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LIE TRANSFORM PERTURBATION THEORY FOR HAMILTONIAN SYSTEMS

John R. Cary
Institute for Fusion Studies
University of Texas
Austin, Texas 78712

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A review of the theory of Lie transform perturbation theory for Hamiltonian systems is presented. The operator theory of Dewar for continuous families of canonical transformations is discussed. It is then used to derive the perturbation method of Deprit. Two examples of the use of this method are provided. In addition, the more efficient perturbation method of Dragt and Finn is discussed.

Contents

1. Introduction

In the last twenty years there has been a great resurgence in the study of classical mechanics by physicists. This research has led to developments in perturbation theory in two areas. The first is the study of systems for which perturbation theory fails, while the second has been the development of more efficient perturbation methods for systems in which they succeed. A review of the former is contained in the excellent review article by Chirikov [1]. Here we study the latter.

Perturbation methods follow a basic pattern. To analyze a complicated system, one first neglects effects until the system is simple. Then the neglected effects are restored, and the difference in the motion between the simple system and the more complicated system is calculated as a perturbation series in the strength of the effect. Thus, in the classic problem of the earth's motion in the gravitational field of the sun and the planets, one ignores at first the effects of other planets and solves the problem of the earth's motion in the field of the sun. Then the effects of the weaker gravitational fields of the other planets, such as Saturn, are calculated as a perturbation series. In essence, this is a calculation of a transformation from the natural variables of the simple system to natural variables of the complicated system.

For secular perturbation theory, the transformation sought is the difference between the time-development transformation of the simple system and that of the perturbed system. (This perturbation theory is similar to the method of finding the S-matrix in quantum

mechanics in the interaction picture [2].) As a result, the transformed coordinates for the perturbed system evolve exactly as did the original coordinates for the unperturbed system. This method works well for short times, but in general, secular terms, which are unbounded in time, appear causing the perturbation solution to diverge from the true solution at late times.

This primary failing of secular perturbation theory is not present in the perturbation method of Poincaré and Von Zeipel [3, sec. VII-5, and 4]. In the latter method one seeks a transformation which describes only the oscillatory part of the difference between the two variable sets. By this means, the perturbed system is transformed to a system which is simple and solvable, but not identical to the unperturbed system.

The transformation theory used in the perturbation method is the Hamilton-Jacobi theory [3,5,6]. In this theory, the canonical transformation is specified by the global generating function, $F(q,P,t,\varepsilon)$, of mixed variables. The perturbation method consists of a "brute-force" expansion of the Hamilton-Jacobi equation, $K(Q,P,t) = h(q,p,t) + F(q,P,t)$, in the variable difference $q-Q = \varepsilon q,(Q,P,t) + \varepsilon^2 q_2(Q,P,t) + \ldots$. This method is clumsy because it does not utilize the natural structure of infinitesimal canonical transformations.

More efficient perturbation methods utilizing the natural structure of canonical transformations have been introduced only recently. The transformations involved have been lumped under the

generic name, Lie transform. Garrido [7,8] has formulated the time development transformation and secular perturbation theory as Lie transforms. Hori [9] introduced the Lie transform as the exponentiation of a Poisson bracket operator. Deprit [10] improved upon Hori's idea by redefining the transformation in a way that made time dependent theory more natural and yielded recursion relations for the terms in the perturbation series. Later Dragt and Finn [11] introduced a perturbation method which appears to be even more efficient than the others. Dewar [12] unified these papers by developing an operator framework for general, non-analytic continuous families of canonical transformations.

The development of Lie transforms has led to new attacks on old problems and the generation of new ideas. Schmidt [13] and Cheng [14] have used Lie transforms to discuss problems in celestial mechanics. McNamara [15] has used Lie transforms to generate invariants to high order even in the presence of resonant denominators. Cary and Kaufman [16,17] and Johnston and Kaufman [18] have shown that Lie transforms lead to unexpected relationships between the ponderomotive force and the linear response of a plasma. Mynick [19] and Littlejohn [20] have used Lie transforms to analyze guiding center motion.

The purpose of this paper is to review Lie transform perturbation theory. Chapter 2 consists of a review of some of the basic elements of classical mechanics, including the perturbation method of Poincaré and Von Zeipel. Chapter 3 is a discussion of Dewar's general transformation theory. Chapter 4 illustrates how Deprit's perturbation method arises from Dewar's transformation theory.

(Historically the development was in the opposite order.)

Applications of the perturbation technique are presented. Finally, in Chapter 5 the more efficient perturbation theory of Dragt and Finn is presented.

2. Hamiltonian Mechanics

2.1. Hamilton's equation, Poisson brackets, and canonical transformations.

In Hamiltonian mechanics, the state of a system is determined by a 2N dimensional vector $\underset{\sim}{z}$. The components of this vector are of two types, the coordinates, $q_1,\ldots,q_N = z_1,\ldots,z_N$, and the momenta, $p_1,\ldots,p_N = z_{N+1},\ldots,z_{2N}$. The evolution of the state is determined by a single function, the Hamiltonian $h(\underset{\sim}{z},t)$, according to Hamilton's equations:

$$\dot{q}_i = \frac{\partial h}{\partial p_i} \tag{2.1a}$$

$$\dot{p}_i = -\frac{\partial h}{\partial q_i} \quad . \tag{2.1b}$$

Hamilton's equations and much of Hamiltonian theory are expressed neatly in terms of Poisson brackets. The Poisson bracket is an operation which takes two functions of $\underset{\sim}{z}$ and forms a third according to the prescription,

$$\{f, g\} = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \quad . \tag{2.2}$$

The Poisson bracket may also be written as

$$\{f, g\} = \sum_{\substack{i=1 \\ j=1}}^{2N} \frac{\partial f}{\partial z_i}\,\gamma_{ij}\,\frac{\partial g}{\partial z_j} \quad , \tag{2.3}$$

in terms of the matrix γ which expresses the Poisson bracket relations for the variables,

$$\gamma_{ij} \equiv \{z_i, z_j\} = \begin{cases} 1 \text{ for } j = i+N \\ -1 \text{ for } i = j+N \\ 0 \text{ otherwise} \end{cases} \quad . \tag{2.4}$$

In terms of the Poisson bracket, Hamilton's equations (2.1) are

$$\dot{z}_i = \{z_i, h\} \quad . \tag{2.5}$$

We note in passing that since the Poisson bracket satisfies the Jacobi identity,

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0 \quad , \tag{2.6}$$

the space of functions of $\underset{\sim}{z}$ comprises a Lie algebra with the Poisson bracket being the Lie product.

Another concept which must be introduced is that of a canonical transformation. A general transformation is a set of relations $\underset{\sim}{Z}(\underset{\sim}{z},t)$ between the new coordinate set $\underset{\sim}{z}$ and the old. Such a transformation is canonical if it preserves the Poisson bracket, i.e. if

$$\{Z_i, Z_j\} = \{z_i, z_j\} \tag{2.7}$$

or

$$\sum_{k,\ell} \frac{\partial Z_i}{\partial z_k} \gamma_{k\ell} \frac{\partial Z_j}{\partial z_\ell} = \gamma_{ij} \tag{2.8}$$

Canonical transformations are important because they preserve the Hamiltonian nature of the system [3, VI]. That is, there exists a new Hamiltonian K such that

$$\dot{Z}_m = \{Z_m, K(\underset{\sim}{Z}(\underset{\sim}{z},t),t)\} \tag{2.9}$$

The above notions are conveniently expressed in terms of operator notations of Dewar [12]. For the Poisson bracket, we introduce the notation L(f) to denote the Poisson bracket operator associated with f. Thus,

$$L(f)g \equiv \{f,g\} \tag{2.10}$$

With this choice, Jacobi's identity is

$$[L(f),L(g)] = L(\{f,g\}) \quad , \tag{2.11}$$

where the square brackets denote the commutator of the two operators.

Similarly, we identify a canonical transformation operator T with a canonical transformation $\underset{\sim}{Z}(\underset{\sim}{z},t)$ according to the prescription

$$(Tf)(\underset{\sim}{z},t) = f(\underset{\sim}{Z}(\underset{\sim}{z},t)) \quad . \tag{2.12}$$

In words, the action of this operator is to evaluate the function at the mapped point. In particular, we have

$$(Tz_m)(\underset{\sim}{z},t) = Z_m(\underset{\sim}{z},t) \quad . \tag{2.13}$$

To illustrate the use of the operator notation, we consider the statement that Poisson brackets are preserved by canonical transformation. In ordinary notation this statement is

$$\left\{f[\underset{\sim}{Z}(\underset{\sim}{z},t),t],g[\underset{\sim}{Z}(\underset{\sim}{z},t),t]\right\} = \{f,g\}\Big|_{\underset{\sim}{Z}(\underset{\sim}{z},t),t} \quad . \tag{2.14}$$

where the notation on the right side means: first find the Poisson bracket of the two functions, then substitute $\underset{\sim}{Z}(\underset{\sim}{z},t)$ for $\underset{\sim}{z}$. In operator notation, Eq. (2.14) is

$$TL(f)g = L(Tf)Tg \quad . \tag{2.15}$$

In fact, since Eq. (2.15) is true for all g, we have the operator relation,

$$TL(f)T^{-1} = L(Tf) \quad . \tag{2.16}$$

2.2 The time development transformation.

The objective of Hamiltonian mechanics is to solve for the trajectories of a Hamiltonian, i.e. the vector functions $\underset{\sim}{z}(t)$ which satisfy Hamilton's equations (2.5). Knowledge of these trajectories allows us to define a mapping, $\underset{\sim}{\mathcal{Z}}(\underset{\sim}{z},t,t')$, which gives the position at time t of a particle which was at $\underset{\sim}{z}$ at time t'. That is the mapping is defined by

$$\frac{\partial \mathcal{Z}_i}{\partial t} = \sum_j \gamma_{ij} \frac{\partial h}{\partial z_j}\bigg|_{\mathcal{Z}(z,t,t'),t} \tag{2.17}$$

and

$$\mathcal{Z}_i(z,t,t') = z_i \quad . \tag{2.18}$$

This mapping, which is canonical, is known as the time development mapping. The corresponding time development transformation operator, $S(t,t')$, is defined via

$$(S(t,t')f)(z,t,t') = f(\mathcal{Z}(z,t,t'),t) \quad . \tag{2.19}$$

The effect of the operator S is to develop the observables in time. As an example, suppose we are given the potential energy $V(z,t)$ at time t as a function of the position of a particle at time t. By applying S, we obtain a new function,

$$\mathcal{V}(z,t,t') \equiv (S(t,t')V) = V(\mathcal{Z}(z,t,t'),t) \quad ,$$

which gives the potential energy at time t of a particle at z at time t'.

From its definition, the map S is seen to have the property of connectivity,

$$S(t,t') = S(t,t'')S(t'',t') \quad . \tag{2.20}$$

Furthermore, the inverse of S is given by

$$S^{-1}(t,t') = S(t',t) \quad . \tag{2.21}$$

Since the mapping $\underset{\sim}{\mathcal{G}}(z,t)$ is related to h by Eq. (2.17), we expect the time derivative of S to be related to h. The time derivative of S is defined in the usual way,

$$\frac{\partial S}{\partial t} g = \lim_{\tau \to 0} \frac{S(t+\tau,t')g - S(t,t')g}{\tau} \quad . \tag{2.22}$$

By straightforward evaluation we obtain

$$\frac{\partial}{\partial t} S(t,t') = - S(t,t')L(h(t)) \quad , \tag{2.23}$$

and by using the relationship $S(t,t')S(t',t) = I$, the identity, we obtain

$$\frac{\partial}{\partial t'} S(t,t') = L(h(t'))S(t,t') \quad . \tag{2.24}$$

In the special case where h is time independent, many of the results simplify. For example, the time development operator must be time translation invariant,

$$S(t,t') = S(t - t',0) \quad .$$

In such cases we will drop the second temporal variable. In addition, L(h) and S commute in this case,

$$L(h)S(t) = S(t)L(h) \quad . \tag{2.25}$$

To see this we use the fact that the time development transformation preserves the Hamiltonian, $S(t)h = h$, if h is time independent. This implies $L(S(t)h) = L(h)$, which together with Eq. (2.16) implies Eq. (2.25).

Knowledge of the trajectories or, equivalently, the time development operator, S, allows one to immediately solve the inhomogeneous Liouville equation,

$$\frac{\partial f}{\partial t} + \{f,h\} = g \quad , \tag{2.26}$$

which occurs again and again in Hamiltonian perturbation theory. To find this solution we introduce a new function f' which satisfies

$$f(t) \equiv S(t',t)f'(t,t') \quad . \tag{2.27}$$

Combining Eqs. (2.23-2.26), we find

$$\frac{\partial}{\partial t} f'(t,t') = S^{-1}(t',t)g(t) \quad , \tag{2.28}$$

from which a simple integration yields

$$f'(t,t') = \int_{t_0}^{t} dt_1 \, S^{-1}(t',t_1)g(t_1) + f'(t_0,t') \quad . \tag{2.29}$$

Reinvoking the definition (2.27), we obtain

$$\begin{aligned} f(t) &= S(t',t) \int_{t_0}^{t} dt_1 \, S^{-1}(t',t_1)g(t_1) + S(t',t)f'(t_0,t') \\ &= \int_{t_0}^{t} dt_1 \, S^{-1}(t,t_1)g(t_1) + S^{-1}(t,t_0)f(t_0) \quad . \end{aligned} \tag{2.30}$$

This method of solution is known as finding f by integrating g along the trajectories of h.

2.3 Secular perturbation theory.

Suppose we wish to find the time development transformation of a system whose Hamiltonian is the sum of a term h_0, for which the trajectories are known, and a small additional term h_1. It is natural to assume that the time development operator for the combined system, $S(t,t')$, can be written as the time development operator for the unperturbed system, S_0, combined with a small additional operator:

$$S(t,t') = s(t,t')S_0(t,t') \quad . \tag{2.31}$$

Using Eq.(2.23), we find

$$\frac{\partial S}{\partial t} = - SL(h(t))$$

$$= \frac{\partial s}{\partial t} S_0 - sS_0L(h_0)$$

which implies that s satisfies the differential equation,

$$\frac{\partial}{\partial t} s(t,t') = - sS_0L(h_1(t))S_0^{-1}$$

$$= - s(t,t')L(S_0(t,t')h_1(t)) \quad . \tag{2.32}$$

To obtain the solution to this equation, we order s in the perturbation,

$$S = I + S_1 + S_2 + \ldots \quad , \tag{2.33}$$

resulting in the equation,

$$\frac{\partial S_{n+1}}{\partial t} = - S_n L(S_0 h_1) \quad , \tag{2.34}$$

which can be immediately solved,

$$S_{n+1}(t,t') = \int_{t'}^{t} dt_1 \; S_n(t_1,t') L(S_0(t_1,t') h_1(t_1)) \quad . \tag{2.35}$$

This is known as secular perturbation theory.

