq. (50) also follows from Eq. (2) of Ref. 3 by letting χ=0 and expanding the electrostatic potential $(\omega_1 B/m) r\Psi$ in terms of $\phi_n(r)$. Except for the difference in the definition of p, Eq. (52) is equal to Eq. (11) of Ref. 3. Note that by definition [Eq. (12)] our ω_{*i} is negative. Calculating $p = 1/2(\lambda_n b)^2/(b/a)^2 - 1$ where $\{\lambda_n b\}$ are solutions of $J_\ell(\lambda_n b) = 0$, we find that p is approximately equal to p given in Fig. 1 of Ref. 3. The difference between p and p is obviously due to neglect of the off-diagonal $R(n',n,;\ell)$ terms in the reduction of Eq. (50).

E. New flute modes in the central cell

We assume $v_{zi}/L >> |\tilde{\omega}_i| = |\omega - \ell\bar{\omega}_i|$, where v_{zi} is the thermal velocity of ions parallel to the magnetic field. We keep the leading terms of Eq. (33) and use Eqs. (41) and (46) to derive

$$\det\left[\frac{T_{\perp i}}{T_{\perp e}}\left\{\frac{T_{\perp e}}{T_{\parallel e}}\,\delta_{m',m} - \frac{2(-1)^{m'+m}}{K_{m'}K_m L^2}\,\frac{\omega - \ell\Omega_e}{\omega - \ell\bar{\omega}_e}\right\}\delta_{n',n} + \delta_{m',m}\,\delta_{n',n}\right.$$

$$- \left\{\left(1 - \lambda_n^2 \rho_{\perp i}^2 - \frac{\omega_{*i}}{\omega_{ci}}\right)\delta_{n',n} + \frac{\omega_{*i}}{\omega_{ci}}\,R(n',n;\ell)\right\} \times$$

$$\times \{-(\frac{T_{\perp i}}{T_{\parallel i}} - 1)\delta_{m',m} + \frac{2(-1)^{m'+m}}{K_{m'}K_m L^2} \frac{\omega - \ell\Omega_i}{\omega - \ell\bar{\omega}_i}\}] = 0 \ . \tag{53}$$

As in the previous section, we neglect the off-diagonal terms of $R(n',n;\ell)$ in the above equation.

The low frequency dispersion relation (53), including the bounce effects, is

$$\det[\delta_{m',m} - \frac{2(-1)^{m'+m}}{K_{m'}K_m L^2} U] = 0 \ , \tag{54}$$

where

$$U = \frac{\dfrac{T_{\perp i}}{T_{\perp e}} \dfrac{\omega - \ell\Omega_e}{\omega - \ell\bar{\omega}_e} + \left(1 - \lambda_n^2 \rho_{\perp i}^2 - \dfrac{\omega_{*i}}{\omega_{ci}}\right) \dfrac{\omega - \ell\Omega_i}{\omega - \ell\bar{\omega}_i}}{\dfrac{T_{\perp i}}{T_{\parallel e}} + 1 + \left(\dfrac{T_{\perp i}}{T_{\parallel i}} - 1\right)\left(1 - \lambda_n^2 \rho_{\perp i}^2 - \dfrac{\omega_{*i}}{\omega_{ci}}\right)} \ . \tag{55}$$

Using the relation

$$\sum_{m=0}^{\infty} \frac{2}{K_m^2 L^2} = 1 \ , \tag{56}$$

it is found that the solution of the matrix dispersion relation (54) is

$$U = 1 \ . \tag{57}$$

Equation (57) yields the quadratic dispersion relation

$$(p+q)\tilde{\omega}^2 + \ell\{2\omega_E + \omega_{*i}(1-p) + q\frac{\omega_E^2}{\omega_{ci}}\}\tilde{\omega} + \ell^2\omega_E^2 = 0 , \quad (58)$$

where $\tilde{\omega} = \omega - \ell\omega_E$, p is defined by Eq. (51) and

$$q = -\frac{\omega_{ci}}{\omega_{*i}}\left[\frac{T_{\perp i}}{T_{\perp e}}\frac{T_{\perp e}-T_{\parallel e}}{T_{\parallel e}} + \frac{T_{\perp i}-T_{\parallel i}}{T_{\parallel i}}\left(1 - \lambda_n^2\rho_{\perp i}^2 - \frac{\omega_{*i}}{\omega_{ci}}\right)\right] . \quad (59)$$

