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A STROLL THROUGH THE SOLITON THEORY

(Lecture Notes)

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Abstract

The phenomenon of the nonlinear solitary wave and the role of dissipation and dispersion are discussed in a series of lecture notes. KdV equation is derived in detail for ion plasma waves. The symmetries of the KdV equation are compared with the invariances of Burgers' equation and the important role of time invariance is emphasized. The solitary waves and the cnoidal wave solutions are derived for the KdV equation. Bäcklund, Miura and Gardner transformations are derived and their relation to the conservation laws and the "superposition principle" (multisoliton solution) are obtained. The relation between the KdV equation and the time independent Schrödinger equation is given and the method of inverse scattering theory is discussed.

These lecture notes introduce the theoretical ideas and the techniques applied to a class of nonlinear equations mainly through the context of the KdV equation.

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1. INTRODUCTION

An Historical Note

The phenomenon of solitary wave was first realized in 1834 by J. Scott Russell who made observations on water waves on the Edinburgh to Glasgow canal. These observations together with laboratory experiments (where solitary waves were generated by dropping a weight at the one end of a shallow water channel) where published[1] by Russell in 1844 before the British Association for the Advancement of Science, including the following paragraph:

> "I believe I shall best introduce this phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually

diminished, and after a chase of one or two miles I lost it in the windings of the channel."

Moreover, in an earlier publication[2] Russell reported the observation of solitons

"The great primary waves of translation cross each other without change of any kind in the same manner as the small oscillations produced on the surface of a pool by a falling stone."

Russell suggested the following empirical formula in order to explain the steady state velocity c of the solitary waves

$$c^2 = g(h + a) \tag{1.1}$$

where h is the height of the undisturbed water, a is the amplitude of the wave (see Fig. 1) and g is the gravitational acceleration constant. Equation (1.1) was later deduced by Boussinesq[3] in 1871 and independently by Rayleigh[4] in 1876 by assuming a wave length (~D) much greater than the height of the water in the channel, h (see Fig. 1) and by using the equations of motion for an incompressible fluid. They also derived the displacement of the wave above the level h,

$$y(x,t) - h = a\ \mathrm{sech}^2[(x-ct)/D] \tag{1.2}$$

where

$$D^2 = 4h^2(h+a)/3a \quad . \tag{1.3}$$

In 1985 Korteweg and de Vries (KdV) developed the following equation[5]

$$\frac{\partial y}{\partial t} = \frac{3}{2} \sqrt{\frac{g}{h}} \left[(y-h) \frac{\partial y}{\partial x} + \frac{2}{3} \alpha \frac{\partial y}{\partial x} + \frac{1}{3} \beta \frac{\partial^3 y}{\partial x^3} \right] \qquad (1.4)$$

in order to describe the motion of weakly nonlinear long transversal waves. In Eq. (1.4) α is a small arbitrary parameter and β is given by

$$\beta = \frac{h^3}{3} - \frac{Th}{g\rho} \qquad (1.5)$$

where T is the surface tension of the liquid and ρ is the density.

The modern developments in the theory of KdV equation starts with the work[6] of Fermi, Pasta and Ulam in 1955. They study a nonlinear discrete mass string and the thermalization of the energy in such a nonlinear system. Using computer calculations no thermalization was found. This paradox was solved in 1965 by Kruskal and Zabusky[7] who were able to model the nonlinear spring by the KdV equation and to obtain (numerically) the soliton solutions. The Fermi-Pasta-Ulam problem suggests that a nonlinear interaction does not necessarily cause the destruction of oscillations causing randomization. This type of phenomenon and the KdV equation has since been derived in many physical problems. For example, ion acoustic waves in a cold plasma[8], pressure waves in liquid-gas mixtures[9], rotating flow in a tube[10], drift wave turbulence in plasmas[11], etc., can be described to a first approximation by KdV equation with soliton solutions.

What is a soliton?

A soliton is a localized nonlinear wave of steady form. If it interacts with other solitons, after the interaction these structures separate in such a way[12] that their original structure is preserved and their velocities are unchanged. The positions of the solitons are slightly shifted from where they would have been without the existence of the interaction. The solitons are solitary waves but the opposite statement is not always true. (A general review of solitons is given by Scott, Chu and McLaughlin.[13])

What is a solitary wave?

The solitary waves are a one parameter family of shaped pulses moving with a velocity (c) proportional to the wave amplitude (a) and their width (D) is inversely proportional to the square root of the amplitude (see Fig. 1). Therefore, the larger the amplitude the faster is their velocity and the narrower are their width.

The Wave Equation

The wave equation (in one space—one time dimensions)

$$\frac{\partial^2\psi}{\partial t^2} = c^2 \frac{\partial^2\psi}{\partial x^2} \tag{1.6}$$

describes the propagation of waves (in an homogeneous media) with a constant velocity c. In deriving this equation one has to make three main assumptions: (a) There are no dissipations; (b) The amplitude of oscillations (ψ) are small so that one can neglect nonlinear terms; (c) There are no dispersion, namely the velocity of the wave propagation does not depend on the frequency of the wavelength. If one does not assume the absence of dissipation, nonlinearity and dispersion, then the universal wave equation (1.6) is not applicable and each medium has to be described by a different system of equations.

The wave equation (1.6) is a well defined initial value problem. Therefore, this equation can be solved if the initial values for ψ and its time derivatives are given, i.e., $\psi(x,o)$ and $\frac{\partial\psi}{\partial t}(x,o)$ are known. This equation, like the other linear differential equations, obeys the superposition principle. Namely, if $f(x-ct)$ and $g(x+ct)$ solve Eq. (1.6), then

$$\psi(x,t) = f(x-ct) + g(x+ct) \tag{1.7}$$

is also a solution of Eq. (1.6). $f(x-ct)$ and $g(x+ct)$ represent waves of fixed shape travelling to the right and left respectively (see Fig. 2). The general solution $\psi = f+g$ describes two waves that can pass through each other without changing their shape. In general this superposition principle does not occur in physical systems described by nonlinear differential equations. However, in recent years a number of <u>nonlinear differential equations</u> have been discovered which allow waves to pass through each other without changing their shapes and velocities. Moreover, some principle of superposition can be defined for these nonlinear equations. We shall call these equations "solitary wave equations".

Solitary Wave Equation

These are nonlinear differential equations which have a variety of explicit exact solutions. The main equations considered in the literature are:

(a) The KdV equation, normalized in a way that one can write

$$\psi_t + \psi\psi_x + \psi_{xxx} = 0 \tag{1.8}$$

where from now on we are using the notation:

$$\psi_t = \frac{\partial\psi}{\partial t}\,, \quad \psi_x = \frac{\partial\psi}{\partial x}\,, \quad \psi_{xx} = \frac{\partial^2\psi}{\partial x^2}\,, \quad \text{etc.} \tag{1.9}$$

(b) The cubic Schrödinger equation

$$i\psi_t + \psi_{xx} + \alpha|\psi|^2\psi = 0 \tag{1.10}$$

where α is a coupling constant describing the interaction strength.

(c) The Sine-Gordon equation

$$\psi_{tt} - \psi_{xx} + \sin\psi = 0. \tag{1.11}$$

All these equations combine simple forms of dispersion with the simplest structure of nonlinearity.

In these lecture notes we introduce the theoretical ideas and techniques applied to this class of nonlinear equations through the context of KdV equation.

For additional background material we refer to the following books: "Elements of Soliton Theory" by G.L. Lamb Jr.[14], "Solitons" by P.G. Drazin[15], "Theory of Solitons – The Inverse Scattering Method" by Novikov, Manakov, Pitaevskii and Zakharov[16], "Soliton in Action"[17] and "Solitons"[18]. Some useful review articles are given in Refs. (13), (19), (20) and (21). The extensive bibliographies may be found in references (13), (15), (18), (19), and (20).

The structure of these lecture notes is given by the following sequence: In Chapter 2, the KdV equation is obtained for ion plasma waves. One can see, following the detailed algebra, how a fluid model implies the solitary waves described by the KdV equation. In Chapter 3 the Burgers' equation is introduced and solved[22] by the linearizing tranformation (the Cole[23] and Hopf[24] transformation.) The symmetries of KdV equation are summarized in Chapter 4. These symmetries are compared with the invariances of Burgers' equation and the important role of time invariance is emphasized. In Chapter 5 a short discussion of nonlinearity, dissipation and dispersion is given. The KdV equation is solved in Chapter 6 where the solitary waves and the cnoidal wave solutions are obtained. Bäcklund and Miura transformation are derived in Chapter 7. Gardner's transformation is used in Chapter 8 to derive conservation laws, while Bäcklund transformation is used in Chapter 9 to derive the multisoliton solution and the "superposition" principle for the KdV equation. The relation between the KdV equation and the Schrödinger time independent equation is given in Chapter 10. The last, but not least, subject discussed is the inverse scattering theory, (Chapter 11). The present status of solitons in physics is summarized in Chapter 12 by a short story taken from Turkish folklore.

2. AN EXAMPLE: ION PLASMA WAVES AND KdV EQUATION

We consider the fluctuation in the ion density of a two component plasma (electrons and ions), namely the <u>ion acoustic waves</u>. One has to solve the following fluid equations:

The conservation of mass:

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v_i) = 0 \tag{2.1}$$

where n_i is the ion density and v_i their velocity. The conservation of ion and electron momentum

$$n_i m_i \frac{dv_i}{dt} = zen_i E - \frac{\partial P_i}{\partial x} \tag{2.2}$$

$$n_e m_e \frac{dv_e}{dt} = - en_e E - \frac{\partial P_e}{\partial x}\ , \tag{2.3}$$

where n_e is the electron density, v_e their velocity, E is the electric field, P_i and P_e are the ion and electron pressure respectively, m_i and m_e are the ion mass and the electron mass and z is the charge ionization of the ions. The pressures of the electron and ion fluids are given by the ideal equation of states

$$P_e = n_e T_e \tag{2.4}$$

$$P_i = n_i T_i \tag{2.5}$$

where the ion temperature T_i and the electron temperature T_e are given in energy units (i.e. the Boltzmann constant $k_B = 1$). We assume that the electron temperature is much higher than the ion temperature; for simplicity we can write for this case

$$T_i = 0 \ . \tag{2.6}$$

The electric field E satisfies Poisson's equation (we use cgs units)

$$\frac{\partial E}{\partial x} = 4\pi e(zn_i - n_e) \quad . \tag{2.7}$$

Neglecting the inertia of the electrons relative to that of the ions, one gets

$$E = -\frac{1}{en_e}\frac{\partial P_e}{\partial x} \quad . \tag{2.8}$$

Since $m_i \gg zm_e$ the inertia term in the electron momentum equation can be neglected when $dv_e/dt \simeq dv_i/dt$, i.e. when the high frequency plasma oscillations are neglected. Since we are interested in the regime of density and velocity fluctuations near the ion plasma frequency the above approximation is justified.