The primary shortcoming of secular perturbation theory is that it gives answers which diverge rapidly from the true answer. As an example, consider the harmonic oscillator,

$$h_0 = \frac{1}{2} p^2 + \frac{1}{2} \omega_0^2 q^2 \quad , \tag{2.36}$$

in the presence of a harmonic perturbation,

$$\varepsilon h_1 = \frac{1}{2} \varepsilon q^2 \quad . \tag{2.37}$$

The time development mapping of the harmonic oscillator is easily found to be

$$q_0(q,p,t) = q \cos(\omega_0 t) + (p/\omega_0) \sin(\omega_0 t) \tag{2.38}$$

$$p_0(q,p,t) = p\cos(\omega_0 t) - \omega_0 q\cos(\omega_0 t) \quad . \tag{2.39}$$

Hence, Eqs. (2.31-34) yield the perturbed trajectories

$$\begin{aligned} \underset{\sim}{\mathcal{Z}}(t,0) &= s(t,0)S_0(t,0)\underset{\sim}{z} \\ &= \underset{\sim}{\mathcal{Z}}_0(t,0) - \int_0^t dt_1 \{S_0(t_1,0)h_1(t_1), \underset{\sim}{\mathcal{Z}}_0(t,0)\} \quad . \end{aligned} \tag{2.40}$$

to first order in ε. Straightforward integration gives the following result for the trajectory:

$$\begin{aligned} q(q,p,t,0) = q\Bigg\{&\cos\omega_0 t + \frac{\varepsilon}{2\omega_0}\left[\left(t + \frac{\sin(2\omega_0 t)}{2\omega_0}\right)\sin(\omega_0 t)\right. \\ &\left. + \left(\frac{\cos(2\omega_0 t) - 1}{2\omega_0}\right)\cos(\omega_0 t)\right]\Bigg\} \\ + \frac{p}{\omega_0}\Bigg\{&\sin\omega_0 t - \frac{\varepsilon}{2\omega_0}\left[\left(\frac{\cos(2\omega_0 t) - 1}{2\omega_0}\right)\sin(\omega_0 t)\right. \\ &\left. + \left(t - \frac{\sin(2\omega_0 t)}{2\omega_0}\right)\cos(\omega_0 t)\right]\Bigg\} \end{aligned} \tag{2.41}$$

This formula predicts that q is unbounded in time, where, in fact the perturbed system actually has bounded periodic orbits.

The perturbation theory developed in later chapters of this paper will be able to handle this problem successfully.

2.4 Poincare-Von Zeipel perturbation theory.

This perturbation theory is based on the canonical transformation theory of Hamilton and Jacobi [5, ch. 8], which states that any function $F(\underset{\sim}{q},\underset{\sim}{P},t)$ of the old coordinates and the new momenta generates a canonical transformation via the equations,

$$p_i = \frac{\partial F}{\partial q_i} \quad (\underset{\sim}{q},\underset{\sim}{P},t) \tag{2.42a}$$

$$Q_i = \frac{\partial F}{\partial P_i} \quad (\underset{\sim}{q},\underset{\sim}{P},t) \tag{2.42b}$$

Of course, Eqs. (2.42) do not give the transformation directly. Eq. (2.42a) must be inverted to obtain $\underset{\sim}{P}(\underset{\sim}{q},\underset{\sim}{p},t)$, which may then be substituted into Eq.(2.42b) to obtain $\underset{\sim}{Q}(\underset{\sim}{q},\underset{\sim}{p},t)$. The function $F(\underset{\sim}{q},\underset{\sim}{P},t)$ is called the global generating function. (Actually, as discussed by Goldstein, there are several variants of the global generating function.) The Hamiltonian for the new variables is given by the relationship

$$K(\underset{\sim}{Q},\underset{\sim}{P},t) = h(\underset{\sim}{q},\underset{\sim}{p},t) + \frac{\partial F}{\partial t} \quad (\underset{\sim}{q},\underset{\sim}{P},t) \quad . \tag{2.43}$$

To develop the perturbation theory, let us consider the transformation equations for a Hamiltonian which consists of a solvable part h_0 and a perturbation h_1,

$$h(\underset{\sim}{q},\underset{\sim}{p},t) = h_0(\underset{\sim}{q},\underset{\sim}{p},t) + \varepsilon h_1(\underset{\sim}{q},\underset{\sim}{p},t) \quad , \tag{2.44}$$

and a similarly ordered global generating function,

$$F(\underset{\sim}{q},\underset{\sim}{P},t) = \underset{\sim}{q}\cdot\underset{\sim}{P} + \varepsilon F_1(\underset{\sim}{q},\underset{\sim}{P},t) + \varepsilon^2 F_2(\underset{\sim}{q},\underset{\sim}{P},t) + \ldots \quad . \tag{2.45}$$

These quantities can be immediately substituted into the Hamilton-Jacobi equation (2.43). However, that does not tell us the functional form of $K(\underset{\sim}{Q},\underset{\sim}{P},t)$ until we substitute into h and F the transformations, $\underset{\sim}{q}(\underset{\sim}{Q},\underset{\sim}{P},t,\varepsilon)$ and $\underset{\sim}{p}(\underset{\sim}{Q},\underset{\sim}{P},t,\varepsilon)$.

To this end, we also order the transformation,

$$\underset{\sim}{q}(\underset{\sim}{Q},\underset{\sim}{P},t,\varepsilon) = \underset{\sim}{Q} + \varepsilon\underset{\sim}{q}_1(\underset{\sim}{Q},\underset{\sim}{P},t) + \varepsilon^2\underset{\sim}{q}_2(\underset{\sim}{Q},\underset{\sim}{P},t) + \ldots \tag{2.46a}$$

and

$$\underset{\sim}{p}(\underset{\sim}{Q},\underset{\sim}{P},t,\varepsilon) = \underset{\sim}{P} + \varepsilon\underset{\sim}{p}_1(\underset{\sim}{Q},\underset{\sim}{P},t) + \varepsilon^2 p_2(\underset{\sim}{Q},\underset{\sim}{P},t) + \ldots \quad . \tag{2.46b}$$

We insert these equations into Eqs. (2.42) and calculate the Taylor expansion in the difference q-Q. We thereby obtain the first-order results,

$$\underset{\sim}{q}_1(\underset{\sim}{Q},\underset{\sim}{P},t) = -\left.\frac{\partial F_1}{\partial \underset{\sim}{P}}\right|_{\underset{\sim}{Q},\underset{\sim}{P},t} \tag{2.47a}$$

and

$$\underset{\sim}{p}_1(\underset{\sim}{Q},\underset{\sim}{P},t) = \left.\frac{\partial F_1}{\partial \underset{\sim}{q}}\right|_{\underset{\sim}{Q},\underset{\sim}{P},t} . \tag{2.47b}$$

The expressions on the right of Eqs. (2.47) mean: take the derivative of F, then insert $\underset{\sim}{Q}$, not $\underset{\sim}{q}(\underset{\sim}{Q},\underset{\sim}{P})$, into the slot of F_1 where $\underset{\sim}{q}$ normally goes

We now expand Eq. (2.43) in the variable differences. We find that the new Hamiltonian,

$$K(\underset{\sim}{Q},\underset{\sim}{P},t) = h_0(\underset{\sim}{Q},\underset{\sim}{P},t) + \varepsilon K_1(\underset{\sim}{Q},\underset{\sim}{P},t) + \ldots , \tag{2.48}$$

satisfies the equation,

$$K_1(\underset{\sim}{Q},\underset{\sim}{P},t) = h_1(\underset{\sim}{Q},\underset{\sim}{P},t) - \left.\frac{\partial h_0}{\partial \underset{\sim}{q}}\right|_{\underset{\sim}{Q},\underset{\sim}{P},t} \cdot \left.\frac{\partial F_1}{\partial \underset{\sim}{P}}\right|_{\underset{\sim}{Q},\underset{\sim}{P},t}$$
$$+ \left.\frac{\partial h_0}{\partial \underset{\sim}{P}}\right|_{\underset{\sim}{Q},\underset{\sim}{P},t} \cdot \left.\frac{\partial F_1}{\partial \underset{\sim}{q}}\right|_{\underset{\sim}{Q},\underset{\sim}{P},t} + \left.\frac{\partial F_1}{\partial t}\right|_{\underset{\sim}{Q},\underset{\sim}{P},t} \tag{2.49}$$

Again we stress that Eq. (2.49) expresses a relationship between functions; the same dummy variable is used on both sides of the equation. This is quite different from Eq. (2.43), which expresses a relationship between values of functions. Finally, we recognize that Eq.(2.49) may be written in the form,

$$\frac{\partial F_1}{\partial t} + \{F_1, h_0\} = K_1 - h_1 \, . \tag{2.50}$$

So far, our transformation is arbitrary. At this point we must choose F_1 to make the new system more easily solved than the old. Recognizing that this criterion is vague, we offer the following points of strategy for choosing the transformation.

Comparing Eq. (2.50) with Eq. (2.26), we see that F_1 is the integral of $K_1 - h_1$ along an orbit of h_0. Therefore, we must pick K_1 such that the average of $K_1 - h_1$ along an orbit of h_0 vanishes. Otherwise, the integration to obtain F_1 will yield terms that are unbounded in time. A particular choice is to set K_1 to be the orbit average of h_1.

More specifically, suppose we have transformed to the natural variables of the problem, in which h_0 is a function of the momenta only. The unperturbed motion is simple; the momenta are constants and the coordinates change linearly in time,

$$\underset{\sim}{q} = \underset{\sim}{q}_0 + \frac{\partial h_0}{\partial \underset{\sim}{P}}(\underset{\sim}{P})t \ . \tag{2.51}$$

For this system, our strategy might be to choose the new Hamiltonian K_1 to be the orbit average of h_1:

$$K_1 = \langle h_1 \rangle_{\underset{\sim}{Q}} \ . \tag{2.52}$$

If this choice also causes the orbit average of $K_1 - h_1$ to vanish, we have been successful: we have a nonsecular F_1, and we have a new Hamiltonian,

$$K = h_0(\underset{\sim}{P}) + \varepsilon K_1(\underset{\sim}{P}) \ , \tag{2.53}$$

which is trivially solved.

As an example of a successful application of this technique, let us consider the perturbed harmonic oscillator, which was unsuccessfully treated by secular perturbation theory of the last section. The natural canonical coordinates for this problem are the action,

$$j = \frac{1}{2}(p^2/\omega_0 + \omega_0 q^2) \quad , \tag{2.54a}$$

and the angle,

$$\phi = \arctan(p/\omega_0 q) \tag{2.54b}$$

in terms of which the coordinates are,

$$q = \sqrt{2j/\omega_0}\,\cos\phi \tag{2.55a}$$

and

$$p = \sqrt{2\omega_0 j}\,\sin\phi \ . \tag{2.55b}$$

The unperturbed Hamiltonian and the perturbation are given by

$$h_0 = \omega_0 j \ , \tag{2.56}$$

and

$$h_1 = \frac{1}{2}(j/\omega_0)(1 + \cos 2\phi) \ . \tag{2.57}$$

We now choose our new Hamiltonian K_1 to be the coordinate (or angle) average of h_1,

$$K_1 = \frac{1}{2}(J/\omega_0) \ . \tag{2.58}$$

We may then solve Eq. (2.50) to obtain the global generating function,

$$F_1 = \frac{1}{4}(J/\omega_0)\sin 2\Phi \ , \tag{2.59}$$

which we insert into Eqs. (2.46-2.47) to get the transformation,

$$\phi = \Phi - \frac{\varepsilon}{4\omega_0} \sin 2\Phi$$

$$j = J + \frac{\varepsilon J}{2\omega_0} \cos 2\Phi \ . \tag{2.60}$$

To complete the solution, we must solve for the new onsets of the Hamiltonian,

$$K = (\omega_0 + \varepsilon/2\omega_0)J \ . \tag{2.61}$$

This Hamiltonian implies that the transformed action is conserved, and the frequency of motion is

$$\dot{\Phi} = \frac{\partial K}{\partial J} = \omega_0 + \frac{\varepsilon}{2\omega_0} \tag{2.62}$$

The result is simply the lowest order part of the exact frequency $\omega = (\omega_0^2 + \varepsilon)^{1/2}$.

The Poincare-Von Zeipel perturbation theory is seen to be an improvement over the secular perturbation theory of the last section. However, its application tends to be rather clumsy. To be able to apply the theory, one must expand Eqs. (2.42) and (2.43) to the desired order to obtain functional relations. At the end of the procedure, one obtains, at least in first order c.f. Eq. (2.45), not all arbitrary combinations, but only those combinations which are Poisson brackets.

The Lie transform theories to be discussed in the following sections are identical to the Poincare-Von Zeipel perturbation theory in spirit. One seeks a transformation which simplifies the Hamiltonian. However, the transformations themselves are much simpler. Closed form expressions in which only Poisson brackets appear can be explicitly written down for both the transformation and the transformation equation to arbitrary order.

3. Continuous Families of Canonical Transformations

We now consider canonical mappings which depend on an additional parameter θ. That is, the mapping $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$ is canonical for all values of t and for θ in some domain. Further, we require that $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$ be continuous, and twice differentiable in all variables simultaneously. We call this set of mappings (all θ values) a continuous family of canonical transformations.

The Jacobian of the transformation $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$ does not vanish, since the transformation is canonical. Therefore $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$ is locally invertible with inverse $\underset{\sim}{Z}^{-1}(\underset{\sim}{z},t,\theta)$. We will assume that $\underset{\sim}{Z}^{-1}$ is well defined throughout the region of interest.

In this chapter we will examine the consequences of the parametric dependence of $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$. We first establish the fact that the mapping can be specified by a single function $w(\underset{\sim}{z},t,\theta)$, in place of the 2N functions $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$. In the second section, we derive the expression for the transformed system. For the case where w is a power series in θ, this was first done by Deprit [10]. Later, Dewar [12] found the general expression for the new Hamiltonian when w is not a power series in θ.