The solution of Eq. (58) is

$$\omega - \ell\omega_E = \ell\omega_E\left[-\frac{1 + \frac{\omega_{*i}}{2\omega_E}(1-p) + \frac{q\omega_E}{2\omega_{ci}}}{p+q}\right.$$

$$\left.\pm i\left\{\frac{q - \left(1 + \frac{\omega_{*i}}{2\omega_E}\right)^2(1-p) + \left(\frac{\omega_{*i}}{2\omega_E}\right)^2 p(1-p)}{(p+q)^2}\right\}^{1/2}\right] , \quad (60)$$

where the terms of order $|\omega_E/\omega_{ci}|$, $|\omega_{*i}/\omega_{ci}|$, and $q\omega_E^2/\omega_{ci}^2$ are neglected when compared with unity. In the case of $|q| << 1$, Eq. (60) reduces to Eq. (52). This means that although $v_{zj}/L >> |\tilde{\omega}_j|$ for j=e and i, the bounce motion does not affect the dispersion relation in case of $|q| << 1$ and the conventional flute modes are the eigenmodes in the system with the finite axial length along the magnetic field.

Next we consider the case of $|q| >> 1$, which may be more realistic since the anisotropy may easily exceed the ratio of the ion diamagnetic frequency to the ion cyclotron frequency. From Eq. (60), we have easily

$$\omega = \ell\omega_E\left(1 + i\left(\frac{\omega_{*e}}{\xi\omega_{ci}}\right)^{1/2}\right) , \tag{61}$$

where

$$\xi = \frac{T_{\perp e}-T_{\parallel e}}{T_{\parallel e}} + \frac{T_{\perp i}-T_{\parallel i}}{T_{\parallel i}}\frac{T_{\perp e}}{T_{\perp i}} \tag{62}$$

and the small correction terms have been neglected. The necessary condition for the above modes is

$$\frac{|\omega-\ell\bar{\omega}_i|L}{v_{zi}} \sim \ell\frac{\omega_E L}{v_{zi}}\left(\frac{\omega_{*e}}{\xi\omega_{ci}}\right)^{1/2} \ll 1 . \tag{63}$$

Note that the terms of order $\lambda_n^2\rho_{\perp i}^2 \sim |\omega_{*i}/\omega_{ci}|$ in Eq. (53) are important for the conventional flute modes but are not important for the new modes of Eq. (61). Neglecting these terms in Eq. (53) and using the approximation $\Omega_i \cong \omega_E + \omega_{*i}$ and $\bar{\omega}_i \cong \omega_E$ except for $\bar{\omega}_i$ in the denominator, we have

$$\xi + \frac{\ell\omega_{*e}}{\omega-\ell\omega_E} - \frac{\ell\omega_{*e}}{\omega-\ell\omega_E+\frac{\ell\omega_E^2}{\omega_{ci}}} = 0 . \tag{64}$$

Neglecting the small terms, Eq. (64) then gives the approximate mode given in Eq. (61). In order to drive the new modes unstable, it is necessary that the centrifugal force acts on the ions, causing the difference of the rotation frequency between electrons and ions and that ξ be positive, which requires that the perpendicular temperature is higher than the parallel temperature.

We now show that the modes given by Eq. (57) are flute modes. From Eqs. (29), (35), and (54) with U = 1, we have

$$\sum_{m,n=0}^{\infty} \alpha_{m,n}\left(\delta_{m',m} - \frac{2(-1)^{m'+m}}{K_{m'}K_m L^2}\right)\delta_{n',n} = 0 \ . \tag{65}$$

Equation (65) is rewritten as

$$\alpha_{m',n'} = \frac{2(-1)^{m'}}{K_{m'}L} \sum_{m=0}^{\infty} \frac{(-1)^m}{K_m L} \alpha_{m,n'} \ . \tag{66}$$

Noting Eq. (56), the solution of Eq. (66) is

$$\alpha_{m',n'} = \eta \frac{2(-1)^{m'}}{K_{m'}L} \ , \tag{67}$$

where η is constant and the above expression is independent of n'. Substituting Eq. (67) in Eq. (20) and using the relation

$$1 = \sum_{m=0}^{\infty} \frac{2(-1)^m}{K_m L} \cos(K_m z) \ ,$$

it is found that the symmetric modes given by Eq. (57) are independent of z. Hence the new modes of Eq. (61) are flute modes.