In order to solve Eqs. (2.1), (2.2), (2.3) and (2.7) one introduces the following dimensionless quantities: the dimensionless space coordinate ξ,

$$\xi = \frac{x}{\lambda_D} \tag{2.9}$$

where λ_D is the Debye length defined by

$$\lambda_D^2 = \frac{T_e}{4\pi z e^2 n_o} \tag{2.10}$$

where n_o is the average ion density, the dimensionless time coordinate τ.

$$\tau = \omega_o t \tag{2.11}$$

where ω_o is the ion plasma frequency

$$\omega_o^2 = \frac{4\pi n_o z^2 e^2}{m_i} \;,, \tag{2.12}$$

the dimensionless velocity u,

$$u = \frac{v_i}{\omega_o \lambda_D} \;, \tag{2.13}$$

the dimensionless ion density N and electron density N_e,

$$N = \frac{n_i}{n_o} \tag{2.14}$$

$$N = \frac{n_e}{z n_o} \tag{2.15}$$

and the dimensionless electric field $\tilde{E}$

$$\tilde{E} = \frac{e\lambda_D E}{T_e} . \tag{2.16}$$

The fluid-Poisson equations (2.1), (2.2), (2.8) and (2.7), in these dimensionless variables are accordingly,

$$\frac{\partial N}{\partial \tau} + \frac{\partial}{\partial \xi} (Nu) = 0 \tag{2.17}$$

$$\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial \xi} = \tilde{E} \tag{2.18}$$

$$\tilde{E} + \frac{1}{N_e} \frac{\partial N_e}{\partial \xi} = 0 \tag{2.19}$$

$$\frac{\partial \tilde{E}}{\partial \xi} = N - N_e \tag{2.20}$$

where N, N_e, u and $\tilde{E}$ are functions of ξ and τ. One has a system of four equations with four unknowns. At this stage it is convenient to define the Fourier modes,

$$N_e = 1 + (\delta N_e)\exp[i(\kappa\xi - \Omega\tau)] \tag{2.21}$$

$$N = 1 + (\delta N)\exp[i(\kappa\xi - \Omega\tau)] \tag{2.22}$$

$$u = (\delta u)\exp[i(\kappa\xi - \Omega\tau)] \tag{2.23}$$

$$\tilde{E} = (\delta\tilde{E})\exp[i(\kappa\xi - \Omega\tau)] \tag{2.24}$$

where κ and Ω are the dimensionless momentum $(k\lambda_D)$ and frequency (ω/ω_o)

$$\omega t = \left(\frac{\omega}{\omega_o}\right) \cdot (\omega_o t) = \Omega \cdot \tau \tag{2.25}$$

$$k \cdot x = (\lambda_D \cdot k) \cdot \left(\frac{x}{\lambda_D}\right) = \kappa\xi \tag{2.26}$$

where k is the usual wave number $k=2\pi/\lambda$ and λ is the wavelength. Substituting Eqs. (2.21)-(2.24) into (2.17)-(2.20) and linearizing the equations one gets

$$-\Omega(\delta N) + \kappa(\delta u) = 0 \tag{2.27}$$

$$-i\Omega(\delta u) - (\delta\tilde{E}) = 0 \tag{2.28}$$

$$-i\kappa(\delta N_e) - (\delta\tilde{E}) = 0 \tag{2.29}$$

$$i\kappa(\delta\tilde{E}) - (\delta N) + (\delta N_e) = 0 \quad . \tag{2.30}$$

Solving the linear set of equations (2.27)-(2.30) one gets the dispersion relation

$$\Omega^2\kappa^2 = \kappa^2 - \Omega^2 \quad . \tag{2.31}$$

The dimensionless phase velocity u_{ph} is given by

$$u_{ph}^2 = \left(\frac{\Omega}{\kappa}\right)^2 = 1-\Omega^2 \quad . \tag{2.32}$$

From (2.32) one concludes that for small Ω one has small κ which in turn is related to large wavelength λ (see Fig. 1 of Chapter 1).

$$\Omega \ll 1, \qquad \Omega \sim \kappa \sim \frac{\lambda_D}{\lambda} \ll 1 \quad . \tag{2.33}$$

The dispersion relation (2.31) for $\Omega<<1$ can be written,

$$\kappa = \frac{\Omega}{\sqrt{1-\Omega^2}} \simeq \Omega + \frac{1}{2}\Omega^3 \quad . \tag{2.34}$$

The phase term in the Fourier transform of Eqs. (2.21)–(2.24) is

$$i(\kappa\xi-\Omega\tau) = i\left[\Omega(\xi-\tau) + \frac{1}{2}\Omega^3\xi\right] \quad . \tag{2.35}$$

The term $\frac{1}{2}\Omega^3\xi$ is called the dispersive term. It is convenient to introduce new variables that incorporate this effect,

$$\begin{aligned} \zeta &= \Omega(\xi-\tau) \\ \eta &= \Omega^3\xi \quad . \end{aligned} \tag{2.36}$$

We rewrite Eqs. (2.17)–(2.20) in terms of these new variables. For this purpose we have to use

$$\begin{aligned} \frac{\partial}{\partial\xi} &= \Omega\frac{\partial}{\partial\zeta} + \Omega^3\frac{\partial}{\partial\eta} \\ \frac{\partial}{\partial\tau} &= -\Omega\frac{\partial}{\partial\zeta} \end{aligned} \tag{2.37}$$

so that the "fluid–Poisson" equations are

$$-\frac{\partial N}{\partial\zeta} + \frac{\partial}{\partial\zeta}(Nu) + \Omega^2\frac{\partial}{\partial\eta}(Nu) = 0 \tag{2.38}$$

$$-\Omega\frac{\partial u}{\partial\zeta} + \Omega u\left(\frac{\partial u}{\partial\zeta} + \Omega^2\frac{\partial u}{\partial\eta}\right) = \tilde{E} \tag{2.39}$$

$$\Omega \frac{\partial N_e}{\partial \zeta} + \Omega^3 \frac{\partial N_e}{\partial \eta} = -\tilde{E}\, N_e \tag{2.40}$$

$$\Omega \frac{\partial \tilde{E}}{\partial \zeta} + \Omega^3 \frac{\partial \tilde{E}}{\partial \eta} = N - N_e \quad . \tag{2.41}$$

We solve Eqs. (2.38)-(2.41) by the following perturbation series:

$$N = 1 + \Omega^2 N^{(1)} + \Omega^4 N^{(2)} + \ldots$$

$$N_e = 1 + \Omega^2 N_e^{(1)} + \Omega^4 N_e^{(2)} + \ldots$$

$$u = \Omega^2 u^{(1)} + \Omega^4 u^{(2)} + \ldots$$

$$\tilde{E} = \Omega^3 \tilde{E}^{(1)} + \Omega^5 \tilde{E}^{(2)} + \ldots \quad . \tag{2.42}$$

The leading terms in the expansion of $\tilde{E}$ and u are evident using Eqs. (2.39) and (2.40) and the Ω^2 expansion of N and N_e. From (2.40) one has $\tilde{E} \sim \Omega^3$ and therefore (2.39) implies $u \sim \Omega^2$. The Ω^2 expansion of N and N_e is necessary in order to have a consistent perturbation theory in the small parameter Ω. Equations (2.38) - (2.41) to lowest order in Ω are [(2.38) and (2.41) to order Ω^2, and (2.39) - (2.40) to order Ω^3]

$$-\frac{\partial N^{(1)}}{\partial \zeta} + \frac{\partial u^{(1)}}{\partial \zeta} = 0 \tag{2.43}$$

$$-\frac{\partial u^{(1)}}{\partial \zeta} = \tilde{E}^{(1)} \tag{2.44}$$

$$\frac{\partial N_e^{(1)}}{\partial \zeta} = -\tilde{E}^{(1)} \tag{2.45}$$

$$N^{(1)} = N_e^{(1)} \quad . \tag{2.46}$$

The solution of Eqs. (2.43)-(2.46) is

$$N^{(1)} = N_e^{(1)} = u^{(1)} \equiv \phi(\zeta,\eta) \tag{2.47}$$

$$\tilde{E}^{(1)} = -\frac{\partial\phi}{\partial\zeta} \quad . \tag{2.48}$$

An arbitrary function of η was neglected by integrating Eq. (2.43).

In the next order in Ω Eqs. (2.38) - (2.41) are [(2.38) and (2.41) to order Ω^4, and (2.39) - (2.40) to order Ω^5]

$$-\frac{\partial N^{(2)}}{\partial\zeta} + \frac{\partial}{\partial\zeta}\left[u^{(2)} + \phi\cdot\phi\right] + \frac{\partial\phi}{\partial\eta} = 0 \tag{2.49}$$

$$-\frac{\partial u^{(2)}}{\partial\zeta} + \phi\,\frac{\partial\phi}{\partial\zeta} - \tilde{E}^{(2)} = 0 \tag{2.50}$$

$$\frac{\partial N_e^{(2)}}{\partial\zeta} + \frac{\partial\phi}{\partial\eta} + \phi\,\tilde{E}^{(1)} + \tilde{E}^{(2)} = 0 \tag{2.51}$$

$$\frac{\partial\tilde{E}^{(1)}}{\partial\zeta} - N^{(2)} + N_e^{(2)} = 0 \quad . \tag{2.52}$$

Equations (2.49) - (2.52) are linearily combined $\left(\text{through } \frac{\partial}{\partial\zeta}\right.$ (2.52) + (2.50) - (2.49) - (2.51)$\left.\right)$ to yield an equation for ϕ

$$\frac{\partial^2\tilde{E}^{(1)}}{\partial\zeta^2} - \frac{\partial}{\partial\zeta}(\phi\cdot\phi) - 2\,\frac{\partial\phi}{\partial\eta} - \phi\tilde{E}^{(1)} - \phi\,\frac{\partial\phi}{\partial\zeta} = 0 \quad . \tag{2.53}$$

Substituting $\tilde{E}^{(1)}$ from (2.48), Eq. (2.53) results into the desired KdV equation

$$\frac{\partial\phi}{\partial\eta} + \phi\,\frac{\partial\phi}{\partial\zeta} + \frac{1}{2}\,\frac{\partial^3\phi}{\partial\zeta^3} = 0\ . \tag{2.54}$$

Using the notation

$$\frac{\partial\phi}{\partial\eta} = \phi_\eta\ ,\ \frac{\partial\phi}{\partial\zeta} = \phi_\zeta,\ \ \frac{\partial^3\phi}{\partial\zeta^3} = \phi_{\zeta\zeta\zeta} \tag{2.55}$$

the KdV equation (2.54) is rewritten by

$$\phi_\eta + \phi\cdot\phi_\zeta + \frac{1}{2}\,\phi_{\zeta\zeta\zeta} = 0\ . \tag{2.56}$$

The KdV equation (2.56) admits a solution of a solitary wave that is a single pulse which retains its shape as it propagates with a velocity $\tilde{c}$ in (ζ,η) space. Thus we are looking for a solution of the variable $(\zeta-\tilde{c}\eta)$, or equivalently in the usual space (x,t) the solution is of the form $\Phi(x-ct)$, where c is the speed of the solitary wave. Since we have the following relation between the coordinates

$$\zeta = \Omega\xi - \Omega\tau = \frac{\Omega}{\lambda_D}\,x - \Omega\omega_o t$$

$$\eta = \Omega^3\xi = \frac{\Omega^3}{\lambda_D}\,x \tag{2.57}$$

we get

$$x - ct = \frac{c}{\Omega\omega_o} (\zeta - \tilde{c}\eta) \tag{2.58}$$

which implies the relation between c and $\tilde{c}$

$$c = \frac{\omega_o \lambda_D}{1-\Omega^2\tilde{c}} \tag{2.59}$$

$$\tilde{c} = \frac{1}{\Omega^2} \left(1 - \frac{\omega_o \lambda_D}{c}\right) \quad . \tag{2.60}$$

Looking now for a solution of Eq. (2.56) in variable z

$$z = \zeta - \tilde{c}\eta \tag{2.61}$$

one gets from (2.56),

$$-\tilde{c}\phi_z + \phi\phi_z + \frac{1}{2}\phi_{zzz} = 0 \quad . \tag{2.62}$$

This equation can be integrated to yield

$$\tilde{c}\phi - \frac{1}{2}\phi^2 - \frac{1}{2}\frac{d^2\phi}{dz^2} = 0 \tag{2.63}$$

where the constant of integration is assumed to vanish. Multiplying Eq. (2.63) by $d\phi/dz$ and integrating once more (again with a zero constant of integration), one has

$$\frac{1}{2}\,\tilde{c}\phi^2 - \frac{1}{6}\,\phi^3 - \frac{1}{4}\left(\frac{d\phi}{dz}\right)^2 = 0 \tag{2.64}$$

or equivalently

$$\left(\frac{d\phi}{dz}\right)^2 = \frac{2}{3}\,\phi^2(3\tilde{c}-\phi). \tag{2.65}$$

The solution of this equation is

$$\phi(z) = 3\,\tilde{c}\,\operatorname{sech}^2\!\left(\sqrt{\frac{\tilde{c}}{2}}\,z\right) \quad . \tag{2.66}$$

One can verify the solution (2.66) by direct substitution into (2.65) by using the identities

$$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \cdot \tanh z \tag{2.67}$$

$$\operatorname{sech}^2 z + \tanh^2 z = 1 \quad . \tag{2.68}$$

The solution (2.66) has a peak at $z=0$ with a maximum ϕ of $3\tilde{c}$, it vanishes at $z = \pm\infty$, it moves with a velocity $\tilde{c}$ (in ζ, η coordinates) and it has a width of $\sqrt{2/\tilde{c}}$. This solution has only one parameter, $\tilde{c}$ and has the characteristic properties of a solitary wave, namely, the larger the speed the greater is the amplitude and the narrower is the width.