3.1 The local generating function.

An important property of canonical transformations is that they can be specified by a single function. This property is helpful since it means that one can work with this single function rather than the 2N functions $\underset{\sim}{Z}(\underset{\sim}{z},t)$. In Hamilton-Jacobi theory this function is $F(\underset{\sim}{q},\underset{\sim}{P},t)$, the global generating function. For continuous families of canonical transformations, $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$, there is [3] a function $w(\underset{\sim}{z},t,\theta)$ satisfying

$$\frac{\partial Z_i}{\partial \theta} = \left\{ Z_i,\ w(\underset{\sim}{Z}(\underset{\sim}{z},t,\theta),\ t,\ \theta) \right\}$$

$$= \sum_j \gamma_{ij} \frac{\partial w}{\partial z_j} \Bigg|_{\underset{\sim}{Z}(\underset{\sim}{z},t,\theta),\ t,\ \theta} \quad . \tag{3.1}$$

If w has appropriate properties (say $w \in C_\infty$), we can specify $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$ as the unique solution of (3.1) for some given boundary conditions. In our work we will always require that the mapping reduce to the identity when $\theta = 0$:

$$\underset{\sim}{Z}(\underset{\sim}{z},t,0) = \underset{\sim}{z} \quad . \tag{3.2}$$

We note that equation (3.1) is analogous to (2.17). Just as the Hamiltonian h generates the time development mapping $\underset{\sim}{\mathcal{Z}}$ in the variable t, the function w generates the mapping $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$ in the variable θ. Following this analogy, we introduce the canonical transformation T obtained from the mapping $\underset{\sim}{Z}$, just as S came from $\underset{\sim}{\mathcal{Z}}$. From (2.23) we see that T must satisfy

$$\frac{\partial T}{\partial \theta} = - TL(w) \quad . \tag{3.3}$$

We also note that T^{-1} must satisfy

$$\frac{\partial T^{-1}}{\partial \theta} = L(w) T^{-1} \quad , \tag{3.4}$$

which is analogous to (2.24). Finally, any function f satisfying

$$\frac{\partial f}{\partial \theta} + \{f,w\} = g \qquad (3.5)$$

can be solved using

$$f(\theta) = T^{-1}(\theta)\int_{\theta_0}^{\theta} d\theta' T(\theta')g(\theta') + T^{-1}(\theta)T(\theta_0)f(\theta_0) \qquad (3.6)$$

which is analogous to (2.30).

3.2 The New Hamiltonian.

The reason we introduce transformation theory is that we hope to be able to transform to a new system where the Hamiltonian has a simpler form. Upon solving the equations in the simpler system, we can transform back to obtain the solutions in the original system.

The new Hamiltonian K must have the property of giving the equations of motion for the transformed variables $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$:

$$\dot{Z}_i = \left\{Z_i, K\left(\underset{\sim}{Z}(\underset{\sim}{z},t,\theta),t,\theta\right)\right\} \quad . \qquad (3.7)$$

In this equation, overdot refers to the time derivative along a trajectory. Using the old Hamiltonian, we can also find Z_i:

$$\dot{Z}_i = \frac{\partial Z_i}{\partial t} + \left\{Z_i, h(\underset{\sim}{z},t)\right\} \quad . \qquad (3.8)$$

Introducing the function $H(\underset{\sim}{z},t,\theta) = h\left(\underset{\sim}{Z}^{-1}(\underset{\sim}{z},t,\theta),t\right)$, we can write (3.8) in the form

$$\dot{Z}_i = \left\{ Z_i, H\left(\underset{\sim}{Z}(\underset{\sim}{z},t,\theta),t,\theta\right)\right\} + \frac{\partial Z_i}{\partial t} . \qquad (3.9)$$

Hence we need only find a function R such that

$$\frac{\partial Z_i}{\partial t} = \left\{ Z_i, R\left(\underset{\sim}{Z}(\underset{\sim}{z},t,\theta),t,\theta\right)\right\} . \qquad (3.10)$$

Then the new Hamiltonian K is given by $K = H + R$.

That such an R exists is evident from the following considerations. $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$ is a canonical transformation for all t and θ. Since it is canonical for all θ, we were able to conclude that w exists satisfying (3.1). Likewise, since $\underset{\sim}{Z}(\underset{\sim}{z},t,\theta)$ is canonical for all t, we conclude that an R exists satisfying (3.10). Knowing that R exists, we can be more secure about finding it.

To find R, we consider equations (3.1) and (3.10) written in operator form.

$$\frac{\partial T}{\partial \theta} = - TL(w) \qquad (3.11)$$

$$\frac{\partial T}{\partial t} = - TL(R) \qquad (3.12)$$

Now we equate the second partials of T calculated from the two above equations.

$$\frac{\partial T}{\partial t} L(w) + T L\left(\frac{\partial w}{\partial t}\right) = \frac{\partial T}{\partial \theta} L(R) + T L\left(\frac{\partial R}{\partial \theta}\right) \qquad (3.13)$$

By using (3.11) and (3.12) again we get

$$L\left(\frac{\partial R}{\partial \theta}\right) + L\left(\{R,w\}\right) = L\left(\frac{\partial w}{\partial t}\right) \quad . \tag{3.14}$$

From equation (3.14) we deduce

$$\frac{\partial R}{\partial \theta} + \{R,w\} = \frac{\partial w}{\partial t} + g \quad , \tag{3.15}$$

where g is any function independent of $\underset{\sim}{z}$. However, we can set g equal to zero here, since R has relevance only through its Poisson bracket relations as in (3.10). The addition of a function independent of $\underset{\sim}{z}$ to R does not affect any Poisson bracket relations. Setting g = 0, we get

$$\frac{\partial R}{\partial \theta} + \{R,w\} = \frac{\partial w}{\partial t} \quad . \tag{3.16}$$

Following the discussion in the last section, this equation can be integrated using (3.6). We use $\theta_0 = 0$ since then T reduces to the indentity implying that $R(\theta_0 = 0)$ vanishes. This gives the result for R:

$$R(\theta) = T^{-1}(\theta) \int_0^\theta d\theta' T(\theta') \frac{\partial w}{\partial t}(\theta') \quad . \tag{3.17}$$

If we now use K = H + R, we get the new Hamiltonian.

$$K(\theta) = T^{-1}(\theta)h + T^{-1}(\theta) \int_0^\theta d\theta' T(\theta') \frac{\partial w}{\partial t}(\theta') \quad . \tag{3.18}$$

$$\frac{\partial Q}{\partial \theta} = - 2\theta t \; P \tag{3.22a}$$

and

$$\frac{\partial P}{\partial \theta} = 0 \tag{3.22b}$$

These equations have the solutions,

$$Q = q - \theta^2 t \; P \tag{3.23a}$$

$$P = p \tag{3.23b}$$

We note that the boundary conditions (3.2) have been satisfied.

The operator T corresponding to (3.23) acts on a function f as follows. If Tf = g, then

$$g(q,p,t,\theta) = f(q - \theta^2 tp, p, t, \theta) \quad . \tag{3.24}$$

We could also use the operator notation to deduce (3.24). The Lie operator in this case is given by

$$L_w = + 2\theta tp \frac{\partial}{\partial q} \quad . \tag{3.25}$$

The equation for T is

$$\frac{\partial T}{\partial \theta} = - T \; 2\theta tp \frac{\partial}{\partial q} \quad . \tag{3.26}$$

The solution for T is

$$T = \exp\left(- \theta^2 tp \frac{\partial}{\partial q} \right) \tag{3.27}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (- \theta^2 tp)^n \left(\frac{\partial}{\partial q}\right)^n \quad ,$$

and this is just the Taylor series expression for the shift operator in (3.24).

We now calculate the new Hamiltonian using (3.18) for the case of a free particle, $h = p^2/2m$. For this case, $T^{-1}h = h$ and $T\frac{\partial w}{\partial t} = \frac{\partial w}{\partial t}$. Thus, K is given by

$$K = \frac{1}{2m} p^2 - \frac{1}{2} \theta^2 p^2 \tag{3.28}$$

This Hamiltonian is that of a free particle with mass

$$M(\theta) = \frac{m}{1 - \theta^2 m} \quad . \tag{3.29}$$

We see that we can transform away the Hamiltonian entirely in this simple case, by choosing $\theta^2 = \frac{1}{m}$. In this new system q and p are constant. To find the motion in the original system, we first apply the inverse transformation

$$Q^{-1} = q + pt/m \tag{3.30a}$$

$$P^{-1} = p \tag{3.30b}$$

Then we substitute the trajectories in the transformed system $q_K(t) = q_0$ and $p_K(t) = p_0$ to get the trajectories in the original system

$$\mathcal{Q}_h(t) = \mathcal{Q}_K(t) + \frac{1}{m}t\,\mathcal{P}_K(t) = q_0 + \frac{1}{m}tp_0 \tag{3.31a}$$

$$\mathcal{P}_h(t) = \mathcal{P}_K(t) = p_0 \tag{3.31b}$$

3.4. Noether's Theorem

Noether's theorem [3, p. 83] in Lagrangian dynamics states that if one is given a family of transformations which leaves the form of the Lagrangian unchanged except for the addition of a total time derivative, then one can construct an invariant of the motion. Conversely, given an invariant of the motion, one can construct a family of transformations which leave the Lagrangian unchanged except for the addition of a total time derivative.

Since there is a direct correlation between Lagrangian dynamics and Hamiltonian dynamics, one expects a similar theorem to hold in Hamiltonian mechanics. This theorem has in fact been discussed by Whittaker [6], Anderson [22], and Cary [23]. The advantage of the Hamiltonian formulation is that the relevant transformations are the time dependent canonical transformations, the natural transformations which occur in Hamiltonian mechanics. In contrast, the Lagrangian theorem requires the use of path transformations, not just time dependent coordinate transformations.

Given a system which evolves according to a Hamiltonian h, we say that h has a symmetry if there exists a transformation family $T(\theta)$ which leaves the Hamiltonian unchanged up to the addition of a phase space independent function:

$$K = h + f \tag{3.32}$$

where

$$\frac{\partial f}{\partial z_i} = 0 \quad \text{for all } i. \tag{3.33}$$

The reason we allow the addition of f is that it will not affect the motion in the new system, since its Poisson bracket with any function vanishes.

We define an invariant to be any function g whose total time derivative along a trajectory vanishes:

$$\frac{\partial g}{\partial t} + \{g,h\} = 0 \tag{3.34}$$

We will prove that given a symmetry, one can construct an invariant, and given an invariant, one can construct a symmetry.

First we suppose that we are given a symmetry. Using (3.32) in (3.18), we have

$$h + f = T^{-1}(\theta)h + T^{-1}(\theta)\int_0^\theta d\theta' T(\theta') \frac{\partial w}{\partial t}(\theta'), \tag{3.35}$$

where w is constructed from T as discussed in sec. 3.1. We premultiply this equation by T, and note that Tf = f, since f does not depend on the phase space variables.

$$T(\theta)h + f = h + \int_0^\theta d\theta' T(\theta') \frac{\partial w}{\partial t}(\theta') \tag{3.36}$$

Differentiating (3.36) with respect to θ, and premultiplying by T^{-1} gives

$$\frac{\partial w}{\partial t} - \frac{\partial f}{\partial \theta} + \{w,h\} = 0. \tag{3.37}$$

Thus the function $g(\underset{\sim}{z},t,\theta) = w - \int^t d\tau\ \partial f/\partial\theta$ is an invariant for all θ.

To prove the other half of the theorem, we assume we have an invariant g. From this invariant we construct a transformation $T(\theta)$ using $w = g$.

The new Hamiltonian K is then given by

$$K(\theta) = T^{-1}(\theta)h + T^{-1}(\theta)\int_0^\theta d\theta' T(\theta') \frac{\partial g}{\partial t} . \tag{3.38}$$

Premultiplying T and differentiating with respect to θ gives

$$-T\, L(g)K + T \frac{\partial K}{\partial \theta} = T \frac{\partial g}{\partial t} , \tag{3.39}$$

which implies that

$$\frac{\partial K}{\partial \theta} + \{K,g\} = \frac{\partial g}{\partial t} . \tag{3.40}$$

We now use the fact that g is an invariant. This gives

$$\frac{\partial (K-h)}{\partial \theta} + \{K-h,g\} = 0 \tag{3.41}$$

since h is independent of θ. We define b by $b = T^{-1}(K-h)$. In terms of b, (3.41) reads

$$\frac{\partial b}{\partial \theta} = 0. \tag{3.42}$$

Since $b = 0$ at $\theta = 0$, b vanishes for all θ, from which we deduce $K = h$, proving the theorem.

As stated here, the theorem is more powerful than it is in its usual form. Here we find that the generating function g is an invariant for all values of θ, whereas usually only g at $\theta = 0$ is shown to be an invariant.

We are used to thinking of the explicitly time independent invariants such as momentum and angular momentum. This theorem shows that symmetries are associated even with time dependent invariants such as the function

$$g = p \cos\omega_0 t + q \sin\omega_0 t \tag{3.43}$$

for the harmonic oscillator $h = \frac{1}{2}\omega_0 (q^2 + p^2)$.

4. Deprit Perturbation Theory

In this section we show how to construct a perturbation theory from the general transformation theory of the previous chapter. Historically the development was in the opposite order. Deprit constructed power series representations of the transformation, and later Dewar gave the general representation of nonanalytic transformations.

A prerequisite of perturbation theory is that the Hamiltonian consists of a solvable term plus unsolvable terms which are ordered in a small parameter. We equate this small parameter to the parameter θ of a Lie transform which is to be determined. We then expand the Lie transform in powers of the parameter θ, and use the expansions in equation (3.18). We end up with an equation for each order relating the new Hamiltonian to the transformation and the old Hamiltonian.

The next stage is to pick the transformation, thereby choosing the form of the new Hamiltonian. Ideally we would like to pick the transformation so that K vanishes in all the higher orders. However, this choice is not always best since it may lead to secular terms or small denominators in the transformation, making it useless for a discussion of long time effects. The scheme we adopt then is to transform away as many terms as possible. The slowly varying terms, which give rise to secularities and small denominators, must be kept in the new Hamiltonian. This scheme usually simplifies the analysis to some degree since at least the rapidly varying terms are transformed away. Sometimes this scheme in effect solves the problem when the slowly

varying terms depend only on the momenta.

The structure of this chapter is as follows. In the first section we derive the power series expressions for the Lie transform theory. Then we illustrate the method of choosing the transformations outlined in the previous paragraph by doing two examples. However, it must be noted that the described method of choosing is not the only one. A nice feature of the Deprit perturbation theory is that one can use any method to select the transformation. The essential criterion is to choose the transformation so that the new Hamiltonian is as easy to analyze as possible.

4.1. Deprit Perturbation Series Relations

This perturbation theory relies on the power series expansions of the Lie transforms. In this section we assume that the Hamiltonian h is ordered in a parameter which we equate to the Lie transform parameter. We then insert the power series expansions into (3.18) to obtain the expression for the n^{th} order term in the new Hamiltonian.

The starting point is the assumption that the objects h,w,K, and T can all be expanded in power series. For h,K, and T we have

$$h(\underset{\sim}{z},t,\theta) = \sum_{n=0}^{\infty} \theta^n h_n(\underset{\sim}{z},t), \tag{4.1}$$

$$K(\underset{\sim}{z},t,\theta) = \sum_{n=0}^{\infty} \theta^n K_n(\underset{\sim}{z},t), \tag{4.2}$$

and $$T(t,\theta) = \sum_{n=0}^{\infty} \theta^n T_n(t). \tag{4.3}$$

However, for w we assume a slightly different form

$$w(\underset{\sim}{z},t,\theta) = \sum_{n=0}^{\infty} \theta^n w_{n+1}(\underset{\sim}{z},t). \tag{4.4}$$

The reason for this choice is that w occurs in (3.18) along with an integral over $d\theta$ which effectively boosts the order to w by 1 in all equations.