The radial structure for the $\ell=1$ fundamental flute mode with n=0 is shown in Fig. 2. As is shown by the derivation of Eq. (61), which neglects $\lambda_n^2\rho_{\perp i}^2$, the properties of the new $\ell=1$ flute mode are insensitive to the position of the conducting wall in contrast to the stability of the conventional $\ell=1$ flute mode given by Eq. (52), which is sensitive to b/a through the parameter p.

IV. Summary and discussions

Using the model equilibrium magnetic field shown in Fig. 1, we derive the dispersion relation $\det|\sum Q_j| = 0$ for the electrostatic modes satisfying the quasi-neutrality [see Eqs. (31)-(35)]. For the low frequency, radially global modes, Q_e is given by Eq. (38) and Q_i by Eq. (41). The dispersion relation contains the effects of (1) the diamagnetic drifts, (2) the $\underset{\sim}{E}\times\underset{\sim}{B}$ drift, (3) the lowest order ion Larmor radius, (4) the bounce motion of particles along the magnetic field, and (5) the anisotropy of the temperatures.

In Sec. III.C the influence of the bounce motion of the bulk electrons is studied. It is shown that the axially symmetric modes with low axial mode numbers are significantly influenced by the bounce motion along the magnetic field. In Secs. III.D and E the conventional flute mode and the new flute mode are derived from the $\det|\sum Q_j| = 0$. It is found that the conventional flute modes are the eigenmodes in the central cell with the finite length if the anisotropy parameter defined by Eq. (59) is $|q| \ll 1$. On the other hand, if $|q| \gg 1$ the new flute modes given by Eq. (61) are the eigenmodes of the central cell plasmas. These modes are driven unstable by the combination of the centrifugal force acting on the ions and the perpendicular temperature being higher than the parallel temperatures. In order for the anisotropy of temperature to survive in the new flute mode, it is necessary that the axial length of the central cell be finite. The necessary condition on the anisotropy and axial length for the new flute mode is given by inequality (63).

In the TMX tandem mirror experiment,[1] $\omega_E = 31.4 \times 10^3$(rad/sec), $\omega_{*e} = 8.2 \times 10^3$(rad/sec), $\omega_{ci} = 4.8 \times 10^6$(rad/sec), and $v_{zi}/L = 13 \times 10^3$(1/sec) for $T_i \sim 25$ eV at $r = 30$ cm. Substituting these parameters into inequality (63), we have

$$|\omega - \ell\bar{\omega_i}|L/v_{zi} \sim 0.1\ell/\xi^{1/2} << 1 \ .$$

Although there is no experimental observation of the temperature anisotropy parameter, ξ, the condition is well satisfied for $\xi \geqslant 0.1$ for the ℓ=1 flute mode. The observed frequency $\omega = 44 \times 10^3$(rad/sec) is close to the observed ω_E at $r = 30$ cm. The radial structure for the measured ℓ=1 density oscillation shown in Fig. 10 of Ref. 1 is similar to that for the perturbed density of the ℓ=1 fundamental mode shown in Fig. 2. It is our opinion that the ℓ=1 flute mode given by Eq. (61) explains the essential features of the ℓ=1 mode observed in the TMX experiment.[1]

Although both variations of the $\underset{\sim}{E}\times\underset{\sim}{B}$ rotation frequency and the temperatures exist in the radial direction, the measured oscillation amplitude is sufficiently large that the nonlinear $\underset{\sim}{E}\times\underset{\sim}{B}$ convection may be strong enough to overcome this radial dispersion.

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Figure Captions

Fig. 1 The model equilibrium magnetic field. The magnitude of the magnetic field is a finite constant in the region $|z| \leqslant L$, where L is the half length of the central cell. All of the particles are reflected at $z = \pm L$.

Fig. 2 The radial structure of the perturbed density, $\tilde{n}(r)$, and the perturbed electrostatic potential, $\tilde{\phi}(r)$, for the $\ell=1$ flute mode given by Eq. (61). The parameter a and b are the radius of the plasma and conducting wall, respectively.