Recalling that $\phi = u^{(1)} = N^{(1)}$ we return to the dimensional variables. From Eq. (2.42) and (2.66) one gets,

$$n_i - n_o = \Omega^2 n_o N^{(1)} = (\delta n)\ \mathrm{sech}^2\left(\frac{x-ct}{D}\right) \tag{2.69}$$

where

$$\delta n = 3\Omega^2 n_o \tilde{c} \ . \tag{2.70}$$

Using (2.59) and (2.70),

$$c = \frac{\omega_o \lambda_D}{1-\Omega^2 \tilde{c}} = \frac{\omega_o \lambda_D}{1 - \dfrac{\delta n}{3n_o}} \simeq \omega_o \lambda_D \left(1 + \frac{\delta n}{3n_o}\right) \ . \tag{2.71}$$

Using the relation (2.58), the solution (2.66), the definition of δn in (2.70) and Eq. (2.59) one gets,

$$\frac{x-ct}{D} = \sqrt{\frac{\tilde{c}}{2}}\ (\zeta - \tilde{c}\eta) = \sqrt{\frac{\tilde{c}}{2}} \cdot \frac{\Omega\omega_o}{c}\ (x-ct)$$

so that

$$\frac{1}{D^2} = \frac{\tilde{c}\Omega^2\omega_o^2}{2c^2} = \frac{\delta n \omega_o^2}{6n_o c^2} = \frac{\delta n(1-\Omega^2\tilde{c})^2}{6n_o\lambda_D} = \frac{\delta n\left(1 - \dfrac{\delta n}{3n_o}\right)^2}{6\lambda_D^2 n_o} \ ,$$

and the width D of the solitary wave in (x,t) space is given by

$$\frac{\lambda_D^2}{D^2} = \frac{\delta n}{6n_o}\left(1 - \frac{\delta n}{3n_o}\right)^2 \simeq \frac{\delta n}{6n_o} \ . \tag{2.72}$$

We conclude this chapter with the standard properties of a solitary wave as can be seen from Eqs. (2.71) and (2.72):

(a) The width of the solitary wave (D) <u>decreases</u> with <u>increasing</u> δn [see Eq. (2.72)].

(b) The velocity of the solitary wave (c) <u>increases</u> with <u>increasing</u> δn [see Eq. (2.71)].

3. BURGERS' EQUATION

Burgers' equation is the simplest equation combining both nonlinear propagation effects and diffusive effects, and it is given by

$$u_t + uu_x - \nu u_{xx} = 0 \quad . \tag{3.1}$$

The equation is obtained from the heat equation

$$u_t - \nu u_{xx} = 0 \tag{3.2}$$

by adding the nonlinear term uu_x. The unique property of this nonlinear equation is that it is possible to linearize it by the following transformation[23,24]

$$u = -2\nu \frac{\varphi_x}{\varphi} \quad . \tag{3.3}$$

This transformation reduces Burgers' equation to the linear heat equation (3.2). It is convenient to perform the transformation (3.3) in two steps. First transformation,

$$u = \psi_x , \tag{3.4}$$

then Eq. (3.1) can be rewritten by

$$\psi_{xt} + \psi_x \psi_{xx} = \nu \psi_{xxx} \ . \tag{3.5}$$

Integrating (3.5) with respect to x,

$$\psi_t + \frac{1}{2} \psi_x^2 = \nu \psi_{xx} \ . \tag{3.6}$$

The second transformation which will complete the transformation (3.3) is

$$\psi = -2\nu \ \ell n \ \varphi \tag{3.7}$$

so that $u = \psi_x = -2\nu\varphi_x/\varphi$. Substituting Eq. (3.7) into (3.6) and using the relations

$$\psi_t = -2\nu\varphi_t/\varphi$$

$$\psi_x^2 = 4\nu^2\varphi_x^2/\varphi^2$$

$$\psi_{xx} = -2\nu(\varphi_{xx}-\varphi_x^2)/\varphi^2 \ ,$$

gives the linear heat equation

$$\varphi_t - \nu\varphi_{xx} = 0 \ . \tag{3.8}$$

The general solution of the linear heat equation is well known in the literature. For example if one has an initial value problem

$$u(x,o) = F(x) \tag{3.9}$$

where F is a known function, then by using the transformation (3.3) one obtains the initial value function for the heat equation (3.8). In particular, for this example

$$\varphi(x,o) = \Phi(x) = \exp\{-\frac{1}{2\nu}\int_0^x F(x')dx'\} \quad . \tag{3.10}$$

The solution of (3.8) with the initial conditions (3.10) is given by

$$\varphi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \Phi(x') \exp[-\frac{(x-x')^2}{4\nu t}] \, dx' \quad . \tag{3.11}$$

Using once more the transformation (3.3) one derives the solution of the nonlinear Burgers' equation,

$$u(x,t) = \frac{\int_{-\infty}^{\infty} dx' \, (\frac{x-x'}{t}) \exp[-G(x';x,t)/2\nu]}{\int_{-\infty}^{\infty} dx' \exp[-G(x';x,t)/2\nu]}$$

$$G(x';x,t) = \int_0^{x'} F(x'')dx'' + \frac{(x-x')^2}{2t} \tag{3.12}$$

Burgers' equation is a model for the study of shock waves. For example, one can take for F(x) in Eq. (3.9) a step function and to see how this initial condition diffuses into a steady profile.

4. SYMMETRIES

In this section we check the symmetries of the KdV equation

$$u_t + uu_x + u_{xxx} = 0 \; . \tag{4.1}$$

(a) The Galilean Invariance

Galilean transformations are defined by

$$t' = t$$

$$x' = x + ct$$

$$u' = u + c \; .$$

Galilean invariance of KdV means that if u satisfies Eq. (4.1) then u′ satisfies the same equation. Since one has the transformation

$$x = x' - ct', \; t = t' \; , \tag{4.3}$$

the following relations are obtained

$$u_{x'} = u_x, \quad u_{x'x'x'} = u_{xxx}$$

$$u_{t'} = u_t - cu_x \tag{4.4}$$

and using (4.4) one can prove that u′ satisfies KdV, namely

$$u'_{t'} + u'u'_{x'} + u'_{x'x'x'} = (u+c)_{t'} + (u+c)(u+c)_{x'} + (u+c)_{x'x'x'}$$

$$= u_{t'} + (u+c)_{x'} + u_{x'x'x'} = u_t - cu_x + (u+c)\ u_x + u_{xxx}$$

$$= u_t + uu_x + u_{xxx} = 0$$

since u satifies KdV.

In a similar way, one can show that Eq. (4.1) satisfies

(b) Translation Invariance

$$x' = x + x_o$$

$$t' = t + t_o \ , \tag{4.5}$$

and

(c) Scale Invariance

$$x' = cx$$

$$t' = c^3 t$$

$$u' = c^{-2} u \ . \tag{4.6}$$

Last relation follows from

$$u' = \frac{dx'}{dt'} = \frac{cdx}{c^3 dt} = c^{-2} u \ .$$

The scaling between t' and t is implemented from the KdV equation, since $\partial/\partial t$ and $\partial^3/\partial x^3$ have to scale in a similar way. The last, but most important symmetry that we like to mention is

(d) Time-Space Invariance

$$t' = -t$$

$$x' = -x \quad . \tag{4.7}$$

The KdV equation is invariant under the transformation (4.7). Time invariance ($t' = -t$) is related to dissipation, in particular if an equation describing a physical system is invariant under the transformation (i.e. time reversibility),

$$t' = -t \tag{4.8}$$

then the system is not dissipative. For most of the equations with first order time derivatives the symmetry (4.8) is not satisfied. For example, the Burgers' equation, (3.1), is not invariant under the transformation (4.8) and therefore, as everyone expects from a heat equation, the solutions of Burgers' equations are dissipative. However, although KdV is also an equation with first order time derivative, it satisfies the symmetry time-space given by (4.7), and as a consequence the solutions of the KdV equation are not dissipative.

5. NONLINEARITY, DISSIPATION AND DISPERSION

The simplest nonlinear equation is

$$u_t + uu_x = 0 \quad . \tag{5.1}$$

This nonlinearity is included in KdV equation as well as in Burgers' equation and in many other nonlinear differential equations related to fluid systems. The total differential of u is

$$du = u_t\, dt + u_x\, dx \ , \tag{5.2}$$

$du = 0$ implies $u(x,t)$ constant along the straight lines c (called characteristics) in the x,t plane. Therefore, $u(x,t)$ is known from the initial values $u(x,o)$ by drawing straight lines in x,t plane with a slope $1/u(x,o)$, (see Fig. 3). Starting with smooth initial values given by $u(x,o)$, one gets that point A in Fig. 3 is moving faster than point B and therefore after a finite time, at point P in Fig. 3, the solution becomes double value. This occurence of singularity is typical for nonlinear equations of the type given by (5.1). As a result of these singularities, a stable shock solution is developed. Burgers' equation [Eq. (3.1)] prevents the creation of the steep shock wave structure by the term νu_{xx} -the dissipation term. The KdV equation, Eq. (4.1), prevents the occurence of sharp shocks by dispersion, since due to the time reversal invariance of this equation there is no dissipation in this case.

In order to understand qualitatively the dispersion effects it is useful to look into the linear KdV equation

$$u_t + u_{xxx} = 0 \quad . \tag{5.3}$$

The dispersion relation is obtained, as usual, by looking for Fourier transform solutions

$$u \sim \exp[i(\omega t - kx)] \quad . \tag{5.4}$$

Substituting (5.4) into (5.3) gives the dispersion relation,

$$\omega + k^3 = 0 \tag{5.5}$$

or, equivalently

$$v_{phase} = \frac{\omega}{k} = -k^2 \quad . \tag{5.6}$$

Equation (5.6) shows that the phase velocity changes with the wave number k (or the wavelength λ); this phenomenon is called dispersion. From the above linear KdV equation, Eq. (5.3), and its dispersion relation it is evident that the u_{xxx} term can change (in time) the shape of the wave due to the dispersive mechanism. It is worthwhile to mention that the linear KdV equation in the more general form is

$$u_t + c_o u_x + \nu u_{xxx} = 0 \quad . \tag{5.7}$$

By using Eq. (5.4) the following dispersion relation is obtained for the linear KdV equation,

$$\omega = c_o k - \nu k^3 \quad . \tag{5.8}$$

6. SOLITONS AND CNOIDAL WAVES

For the following KdV equation,

$$u_t + uu_x + u_{xxx} = 0 \tag{6.1}$$

we look for traveling waves, namely

$$u(x,t) = U(z), \quad z = x - ct \quad . \tag{6.2}$$

In this case one obtains an ordinary differential equation

$$-cU_z + UU_z + U_{zzz} = 0 \quad . \tag{6.3}$$

This equation can be integrated once to give

$$-cU + \frac{1}{2} U^2 + U_{zz} = \frac{a}{\sqrt{2}} \tag{6.4}$$

where $a/\sqrt{2}$ is the constant of integration. Multiplying Eq. (6.4) by U_z and integrating once more, one gets

$$-\frac{c}{2}U^2 + \frac{1}{6}U^3 + \frac{1}{2}U_z^2 = \frac{a}{\sqrt{2}}U + b \tag{6.5}$$

where b is the constant of this integration. Integrating Eq. (6.5) yields,

$$\int_{U_o}^{U} \left(b + \frac{a}{\sqrt{2}}V + \frac{c}{2}V^2 - \frac{1}{6}V^3\right)^{-1/2} dV = z - z_o \quad . \tag{6.6}$$