We also assume that the transformation is close to the identity. This is necessary for perturbation theory to work at all. This implies that $T_0 = I$, putting $\theta = 0$. It also implies $K_0 = h_0$ since in (3.18) the integral is at least of order θ.

We first find the form for the operator T to all orders. We start with the relation

$$\frac{\partial T}{\partial \theta} = -T\, L_w. \tag{4.5}$$

To use this relation we will need to know the series for L(w). Since the map $g \to L(g)$ is linear, we have

$$L(w) = \sum_{n=0}^{\infty} \theta^n L(w_{n+1}) \tag{4.6}$$

We abbreviate the operator $L(w_n)$ by L_n. Now we simply take the series representations for L(w) and T and insert them into (4.5). As a result we calculate the following recursion relation:

$$T_n = \frac{-1}{n} \sum_{m=0}^{n-1} T_m L_{n-m} . \tag{4.7}$$

By iterating this relation we arrive at

$$T_n = \sum_{(m_1,\ldots,m_r)} \left(\frac{-1}{n}\right)\left(\frac{-1}{m_1}\right)\cdots\left(\frac{-1}{m_r}\right) L_{m_r} \ldots L_{m_2-m_3} L_{m_1-m_2} L_{n-m_1} . \tag{4.8}$$

$$n>m_1>m_2 \ldots>m_r>0$$

The sum is over all sets of integers $m_1,\ldots,m_r$ satisfying $n>m_1>m_2\ldots>m_r>0$.

From the expression $\frac{\partial T^{-1}}{\partial \theta} = L_w T^{-1}$, we can derive similar relations for T^{-1}. The results are

$$T_n^{-1} = \frac{1}{n} \sum_{m=0}^{n-1} L_{n-m} T_m^{-1} \tag{4.9}$$

which gives

$$T_n^{-1} = \sum_{(m_1,\ldots,m_r)} \left(\frac{1}{n}\right)\left(\frac{1}{m_1}\right)\cdots\left(\frac{1}{m_r}\right) L_{n-m_1} L_{m_1-m_2}\cdots L_{m_r} . \tag{4.10}$$

$$n>m_1>m_2\ldots>m_r>0$$

To find the expression for K, we premultiply (3.18) by T and differentiate with respect to θ to obtain

$$\frac{\partial T}{\partial \theta} K + T \frac{\partial K}{\partial \theta} = \frac{\partial h}{\partial \theta} + T \frac{\partial w}{\partial t} . \tag{4.11}$$

Using (3.11), and premultiplying by T^{-1}, we find

$$\frac{\partial w}{\partial t} = \frac{\partial K}{\partial \theta} - L(w) K - T^{-1} \frac{\partial h}{\partial \theta} . \tag{4.12}$$

Inserting the series expressions for K, T, and h, we obtain (in n^{th}

order)

$$\frac{\partial w_n}{\partial t} = n K_n - \sum_{m=0}^{n-1} L_{n-m} K_m - \sum_{m=1}^{n} T^{-1}_{n-m} m h_m . \qquad (4.13)$$

By writing out the first term of the first sum, and the last term of the sum we get the final answer (note $K_0 = h_0$):

$$\frac{\partial w_n}{\partial t} + \left\{ w_n, h_0 \right\} = n K_n - n h_n - \sum_{m=1}^{n-1} (L_{n-m} K_m + m T^{-1}_{n-m} h_m) . \qquad (4.14)$$

This is the form with which we work when doing n^{th} order perturbation theory. After calculating through order (n-1), we know all of the quantities in the sum in the above equation. We also know h_n. We then pick K_n in a manner described in the next section. Finally we find w_n by integrating the right side of (4.14) along a trajectory. At this point all of the quantities are known through order n, hence we know the motion through n^{th} order, and we can begin the order (n+1) calculation.

For future reference, we give the formulas for the various quantities through fourth order.

$$T_0 = I \qquad (4.15a)$$

$$T_1 = -L_1 \qquad (4.15b)$$

$$T_2 = -\frac{1}{2} L_2 + \frac{1}{2} L_1^2 \qquad (4.15c)$$

$$T_3 = -\frac{1}{3} L_3 + \frac{1}{6} L_2 L_1 + \frac{1}{3} L_1 L_2 - \frac{1}{6} L_1^3 \qquad (4.15d)$$

$$T_4 = -\frac{1}{4} L_4 + \frac{1}{12} L_3 L_1 + \frac{1}{8} L_2^2 + \frac{1}{4} L_1 L_3 - \frac{1}{24} L_2 L_1^2 - \frac{1}{12} L_1 L_2 L_1 - \frac{1}{8} L_1^2 L_2 + \frac{1}{24} L_1^4 \qquad (4.15e)$$

$$T_0^{-1} = I \qquad (4.16a)$$

$$T_1^{-1} = L_1 \qquad (4.16b)$$

$$T_2^{-1} = \frac{1}{2} L_2 + \frac{1}{2} L_1^2 \tag{4.16c}$$

$$T_3^{-1} = \frac{1}{3} L_3 + \frac{1}{6} L_1 L_2 + \frac{1}{3} L_2 L_1 + \frac{1}{6} L_1^3 \tag{4.16d}$$

$$T_4^{-1} = \frac{1}{4} L_4 + \frac{1}{12} L_1 L_3 + \frac{1}{8} L_2^2 + \frac{1}{4} L_3 L_1 + \frac{1}{24} L_1^2 L_2 + \frac{1}{12} L_1 L_2 L_1 + \frac{1}{8} L_2 L_1^2 + \frac{1}{24} L_1^4 \tag{4.16e}$$

We note that $T_n^{-1} = T_n^{+}$, where the dagger indicates hermitian conjugate. To find the hermitian conjugate of a product of operators, we reverse the order of the product, and then take the hermitian conjugate of each of the multipliers;

$$(AB)^{+} = B^{+}A^{+} . \tag{4.17}$$

Since $L_n^{+} = -L_n$ (L_n is antihermitian), we see that $T_n^{-1} = T_n^{+}$ is true in (4.16). This fact is proven in general by noting that the differential equations for T and T^{-1} are hermitian conjugates.

The equations for K_n to fourth order are:

$$K_0 = h_0 \tag{4.17a}$$

$$\frac{\partial w_1}{\partial t} + \{w_1, h_0\} = K_1 - h_1 \tag{4.17b}$$

$$\frac{\partial w_2}{\partial t} + \{w_2, h_0\} = 2(K_2 - h_2) - L_1(K_1 + h_1) \tag{4.17c}$$

$$\frac{\partial w_3}{\partial t} + \{w_3, h_0\} = 3(K_3 - h_3) - L_1(K_2 + 2h_2) - L_2(K_1 + \tfrac{1}{2}h_1) - \tfrac{1}{2}L_1^2 h_1 \tag{4.17d}$$

$$\frac{\partial w_4}{\partial t} + \{w_4, h_0\} = 4(K_4 - h_4) - L_1(K_3 + 3h_3) - L_2(K_2 + h_2) - L_1^2 h_2 - L_3(K_1 + \tfrac{1}{3}h_1) - \tfrac{1}{6}(L_1 L_2 + 2L_2 L_1 + L_1^3)h_1 . \tag{4.17e}$$

4.2. The Anharmonic Oscillator

In this section we apply the Deprit perturbation technique to a sample problem as an illustration. The problem we consider is that of a particle moving in a potential with cubic and fifth power force nonlinearities. The Hamiltonian in this case is given by

$$h = \frac{1}{2} p^2 + \frac{1}{2} \omega_0^2 q^2 + \frac{\varepsilon}{4}\omega_0^2 q^4 + \frac{\varepsilon^2}{8} a\omega_0^3 q^6 \tag{4.18}$$

We solve for the motion near the bottom of the well where the nonlinearity is small.

The unperturbed Hamiltonian is a simple harmonic oscillator. For convenience we use the (unperturbed) action angle variables defined by Eqs. (2.69-2.70). In terms of these variables the Hamiltonian takes the form

$$h = \omega_0 j + \varepsilon j^2 \cos^4\phi + \varepsilon^2 a j^3 \cos^6\phi \quad . \tag{4.19}$$

The ordering scheme for this Hamiltonian h is of course

$$\begin{aligned} h_0 &= \omega_0 j \\ h_1 &= j^2 \cos^4\phi \\ h_2 &= a j^3 \cos^6\phi \\ h_n &= 0 \text{ for } n > 2 \end{aligned} \tag{4.20}$$

We equate the Lie transform parameter θ to the small parameter ε .

The first step in perturbation theory is to solve the unperturbed problem; i.e. we must find the time development operator. In this case the time development mapping is

$$\varphi(\phi,j,t) = \phi + \omega_0 t \tag{4.21a}$$

$$\dot{j}(\phi,j,t) = j \ . \tag{4.21b}$$

The time development mapping M is therefore given by

$$(S_0 g)(\phi,j,t) = g(\phi + \omega_0 t, j, t) \ . \tag{4.22}$$

To do first order perturbation theory we consider the first order Deprit equation (4.16b). In this case we have

$$\frac{\partial w_1}{\partial t} + \{w_1, h_0\} = K_1(\phi,j,t) - j^2\cos^4\phi \ ,$$

$$= K_1 - j^2\left(\frac{3}{8} + \frac{1}{2}\cos 2\phi + \frac{1}{8}\cos 4\phi\right) \ . \tag{4.23}$$

We can solve for w_1 using (2.30). Since we are looking for any solution to (4.23), we use the indefinite integral:

$$w_1 = -\int^t d\tau \left[j^2\left(\frac{3}{8} + \frac{1}{2}\cos[2\phi + \omega_0(\tau - t)] + \frac{1}{8}\cos[4\phi + 4\omega_0(\tau - t)]\right) - K_1\left(j, \phi + \omega_0(\tau - t), \tau\right)\right] \tag{4.24}$$

We would like to pick $K_1 = 0$, but since h_1 has a nonzero time average, this would cause w_1 to be secular (unbounded in time). Therefore we pick K_1 to be the time average of h_1 along a particle orbit:

$$K_1(j) = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} d\tau h_1(j, \phi + \omega_0\tau)$$

$$= \frac{3}{8} j^2 \ . \tag{4.25}$$

Inserting this result into (4.24) gives

$$w_1 = -j^2\left(\frac{1}{32}\sin 4\phi + \frac{1}{4}\sin 2\phi\right)/\omega_0 \ . \tag{4.26}$$

We have completed our calculation to first order since we have w_1 and K_1. Introducing the transformed variables Φ and J we know that they evolve in time according to

$$J = J_0 \tag{4.27}$$

$$\Phi = \Phi_0 + (\omega_0 + \frac{3}{4}\varepsilon J_0)t \quad , \tag{4.28}$$

since to first order the transformed Hamiltonian is

$$K(J) = \omega_0 J + \frac{3}{8}\varepsilon J^2 \quad . \tag{4.29}$$

Also, by applying the inverse transform (4.16) we know that the old and new variables are related by $z_i = Z_i + \{w_1, Z_i\}$ giving

$$\phi = \Phi + \varepsilon J\left(\frac{1}{16}\sin 4\Phi + \frac{1}{2}\sin 2\Phi\right)/\omega_0 \tag{4.30}$$

$$j = J - \varepsilon J^2\left(\frac{1}{8}\cos 4\Phi + \frac{1}{2}\cos 2\Phi\right)/\omega_0 \quad . \tag{4.31}$$

Inserting the time development of J and Φ (4.27) and (4.28) into (4.30) and (4.31) gives the time development of j and ϕ .

Going to second order, we insert our results for K_1 and w_1 into (4.17c). The result is

$$\frac{\partial w_2}{\partial t} + \{w_2, h_0\} = 2\left[K_2 - \frac{aj^3}{32}\left(10 + 15\cos 2\phi + 6\cos 4\phi + \cos 6\phi\right) + \frac{j^3}{64\omega_0}\left(17 + 33\cos 2\phi + 6\cos 4\phi - 2\cos 6\phi\right)\right] \quad . \tag{4.32}$$

Since the average along a particle orbit is essentially a phase average, we can pick off the terms in K_2 by inspection.

$$K_2(j) = \left(\frac{5a}{16} - \frac{17}{64\omega_0}\right)j^3 \tag{4.33}$$

We could now find w_2 from (4.31) and proceed to third order.

To understand the physical meaning of this transformation, we note that these transformations do not change the topology of the variables. Φ and ϕ are both defined modulo 2π. Hence we can calculate the action I of the oscillator

$$I = \frac{1}{2\pi} \oint p(q)dq = \frac{1}{2\pi} \int_0^{2\pi} J(\phi)\, d\phi \quad . \tag{4.34}$$

Since J is constant along an orbit, it follows that the action is J. The transformation T takes the action angle variables of the unperturbed oscillator to the action angle variables of the perturbed system.

The physical relationship which one usually desires for an oscillator is the frequency versus energy function. The numerical value of the Hamiltonian is unchanged in a time independent transformation, so we can get this relationship by eliminating J from the expressions for the new Hamiltonian and the frequency:

$$K = \omega_o J + \frac{3}{8}\varepsilon J^2 + \varepsilon^2\left(\frac{5a}{16} - \frac{17}{64\omega_o}\right)J^3 \tag{4.35}$$

$$\omega = \omega_o + \frac{3}{4}\varepsilon J + \varepsilon^2\left(\frac{15a}{16} - \frac{51}{64\omega_o}\right)J^2. \tag{4.36}$$

To second order in ε this result is

$$\omega = \omega_o + \frac{3}{4}\varepsilon\left(\frac{E}{\omega_o}\right) + \varepsilon^2\left(\frac{E}{\omega_o}\right)^2\left(\frac{15a}{16} - \frac{69}{64\omega_o}\right) \tag{4.37}$$

We have set $K = E$.

We now compare our results to a known solvable system, a pendulum. In this case the Hamiltonian is

$$h = \frac{1}{2} p^2 + \omega_o^2 (1 - \cos q) \tag{4.38}$$

$$= \frac{1}{2} p^2 + \frac{1}{2} \omega_o^2 q^2 - \frac{1}{4!} \omega_o^2 q^4 + \frac{1}{6!} \omega_o^2 q^6 , \tag{4.39}$$

Comparing with (4.18), we see that $\varepsilon = -\frac{1}{6}$ and $a = \frac{2}{5\omega_o}$. Perturbation theory thus gives

$$\omega = \omega_o \left(1 - \frac{1}{8} \frac{E}{\omega_o^2} - \frac{5}{256} \left(\frac{E}{\omega_o^2}\right)^2 \right) \tag{4.40}$$

The exact result (ref. 25, p. 112) is

$$\omega = \omega_o \left[2K\left(\sqrt{\frac{E}{2\omega_o^2}}\right)\right]^{-1} , \tag{4.41}$$

which yields (4.40) upon expansion (K is the complete elliptic integral).

A point to note in this example is that we generate the exact solution as a power series in ε. It could happen that there are singularities at $\varepsilon=0$, e.g. $\omega(\varepsilon)$ in (4.36) could contain a term like e^{-1/ε^2}. Perturbation theory cannot pick these terms up since analyticity in ε is assumed from the outset. But, if the exact solution is a power series, we should be able to obtain it term by term using perturbation theory.