Fig. 3 The particle trajectories as a function of $\tau = t'-t$ for a arriving at z with a velocity v_z at time $t'=t$.

Appendix I. Derivation of Eq. (24)

The perpendicular motion of a particle in the equilibrium fields is given by the equations

$$\frac{d^2x'}{d\tau^2} = -\omega_{cj}\omega_E x' + \omega_{cj}\frac{dy'}{d\tau}$$

$$\frac{d^2y'}{d\tau^2} = -\omega_{cj}\omega_E y' - \omega_{cj}\frac{dx'}{d\tau}, \qquad (A.1)$$

where $x'(\tau) = r'(\tau)\cos\theta'(\tau)$, $y'(\tau) = r'(\tau)\sin\theta'(\tau)$, $\tau = t'-t$, and ω_E is constant by assumption. The characteristic frequencies of the motion of a particle from Eqs. (A.1) are given by

$$\omega_j^{\pm} = -\frac{1}{2}\,\omega_{cj}\left\{1 \pm \left(1 + \frac{4\omega_E}{\omega_{cj}}\right)^{1/2}\right\}. \qquad (A.2)$$

The solutions of Eqs. (A.1) are

$$x'(\tau) = \frac{1}{\Delta_j}\{v_\perp[\sin(\zeta + \omega_j^+\tau) - \sin(\zeta + \omega_j^-\tau)] +$$

$$+ r(\Omega_j - \omega_j^-)\cos(\theta + \omega_j^+\tau) - r(\Omega_j - \omega_j^+)\cos(\theta + \omega_j^-\tau)\} \qquad (A.3a)$$

$$y'(\tau) = \frac{1}{\Delta_j}\{v_\perp[\cos(\zeta + \omega_j^-\tau) - \cos(\zeta + \omega_j^+\tau)] +$$

$$+ r(\Omega_j - \omega_j^-)\sin(\theta + \omega_j^+\tau) - r(\Omega_j - \omega_j^+)\sin(\theta + \omega_j^-\tau)\}, \qquad (A.3b)$$

where $\Delta_j = \omega_j^+ - \omega_j^-$, $v_\perp = (v_r^2 + v_\theta^2)^{1/2}$ [see Eq. (9)], and $r = r'(\tau=0)$, as usual. Substitution of Eqs. (A.3) into the identity[8]

$$e^{i\ell\theta'} J_\ell(\lambda_n r') = \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{i\ell\left(\alpha - \frac{\pi}{\tau}\right)}\, e^{i\left(\lambda_x x' + \lambda_y y'\right)} \;, \qquad \text{(A.4)}$$

where $\lambda_x = \lambda_n \cos\alpha$ and $\lambda_y = \lambda_n \sin\alpha$ leads to Eq. (24).

Appendix II. Derivations of Eqs. (27) and (28)

Define $A_1(v_z,K)$ and $A_2(v_z,K)$ as

$$A_1(v_z,K) = \int_{-\infty}^{0} d\tau\, e^{-i\tilde{\omega}\tau}\, e^{iKz'} = \left(1 - e^{i\frac{4\tilde{\omega}L}{|v_z|}}\right)^{-1} I_1(v_z,K) \tag{A.5a}$$

and

$$A_2(v_z,K) = \int_{-\infty}^{0} d\tau\, e^{-i\tilde{\omega}\tau} v_z'\, e^{iKz'} = \left(1 - e^{i\frac{4\tilde{\omega}L}{|v_z|}}\right)^{-1} I_2(v_z,K)\ , \tag{A.5b}$$

where

$$\begin{Bmatrix} I_1(v_z,K) \\ I_2(v_z,K) \end{Bmatrix} = \int_{-\frac{4L}{|v_z|}}^{0} d\tau\, e^{-i\tilde{\omega}\tau}\, e^{iKz'} \begin{Bmatrix} 1 \\ v_z' \end{Bmatrix} . \tag{A.6}$$