The integral in Eq. (6.6) can be represented in terms of elliptic functions. In order to see this, it is necessary to perform a transformation, which for convenience we shall do it in three steps. First, define the function φ by

$$U = \sqrt{2}\, f \tag{6.7}$$

so that Eq. (6.5) can be rewritten as

$$f_z^2 = -\frac{\sqrt{2}}{3}f^3 + cf^2 + af + b \ . \tag{6.8}$$

Since this expression is a cubic polynom, one can write it in the following way

$$f_z^2 = -\frac{\sqrt{2}}{3}(f-\alpha_1)(f-\alpha_2)(f-\alpha_3) \equiv \frac{\sqrt{2}}{3}F(f) \ , \tag{6.9}$$

where α_1, α_2 and α_3 are the roots of the polynom and we choose for convenience (see Fig. 4).

$$\alpha_1 < \alpha_2 < \alpha_3 \ . \tag{6.10}$$

For finite amplitude motion there are three real roots to Eq. (6.9). Since f_z has to be real, f_z^2 is positive and therefore $F(f) \geqslant 0$. Thus for a bounded motion one has

$$\alpha_2 \leqslant f \leqslant \alpha_3 \tag{611}$$

which implies at least two real roots for Eq. (6.9). However, the cubic polynom has real coefficients, therefore all the three roots should be real, (see Fig. 4). Denoting the only negative factor in Eq. (6.9) by $-g(g \geqslant 0)$, we perform now the second step in our transformation,

$$-g = f - \alpha_3 \ . \tag{6.12}$$

Substituting Eq. (6.12) into (6.9) yields the following equation for g

$$g_z^2 = \frac{\sqrt{2}}{3}\, g(\alpha_3 - \alpha_2 - g)(\alpha_3 - \alpha_1 - g) \tag{6.13}$$

Doing now the last step in our transformation,

$$g = (\alpha_3 - \alpha_2)h^2 \tag{6.14}$$

we get the following equation for h, from Eq. (6.13),

$$h_z^2 = \frac{\sqrt{2}}{12}\,(\alpha_3 - \alpha_1)(1 - h^2)(1 - k^2h^2) \tag{6.15}$$

where

$$k^2 = \frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1} \quad . \tag{6.16}$$

The solution of Eq. (6.15) should be of the argument y

$$y = [\ \frac{\sqrt{2}}{12} (\alpha_3 - \alpha_1)]^{1/2} z \ , \tag{6.17}$$

so that Eq. (6.15) can be rewritten,

$$h_y^2 = (1 - h^2)(1 - k^2h^2) \quad . \tag{6.18}$$

The solution of Eq. (6.18) is given by the elliptic integral

$$y - y_o = \int_0^h \frac{dx}{(1-x^2)^{1/2}(1-k^2x^2)^{1/2}} \ , \tag{6.19}$$

or in other terms, h(y) is given by the Jacobian elliptic function sn(y,k),

$$h(y) = sn(y,k) \tag{6.20}$$

where k is the modulus of the elliptic function. The function sn(y,k) has two limiting forms

$$sn(y,0) = \sin y$$

$$sn(y,1) = \tanh y \quad . \tag{6.21}$$

In general, the solution of Eq. (6.15) is

$$h(z) = \mathrm{sn}\{[\ \frac{\sqrt{2}}{12}\ (\alpha_3-\alpha_1)]^{1/2}\ z,\ k\}. \tag{6.22}$$

Going back through the transformations given by Eqs. (6.14) and (6.12) one gets

$$\frac{u(x-ct)}{\sqrt{2}} = f(x-ct) = \alpha_3 - (\alpha_3 - \alpha_2)\ \mathrm{sn}^2\{[\frac{\sqrt{2}}{12}\ (\alpha_3 - \alpha_1)]^{1/2}(x-ct),\ k\}. \tag{6.23}$$

These are in general called the cnoidal waves. We would like to note that if one starts with the KdV equation written in the form

$$u_t - 6uu_x + u_{xxx} = 0 \tag{6.24}$$

instead of Eq. (6.1), then the messy factor $\sqrt{2}/12$ will change to unity. This serves as a good reason to start with Eq. (6.24) rather than with Eq. (6.1).

The function $\mathrm{sn}^2(y,k)$ oscillates between 0 and 1 with a period equal to 2K

$$2K = 2 \int_0^1 dx\{(1-x^2)(1-k^2x^2)\}^{-1/2}\ . \tag{6.25}$$

Therefore, $f(x-ct)$ oscillates between α_2 and α_3 with a period T

$$T = \frac{2K}{(\sqrt{2}/12)^{1/2}(\alpha_3-\alpha_1)^{1/2}}\ . \tag{6.26}$$

As α_2 approaches α_1 the value of the modulus k goes to 1 [see Eq. (6.16)] and in this case $K \to \infty$, so that the period of oscillation is

$$T = \infty \qquad \text{for} \qquad k = 1 \; (\alpha_2 = \alpha_1) \; . \tag{6.27}$$

In this case the solution (6.23) is*

$$f(x-ct) = \alpha_2 + (\alpha_3-\alpha_2)\mathrm{sech}^2\{[\frac{\sqrt{2}}{12}\,(\alpha_3-\alpha_2)]^{1/2}(x-ct)\} \; . \tag{6.28}$$

This solution has the form of a pulse that rises to an amplitude $\alpha_3-\alpha_2$ above a reference level α_2. The width of the pulse is given by $\sqrt{12}/[\sqrt{2}\,(\alpha_3-\alpha_2)]^{1/2}$. Since the coefficient of the squared term in a cubic equation is equal to minus the sum of the roots, by comparing Eqs. (6.8) and (6.9) one gets,

$$\alpha_1 + \alpha_2 + \alpha_3 = \frac{3c}{\sqrt{2}} \; . \tag{6.29}$$

For the solution given in Eq. (6.28) one has $\alpha_1=\alpha_2$. Moreover, since α_2 is the reference level of the pulse, it is allowed to choose $\alpha_2=0$ without loss of generality. In this case one has

$$\alpha_1 = \alpha_2 = 0, \quad \alpha_3 = \frac{3c}{\sqrt{2}} \tag{6.30}$$

so that Eq. (6.28) takes the form

*For k=1, one gets $f=\tanh^2$ but using the identity $\tanh^2 = 1-\mathrm{sech}^2$ one has $\alpha_3-(\alpha_3-\alpha_2)\tanh^2 = \alpha_3+(\alpha_3-\alpha_2)\mathrm{sech}^2$.

$$\frac{1}{\sqrt{2}}\, u(x-ct) = f(x-ct) = \frac{3c}{\sqrt{2}} \operatorname{sech}^2 \left\{ \sqrt{\frac{c}{4}}\,(x-ct) \right\} . \tag{6.31}$$

This solution can be obtained directly from Eq. (6.6) for $a=b=0$. This is considered the soliton solution.

Since the KdV equation is written with various numerical coefficients [for different physical problems, see e.g. Eq. (2.56)] it is useful to have the following relations for the soliton type solutions. For the KdV equation

$$u_t + c_o u u_x + \nu u_{xxx} = 0 \tag{6.32}$$

the single soliton solution is

$$u = u_o \operatorname{sech}^2[(x-Vt)/L] \tag{6.33}$$

where

$$L = 2\left(\frac{3\nu}{u_o c_o}\right)^{1/2} \tag{6.34}$$

$$V = \frac{u_o c_o}{3} \tag{6.35}$$

For example,

$$c_o = \nu = 1 \; ; \quad u_o = 3V, \quad L = \sqrt{4/V} \text{ [see Eq. (6.31)]} \tag{6.36}$$

$$c_o = -6, \ \nu = 1 \ ; \quad u_o = -\frac{V}{2}, \quad L = \sqrt{4/V} \tag{6.37}$$

$$c_o = 1, \ \nu = 1/2 \ ; \quad u_o = 3V, \quad L = \sqrt{2/V} \quad \text{[see Eq. (2.66)]} \tag{6.38}$$

Another interesting limit for the solutions (6.23) is obtained when [see Eq. (6.16)]

$$\alpha_3 \to \alpha_2 \quad , \quad k \to 0 \ . \tag{6.39}$$

In this case one gets

$$\frac{u(x-ct)}{\sqrt{2}} = f(x-ct) = \alpha_3 - (\alpha_3-\alpha_2) \ \sin^2 \left[\frac{\sqrt{2}}{12} (\alpha_3-\alpha_1) \right]^{1/2} (x-ct) \tag{6.40}$$

where

$$c = \frac{\sqrt{2}}{3} (\alpha_1 + 2\alpha_3) \quad . \tag{6.41}$$

Before ending this chapter, it is useful to point out an analogy with "classical mechanics." Newton's law of mechanics is given by (F=ma)

$$\frac{d^2x}{dt^2} = - \frac{dV}{dx} \tag{6.42}$$

where x describes the position of a particle at a time t and V is the classical potential energy per unit mass. Comparing Eq. (6.42) with a typical field equation, for a field $\psi = \psi(z)$,

$$\frac{d^2\psi}{dz^2} = -\frac{dV_{eff}}{d\psi} \tag{6.43}$$

one can make an analogy

$$\begin{aligned} x &\leftrightarrow \psi \\ t &\leftrightarrow z \\ V &\leftrightarrow V_{eff} . \end{aligned} \tag{6.44}$$

Eq. (6.4) can be written in the form

$$U_{zz} = -\frac{1}{2} U^2 + cU + \frac{a}{\sqrt{2}} \equiv -\frac{dV_{eff}}{dU} . \tag{6.45}$$

Integrating this equation,

$$V_{eff}(U) = \frac{1}{6} U^3 - \frac{c}{2} U^2 - \frac{a}{\sqrt{2}} U - b , \tag{6.46}$$

and changing variables acording to Eq. (6.7),

$$V_{eff}(f) = \frac{\sqrt{2}}{3} f^3 - cf^2 - af - b \tag{6.47}$$

one gets that the function $-F(f)$ (up to a factor $\sqrt{2}/3$) of Eq. (6.9) is the effective potential describing a "particle" with a wavefunction f (see Fig. 5). For states with an effective potential equal to V_B one gets the cnoidal wave solutions for the KdV equation. In this case the "particle" moves back and forth inside the well with a time period given by Eq. (6.26), so that these solutions are periodic in $z(= x - ct)$. Two special cnoidal waves are the soliton solution given by $V_{eff} = V_A$. The soliton wave solution corresponds to the largest possible cnoidal wave where it takes the particle an infinite time to roll across the potential well. In order for the particle to be bounded (and not to roll to $-\infty$), the velocity at the maximum of the well should be zero. This analysis, done in analogy with classical mechanics, is useful in analyzing the physical context of the solutions for the KdV equation, (as well as for other equations describing physical systems).

7. MIURA, GARDNER AND BÄCKLUND TRANSFORMATIONS

Miura Transformation[25]

Consider the KdV equation,

$$u_t - 6uu_x + u_{xxx} = 0 \quad , \tag{7.1}$$

and the so-called modified KdV (MKdV) equation,

$$v_t - 6v^2v_x + v_{xxx} = 0 \quad . \tag{7.2}$$

Miura found a transformation given by*

*Note that if one starts with the KdV equation $u_t + uu_x + u_{xxx} = 0$ and defined the MKdV equation by $v_t + v^2v_x + v_{xxx} = 0$ then Miura's transformation is $u = v^2 + \sqrt{-6}\ v_x$.

$$u = v^2 + v_x \tag{7.3}$$

which relates the KdV and the MKdV equations. In particular one has

$$u_x = 2vv_x + v_{xx}$$

$$u_{xx} = 2v_x^2 + 2vv_{xx} + v_{xxx}$$

$$u_{xxx} = 6v_x v_{xx} + 2vv_{xxx} + v_{xxxx} \tag{7.4}$$

so that KdV and MKdV are related by

$$u_t - 6uu_x + u_{xxx} = \left(2v + \frac{\partial}{\partial x}\right)\left(v_t - 6v^2 v_x + v_{xxx}\right) \quad . \tag{7.5}$$

The KdV equation is satisfied by u if the MKdV equation is satisfied by v (but not necessarily vice versa) Miura's transformation is a special case of a more general transformation, to be defined later, the Bäcklund transformation.