4.3 The parametrically driven anharmonic oscillator

We now show what happens when we add a time dependent perturbation to the problem of the previous section. The particular perturbation we choose is a parametric time dependence, so that h has the form

$$h = \frac{1}{2} p^2 + \frac{1}{2} \omega_o^2 (1 + \alpha \sin\Omega t) q^2 + \frac{\varepsilon}{4} \omega_o^2 q^4 \tag{4.42}$$

Before embarking on this problem, we discuss the ordering

scheme. If we wish to be careful, in any calculation we must assign an order to each of the various terms in a Hamiltonian. Then we can approach the question of errors. However, in most work in physics only very rough error estimates are given. Secondly, a natural small parameter may not appear obvious. For example, in the problem of the last section we could have set $\varepsilon = 1$ and followed the same perturbation calculation. The Deprit equations would keep track of the orders, and we would have to keep in mind that our solution is good only when the term treated as a perturbation is small. In the previous calculation this would mean small energy. Hence in doing a calculation it is not necessary that h_2 have an ε^2 in front of it, instead we know that upon doing a calculation to first order, errors of the size of h_2 remain.

In the above Hamiltonian we have two small parameters, α and ε. We want to solve it in the case where the time dependent perturbation is smaller than the nonlinearity, so we choose

$$h_o = \frac{1}{2} p^2 + \frac{1}{2} \omega_o^2 q^2$$

$$h_1 = \frac{\varepsilon}{4} \omega_o^2 q^4$$

$$h_2 = \frac{1}{2} \alpha \omega_o^2 \sin\Omega t \; q^2$$

$$h_n = 0 \text{ for } n > 0. \qquad (4.43)$$

In this calculation we set $\theta = 1$, letting the Deprit perturbation equations keep track of the ordering.

We again switch to the more convenient variables. We have done the first order analysis in the previous section, and so we know h_1 and K_1. Here we start with h_2 (and the more convenient variables).

$$h_2 = \frac{\alpha\omega_o j}{2}\left[\sin\Omega t + \frac{1}{2}\sin(\Omega t + 2\phi) + \frac{1}{2}\sin(\Omega t - 2\phi)\right] \qquad (4.44)$$

This term modifies (4.32) to be

$$\frac{\partial w_2}{\partial t} + \{w_2, h_o\} = 2K_2 - \alpha\omega_o j\left[\sin\Omega t + \frac{1}{2}\sin(\Omega t + 2\phi) + \frac{1}{2}\sin(\Omega t - 2\phi)\right] + \frac{j^3}{\omega_o}\left(\frac{17}{32} + \frac{51}{16}\cos 2\phi + \frac{3}{4}\cos 4\phi - \frac{1}{8}\cos 6\phi\right). \qquad (4.45)$$

The analysis of the last four terms was done in the previous section. Since (4.45) is linear in w_2 and K_2, we can find w_2 and K_2 by analyzing the equation

$$\frac{\partial w_2}{\partial t} + \{w_2, h_o\} = 2K_2 - \alpha\omega_o j\left[\sin\Omega t + \frac{1}{2}\sin(\Omega t + 2\phi) + \frac{1}{2}\sin(\Omega t - 2\phi)\right]. \qquad (4.46)$$

Addition the results of this analysis to the w_2 and K_2 of the previous section will give the total w_2 and K_2.

In analyzing (4.46) we first set $K_2 = 0$ and integrate to find w_2.

This gives

$$w_2 = +\alpha\omega_o j\left(\frac{\cos\Omega t}{\Omega} + \frac{1}{2}\frac{\cos(\Omega t + 2\phi)}{(\Omega + 2\omega_o)} + \frac{1}{2}\frac{\cos(\Omega t - 2\phi)}{(\Omega - 2\omega_o)}\right). \qquad (4.47)$$

We see that w_2 is very large if either Ω, $\Omega + 2\omega_o$, or $\Omega - 2\omega_o$ is small. This is the familiar problem of small denominators. For $\Omega >> \omega_o$ we pick w_2 as in (4.47), but we must treat the other cases individually.

Here we treat the case where $\Omega \sim 2\omega_o$. Then we choose K_2 to cancel the only term which we can not transform away. Thus we have

$$W_2 = \alpha\omega_o j \left(\frac{\cos\Omega t}{\Omega} + \frac{1}{2} \frac{\cos(\Omega t + 2\phi)}{\Omega + 2\omega_o} \right) \tag{4.48}$$

$$K_2 = \frac{\alpha\omega_o j}{4} \sin(\Omega t - 2\phi) \quad . \tag{4.49}$$

Collecting all the terms in K including those parts of K_2 calculated in the previous section we have

$$K = \omega_o j + \frac{3}{2} \varepsilon j^2 - \frac{17}{64} \varepsilon^2 \frac{j^3}{\omega_o} + \frac{\alpha\omega_o j}{4} \sin(\Omega t - 2\phi). \tag{4.50}$$

In this case we have not been able to transform to a Hamiltonian independent of the angle, but we have simplified the analysis by eliminating two terms. In fact, this Hamiltonian can now be solved by transforming to the new variables

$$I = j \tag{4.51}$$

$$\theta = \frac{\Omega}{2} t + \phi \quad . \tag{4.52}$$

Using this transformation K becomes a time independent one degree of freedom Hamiltonian,

$$K = \omega_o I + \frac{3}{2} \varepsilon I^2 - \frac{17}{64} \varepsilon^2 \frac{I^3}{\omega_o} + \frac{\alpha\omega_o I}{4} \sin 2\theta, \tag{4.53}$$

which can now be solved by quadrature.

For large perturbations, the technique of treating perturbations individually does not work since several may "overlap" [1], i.e., several resonance denominators may be small in a region of phase space. In this case one must often resort to numerical integration of the equations. However, perturbation theory may still be of use by removing the nonrelevant resonances from the problem.

The inverse mapping is given by

$$M^{-1} = \ldots \exp[\varepsilon^3 L(g_3)] \exp[\varepsilon^2 L(g_2)] \exp[\varepsilon L(g_1)]. \tag{5.4}$$

If we expand this in a series,

$$M^{-1} = I + \sum_{n=1}^{\infty} \varepsilon^n M_n^{-1}, \tag{5.5}$$

we find

$$M_n^{-1} = \sum_{m_1+2m_2+\ldots+nm_n=n} \frac{[L(g_n)]^{m_n}[L(g_{n-1})]^{m_{n-1}}[L(g_1)]^{m_1}}{m_n! m_{n-1}! \ldots m_1!}. \tag{5.6}$$

The relation between the new Hamiltonian and the old Hamiltonian is easily obtained when the transformation is time-independent. We simply insert the transformation (5.1) into Eq. (3.19). We find

$$K_n = \sum_{m=0}^{n} M_{n-m}^{-1} h_m . \tag{5.7}$$

We see that the new Hamiltonian has

$$1 + \sum_{m=1}^{n} p(m)$$

terms of order n. In contrast, Eq. (4.14) gives this part of the Hamiltonian as a sum of $2^n + n-1$ terms, even if $\partial w_n/\partial t + \{w_n, h_o\}$ and the K_m's are counted as one term each. These two formulas are also compared in Table I.

As it stands, this perturbation method cannot be used to analyze time-dependent systems. However, this shortcoming is easily remedied by introducing the extended set of canonical variables $(\underset{\sim}{q},t,\underset{\sim}{p},e)$, whose evolution is given by the extended Hamiltonian,

$$h^x(\underset{\sim}{z},t,e) \equiv e+h(\underset{\sim}{z},t). \tag{5.8}$$

5. Perturbation Theory of Dragt and Finn

The perturbation theory of Dragt and Finn uses a different form for the canonical transformation perturbation series. This form does not easily fit into the general formulation of Dewar (sec. 3.2). However, it may be more useful in practical calculation, since there are fewer terms in the perturbation series.

The transformation introduced by Dragt and Finn has the form,

$$M = \exp[-\varepsilon L(g_1)] \exp[-\varepsilon^2 L(g_2)] \exp[-\varepsilon^3 L(g_3)] \ldots \quad . \tag{5.1}$$

Of course, we would like to write M as a series,

$$M = I + \sum_{n=1}^{\infty} \varepsilon^n M_n \quad . \tag{5.2}$$

By inspection of Eq. (5.1) we see that the terms in the series are given by the formula,

$$M_n = \sum_{m_1+2m_2+\ldots+nm_n=n} (-1)^{m_1+m_2+\ldots+m_n} \frac{[L(g_1)]^{m_1} [L(g_2)]^{m_2} \ldots [L(g_n)]^{m_n}}{m_1!\, m_2! \ldots m_n!} \quad , \tag{5.3}$$

where the sum is over all positive integers $(m_1,\ldots,m_n)$ satisfying $m_1+2m_2+\ldots+nm_n = n$.

Eq. (5.3) illustrates the advantage of using the Dragt-Finn transformation. The number of terms in the sum of Eq. (5.3) is $p(n)$, the partition[26] of n, which, for large n, asymptotically approaches the value $\exp[\pi(2n/3)^{1/2}]/4n\sqrt{3}$. In contrast the number of terms in the sum of Eq. (4.8) is 2^{n-1}. Values of those two quantities are given in Table I. One can see that even for fairly low values of n, Dragt-Finn perturbation theory will be easier to use.

dent case except for the term $M_n^{-1} h_o$. Here the rule is now to replace a term of the form

$$L(g_k)\ L(g_m)\ L(g_n)\ h_o\ , \tag{5.13}$$

by the form

$$L(g_k)\ L(g_m)\ \left[\frac{\partial g_n}{\partial t} + L(g_n) h_o\right]\ , \tag{5.14}$$

To obtain the time-dependent perturbation series relations. Terms not depending on h_0 remain unchanged.

For future reference we give the Dragt-Finn formulas up to fourth order.

$$M_0 = I \tag{5.15a}$$

$$M_1 = -L(g_1) \tag{5.15b}$$

$$M_2 = \frac{1}{2} L(g_1)^2 - L(g_2) \tag{5.15c}$$

$$M_3 = -\frac{1}{6} L(g_1)^3 + L(g_1)L(g_2) - L(g_3) \tag{5.15d}$$

$$M_4 = \frac{1}{24} L(g_1)^4 - \frac{1}{2} L(g_1)^2 L(g_2) + L(g_1)L(g_3) + \frac{1}{2} L(g_2)^2 - L(g_4) \tag{5.15e}$$

$$M_0^{-1} = I \tag{5.16a}$$

$$M_1^{-1} = L(g_1) \tag{5.16b}$$

$$M_2^{-1} = \frac{1}{2} L(g_1)^2 + L(g_2) \tag{5.16c}$$

$$M_3^{-1} = \frac{1}{6} L(g_1)^3 + L(g_2)L(g_1) + L(g_3) \tag{5.16d}$$

$$M_4^{-1} = \frac{1}{24} L(g_1)^4 + \frac{1}{2} L(g_2)L(g_1)^2 + L(g_3)L(g_1) + \frac{1}{2} L(g_2)^2 + L(g_4) \tag{5.16e}$$

$$0 = K_o - h_o \tag{5.17a}$$

$$\frac{\partial g_1}{\partial t} + \{g_1, h_o\} = K_1 - h_1 \tag{5.17b}$$

The time is now another coordinate and e is its conjugate momentum. The time evolves according to

$$\dot{t} = \frac{\partial h^x}{\partial e} = 1, \tag{5.9}$$

as it should. The variable e evolves according to

$$\dot{e} = \frac{\partial h^x}{\partial t} = - \frac{\partial h}{\partial t} . \tag{5.10}$$

Hence, e is the negative of the particle energy to within a constant. Our time-dependent Hamiltonian is now formally equivalent to a time-independent system, so we can apply the Dragt-Finn perturbation theory.

Since we are now working in an extended phase space, we could now allow our functions g_n to depend on both t and e. However, we do not want the transformation to change the evolution of t, Eq. (5.9), since it is already simple. Thus, we do not allow the functions g_n to depend on e. This implies

$$L^x(g_n)f = L(g_n)f + \frac{\partial g_n}{\partial t}\frac{\partial f}{\partial e} . \tag{5.11}$$

Only the lowest order part of the Hamiltonian is changed by the extension operation (5.8). Specifically, we have

$$h_o^x = e + h_o$$

$$h_n^x = h_n \text{ for } n \geqslant 1. \tag{5.12}$$

Combining these facts, we see that the terms in the sum (5.7) for the time-dependent case are identical to those for time indepen-

$$\frac{\partial g_2}{\partial t} + \{g_2, h_o\} = K_2 - h_2 - \frac{1}{2} L(g_1)[K_1 + h_1] \tag{5.17c}$$

$$\frac{\partial g_3}{\partial t} + \{g_3, h_o\} = K_3 - h_3 - L(g_1)h_2 - L(g_2)K_1 - \frac{1}{3}L(g_1)^2 h_1 - \frac{1}{6} L(g_1)^2 K_1 \tag{5.17d}$$

$$\begin{aligned}\frac{\partial g_4}{\partial t} + \{g_4, h_o\} = {} & K_4 - h_4 - L(g_1)h_3 - \frac{1}{2} L(g_1)^2 h_2 - \frac{1}{2} L(g_2)h_2 \\ & - \frac{1}{2} L(g_2)K_2 - \frac{1}{8} L(g_1)^3 h_1 - \frac{1}{24} L(g_1)^3 K_1 \\ & - \frac{1}{4} L(g_2)L(g_1)h_1 - \frac{1}{4} L(g_2)L(g_1)K_1 + L(g_3)K_1 .\end{aligned} \tag{5.17e}$$

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n	2^{n-1}	$p(n)$	$2^{n-1}+n$	$1+\sum_{m=1}^{n} p(m)$
1	1	1	2	2
2	2	2	4	4
3	4	3	7	7
4	8	5	12	12
5	16	7	21	19
10	512	42	522	139

TABLE I.

$$k_r(\omega') = k_r(\omega_{k\pm k_o}) + (\omega'-\omega_{k\pm k_o}) \frac{\partial k_r}{\partial \omega_{k\pm k_o}} + \ldots$$

and observe that the argument of the exponent becomes

$$(k-k_r(\omega')\pm k_o)\alpha^2/2 = (\omega'-\omega_{k\pm k_o}) \frac{\partial k}{\partial \omega_{k\pm k_o}} \alpha^2/2.$$

We integrate over ω' as before. Now, however, we obtain a spectrum balance equation which is nonlocal in k due to the periodicity in the scattering spectrum,

$$\overline{\langle \tilde{H}\tilde{H} \rangle}_k = \frac{(Rq)^6}{D_{\parallel}^3} D(4\pi)^{3/2} \frac{\Delta\eta}{\Delta_k^3} \left(\frac{\Delta\eta^2}{4} + \frac{1}{\Delta_k^2}\right)$$

$$\left\{ \frac{(k-k_o)^2\left(\omega_{*e}(k-k_o)-\omega_{k-k_o}\right)\frac{\partial k}{\partial \omega_{k-k_o}}}{\varepsilon_{Im}(k-k_o,\omega_{k-k_o}) \frac{\partial \varepsilon}{\partial (k-k_0)}} \varepsilon_{Im}^{Ion}(k-k_o,\omega_{k-k_o}) \overline{\langle \tilde{H}\tilde{H} \rangle}_{k-k_o} \right.$$

$$\left. + \frac{(k+k_o)^2\left(\omega_{*e}(k+k_o)-\omega_{k+k_o}\right)\frac{\partial k}{\partial \omega_{k+k_o}}}{\varepsilon_{Im}(k+k_o,\omega_{k+k_o}) \frac{\partial \varepsilon}{\partial (k+k_o)}} \varepsilon_{Im}^{Ion}(k+k_o,\omega_{k+k_o}) \overline{\langle \tilde{H}\tilde{H} \rangle}_{k+k_o} \right\} \qquad (77)$$

The spectrum is summed [Eq. (35)] and the periodicity is represented by a single moment of the spectrum k_o. The nonlocality is thus a shift in wave number and the spectrum balance equation is a difference equation in k.