In the second expression of Eqs. (A.5), we have used the fact that the motion of particles is periodic with the period $4L/|v_z|$. Integrating along the particles trajectories shown in Fig. 3, we have

$$I_1(v_z \gtrless 0,\ K) = \frac{i\left(1 - e^{i\frac{4\tilde{\omega}L}{|v_z|}}\right) e^{iKz}}{\tilde{\omega} - Kv_z}$$

$$\mp 2\, e^{i\frac{\tilde{\omega}}{v_z}(z \pm 2L)} \sin\left(\frac{\tilde{\omega}L}{v_z} + KL\right)\left(\frac{1}{\tilde{\omega}-Kv_z} - \frac{1}{\tilde{\omega}+Kv_z}\right) , \tag{A.7a}$$

and

$$I_2(v_z \gtrless 0, K) = \frac{iv_z\left(1 - e^{i\frac{4\tilde{\omega}L}{|v_z|}}\right) e^{iKz}}{\tilde{\omega} - kv_z}$$

$$\mp 2v_z\, e^{i\frac{\tilde{\omega}}{v_z}(z\pm 2L)} \sin\left(\frac{\tilde{\omega}L}{v_z} + KL\right)\left(\frac{1}{\tilde{\omega}-Kv_z} + \frac{1}{\tilde{\omega}+Kv_z}\right) , \qquad (A.7b)$$

where the upper (lower) signs are for $v_z > 0$ ($v_z < 0$). Substituting Eqs. (A.7) in Eqs. (A.5) and using the relation

$$1 - e^{i\frac{4\tilde{\omega}L}{|v_z|}} = \mp 4i\, e^{\pm i\frac{2\tilde{\omega}L}{v_z}} \sin\left(\frac{\tilde{\omega}L}{v_z}\right) \cos\left(\frac{\tilde{\omega}L}{v_z}\right) \quad \text{for } v_z \gtrless 0$$

we have

$$A_1(v_z,K) = \frac{i\, e^{iKz}}{\tilde{\omega}-Kv_z} + \frac{e^{i\frac{\tilde{\omega}z}{v_z}} \sin\left(\frac{\tilde{\omega}L}{v_z} + KL\right)}{2i\, \sin\left(\frac{\tilde{\omega}L}{v_z}\right) \cos\left(\frac{\tilde{\omega}L}{v_z}\right)} \left(\frac{1}{\tilde{\omega}-KV_z} - \frac{1}{\tilde{\omega}+Kv_z}\right) \qquad (A.8a)$$

and

$$A_2(v_z,K) = \frac{iv_z\, e^{iKz}}{\tilde{\omega}-Kv_z} + \frac{v_z\, e^{i\frac{\tilde{\omega}z}{v_z}} \sin\left(\frac{\tilde{\omega}L}{v_z} + KL\right)}{2i\, \sin\left(\frac{\tilde{\omega}L}{v_z}\right) \cos\left(\frac{\tilde{\omega}L}{v_z}\right)} \left(\frac{1}{\tilde{\omega}-Kv_z} + \frac{1}{\tilde{\omega}+Kv_z}\right) . \qquad (A.8b)$$

Equations (A.8) are valid for any v_z. Applying Eqs. (A.8) to the cases of $K = \pm K_m$ and $K = \pm k_m$ and using the four relations below:

$$\cos(K_m L) = \sin(k_m L) = 0 \ ,$$

$$\cos\left(\frac{\tilde{\omega}_j z}{v_z}\right) = \sum_{m=0}^{\infty} (-1)^m \frac{v_z}{L} \cos\left(\frac{\tilde{\omega}_j L}{v_z}\right)\left(\frac{1}{\tilde{\omega}_j + K_m v_z} - \frac{1}{\tilde{\omega}_j - K_m v_z}\right) \cos(K_m z) \ ,$$

and

$$\sin\left(\frac{\tilde{\omega}_j z}{v_z}\right) = \sum_{m=0}^{\infty} (-1)^m \frac{v_z}{L} \sin\left(\frac{\tilde{\omega}_j L}{v_z}\right)\left(\frac{1}{\tilde{\omega}_j + k_m v_z} - \frac{1}{\tilde{\omega}_j - k_m v_z}\right) \sin(k_m z) \ ,$$

one can easily get Eqs. (27) and (28).