<u>Gardner's Transformation</u>[26]

Defining the transformation

$$u = w + \varepsilon w_x + \varepsilon^2 w^2 \tag{7.6}$$

for any well-behaved function and a real parameter ε, one gets

$$u_t - 6uu_x + u_{xxx} = \left(1 + \varepsilon \frac{\partial}{\partial x} + 2\varepsilon^2 w\right) \left[w_t - 6(w + \varepsilon^2 w^2)w_x + w_{xxx}\right] \; . \tag{7.7}$$

Therefore, the KdV equation (7.1) is satisfied by u if w satisfies Gardner's equation,

$$w_t - 6(w + \varepsilon^2 w^2)w_x + w_{xxx} = 0 \tag{7.8}$$

which is a generalization of Eq. (7.2). Note however, that KdV in general does not imply Eq. (7.8). Like Miura's transformation, Gardner's transformation is also a special case of the Bäcklund transformation.

Bäcklund Transformations

Assuming two differential equations for two functions u and v,

$$F\{u\} = 0$$

$$G\{v\} = 0 \tag{7.9}$$

where F and G are local functionals including space and time derivatives. A Bäcklund transformation is a pair of differential equations relating u and v, namely

$$K\{u,v\} = 0$$

$$L\{u,v\} = 0 \tag{7.10}$$

where K and L denote local functionals with space and time derivatives. In general it is very difficult (or unlikely) to find a pair of functionals K and L so that (7.9) is satisfied.

For KdV equation a Bäclund transformation can be found. One of the main results of this transformation is that one can define a "superposition" principle for this nonlinear equation and moreover, a multisoliton solution can be derived from single soliton solutions by the use of algebra only.

In order to construct the Bäcklund transformation[14] for the KdV equation we start with two Sturm-Liouville equations,

$$y_{xx} = [\lambda + f(x,t)]y \tag{7.11}$$

$$w_{xx} = [\lambda + g(x,t)]w \tag{7.12}$$

where f and g will be assumed to satisfy the KdV equation, λ is a constant independent of t and x, and w is related to y by the equation

$$w = A(x,t,\lambda)y + y_x \ . \tag{7.13}$$

Note that Eqs. (7.11) and (7.12) are actually the usual stationary Schrödinger equations if one considers the time t to be a formal parameter. From Eq. (7.13) one derives

$$w_x = A_x y + A y_x + y_{xx}$$

$$w_{xx} = A_{xx}y + 2A_x y_x + A y_{xx} + y_{xxx} \tag{7.14}$$

and using these relations together with Eq. (7.12) for w_{xx} and Eq. (7.11) for y_{xxx}, one gets

$$A_{xx}y + 2A_x y_x + A(\lambda+f)y + (\lambda+f)y_x + f_x y = A(\lambda+g)y + (\lambda+g)y_x$$

In this equation the coefficients of y and y_x are independently compared to yield,

$$2A_x + f - g = 0 \tag{7.15}$$

$$A_{xx} + A(f - g) + f_x = 0. \tag{7.16}$$

Eliminating $f-g$ from these last equations, one has an equation for A

$$A_{xx} - 2AA_x + f_x = 0, \tag{7.17}$$

which can be easily integrated to

$$A^2 - A_x - f = \tilde{\lambda}(t) \ . \tag{7.18}$$

Using the linearizing equation (see Chapter 3)

$$A = - \tilde{y}_x/\tilde{y} \tag{7.19}$$

for Eq. (7.18), one gets

$$\tilde{y}_{xx} = [\tilde{\lambda}(t) + f(x,t)]\tilde{y} \ . \tag{7.20}$$

We shall also assume

$$\tilde{\lambda}(t) = \text{const} \tag{7.21}$$

as it was assumed for Eqs. (7.11) and (7.12). We introduce potential functions φ and ψ defined by

$$f = \varphi_x$$

$$g = \psi_x \ . \tag{7.22}$$

The equation for A, Eq. (7.15), can be written in terms of the potentials,

$$A = \frac{1}{2} (\varphi - \psi) \tag{7.23}$$

and Eq. (7.18) for A takes the form

$$f + g = \mu + \frac{1}{2} (\varphi - \psi)^2 \tag{7.24}$$

where μ is a constant defined by

$$\mu = -2\tilde{\lambda} \ . \tag{7.25}$$

Equation (7.24) can be expressed with the help of the potential functions,

$$\varphi_x + \psi_x = \mu + \frac{1}{2} (\varphi - \psi)^2 . \tag{7.26}$$

If f and g satisfy the KdV equation

$$f_t - 6ff_x + f_{xxx} = 0 \tag{7.27}$$

and similarly for g, then φ and ψ satisfy

$$\varphi_t - 3\varphi_x^2 + \varphi_{xxx} = 0$$

$$\psi_t - 3\psi_x^2 + \psi_{xxx} = 0 \ . \tag{7.28}$$

One has to use these relations in order to obtain an expression relating the time derivatives of the potentials φ and ψ. The relation between the spatial derivatives of the potential is given by Eq. (7.26). Taking the time derivative of Eq. (7.26) and integrating with respect to x, one gets

$$\varphi_t + \psi_t = \int dx(\varphi - \psi) \ (\varphi_t - \psi_t) \ . \tag{7.29}$$

Defining for convenience

$$\alpha = \varphi + \psi$$

$$\beta = \varphi - \psi \tag{7.30}$$

and using Eq. (7.28) one can write,

$$\varphi_t + \psi_t = \int \ [\frac{3}{2} \alpha_x(\beta^2)_x - \beta\beta_{xxx}]dx \ . \tag{7.31}$$

The right hand side of Eq. (7.31) is obtained in the following way,

$$\varphi_t - \psi_t = (\psi - \varphi)_{xxx} + 3(\varphi_x^2 - \psi_x^2), \qquad \text{using (7.28)}$$

$$= (\psi - \varphi)_{xxx} + 3(\varphi_x - \psi_x)(\varphi_x + \psi_x) = -\beta_{xxx} + 3\beta_x\alpha_x$$

$$(\varphi - \psi)(\varphi_t - \psi_t) = -\beta\beta_{xxx} + 3\beta\beta_x\alpha_x = \frac{3}{2} (\beta^2)_x\alpha_x - \beta\beta_{xxx} \ .$$

Integrating now these terms in Eq. (7.31) gives,

$$-\int \beta\beta_{xxx}\,dx = -\beta\beta_{xx} + \int \beta_x\beta_{xx}dx = -\beta\beta_{xx} + \frac{1}{2}\,\beta_x^2 \tag{7.32}$$

$$\frac{3}{2}\int \alpha_x(\beta^2)_x dx = \frac{3}{2}\int \alpha_x{\cdot}2\alpha_{xx}dx = \frac{3}{2}(\alpha_x)^2 \tag{7.33}$$

where in Eq. (7.33) we have used Eq. (7.26)

$$\alpha_x = \mu + \frac{1}{2}\,\beta^2$$

and its derivative

$$\alpha_{xx} = \frac{1}{2}\,(\beta^2)_x\ .$$

Inserting (7.32) and (7.33) into (7.31) we get

$$\varphi_t + \psi_t = \frac{3}{2}\,(\alpha_x)^2 - \beta\beta_{xx} + \frac{1}{2}\,(\beta_x)^2\quad . \tag{7.34}$$

Substituting back Eq. (7.30) one gets

$$\varphi_t + \psi_t = 2(\varphi_x^2 + \varphi_x\psi_x + \psi_x^2) - (\varphi - \psi)(\varphi_{xx} - \psi_{xx})\quad . \tag{7.35}$$

This equation together with (7.26) are the Bäcklund transformations for the KdV equations (satisfied by f and g)

$$\begin{aligned} &f_t - 6ff_x + f_{xxx} = 0\\ &g_t - 6gg_x + g_{xxx} = 0, \end{aligned} \tag{7.36}$$

where f and g are related to the potential φ and ψ through Eq. (7.22).

In the next chapters we shall see how the transformations developed here [(Eqs. (7.6) or (7.3) and Eqs. (7.26) and (7.35)] can be used in order to obtain an infinite number of conservation laws and an interesting class of KdV solutions (the two soliton solution).

8. CONSERVATION LAWS

Assuming that T and X are functions of u, spatial derivative of u, x and t so that the following differential equation is satisfied,

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0, \tag{8.1}$$

where $u(x,t)$ is a solution of a given differential equation, for example KdV. Then by integrating Eq. (8.1) one gets

$$\frac{d}{dt} \int_a^b T \, dx = X(a) - X(b), \tag{8.2}$$

and in particular if $X(-\infty) = X(+\infty) = 0$, one has

$$\frac{d}{dt} \int_{-\infty}^{\infty} T \, dx = 0. \tag{8.3}$$

In this case the Eq. (8.1) is called a conservation law and it is related to the conservation of the quantity defined in Eq. (8.3). In general, T is a density and X is a flux. For example the conservation of mass

$$\frac{d}{dt} \int_{-\infty}^{\infty} \rho \, dx = 0 \tag{8.4}$$

of a fluid is described by the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \,, \tag{8.5}$$

where ρ is the fluid density and ρu is the flux in the x direction. There are similar equations for the conservation of electric charge, momentum, energy, etc.

A conservation law for a partial differential equation given in (8.1) is related to Eq. (8.3) which implies a conserved quantity given by

$$\int_{-\infty}^{\infty} T \, dx = \text{constant}. \tag{8.6}$$

For the KdV equation, like for the other solitary wave equations discussed in the introduction, one can find an infinite number of conservation laws. Two conservation laws follow immediately from KdV equation. In particular the KdV equation can be written as

$$0 = u_t - 6uu_x + u_{xxx} = \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(-3u^2 + u_{xx}\right) \,. \tag{8.7}$$

Therefore one choice of density (T) and flux (X) is

$$T = u \,, \quad X = u_{xx} - 3u^2 \tag{8.8}$$

so that

$$\int_{-\infty}^{\infty} u \, dx = \text{constant} \ , \tag{8.9}$$

where u satisfies KdV equation. In a similar way one can obtain

$$\int_{-\infty}^{\infty} u^2 \, dx = \text{constant} \ . \tag{8.10}$$

This relation follows from

$$0 = u(u_t - 6uu_x + u_{xxx}) = \frac{\partial}{\partial t}(\frac{1}{2} u^2) + \frac{\partial}{\partial x}(-2u^3 + uu_{xx} - \frac{1}{2} u_x^2) . \tag{8.11}$$

In a fluid model Eqs. (8.9) and (8.10) represent the conservation of mass and of momentum respectively.

Further conservation laws can be deduced by "trial and error". However, in order to derive systematically an infinite number of conservation laws one can use the Miura or the Gardner transformations. The Gardner equation is [see Eq. (7.8)]

$$w_t - 6(w + \varepsilon^2 w^2) w_x + w_{xxx} = 0 \ . \tag{8.12}$$

This equation has a conservation law

$$w_t + (-3w^2 - 2\varepsilon^2 w^3 + w_{xx})_x = 0 \tag{8.13}$$

so that

$$\int_{-\infty}^{\infty} w dx = \text{constant} \ . \tag{8.14}$$

Assume that one can expand

$$w = \sum_{n=0}^{\infty} \varepsilon^k w_n(u) \tag{8.15}$$

for a small parameter ε. In this case the Gardner's transformation [see Eq. (7.6)]

$$w = u - \varepsilon w_x - \varepsilon^2 w^2 \tag{8.16}$$

gives from (8.15), by comparing coefficients of ε^0, ε^1, ε^2, etc.,

$$w_o = u \tag{8.17}$$

$$w_1 = -w_{ox} = -u_x \tag{8.18}$$

$$w_2 = -w_{1x} - w_0^2 = u_{xx} - u^2 \tag{8.19}$$

$$w_3 = -u_{xxx} + 4uu_x \ , \tag{8.20}$$

etc. Substituting this expansion into the conservation law (8.13) an infinite number of conservation laws can be derived. For example a lengthy algebra shows, for example, that the following quantities are conserved

$$T(t) = \frac{u^3}{3} - u_x^2$$

$$T(t) = \frac{u^4}{4} - 3uu_x + \frac{9}{5} u_{xx}^2 \ .$$

A more basic approach to these conservation laws can be used by using Noether's theorems. The conservation laws are obtained from the invariance of the Lagrangian (describing the system) under continuous transformation groups such as discussed in Chapter 4.