2. Wavenumber Spectrum

By specifying the details of the ion saturation mechanism it is possible to determine the wavenumber spectrum. The saturation condition, Eq. (74) gives the appropriate relation between the damping and growth in the spectrum. For the damping mechanism we consider nonlinear ion-wave interaction or ion Compton scattering. This damping mechanism is an important and effective saturation mechanism for electron instabilities in drift waves and is appropriate for the small to moderate level of incoherent emission in the dissipation range. Recently, the nonlinear ion-wave interaction has been calculated for modes with toroidicity induced mode structure.[14] The nonlinear dielectric operator is nonlocal and provides an important channel to the ion resoance through the mode structure. In the long wavelength part of the spectrum the ions mediate a transfer to longer wavelength modes where energy in the modes reaches a broadened ion resonance. In addition, the scattering of short wavelength fluctuations to long wavelength fluctuations is strong for ion Compton scattering. The nonlinear transfer rate exceeds the electron excitation rate above wavenumbers on the order of a few tenths ρ_s^{-1}. Such a cutoff is a feature of experimental observations. For simplicity we will neglect the linear shear damping in the spectrum balance as it is small for the toroidicity induced mode structure.

From Similon and Diamond[14] we obtain the nonlinear damping from the ion-wave interaction

$$\frac{\gamma^{Ion}_{N.L.}}{\omega_{*e}} = \frac{(\omega-\omega_{Di})^2}{\omega_{ti}^2}\,\frac{1}{(\omega-\omega_{*i})}\,\frac{1}{\Gamma_o}\int dk'\ \sum_m (2\pi m k_\vartheta \hat{s})^2$$

$$\left[\frac{\omega'_{*i}-\omega'_{Di}}{\omega'-\omega'_{Di}} - \frac{\omega_{*i}-\omega_{Di}}{\omega-\omega_{Di}}\right]\ \rho\ \frac{<|v_{k_\vartheta'}(\vartheta-2\pi m)|^2>}{-i(\omega''-\omega''_{Di}+\Delta\omega'')} \qquad (78)$$

where $<|v_{k_\vartheta}|^2> = c^2k_\vartheta^2<|\varphi_{k_\vartheta}|^2>/B^2$ is the electrostatic field spectrum, $\omega_{Di} = 2\varepsilon_n\omega_{*i}[\cos\vartheta + \hat{s}(\vartheta-\vartheta_k)\sin\vartheta]$ is the magnetic drift, $\Gamma_o = \Gamma_o(\rho_i^2k_\perp^2(\vartheta))$, $\omega_{T_i} = V_{T_i}/Rq$, $\omega_{*i} = -\ \frac{T_i}{T_c}\,\omega_{*e}$, and $\omega''=\omega-\omega'$. $\Delta\omega$ incorporates the linear and nonlinear broadenings of the ion propagator. These include ion parallel motion (ω_{T_i}); the nonlinear decorelation from E×B turbulent motion,

$$d_k = \left(\frac{c}{B}\right)^2 \sum_{k_\vartheta',m'} (2\pi m')^2\ k_\vartheta^2 k_{\vartheta'}^2 \hat{s}^2 [J_o(v_\perp k'_\perp/\Omega_i)]^2$$

$$\frac{1}{-i(\omega''-\omega''_{Di})+d_{k''}}\ <\varphi\varphi>_{k_\vartheta'}(\vartheta-2\pi m')\ ; \qquad (79)$$

and the incoherent frequency broadening $\Delta\omega_k$. The nonlinear transfer rate, Eq. (78), comprises the nonlinear potential of the nonlocal eigen-mode problem. This potential is determined from the quasineutrality condition with linear and nonlinear dynamics incorporated in an ion gyrokinetic equation, and adiabatic electrons with a nonadiabatic part for the collision driven growth. This nonlinear eigenvalue problem is used to determine nonlinear contributions to mode structure which affect energy transfer in the spectrum at finite amplitude. The mode structure it predicts is consistent with the mode structure assumed in obtaining the linear growth and incoherent emission. The linear contribution of ions to the mode structure has been discussed in Section II. The nonlinear ion dynamics

comprising the nonlinear transfer rate are obtained from a coherent direct interaction approximation of the ion gyrokinetic equation. In this renormalization both the driven ion fluctuations $h_{k''}^{(2)}$ and the shielding plasma response $\varphi_{k''}^{(2)}$ are computed. The driven potential $\varphi_{k''}^{(2)}$ produces an induced scattering associated with $h_{k''}^{(2)}$. Similon and Diamond incorporate the induced scattering reduction into the formula for the ion Compton scattering trasfer rate by including the factor ρ which they give as $\rho \sim 0.1$.

Two interactions typify the nonlinear transfer. For a linear dispersion whose frequency is an increasing function of k_ϑ at low k_ϑ and then a decreasing function at higher k_ϑ it is possible to satisfy the condition for similar frequencies ($\omega_{*e} >> \omega - \omega'$) with values of k_ϑ and k_ϑ' which are widely separated. The beating of a high k_ϑ and low k_ϑ' gives a beat fluctuation k'' with large perpendicular wave number and hence large momentum transfer to the ions. In this part of the spectrum ($k_\vartheta > k_m$ where k_m is the wavenumber corresponding to maximum frequency: $d\omega/dk|_{k_m}=0$) the nonlinear transfer rate exceeds the growth rate. As a result the spectrum is only weakly populated for $k_\vartheta > k_m$. From Eq. (70) we calculate k_m to be $k_m\rho_s = (1+\hat{s}^2\pi^2/4)^{-1/2}$. In the long wavelength part of the spectrum the scattering interaction occurs between modes with similar k_ϑ, k_ϑ'. Energy cascades to lower k_ϑ where it is absorbed by shear damping aided by a braodened ion resonance. The spectrum at long wavelength is given by the saturation condition, Eq. (74) with the left hand side given by the ion Compton scattering transfer rate, Eq. (78):

$$\frac{(\omega-\omega_{Di})^2}{\omega_{T_i}^2}\,\frac{1}{\omega-\omega_{*i})}\,\frac{1}{\Gamma_o}\sum_{k_\vartheta',m}(2\pi k_\vartheta \hat{s} m)^2\,\left(\frac{\omega'_{*i}-\omega'_{Di}}{\omega'-\omega'_{Di}}-\frac{\omega_{*i}-\omega_{Di}}{\omega-\omega_{Di}}\right)$$

$$\rho\,\frac{<|V_{k_\vartheta'}(\vartheta-2\pi m)|^2>}{-i(\omega''-\omega''_{Di})+\Delta\omega''}=k_\vartheta^2\rho_s^2(1+\hat{s}^2\pi^2/4)\left(\frac{\nu_{ei}\omega_{*e}(Rq)^2}{v_{te}^2}\right)^{1/2}\left(1-C(k_\vartheta,\omega_{k_\vartheta})\right)^{-1}$$

$$\approx k_\vartheta^2\rho_s^2(1+\hat{s}^2\pi^2/4)\left(\frac{\nu_{ei}\omega_{*e}(Rq)^2}{v_{te}^2}\right)^{1/2}\left(1+C(k_\vartheta,\omega_{k_\vartheta})\right)\ . \qquad (80)$$

We have assumed here that the incoherent emission level is not too strong. This holds over most of the dissipation range, and we approximate $(1-C)^{-1}\approx 1+C$. The function $C(k,\omega_k)$ is proportional to D, the E×B diffusion, Eq. (34). The two-time correlation in D is written as the single particle propagator times the equal time correlation $<|\varphi(\eta+2\pi m)|^2)_{k'}=\mathrm{Re}L_{\substack{k'\\ \omega'}}<|\varphi(\eta+2\pi m)|^2>_{k'}$ with $L_{\substack{k\\ \omega}}=(-i\omega+D_\parallel\Delta_k^2/(Rq)^2)^{-1}$. The ω' integraton is performed yielding

$$D=\frac{1}{4}\,\frac{(Rq)^2}{D_\parallel\Delta_k^2}\int dk'\,\hat{s}^2\sum_m(2\pi m)^2<|v_{k_\vartheta'}(\eta+2\pi m)|^2>\ . \qquad (81)$$

Substituting this result into Eq. (69) and Eq. (69) into Eq. (80), we obtain the integral equation for the saturated spectrum,

$$k_\vartheta^2\rho_s^2(1+\hat{s}^2\pi^2/4)\left(\frac{\nu_{ei}\omega_{*e}(Rq)^2}{v_{te}^2}\right)^{1/2}=\frac{(\omega-\omega_{Di})^2}{\omega_{ti}^2}\,\frac{1}{(\omega-\omega_{*i})}\,\frac{1}{\Gamma_o}$$

$$\int\frac{dk'}{2\pi}\left\{\left[\frac{\omega'_{*i}-\omega'_{Di}}{\omega'-\omega'_{Di}}-\frac{\omega_{*i}-\omega_{Di}}{\omega-\omega_{Di}}\right]\frac{\rho}{(-i(\omega''-\omega''_{Di})+\Delta\omega'')}-\frac{\omega_{ti}^2(\omega-\omega_{*i})\Gamma_o}{(\omega-\omega_{Di})^2}\right.$$

$$\left.k_\vartheta^2\rho_s^2(1+\hat{s}^2\pi^2/4)\left(\frac{\nu_{ei}\omega_{*e}(Rq)^2}{v_{te}^2}\right)^{1/2}T_k\tau_{\|}\right)\sum_m(2\pi mk_\vartheta\hat{s}^2)<|v_{k_\vartheta'}(\eta+2\pi m)|^2> \qquad (82)$$

where $T_k=\frac{5}{16}\frac{(Rq)^6}{D_{\|}^3}(4\pi)^{3/2}\left(\frac{\pi}{2}\right)^6(\omega_*-\omega_k)\left(\frac{\partial k}{\partial\omega_k}\frac{\partial\varepsilon}{\partial k}\right)^{-1}$. With the extended mode structure the interaction is strongest for $\eta=\pm\pi$ where the k_ϑ and k_ϑ' modes have maximum overlap, and $m=\pm1$ contributions dominate in the sum over m. Since $k_\vartheta-k_\vartheta'$ is small, we expand the functions $<|v_{k_\vartheta'}(\pi)|^2>$ and $[(\omega'_{*i}-\omega'_{Di})/(\omega'-\omega'_{Di})-(\omega_{*i}-\omega_{Di})/(\omega-\omega_{Di})]$ about k_ϑ. The integration over k_ϑ' is then trivial and the spectral equation becomes an ordinary differential equation:

$$k_\vartheta^2\rho_s^2(1+\hat{s}^2\pi^2/4)\left(\frac{\nu_{ei}\omega_{*e}(Rq)^2}{v_{te}^2}\right)^{1/2}=\left\{\frac{(\omega-\omega_{Di})^2}{\omega_{t_i}^2}\frac{1}{(\omega-\omega_{*i})}\frac{1}{\Gamma_o}\right.$$

$$\frac{\frac{1}{3}\rho\Delta k_\vartheta^3}{\Delta\omega''}\frac{d}{dk_\vartheta}\left(\frac{\omega_{*i}-\omega_{Di}}{\omega-\omega_{Di}}\right)\frac{d}{dk_\vartheta}<|v_{k_\vartheta}(\pi)|^2>-$$

$$\left.\Delta k_\vartheta T_k\tau_{\|}\;k_\vartheta^2\rho_s^2(1+\hat{s}^2\pi^2/4)\left(\frac{\nu_{ei}\omega_{*e}(Rq)^2}{v_{te}^2}\right)^{1/2}<|v_{k_\vartheta}(\rho)|^2>\right\}(2\pi k_\vartheta\hat{s})^2. \qquad (83)$$

The function $\frac{1}{3}\Delta k_\vartheta^3$ comes from the integration of $(k_\vartheta'-k_\vartheta)^2$ and is the wavenumber range of the mode interaction. The function $\Delta\omega$ is the particle decorrelation. The particle decorrelation is determined by ion parallel motion ($\Delta\omega''=\omega_{ti}$) at the lowest turbulence levels. For higher turbulence

levels the turbulent decorrelation rate is dominant and $\Delta\omega''=d_{k''}$. The wavenumber range Δk_ϑ is given by the wavenumber corresponding to the distance travelled by a wave in the correlation time, $\Delta k_\vartheta = \Delta\omega/v_*$. This is limited, however by the spectrum width so that Δk_ϑ is the smaller of $\Delta\omega/v_*$ and k_m. For the low Reynold's number regime with moderate linear excitation but relatively weak incoherent emission the turbulent decorrelation d_k is sufficiently strong to dominate the particle decorrelaton but still give a wavenumber range within the spectrum width. Thus $\Delta\omega'' = d_{k''}$ and $\Delta k_\vartheta = d_{k'}/v_*$ where

$$d_{k''} \approx \frac{1}{2\pi} k_\vartheta^2 (2\pi)^2 \hat{s}^3(0.2) \frac{1}{v_*} <|v_{k_\vartheta}(0)|^2> . \tag{84}$$

To lowest order in the Reynold's number, solution of Eq. (83) yields the following spectrum,

$$\frac{<|v_{k_\vartheta}(0)|^2>}{v_*^2} \approx \frac{13}{\hat{s}^2(2\pi)^{4/3}} \left(\frac{1-\pi\varepsilon_n\hat{s}}{1+\hat{s}^2\pi^2/4}\right)^{1/3} \left(\frac{\rho_s}{k_m^5}\right)^{1/6} \left(\frac{\nu_{ei}c_s}{\omega_{te}^2 L_n}\right)^{1/6} \left[\frac{k_m^{5/2}}{k_\vartheta^{5/2}} - 1\right]^{1/3} \tag{85}$$

The next order correction, nominally first order in the Reynold's number,

$$\frac{<|v_{k_\vartheta}(0)|^2>^{(1)}}{v_*^2} \approx \frac{144}{3} \left(\frac{\pi}{2}\right)^7 (1-\pi\varepsilon_n\hat{s})\Delta\eta_c^2\rho_s^{9/2}k_m^{7/2}\left(\frac{k_\vartheta}{k_m}\right)^{5/2}$$

$$\left[1-\left(\frac{k_\vartheta}{k_m}\right)^{5/2}\right]^{-2/3}\left[1-\left(\frac{k_\vartheta}{k_m}\right)^{11/6}\right]\left(\frac{\nu_{ei}c_s}{\omega_{te}^2 L_n}\right)^4$$

is actually proportional to R_e^2 when the turbulent diffusion coefficient D is evaluated for the integrated intensity given below in Eq. (87):

$$\frac{<|v_{k_\vartheta}(0)|^2>^{(1)}}{v_*^2} \approx R_e^2\left(\frac{k_\vartheta}{k_m}\right)^{5/2}(k_m\rho_s)^{7/2}\left[1-\left(\frac{k_\vartheta}{k_m}\right)^{5/2}\right]^{-2/3}\left[1-\left(\frac{k_\vartheta}{k_m}\right)^{11/6}\right]$$

$$10^4\rho_s(1-\pi\varepsilon_n\hat{s})\Delta\eta_c^2\left(\frac{\nu_{ei}c_s}{\omega_{te}^2 L_n}\right)^{1/6} . \qquad (86)$$

This result indicates that in the dissipation range the wavenumber spectrum and integrated intensity are relatively insensitive to incoherent emission even when there is a broadened line width.