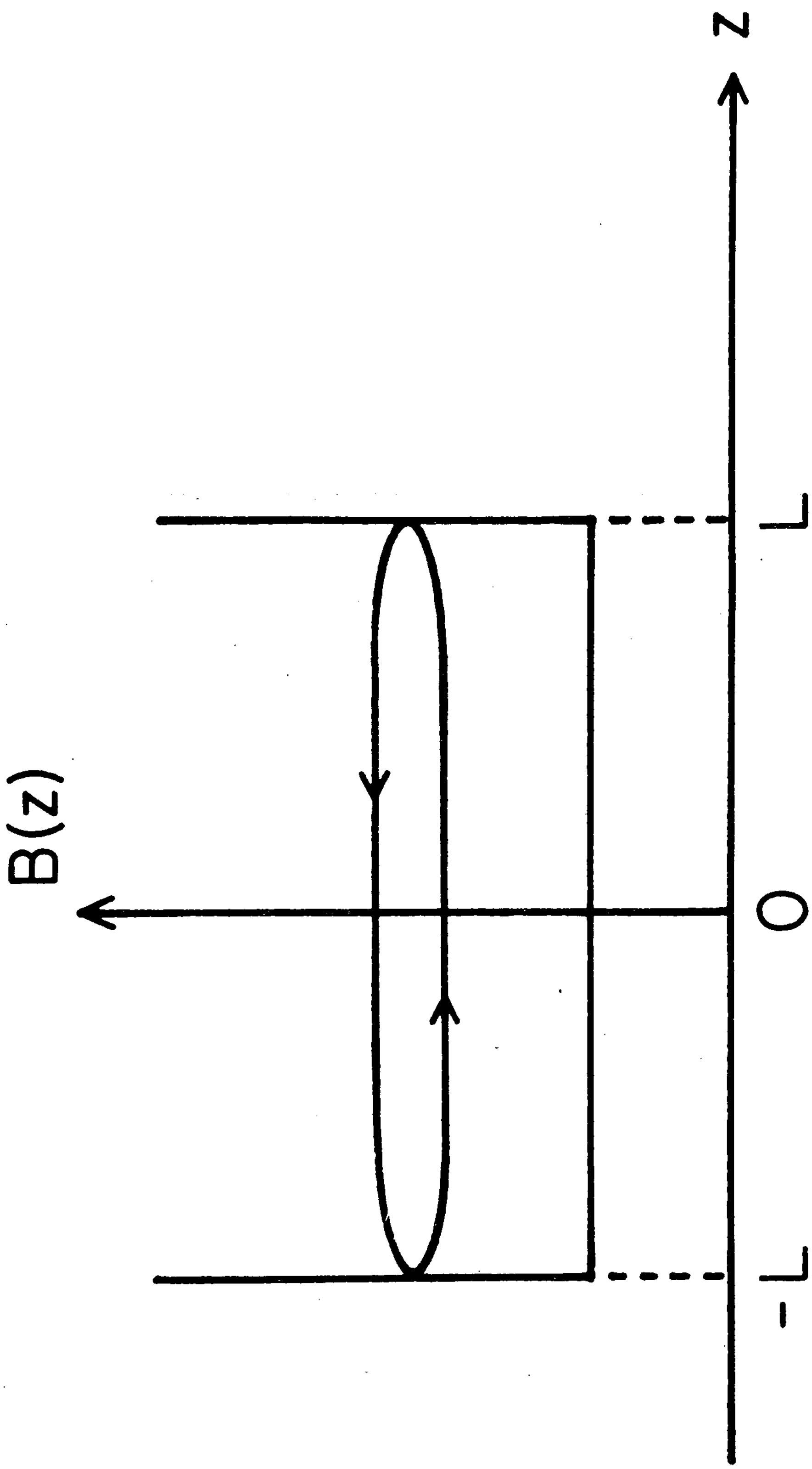


Fig. 1

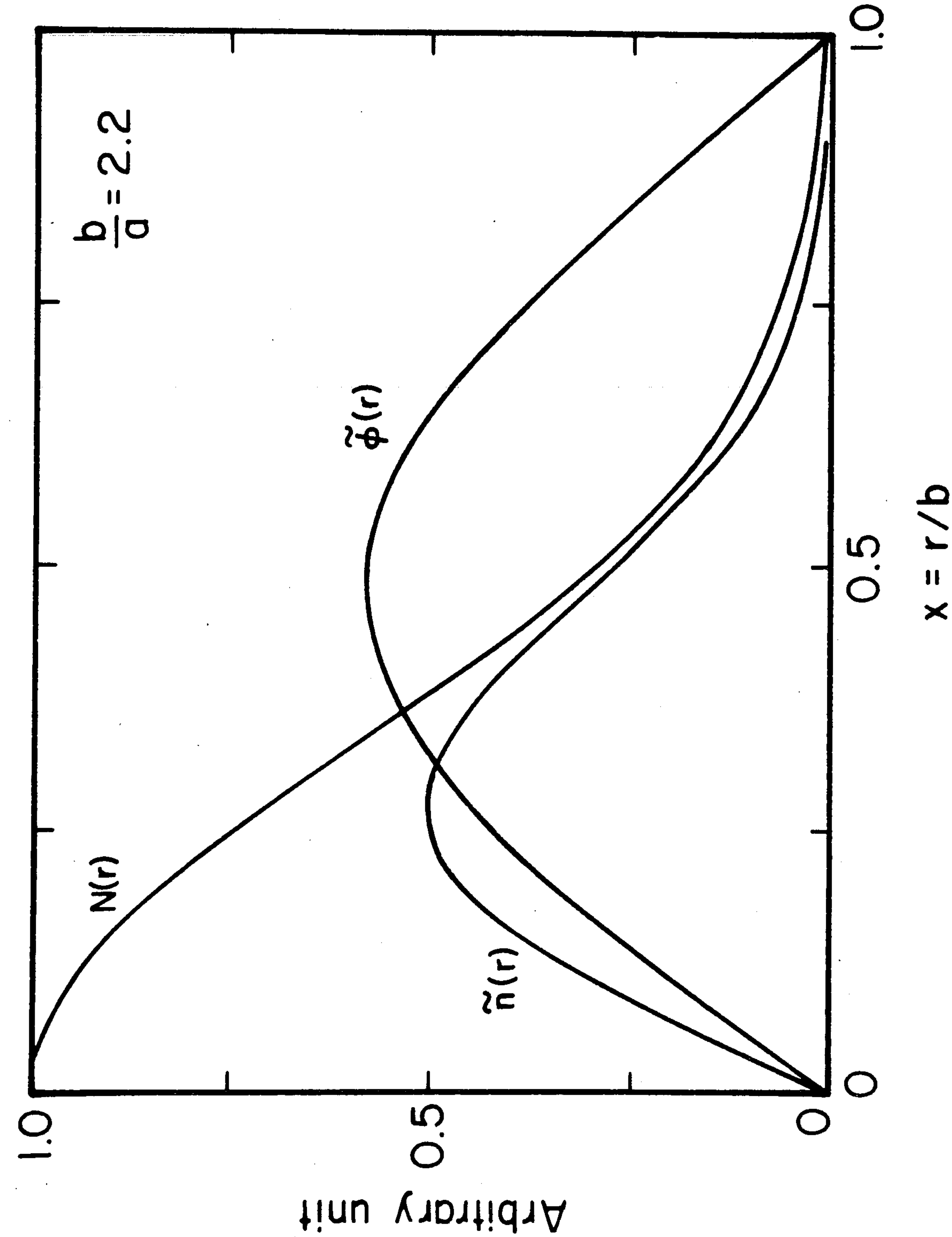
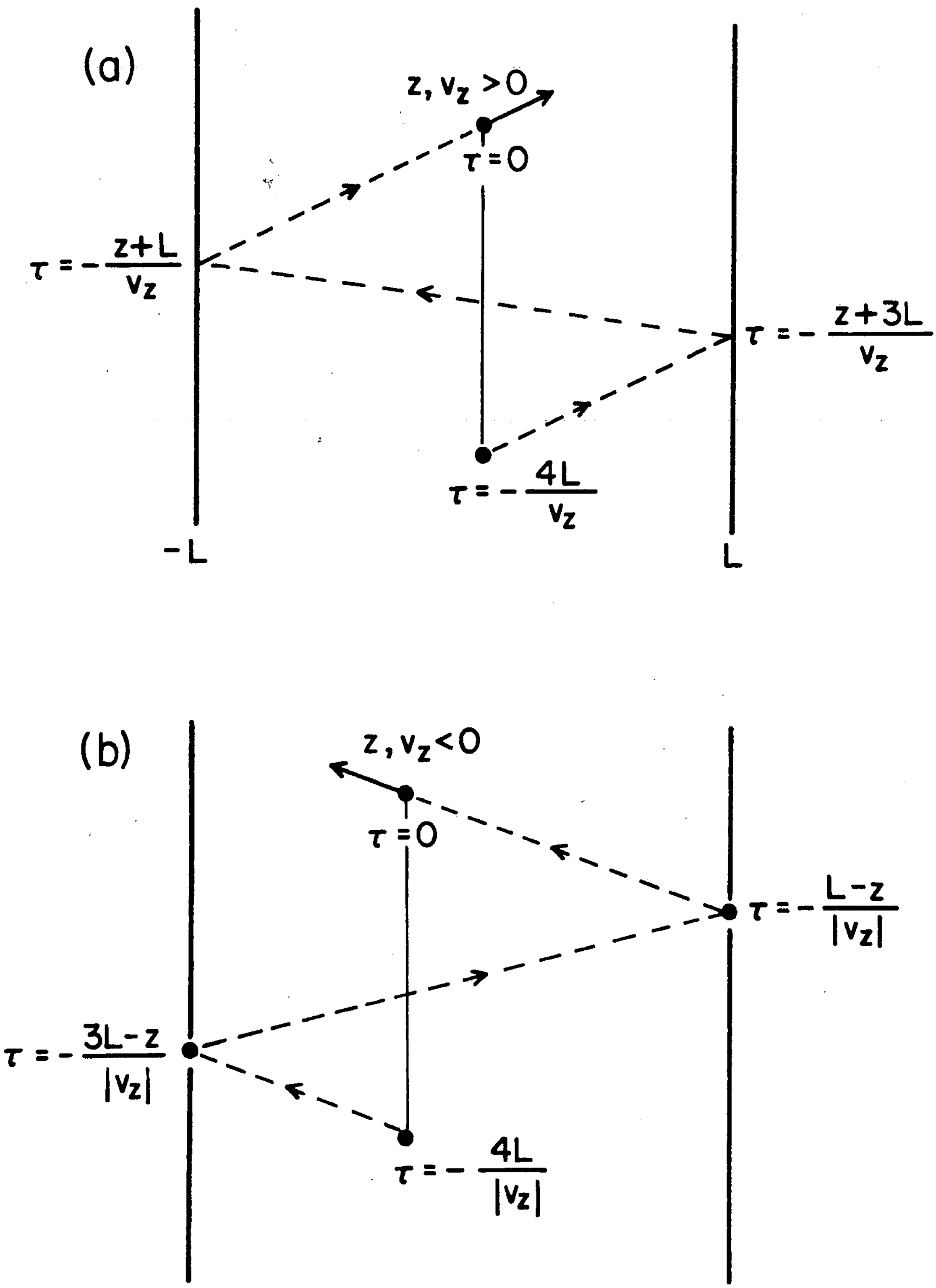


Fig. 2



(a)
z, $v_z > 0$
$\tau = 0$
$\tau = -\frac{z+L}{v_z}$
$\tau = -\frac{z+3L}{v_z}$
$\tau = -\frac{4L}{v_z}$
-L
L
(b)
z, $v_z < 0$
$\tau = 0$
$\tau = -\frac{L-z}{|v_z|}$
$\tau = -\frac{3L-z}{|v_z|}$
$\tau = -\frac{4L}{|v_z|}$

Fig. 3

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Japan

Dear Akio:

Enclosed is a corrected manuscript. I am still hoping you will write a short version giving a simpler physical picture of the instability mechanism, as perhaps with an anisotropic fluid description with a harmonically bound electron motion $\ddot{z} + \omega_{be}^2 z = 0$ of the electrons in z-direction, $\varepsilon^e = 1 - \omega_{pe}^2/(\omega^2 - \omega_{be}^2)$.

I will wait to hear from you with final corrections before making the IFS report.

After the report is out we can correspond again before submitting the work to the Physics of Fluids.

Thank you for coming to the Steering Committee Meeting and traveling back to Tokyo with us.

Best regards,

Wendell

Wendell Horton

/jh
Encl: a/s