9. MULTISOLITON SOLUTIONS AND THE "SUPERPOSITION" PRINCIPLE

First we use the Bäcklund transformation, developed in Chapter 7, in order to find solutions for the KdV equation. In particular, one can choose in Eqs. (7.26) and (7.35):

$$\psi = 0 , \tag{9.1}$$

so that Eq. (7.26) implies

$$\varphi_x = \mu + \frac{1}{2}\varphi^2 \tag{9.2}$$

and Eq. (7.35) gives

$$\varphi_t = 2\varphi_x^2 - \varphi\varphi_{xx} = 2\mu\varphi_x \tag{9.3}$$

where the last step was obtained taking the derivative of Eq. (9.2). Denote

$$\mu = -2\kappa^2, \tag{9.4}$$

and equations (9.2) and (9.3) can be integrated to give the following solutions

$$\varphi = -2\kappa\tanh(\kappa x - 4\kappa^3 t) \ , \quad |\varphi| < 2\kappa \tag{9.5}$$

$$\tilde{\varphi} = -2\kappa\coth(\kappa x - 4\kappa^3 t) \ , \quad |\tilde{\varphi}| < 2\kappa \ . \tag{9.6}$$

The function $f = \varphi_x$ is the single soliton solution of KdV,

$$f = \varphi_x = -2\kappa^2\mathrm{sech}^2(\kappa x - 4\kappa^3 t) \ . \tag{9.7}$$

The solution (9.5) could now be used into Bäcklund transformations in order to obtain a new solution. However, the differential euqations in this case would be more complicated than above. However, there exist a very simple way to find a more sophisticated solution by means of a "superposition" formula that does not require any additional integration.

Let us assume two solutions $\varphi_{(1)}$ and $\varphi_{(2)}$ that are obtained from Bäcklund transformation by starting from a solution $\varphi_{(0)}$ and with two different values of μ. Equation (7.26) for this case is written in the following way,

$$\varphi_{(1)x} + \varphi_{(0)x} = \mu_1 + \frac{1}{2}(\varphi_{(1)} - \varphi_{(0)})^2 \tag{9.8}$$

$$\varphi_{(2)x} + \varphi_{(0)x} = \mu_2 + \frac{1}{2}(\varphi_{(2)} - \varphi_{(0)})^2 \tag{9.9}$$

We use again this equation but for a starting function $\varphi_{(1)}$ (instead of $\varphi_{(0)}$) and an eigenvalue μ_2. The new solution is denoted by $\varphi_{(12)}$,

$$\varphi_{(12)x} + \varphi_{(1)x} = \mu_2 + \frac{1}{2}(\varphi_{(12)} - \varphi_{(1)})^2 \tag{9.10}$$

In a similar way one can start with $\varphi_{(2)}$ and eigenvalue μ_1 to obtain $\varphi_{(21)}$,

$$\varphi_{(21)x} + \varphi_{(2)x} = \mu_1 + \frac{1}{2}(\varphi_{(21)} - \varphi_{(2)})^2 \tag{9.11}$$

We shall assume without proof (see for example, Ref. 14)

$$\varphi_{(12)} = \varphi_{(21)} = \varphi_{(3)} \cdot \tag{9.12}$$

From Eqs. (9.8) and (9.9) one obtains,

$$\varphi_{(2)x} - \varphi_{(1)x} = (\mu_2 - \mu_1) + \frac{1}{2}(\varphi_{(1)} - \varphi_{(2)})(\varphi_{(1)} + \varphi_{(2)} - 2\varphi_{(0)}) \tag{9.13}$$

and similarly from Eqs. (9.10) and (9.11),

$$\varphi_{(2)x} - \varphi_{(1)x} = -(\mu_2 - \mu_1) + \frac{1}{2}(\varphi_{(2)} - \varphi_{(1)})(\varphi_{(1)} + \varphi_{(2)} - 2\varphi_{(3)}) \ . \tag{9.14}$$

From Eqs. (9.13) and (9.14) one derives the "superposition" formula for the (nonlinear) KdV equation,

$$\varphi_{(3)} - \varphi_{(0)} = \frac{2(\mu_2 - \mu_1)}{\varphi_{(1)} - \varphi_{(2)}} \ . \tag{9.15}$$

$\varphi_{(1)}$ and $\varphi_{(2)}$ are the solutions (9.5) and (9.6) if one takes $\varphi_{(0)} = 0$. More specifically, we assume $\mu_1 = -8$ so that $\kappa_1 = 2$ [see Eq. (9.4)] and $\mu_2 = -2$ so that $\kappa_2 = 1$. We have to take $\varphi_{(1)} = \varphi$ from Eq. (9.5) and $\varphi_{(2)} = \tilde{\varphi}$ from (9.6) so that one avoids a zero in the denominator of (9.15). For this particular choice the "superposition" formula gives

$$\varphi_{(3)} = -6\{2\coth(2x - 32t) - \tanh(x - 4t)\}^{-1} . \tag{9.16}$$

The spatial derivative of this equation is the <u>two-soliton</u> solution

$$\varphi_{(3)x} = \frac{-12[4\cosh(2x - 8t) + \cosh(4x - 64t) + 3]}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2} . \tag{9.17}$$

If one looks for the development (in time) of this equation, one gets for $t \to -\infty$ a two soliton structure and for $t \to +\infty$ the soliton structure is reversed, namely the soliton that moves with a larger velocity goes through the soliton moving slower, without changing the shapes of the two solitons.

10. THE TIME INDEPENDENT SCHRÖDINGER EQUATION

Starting with Miura's transformation

$$u = v^2 + v_x \tag{10.1}$$

and using the linearizing tranformation

$$v = \frac{\psi_x}{\psi} \tag{10.2}$$

one gets

$$u = \frac{\psi_{xx}}{\psi} \tag{10.3}$$

where u satisfies the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0. \tag{10.4}$$

Inserting (10.2) into the MKdV equation

$$v_t - 6v^2 v_x + v_{xxx} = 0 , \tag{10.5}$$

using Eq. (10.3) and the relations

$$v_x = \frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2}$$

$$v_{xx} = \frac{\psi_{xxx}}{\psi^2} - \frac{3\psi_x\psi_{xx}}{\psi^2} + \frac{2\psi_x^3}{\psi^3} , \tag{10.6}$$

one gets after an integration with respect to x,

$$\psi_t - 2u\psi_x + u_x\psi = 0 \tag{10.7}$$

where the constant of integration was chosen to vanish. This equation can be rewritten after using (10.3) once more,

$$\psi_t - 3u\psi_x + \psi_{xxx} = 0 . \tag{10.8}$$

Performing a Galilean transformation on Eqs. (10.3) and (10.8) one gets

$$\psi_t - 3(u - c)\psi_x + \psi_{xxx} = 0 \tag{10.9}$$

$$\psi_{xx} - (u + c)\psi = 0 \ . \tag{10.10}$$

Equation (10.10) is the time-independent Schrödinger equation, where u satisfies the KdV equation (10.4) and it plays the role of the potential while c has the role of an eigenvalue. The time dependence of ψ is obtained using Eq. (10.9). It is important to point out that if the constant of integration for deriving Eq. (10.7) does not vanish, one gets for the time dependent equation for ψ the following relation

$$\psi_t - 3(u - c)\psi_x + \psi_{xxx} = A\psi \tag{10.11}$$

instead of Eq. (10.9), where A is the constant of integration.

Scattering by a $\delta(x)$ potential

The time independent Schrödinger equation is usually written in the form

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \tag{10.12}$$

where m is the mass of the particle, $\hbar$ is Planck's constant divided by 2π, V(x) is the potential energy of the particle while E is its total energy. It is convenient to denote

$$\frac{2mE}{\hbar^2} = k^2$$

$$\frac{2mV}{\hbar^2} = u \tag{10.13}$$

so that Eq. (10.12) takes the form

$$\psi_{xx} + [k^2 - u(x)]\psi = 0 \tag{10.14}$$

which can be compared with (10.10) if $c = -k^2$. For negative potential, $u < 0$, bound states will occur in addition to the scattering solutions. As a simple example we choose

$$V(x) = V_o a\delta(x) \ . \tag{10.15}$$

For this potential Schrödinger equation (10.12), or (10.14), can be easily solved. The scattering solution for an incident wave of amplitude ψ_o is given by

$$x \to -\infty : \qquad \psi_{\leftarrow} = \psi_o(e^{ikx} + Re^{-ikx}) \tag{10.16}$$

$$x \to +\infty : \qquad \psi_{\rightarrow} = \psi_o Te^{ikx} \tag{10.17}$$

where the reflection coefficient R and the transmission coefficient T are given by

$$R = \frac{-iu_o a/2}{k + iu_o a/2} \tag{10.18}$$

$$T = \frac{k}{k + iu_o a/2} \tag{10.19}$$

where u_o is defined by

$$u_o = \frac{2mV_o}{\hbar^2} \tag{10.20}$$

and the conservation of probability implies

$$|R|^2 + |T|^2 = 1 \ . \tag{10.21}$$

It is important to note that one can have non-zero solutions even for $\psi_o = 0$ (bound solutions). This can happen if R^{-1} and T^{-1} vanish, which is the case for

$$k = -iu_o a/2 \ . \tag{10.22}$$

Equations (10.16) and (10.17) in this case give

$$\begin{aligned} x \to -\infty : \quad & \psi_{\leftarrow} \simeq e^{-u_o ax/2} \\ x \to +\infty : \quad & \psi_{\rightarrow} \simeq e^{+u_o ax/2} \end{aligned} \tag{10.23}$$

which are localized (i.e. bound) states for $u_o < 0$. For this solution the pole singularities of R and T are in the upper half of the complex plane of k.

Scattering by a sech^2x potential

As we have seen in the previous sections, the KdV equation has a soliton solution given by the function sech^2x. Since we are interested in the Schrödinger equations with a potential satisfying KdV equation, it is useful to solve the Schrödinger equation for this particular (soliton) potential, namely we have in Eq. (10.12)

$$V(x) = -V_o \,\mathrm{sech}^2(x/d) \ , \qquad V_o > 0 \ . \tag{10.24}$$

Defining dimensionless quantities

$$\begin{aligned} \varepsilon &= 2md^2E/\hbar^2 \\ v &= 2md^2V_o/\hbar^2 \\ z &= x/d \ , \end{aligned} \tag{10.25}$$

the Schrödinger equation (10.12) becomes

$$\psi_{zz} + (\varepsilon + v \,\mathrm{sech}^2 z)\psi = 0 \quad . \tag{10.26}$$

This equation can be transformed to the hypergeometric equation by using the transformation[14]

$$\psi = A \,\mathrm{sech}^\beta z \; y(z) \tag{10.27}$$

$$\beta^2 = -\varepsilon \quad . \tag{10.28}$$

The equation for y is

$$y_{zz} - 2\beta \tanh z \, y_z + (v^2 - \beta^2 - \beta) \operatorname{sech}^2 z \, y = 0 \quad . \tag{10.29}$$

Making a second transformation

$$u = \frac{1}{2} (1 - \tanh z) = e^{-z}/(e^z + e^{-z}) \tag{10.30}$$

yields the following equation for y,

$$u(1 - u) \frac{d^2y}{du^2} + [c - (a+b+1)u] \frac{dy}{du} - a \cdot by = 0 \tag{10.31}$$

where

$$c = 1 + \beta, \quad a + b + 1 = 2(1 + \beta), \; ab = \beta^2 + \beta - v \quad . \tag{10.32}$$