The lowest order spectrum, Eq. (85), diverges as $k_\vartheta^{-5/6}$ near the origin and has negligible energy above $k_m \cong \rho_s^{-1}(1+\hat{s}^2\pi^2/4)^{-1/2}$. The spectrum is illustrated in Fig. 3, where $<|\phi_{k_\vartheta}|^2>$ is plotted as a function of k_ϑ. Asymptotically, the spectrum goes as $k_\vartheta^{-17/6}$, for large k_ϑ, in good agreement with experiment. Though divergent at $k_\vartheta = 0$, the spectrum is integrable and the integrated intensity is

$$\frac{<v_E^2>}{v_*^2} \approx 0.5\,\frac{1}{\hat{s}^3}\left(\frac{\nu_{ei}c_s}{\omega_{te}^2 L_n}\right)^{1/6} \qquad (87)$$

where $<v_E^2> = \int\frac{dk_\vartheta'}{2\pi}<|v_{k_\vartheta'}(0)|^2>$. In terms of $e\varphi/T_e$ the saturation level is

$$\frac{e<|\varphi|>}{T_e} \approx 2.2\left(\frac{\rho_s}{L_n}\right)\frac{1}{\hat{s}^{3/2}}\left(\frac{\nu_{ei}c_s}{\omega_{te}^2 L_n}\right)^{1/12}$$

We note that the k_m cutoff of the k spectrum arising from the strong turbulent scattering above the maximum frequency places most of the energy in modes well below the cutoff of $k_\vartheta \rho_s \cong 0.5$.

The asymptotic scaling of the fluctuation spectrum for large ω has also been measured experimentally. For purposes of comparison we recall from Eq. (21) that $|\phi_k|^2 = \frac{T_e^2}{|e|^2} \left(L_k^{-1}\right)^2 \langle \tilde{H}^2 \rangle_k$. We approximate $(L_k^{-1})^2$ with $(\omega_*/\omega - 1)^2$, which goes as a constant asymptotically. The ω dependence of $\langle \tilde{H} \rangle_k^2$ is incorporated in the propagator which takes the one-time correlation into the two-time correlation (see Eq. (65)):

$$\langle \tilde{H}^2 \rangle_k = 2re\left\{-i\omega + \frac{v_{Te}^2}{\nu_{ei}(Rq)^2(\Delta\eta)^2}\right\}^{-1} \langle \tilde{H}^2 \rangle_k$$

Asymptotically, the propagator goes as $v_{Te}^2/(\omega^2 \nu_{ei}(Rq)^2(\Delta\eta)^2)$ implying that $|\varphi_k|^2 \sim 1/\omega^2$ as $\omega \to \infty$.

An important consequence of incoherent emission is the possible deviation from an adiabatic electron response in a regime where the condition for an adiabatic response, $\omega \nu_{ei} < v_{Te}^2 k_\parallel$ is satisfied. This occurs when a significant portion of the density fluctuation is incoherent. The electron density consists of the adiabatic part and the nonadiabatic contribution which in turn is comprized of coherent and incoherent contributions,

$$\hat{n}_E = \frac{|e|}{T_e} \hat{\varphi} + H_E = \frac{|e|}{T_e} \hat{\varphi} + H_E^{(c)} + \tilde{H}_E \ .$$

From quasi-neutrality the electron density may be expressed in terms of the ion operator $L^{Ion}(k,\omega)$,

$$\frac{\hat{n}_E}{n_o} = L^{Ion}(k,\omega)\ \frac{|e|}{T_e}\ \hat{\varphi}_{\substack{k\\ \omega}} .$$

The density fluctuation level $<\delta n^2>_{\substack{k\\ \omega}}$ is thus

$$<(\delta n)^2>_{\substack{k\\ \omega}} = |L^{Ion}(k,\omega)|^2\ \frac{|e|^2}{T_e^2}\ <\hat{\phi}^2_{\substack{k\\ \omega}}>$$

$$= \{|L^{Ion}_{re}(k,\omega)|^2+|L^{Ion}_{im}(k,\omega)|^2\}\ \frac{|e|^2}{T_e^2}\ <\hat{\varphi}^2_{\substack{k\\ \omega}}> . \qquad (88)$$

From the dispersion relation $|L^{Ion}_{re}|^2 \sim 1$ for the small values of k_ϑ typical in the spectrum. From the saturation condition, Eq. (74), we replace L^I_{Im} with $\varepsilon^{Elec}_{Im}/(1-C)$. Thus

$$<\left(\frac{\delta n}{n}\right)^2>_{\substack{k\\ \omega}} \simeq \{1+ \left[\frac{\varepsilon^{Elec}_{Im}(k,\omega_k)}{1-C(k,\omega_k)}\right]^2\}\ \frac{|e|^2}{T_e^2}\ <\hat{\varphi}^2>_{\substack{k\\ \omega}} \qquad (89)$$

and the departure from an adiabatic response is incorporated in the term $[\varepsilon^{Elec}_{Im}/(1-C)]^2$. We note that for C=0, corresponding to no incoherent emission, the departure from adiabaticity is just the term $(\varepsilon^{Elec}_{Im})^2$ which arises from the coherent part of the nonadiabatic density. This term is proportional to $(\gamma/\omega_*)^2$ which is small in the adiabatic regime. With

enhancement by incoherent emission, however, it is possible for $\varepsilon_{Im}^{Elec}/(1-C)$ to exceed one even with $\varepsilon_{Im}^{Elec}<1$.

3. Particle Diffusion

Having obtained the spectrum and fluctuation level it is straightforward to determine the anomalous particle transport. The anomalous particle flux Γ_N from E×B convection of the perturbed density is given in terms of the density-potential cross correlation by

$$\Gamma_n = \sum_{k_\vartheta'} (-i)\, k_{\vartheta'}\, \rho_s c_s \frac{|e|}{T_e} \langle \hat{\varphi}\hat{H} \rangle_{k_{\vartheta'}}$$

$$= \sum_{k_{\vartheta'}} (-i) k_{\vartheta'}\, \rho_s c_s \frac{|e|}{T_e} \left(\langle \hat{\varphi}\hat{H}^c \rangle_{k_{\vartheta'}} + \langle \hat{\varphi}\tilde{H} \rangle_{k_{\vartheta'}} \right) , \qquad (90)$$

where we have expressed the electron density $\hat{H}_{k_{\vartheta'}}$ in terms of its coherent and incoherent components $H^{(c)}_{k_{\vartheta'}}$ and $\tilde{H}_{k_{\vartheta'}}$ respectively. Using Eqs. (18) and (21) we express the coherent density $H^{(c)}$ in terms of its response function and the incoherent density in terms of the shielding potential,

$$\langle \hat{\varphi} H^{(c)} \rangle_{k_{\vartheta'}} + \langle \hat{\varphi}\tilde{H} \rangle_{k_{\vartheta'}} = \langle \hat{\varphi} R_{\substack{k' \\ \omega'}} \frac{|e|}{T_e} \hat{\varphi} \rangle_{k_{\vartheta'}} + \langle \hat{\varphi}\, L_{\substack{k' \\ \omega'}} \frac{|e|}{T_e} \hat{\varphi} \rangle_{k_{\vartheta'}} . \qquad (91)$$

The electron and ion response functions determine the operator, $[L_{\substack{k \\ \omega}}^{Ion} - 1 - R_{\substack{k \\ \omega}}] = L_{\substack{k \\ \omega}}$ allowing us to express the incoherent contribution (second term in Eq. (89)) in terms of $R_{\substack{k \\ \omega}}$ and $L_{\substack{k \\ \omega}}^{Ion}$. The adiabatic density is not convected by E×B drift and as with the source term, the nonadiabatic part of the density cancels out, leaving

$$\langle\hat{\varphi}H^{(c)}\rangle_{k_\vartheta'} + \langle\hat{\varphi}\tilde{H}\rangle_{k_\vartheta'} = \frac{e}{T_e}\langle\hat{\varphi}\ L^{Ion}_{\substack{k'\\ \omega'}}\ \hat{\varphi}\rangle_{k_\vartheta'} .$$

The ion response is the damping which balances the clump-enhanced electron dissipation as expressed in the saturation condition, Eq. (74). Thus $L^{Ion}_{\substack{k'\\ \omega'}} = L^{Elec}_{\substack{k'\\ \omega'}}/[1-C(k,\omega_k)]$ and Eq. (90) becomes

$$\Gamma_N = \sum_{k_\vartheta'} (-i)k_\vartheta'\rho_s c_s \frac{|e|}{T_e}\frac{1}{[1-C(k',\omega_{k'})]}\langle\hat{\varphi}\ L^{Elec}_{\substack{k'\\ \omega_{k'}}}\ \hat{\varphi}\rangle_{k_\vartheta'} . \qquad (92)$$

This formula indicates that the particle flux is esentially the quasilinear flux with an enhancement due to incoherent emission.

Proceeding, we obtain the electron response from Eq. (5),

$$L^{Elec}_{\substack{k\\ \omega}} = \left(-i\omega+\alpha\frac{\partial^2}{\partial\eta^2}\right)^{-1}\frac{(-i)|e|}{T_e}\omega_{*e}\left(\frac{\omega}{\omega_{*e}}-1\right)\hat{\varphi}_{k_\vartheta}$$

where $\alpha=\omega_{T_e}^2/\nu_{ei}$. Substituting into Eq. (92) yields

$$\Gamma_N = \sum_{k_\vartheta'} (k_\vartheta'\rho_s)\ c_s\left(1-\frac{\omega_{*e}}{\omega}\right)\frac{|e|^2}{T_e^2}\frac{1}{[1-C(k',\omega_{k'})]}\langle\hat{\varphi}\left(-i+\frac{\alpha}{\omega}\frac{\partial^2}{\partial\eta^2}\right)^{-1}\hat{\varphi}\rangle_{k_\vartheta'} \quad (93)$$

The average denoted by $\langle\ \rangle$ is accomplished with an integration over η. The integral $\int d\eta\ \varphi(-1+\frac{\alpha}{\omega}\partial^2/\partial\eta^2)^{-1}\varphi$ has been worked out in Sec. II (see Eqs. (9) and (10)). Thus, in the adiabatic limit,

$$\Gamma_N = -\sum_{k_\vartheta'} |k_\vartheta'\rho_s| c_s \left(1-\frac{\omega_{*e}'}{\omega}\right)\pi\left(\frac{\omega_{k'}\nu_{ei}}{\omega_{T_e}^2}\right)^{1/2} \frac{|e^2|}{T_e^2} \frac{1}{[1-C(k',\omega_{k'})]} |\hat{\varphi}_{k_\vartheta'}|^2$$

$$= \sum_{k_\vartheta'} \frac{v_*^2}{c_s} \left\langle \frac{v_{k'}^2}{v_*^2} \right\rangle \pi\left(\frac{\nu_{ei} c_s}{\omega_{T_e}^2 L_n}\right)^{1/2} \frac{1}{(k_\vartheta\rho_s)^{1/2}} \left[\left(1-\frac{\omega'}{\omega_{*e}'}\right)\left(\frac{\omega_{*e}'}{\omega}\right)^{1/2}\right] \frac{1}{[1-C(k',\omega_{k'})]} \tag{94}$$

From Eq. (87) we substitute for the integrated intensity

$$\langle v_e^2/v_*^2\rangle \approx \sum_{k_\vartheta} (v_k^2/v_*^2) = 0.5 \frac{1}{\hat{s}^3}\left(\frac{\nu_e c_s}{\omega_{T_e}^2 L_n}\right)^{1/6} ,$$

obtaining the final form for the particle flux

$$\Gamma_N = \frac{v_*^2}{c_s} \left(\frac{\nu_{ei} c_s}{\omega_{T_e}^2 L_n}\right)^{2/3} \frac{\pi}{2} \frac{1}{\hat{s}^3} \overline{\left\{\left(\frac{1}{k_\vartheta\rho_s} \frac{\omega_{*e}}{\omega_k}\right)^{1/2}\left(1-\frac{\omega_k}{\omega_*}\right) \frac{1}{[1-C(k,\omega_k)]}\right\}} \tag{95}$$

where the bar over the factor in the curved brackets indicates evaluation at a mean wavenumber in the spectrum. The particle diffusion is given by $D=\Gamma_N L_n$. For pretext tokamak parameters[3] with L_n=0.5cm and T_e=25eV the quasilinear part of the diffusion is 7.5×10^3 cm^2/sec. With incoherent emission sufficient to produce a linewidth of $\Delta\omega_k/\omega_{*e}=1$, the particle diffusion is $2.4\times10^4 cm^2/sec.$, which is slightly larger than $D_{Bohm}=T_e/(16eB)\approx1.8\times10^4$. For T_e=25eV, B=8kG. The scaling of the particle diffusion is not Bohm-like, but goes as

$$D \sim T_e^{1/6}\, n^{2/3} \quad . \tag{96}$$

The weak temperature dependence is a natural consequence of the near off set of the intrinsic drift wave particle diffusion scaling with respect to temperature, which goes as ρ_s, and the scaling intrinsic to collisionally driven systems which tends to go as T_e to a negative power. This scaling is in rough accord with recent measurements, which find that the diffusion decreases with increasing minor radius.[3]

B. High Reynold's Number Regime

We now consider the turbulent spectrum in the limit opposite of that treated in the first half of this section. In the inertial range or high Reynold's number regime the E×B turbulent diffusion dominates the parallel collisional viscosity. We also assume that it dominates other linear processes, in particular, the magnetic drifts, ω_{De}. Thus, the high Reynold's number regime is intrinsically a strong turbulence regime and turbulent diffusion determines not only the two-point diffusion, but fixes time scales as well. It is not possible to describe the turbulence with the same detail of weak turbulence theory as many assumptions no longer hold. From mixing length arguments, we can, however, infer features of the spectrum, such as linewidth and wavenumber dependence, estimate fluctuation levels and determine transport scalings. From this emerges a picture which is consistent with results obtained in the dissipation range.