Equation (10.31) is the standard form for the hypergeometric equation. The solution of this equation for $u = 0$ (i.e. $z = \infty$) is

$$u \to 0 : \quad y = F(a,b,c;\, u) = 1 + \frac{abu}{c} + \ldots \tag{10.33}$$

which implies for ψ

$$z \to \infty : \quad \psi = A2^{\beta} e^{-\beta z} \quad . \tag{10.34}$$

Since we are interested in asymptotic plane waves, $\sim e^{ikx}$, the following relations are satisfied, (note $z = x/d$)

$$x \to \infty : \quad \psi \sim e^{ikx} \tag{10.35}$$

$$\beta = -ikd \tag{10.36}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \ . \tag{10.37}$$

Solving now Eqs. (10.32) for a, b and c one gets

$$a = \frac{1}{2} - ikd + \sqrt{v+1/4}$$

$$b = \frac{1}{2} - ikd - \sqrt{v+1/4}$$

$$c = 1 - ikd \ . \tag{10.38}$$

We are interested not only in asymptotic solutions for $x \to \infty$ but also in $x \to -\infty$ (i.e. $u \to 1$). In order to obtain this asymptotic solution we have to use the following identity for the hypergeometric function

$$F(a,b,c;\ u) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b,a+b-c+1;\ 1-u)$$

$$+ \frac{(1-u)^{c-a-b}\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a,c-b,c-a-b+1;\ 1-u) \tag{10.39}$$

where Γ's are the gamma functions. Using the relations

$$(1 - u)^{c-a-b} = e^{-2\beta z} \tag{10.40}$$

$$F(\cdot,\cdot,\cdot;\ 1-u) \to 1 \quad \text{for} \quad 1 - u \to 0 \tag{10.41}$$

[where the last equation is obtained from (10.33)] into Eq. (10.39) one gets

$$\psi(z \to -\infty) = A2^{\beta} \left[\frac{\Gamma(c)\Gamma(c-a-b)e^{\beta z}}{\Gamma(c-a)\Gamma(c-b)} + \frac{\Gamma(c)\Gamma(a+b-c)e^{-\beta z}}{\Gamma(a)\Gamma(b)}\right] . \tag{10.42}$$

Substituting (10.36) for β gives

$$x \to -\infty : \quad \psi = A2^{\beta} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \left[e^{ikx} + \frac{\Gamma(c-a-b)\Gamma(a)\Gamma(b)e^{-ikx}}{\Gamma(c-a)\Gamma(c-b)\Gamma(a+b-c)}\right] \tag{10.43}$$

one can determine the reflection coefficient and the incident wave amplitude. In order to get the transmission coefficient as well, one has to write Eq. (10.34) with the same factor as in Eq. (10.43) [or (10.42)]. Compare with Eqs. (10.16) and (10.17) for a $\delta(x)$ potential. For the $\mathrm{sech}^2 x$ potential we have

$$R = \frac{\Gamma(c-a-b)\Gamma(a)\Gamma(b)}{\Gamma(c-a)\Gamma(c-b)\Gamma(a+b-c)} \tag{10.44}$$

$$T = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(a+b-c)} . \tag{10.45}$$

Using the identity

$$\Gamma(\frac{1}{2} - z)\Gamma(\frac{1}{2} + z) = \frac{\pi}{\cos\pi z} \tag{10.46}$$

and the values of a, b and c from (10.38) one gets

$$R \propto \cos\left[\pi(v + \frac{1}{4})^{1/2}\right] . \tag{10.47}$$

Therefore, the reflection in zero for

$$(v + \frac{1}{4})^{1/2} = n + \frac{1}{2} \ , \ n = 1,2,3,\ldots, \tag{10.48}$$

while R and T have pole singularities for

$$\frac{1}{2} - ikd \pm (v + \frac{1}{4})^{1/2} = -p \ , \qquad p = 0,1,2,\ldots \tag{10.49}$$

The poles in the upper half of the complex k plane are the solutions of

$$kd = i[(v + \frac{1}{4})^{1/2} - (p + \frac{1}{2})] \ . \tag{10.50}$$

Since the imaginary values of k are positives, this equation imposes an upper limit for the values of p. Assuming reflectionless potentials for the scattering states and the bound states to be defined by (10.50), one has

$$(v + \frac{1}{4})^{1/2} = n + \frac{1}{2} \ , \qquad n = 1,2,3,\ldots$$

$$kd = i(n - p) \ , \qquad p = 0,1,2,\ldots, n - 1 \ . \tag{10.51}$$

In summary we can write that the soliton potentials are related to reflectionless potentials

$$R = 0 \tag{10.52}$$

via the Schrödinger equation

$$\frac{d^2\psi}{dz^2} + [-(n - p)^2 + n(n + 1)\mathrm{sech}^2 z]\psi = 0 \tag{10.53}$$

where

$$n = 1,2,3,\ldots$$

$$p = 0,1,\ldots n - 1, \tag{10.54}$$

and the eigenfunctions are derived from hypergeometric functions. The interesting phenomenon that soliton type potentials are relectionless might be an indicator about the collision properties of solitons.

11. INVERSE SCATTERING THEORY

The problem of this chapter is to solve the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{11.1}$$

by calculating the function $u(x,t)$ for all $t > 0$ and $-\infty < x < \infty$ assuming that

$$u(x,o) = g(x) \tag{11.2}$$

is known. For this pupose we shall use the Schrödinger equation

$$\psi_{xx} + (\lambda - u)\psi = 0 \tag{11.3}$$

where the potential u satisfies KdV equation and λ are the eigenvalues. For

$\lambda \geqslant 0$ one has oscillatory solutions for $x = \pm\infty$ since u vanishes reasonably fast at infinity. For $\lambda < 0$ the solution of the Schrödinger equation vanishes at infinity for a discrete set of eigenvalues λ_i $(i = 1, \ldots, N)$. These eigenvalues correspond to bound states $\psi_i(i = 1,\ldots,N)$ of the potential u. We use the notation

$$\lambda = k^2, \quad k \geqslant 0 \qquad \text{if} \qquad \lambda \geqslant 0$$

$$\lambda = -\kappa_i^2, \quad \kappa_i \geqslant 0 \quad \text{if} \quad \lambda = \lambda_i < 0 \quad . \tag{11.4}$$

We consider (for $\lambda \geqslant 0$) a plane wave incident from $x \to -\infty$ which scatters into transmitted and reflected waves. Asymptotically one has

$$x \to -\infty : \quad \psi_{\leftarrow} \sim e^{ikx} + R(k)e^{-ikx} \tag{11.5}$$

$$x \to +\infty : \quad \psi_{\rightarrow} \sim T(k)e^{ikx} \tag{11.6}$$

with the usual probability conservation of

$$|R|^2 + |T|^2 = 1 \quad . \tag{11.7}$$

The bound states are normalized and satisfy

$$\psi_i \sim e^{-\kappa_n |x|} \quad , \quad |x| \to \infty \; , \quad i = 1,\ldots,N \; . \tag{11.8}$$

An important variable in this approach is

$$c_i = \lim_{x\to\infty} \psi_i e^{i\kappa_i x} \quad . \tag{11.9}$$

The Gelfand–Levitan[27] inverse scattering theory expresses the potential u in terms of the quantities c_i, $\kappa_i (i=1,\ldots,N)$ and $R(k)$.

Before going into details we shall sketch the procedure of the inverse scattering theory of calculating $u(x,t)$. The initial value $u(x,o)$ is taken as the scattering potential in the time independent Schrödinger equation. The solution of Schrödinger equation with $u(x,o)$ gives scattering data, namely the eigenvalues $\kappa_i (i=1,\ldots,N)$, their asymptotic eigenfunctions and the reflection coefficient R of the problem at $t = 0$. Using an evolution equation [Eqs. (10.9) or (10.11)] one calculates the scattering data for $t > 0$ in terms of these data at $t = 0$. Finally, the scattering data for $t > 0$ is used to find the potential $u(x,t)$ which satisfies KdV equation. We give the result without proof,[15]

$$u(x,t) = -2 \frac{\partial K(x,x,t)}{\partial x} \tag{11.10}$$

where $K(x,y,t)$ satisfies the integral equation

$$K(x,y,t) + B(x+y,t) + \int_x^\infty dz\, B(y+z,t)K(x,z,t) = 0 \tag{11.11}$$

where the kernel B is defined by

$$B(y,t) = \sum_{i=1}^{N} c_i^2(t)\exp(-\kappa_i(t)y) + \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, R(k,t)\exp(iky). \tag{11.12}$$

In Eq. (11.10), first put $y = x$ in $K(x,y,t)$ and then differentiate with respect to x for a fixed value of t. The advantage of this approach is that $u(x,t)$ is given in terms of c_i, κ_i and R using only linear equations.

However, unfortunately, it is not always possible to solve these linear equations analytically, but this procedure seems to be very useful for computer (numerical) solutions.

Since $u(x,o)$ is given one can solve the Schrödinger equation for $t = 0$ to obtain

$$c_i(0),\ \kappa_i(0) \quad \text{for} \quad i = 1,\ldots,N\ ; \quad R(k,o)\ . \tag{11.13}$$

The time dependence of c_i, κ_i and R are obtained through the evolution equation derived in Chapter 10 [see Eqs. (10.9) and (10.11)] which are rewritten here for convenience,

$$\psi_t - 3(u + \lambda)\psi_x + \psi_{xxx} = 0 \tag{11.14}$$

or

$$\psi_t - 3(u + \lambda)\psi_x + \psi_{xxx} = A\psi\ . \tag{11.15}$$

These equations together with the KdV equation are used to calculate the time dependence of (11.13). In particular, one has

$$\kappa_n(t) = \kappa_n(o) \tag{11.16}$$

since the eigenvalues λ were assumed to be time independent (see Chapters 7 and 10). We derive now the time dependence of c_i which are defined in Eq. (11.9). From ψ_t equation (11.14) and $u(x = \infty) = 0$, one gets

$$\psi_t(x = \infty) = -4\psi_{xxx}(x = \infty) \tag{11.17}$$

since $\lambda = -\kappa_i^2$ by definition [see Eq. (11.4)] and $\psi_{xxx} = -\kappa_i^2\psi_x$ by direct substitution of Eq. (11.8). Using (11.8) into (11.17) the following simple equation for $c_i(t)$ is obtained

$$\frac{d}{dt}\, c_i(t) = 4\kappa_i^3 c_i(t) \ , \qquad i = 1,2,\ldots,N \tag{11.18}$$

which has the solution

$$c_i(t) = c_i(o)\exp(4\kappa_i^3 t) \ , \quad i = 1,2,\ldots,N \ . \tag{11.19}$$

Now we calculate the time dependence of the reflection coefficient $R(k,t)$. The asymptotic solution for ψ is given by (11.5). The equation for the time evolution in this case is given by Eq. (11.15). [Eq. (11.15) is necessary instead of Eq. (11.14) in order to obtain a consistent equation in the following procedure.] Inserting the asymptotic form (11.5) into the evolution equation (11.15) one gets

$$\frac{dR}{dt}\, e^{-ikx} + 4ik^3\left(-e^{ikx} + Re^{-ikx}\right) = A\left(e^{ikx} + Re^{-ikx}\right) \ . \tag{11.20}$$

Equating coefficients of e^{ikx} and e^{-ikx} searately, one gets

$$\frac{dR}{dt} = -4ik^3R + AR$$

$$A = -4ik^3 \ . \tag{11.21}$$

The solution of these (simple!) equations is

$$R(k,t) = R(k,o)\ \exp(-8ik^3t) \quad . \tag{11.22}$$

Using Eqs. (11.16), (11.19) and (11.22) into Eq. (11.12), the kernel of the integral equation (11.11) is known and (hopefully) this implies the solution for $K(x,y,t)$ which in turn gives $u(x,t)$ by Eq. (11.10). However, in general it is difficult to solve these equations starting with a general initial condition $u(x,o)$. However, for a reflectionless potential

$$R(k,t) = R(k,o) = 0 \tag{11.23}$$

the calculations can be performed. In this case, the kernel of Eq. (11.11) is given by

$$B(y) = \sum_{i=1}^{N} c_i^2 \exp(-\kappa_i y) \quad . \tag{11.24}$$

For this kernel Eq. (11.11) is solved by a sum of exponentials [like (11.24)] and the result for $u(x,t)$ is the N soliton solution. For $R = 0$ one gets

$$u(x,t) = -4 \sum_{i=1}^{N} \kappa_i \psi_i^2 \tag{11.25}$$

so that one can interpret $-4\kappa_i\psi_i^2$ as a soliton. In particular, if one takes for the case $R = 0$ [see Eq. (10.53)],

$$u(x) = -N(N + 1)\operatorname{sech}^2 x, \quad N=1,2,3,\ldots \tag{11.26}$$

then for N = 1 we get the one soliton solution, ($\kappa_1 = 1$)

$$N = 1, \qquad u = -2\mathrm{sech}^2(x-4t) \quad . \tag{11.27}$$

For N = 2, the eigenvalues are $\kappa_1 = 1$, $\kappa_2 = 2$ (two solitons) and the solution is identical to that obtained in Eq. (9.17).

To summarize, the aim of the inverse scattering method is to solve nonlinear differential equations for u(x,t), where u is the potential of the time independent Schrödinger equation. u(x,t) is constructed from the asymptotic values of the scattering states and the bound states.

12. CONCLUSION

The research of problems in physics (and scientific problems in general) are described by many (hundreds?) nonlinear equations. In this note we dealt mainly with one equation, the KdV equation and its variations. However, the preceding chapters outline many ideas and techniques which seem to be useful in approaching nonlinear problems. The emphasis is on soliton solutions, which seem to be an exception (in dealing with nonlinear differential equation) rather than a generality. Nonetheless, these soliton type solutions were observed in several other nonlinear equations such as those mentioned in the introduction.

The scientific phenomena which lead to soliton solutions was suggested almost in every field of physics, including fluids, plasmas, nonlinear optics, solids and even particle physics. In the past twenty years (since

Zabusky and Kruskal reintroduced the soliton in 1965) there have been many fascinating and beautiful developments in the subject of nonlinear differential equations related to scientific phenomena, however it seems that a long journey has to be made before we shall be able to really understand the nonlinear physics through the nonlinear differential equations (of one class or another). It looks to me that the following story taken from Turkish folklore summarizes the view of many scientists who "believe" in "solitons".

Hoja Wants to See Better

> One night Hoja woke his wife in a great state of excitement. "Quick!" said he. "Give me my spectacles before I awaken." She handed them to him, but asked the reason for his agitation.
>
> "I am having a beautiful dream", he answered, "but there are one or two things in it I cannot make out very clearly".

However, the field is beautiful even without the spectacles.

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Figure Captions

1. A solitary wave, $D \gg h \gg a$ moving with a steady velocity c.

2. Two waves with shapes described by the functions f and g at (a) a time $t \to -\infty$ and (b) a time $t \to +\infty$.

3. The characteristics C_A and C_B for the initial value $u(x,o)$. $u_A(x,t)$ is constant along C_A while $u_B(x,t)$ is constant along the characteristic C_B.

4. $F(f)$ as a function of f [see Eq. (6.(9)]. The figure A describes the soliton solution while figure B shows the oscillatory solution.

5. The effective potential for a "classical" particle. For $V_{eff} = V_B$ one has cnoidal wave solutions, for $V_{eff} = V_c$ one has the soliton solution while for $V_{eff} = V_A$ one gets the usual sine wave.

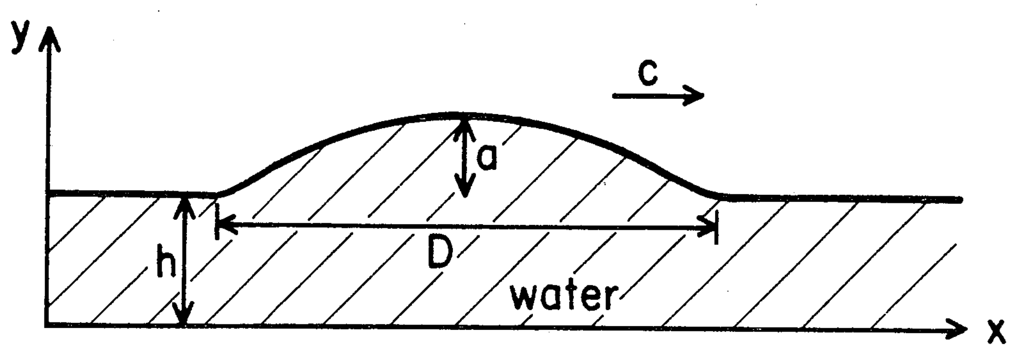
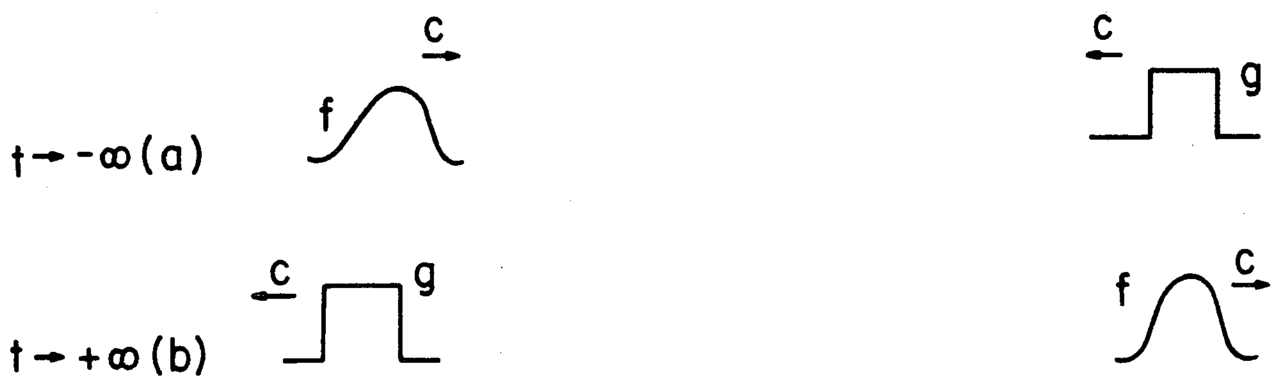


FIGURE 1



FIGURES 2(a)-2(b)

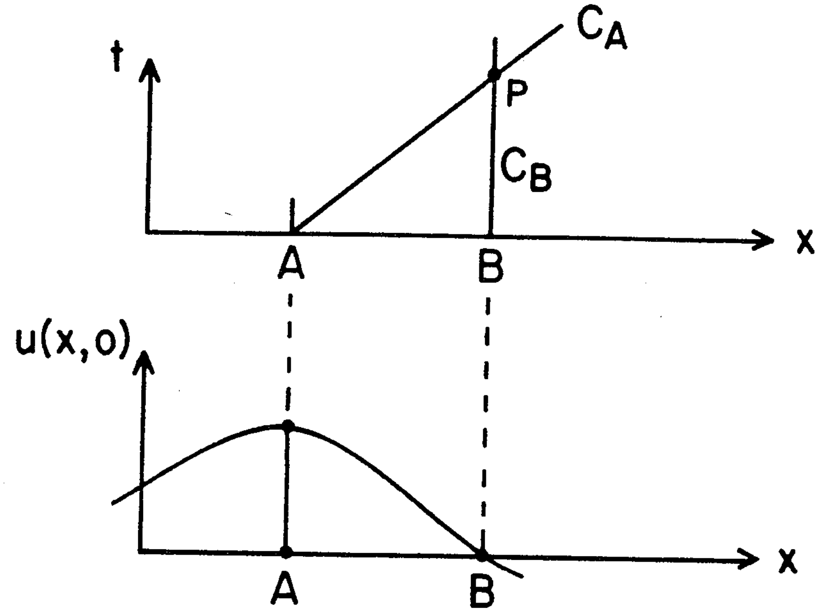


FIGURE 3

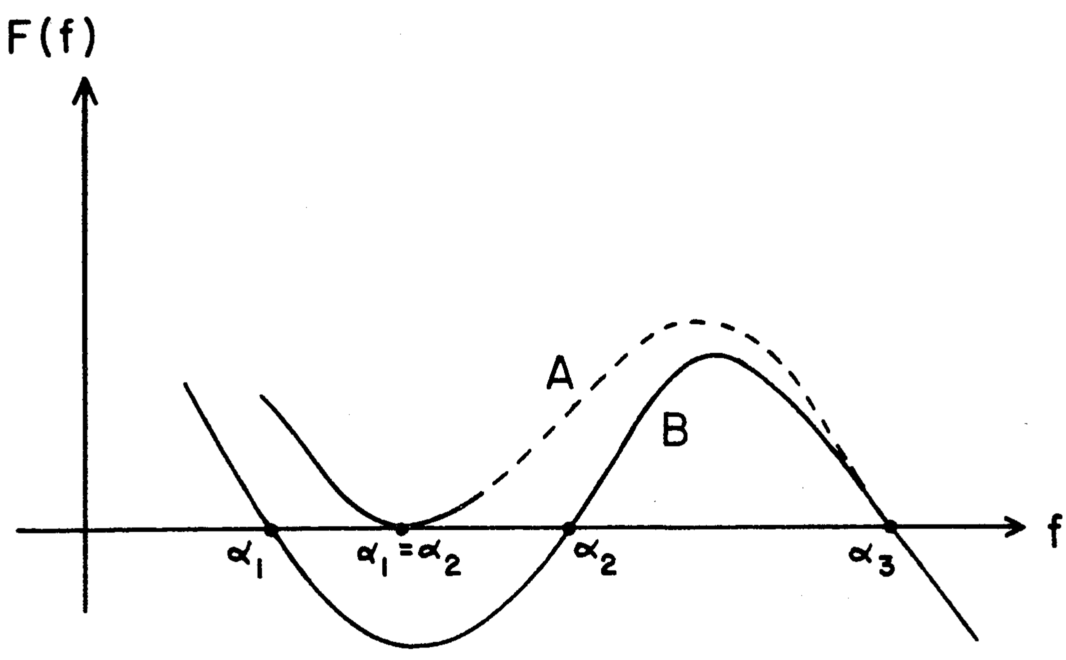


FIGURE 4

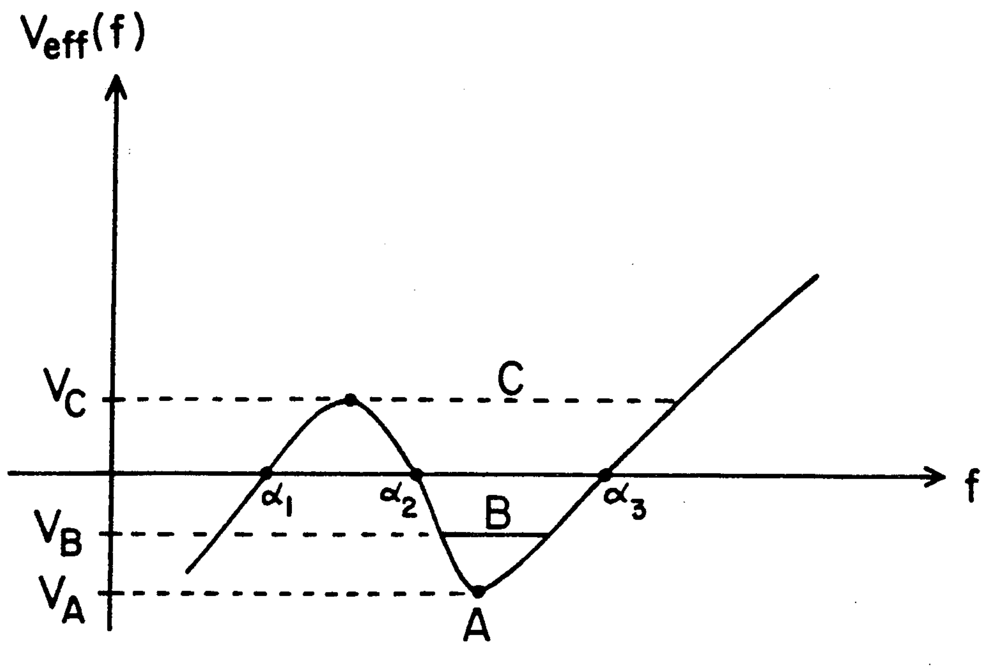


FIGURE 5