1. Spectrum Properties

As noted in the previous section, two-point diffusion is predominantly the toroidal E×B diffusion. The standard approximate solution of the two-point equation applies; $\langle\tilde{H}\tilde{H}\rangle = (\tau_{c\ell}-\tau_c)S$ with $\tau_{c\ell}$ obtained from the inversion of the time evolution of the toroidal separation of neighboring points. From Eq. (53),

$$\tau_{c\ell} = \frac{\tau_{E\times B}}{2}\ \ell n\{1/[k_o^2 y_-^2 + \frac{\eta_-^2}{\Delta\eta^2} + R_e^{-1}]\}, \tag{97}$$

where the terms proportional to magnetic drift have been dropped. With the source term S given in Eq. (25), the incoherent two-point correlation is

$$\langle\tilde{H}(\eta_1,y_1)\tilde{H}(\eta_2,y_2)\rangle = \tau_{E\times B}\ \ell n\{1/[k_o^2 y_-^2 + \eta_-^2/\Delta\eta^2 + R_e^{-1})\}$$

$$\sum_{k',\omega'}\{\frac{(\omega'-\omega'_{*e})\varepsilon_{Im}^{Ion}(k',\omega')|\hat{\varphi}_{k'}(\eta)|^2}{|\varepsilon(k',\omega')|^2}\ \frac{N_{k'}}{(\omega'^2+N^2_{k'})}\ \overline{\langle\tilde{H}^2\rangle}_{k'}\} \tag{98}$$

As before, the two-time, two-point correlation has been expressed as the product of the single density element propagator and the two-point, one time correlation,

$$\langle H(1)H(2)\rangle_{\substack{k\\ \omega}} = 2reR(k,\omega)\langle\tilde{H}(1)\tilde{H}(2)\rangle_k. \tag{99}$$

The single density element propagator,

$$R(k,\omega) = \left[-i\omega - \frac{D_{\parallel}}{(Rq)^2}\frac{\partial^2}{\partial\eta^2} + N_{\substack{k\\ \omega}}\right]^{-1}$$

is dominated by E×B nonlinearity $N_{\substack{k\\ \omega}}$, (Eq. (6)), so that $2reR(k,\omega) \cong 2N_{\substack{k\\ \omega}}/(\omega^2+N^2_{\substack{k\\ \omega}})$. We renormalize the E×B nonlinearity and from the results of Sec. III, we find that $N_{\substack{k'\\ \omega'}} \approx Dk'^2$. We also recall that $\tau_{E\times B}=1/k_o^2D$ where $k_o^2 \equiv \langle k^2\hat{\varphi}^2(k)\rangle/\langle\hat{\varphi}^2(k)\rangle$. The product $\tau_{E\times B}N_{\substack{k'\\ \omega'}}$ is thus approximated as $\tau_{E\times B}N_{\substack{k'\\ \omega'}} \approx k'^2/k_o^2$ and we note that the short wavelength is favored in the turbulent scattering process. Again, we operate on both sides of Eq. (98) in order to express $\langle\tilde{H}(\eta_1 y_1)\tilde{H}(\eta_2,y_2)\rangle$ as $\overline{\langle\tilde{H}^2\rangle}_k$ and obtain a balance condition. The operations include the Fourier transform in y_- and the eigenfunction projections in η_+ and η_-. These integrals have been worked out in previous calculations[4,10,15] and the details will not be repeated here. The η_+ and η_- integrations yield

$$\overline{\langle\tilde{H}^2\rangle}_k = \int dy_- \, e^{-iky_-[(1-\ell^2)^{1/2}-|\ell|\cos^{-1}|\ell|]} \sum_{k'}\sum_{\omega'} \frac{(\omega'-\omega'_{*e})\varepsilon^{Ion}_{Im}(k',\omega')k'^2\overline{\langle\tilde{H}^2\rangle}_{k'}}{|\varepsilon(k',\omega')|^2(\omega'^2+N^2_{\substack{k'\\ \omega'}})k_o^2} \qquad (100)$$

where $\ell^2 = k^2y_-^2+R_e^{-2}$. While the y_- integration may be peformed approximately for R_e finite, it is easier to let $R_e\to\infty$ which is the limit we are considering in any case. In this limit the y_- integration has also been given previously. Using these results we obtain the spectrum balance equation

$$\overline{\langle\tilde{H}(1)\tilde{H}(2)\rangle_k} = \frac{2\pi}{k_o}\left(\frac{k_o^2}{k^2}\right)\left[1-J_o(2k/k_o)\right]\sum_{k',\omega'}\frac{(\omega'-\omega'_{*e})\varepsilon_{Im}^{Ion}(k',\omega')}{|\varepsilon(k',\omega')|^2}\frac{\overline{\langle\tilde{H}(1)\tilde{H}(2)\rangle_{k'}}}{(\omega'^2+N^2_{\substack{k'\\ \omega'}})}\left(\frac{k'^2}{k_o^2}\right) \tag{101}$$

We note from Eq. (101) the absense of any linear dissipative time scale. The only relaxation process occurring is nonlinear. Consequently, the nonlinear relaxation rate and linewidth $\Delta\omega$ are related to the frequency ω which in turn is related to ω_{*e}. This is seen more readily when we eliminate the density correlation from Eq. (101). We multiply both sides of Eq. (101) by $(\omega-\omega_{*e})\varepsilon_{Im}^{Ion}(k,\omega)(k^2/k_o^2)/[|\varepsilon(k,\omega)|^2(\omega^2+N^2_{\substack{k\\ \omega}})]$ and sum over k. The product of this factor and the correlation cancels out, leaving

$$1 = \sum_{\substack{k\\ \omega}}\frac{2\pi}{k_o}\left(\frac{k_o^2}{k^2}\right)\left(1-J_o(2k/k_o)\right)\frac{(\omega-\omega_{*e})\varepsilon_{Im}^{Ion}(k,\omega)}{|\varepsilon(k,\omega)|^2(\omega^2+N^2_{\substack{k\\ \omega}})}\frac{k^2}{k_o^2} \tag{102}$$

Passing to the continuous limit, and replacing ω and k with dimensionless variables $\alpha = \omega/\omega_{*e}$, $d\alpha = d\omega/\omega_{*e}$; $\beta = k\rho_s$, $d\beta = \rho_s dk$,

$$1 = \int d\beta \int d\alpha \frac{A(\beta,\beta_o)(1-\alpha)|\varepsilon_{Im}^{Ion}(\beta,\alpha,N_{\substack{\alpha\\ \beta}}/\omega_{*e},\ \beta_{\substack{\alpha\\ \beta}}/\omega_{*e})|\left(\frac{\beta^2}{\beta_o^2}\right)}{(\alpha^2+N^2_{\substack{\alpha\\ \beta}}/\omega^2_{*e})\left[|\varepsilon_r(\alpha,\beta)|^2 + |\varepsilon_{Im}(\alpha,\beta,N_{\substack{\alpha\\ \beta}}/\omega_{*e},\beta_{\substack{\alpha\\ \beta}}/\omega_{*e})|^2\right]} \tag{103}$$

where $A(\beta,\beta_o) = \frac{2\pi}{\beta_o}\left(\frac{\beta_o^2}{\beta^2}\right)\left(1-J_o(\frac{2\beta}{\beta_o})\right)$. The notation indicates that the imaginary part of the dielectrics ε_{Im}^{Ion} and ε_{Im} are dependent on the variables α and β and the quantities $N_{\substack{\alpha\\ \beta}}/\omega_{*e}$ and $\beta_{\substack{\alpha\\ \beta}}/N_{*e}$ which are the

normalized turbulent decorrelation and polarization respectively. Examples of these dielectrics are given by $\varepsilon^E_{Im} = reR(\beta,\alpha)(\alpha-1) = (\alpha-1)N^2_{\alpha/\beta}/(\alpha^2+N^2_{\alpha/\beta}\omega^2_{*2})$ for the electron contribution and Eq. (80) as illustrative of an ion dielectric. From dimensional analysis of Eq. (103) it follows that the high Reynold's number regime is typified by $\Delta\alpha\sim1$, or equivalently, $\Delta\omega\sim\omega_{*e}$; $\beta\sim\beta_o\sim1$; $N_{k\atop\omega}\sim\omega_{*e}$; and $\beta_{k\atop\omega}\sim\omega_{*e}$. Accordingly, this regime is fundamentally a strongly turbulent regime with intrinsically broad linewidth.

The equivalence of the nonlinear relaxation rate with ω_{*e} effectively determines the saturation level. The nonlinear relaxaton rate is

$$N_{k\atop\omega} = \frac{T_e^2c^2}{e^2B^2}\sum_{k_\vartheta',\omega'}\sum_m k_\vartheta^2k_\vartheta'^2(2\pi m\hat{s})^2\frac{e^2}{T_e^2}<\varphi(\eta+2\pi m)^2>_{k_\vartheta'\atop\omega'}\quad\frac{N_{k+k'\atop\omega+\omega'}}{[\omega''^2+N^2_{k+k'\atop\omega+\omega'}]}\,.$$

From a mixing length approximation ($\omega''\to0$),

$$N^2_{k\atop\omega}\sim k_\vartheta^2\rho_s^2c_s^2\hat{s}^2\bar{k}_\vartheta^2\left(\frac{2\pi\varphi(2\pi)}{\varphi(0)}\right)^2\frac{e^2<\varphi^2>}{T_e^2}\,. \tag{104}$$

The condition $N_{k\atop\omega}\sim\omega_{*e}$ then provides an approximate fluctuation level:

$$\frac{e<\varphi>_{RMS}}{T_e}\sim\frac{\rho_s}{L_n}\left[<k_\vartheta^2\rho_s^2>^{1/2}\,\hat{s}\,\frac{2\pi\varphi(2\pi)}{\varphi(0)}\right]^{-1}\,. \tag{105}$$

This estimate exhibits the standard drift wave scaling $e\varphi/T_e\sim\rho_s/L_n$ and with $\varphi(2\pi)/\varphi(0) = 0.2$, as previously discussed, and for typical wavenumbers $<k_\vartheta^2\rho_s^2>^{1/2}\sim0.3$. The saturation level is somewhat above that of the low

Reynold's number regime and consistent with experimental measurements in the 30% range. Mixing length estimates are based on an assumption of maximal turbulent energy transfer and ignore details of the energy sink. If the sink removes energy more slowly than the turbulent transfer, energy will accumulate and the fluctuation level will be higher than that predicted from a mixing length aproximation. Thus, the mixing length estimate represents a lower bound.

In the mixing length limit, no wavenumber dependence of the amplitude is obtained. The spectrum balance equation provides a further relation from which the asymptotic wavenumber dependences of the saturated spectrum may be obtained. From the spectrum balance equation, Eq. (98) we obtain

$$\overline{\langle\tilde{H}(1)\tilde{H}(2)\rangle}_k \cong A(k)\,\frac{\rho_s^2}{L_n^2}\left(\langle k_\vartheta^2\rho_s^2\rangle^{1/2}\,\hat{s}\,\frac{2\pi\varphi(2\pi)}{\varphi(0)}\right)^{-2} M$$

where we have assumed $(\delta n/n)_{RMS} \sim \frac{\rho_s}{L_n}\left[\langle k_\vartheta^2\rho_s^2\rangle^{1/2}\,\hat{s}\,2\pi\varphi(2\pi)/\varphi(0)\right]^{-1}$ from Eq. (102) and M is a number related to the dimensionless integral, Eq. (100). Referring back to the eigenfunction representation of the operators and the relationship between incoherent density and the potential, i.e., Eqs. (21),(24) and (26), $\langle\tilde{H}(1)\tilde{H}(2)\rangle_k = \langle\varphi(1)\varphi(2)\rangle_k\,\overline{\langle\tilde{H}(1)\tilde{H}(2)\rangle}_k$. With this relationship the spectrum balance equation is written

$$\langle\tilde{H}(1)\tilde{H}(2)\rangle_k = \bar{A}(k_\vartheta)\langle\hat{\varphi}(1)\hat{\varphi}(2)\rangle_{k_\vartheta} \qquad (106)$$

where

$$\bar{A}(k_\vartheta) = \frac{\rho_s^2}{L_n^2}\left[<k_\vartheta^2\rho_s^2>^{1/2}\ \hat{s}\ \frac{2\pi\varphi(2\pi)}{\varphi(0)}\right]^{-2} M\ A(k_\vartheta).$$

We seek a relationship between the electron density $\delta\eta$ and the incoherent nonadiabatic density $\tilde{H}$. By definition, $\delta\eta = (1+R_{k\,\omega})\frac{e\varphi}{T_e} + \tilde{H}_{k\,\omega}$. Replacing φ with $\tilde{H}$ through the quasineutral shielding condition, Eq. (19), yields

$$\delta n = (1+L_{k\,\omega}^{-1} + R_{k\,\omega}L_{k\,\omega}^{-1})\tilde{H}_{k\,\omega}\ . \tag{107}$$

The electron response $R_{k\,\omega}$, together with the ion operator and the adiabatic piece comprize the eigenoperator: $L_{k\,\omega}^{Ion} - (R_{k\,\omega}+1) = L_{k\,\omega}$ or $R_{k\,\omega} = -L_{k\,\omega}+L_{k\,\omega}^{Ion}-1$. Substituting for $R_{k\,\omega}$ in Eq. (107), taking the complex conjugate square of both sides, and substituting for $|\tilde{H}_{k\,\omega}|^2$ from Eqs. (99) and (106) gives

$$<\delta n^2>_{k\,\omega} = |L_{k\,\omega}^{Ion}L_{k\,\omega}^{-1}|^2\left(\frac{N_{k\,\omega}}{\omega^2+N_{k\,\omega}^2}\right)\ |\hat{\varphi}_k(\eta)|^2\ \bar{A}(k_\vartheta)$$

$$= \frac{N_{k\,\omega}}{(\omega^2+N_{k\,\omega}^2)}\ \bar{A}(k_\vartheta)\,|\varphi(\eta)|^2\ \left\{1+\frac{|\varepsilon^E(k_\vartheta\omega)|^2}{|\varepsilon^{Ion}(k_\vartheta,\omega)|^2}\right\}^{-1} \tag{108}$$

Since $|\omega+i\Delta\omega| > v_{Te}^2/(\nu_{ei}\ \Delta\eta_c^2)$, both species are hydrodynamic, $\varepsilon \sim \omega_*/(\omega + iN_{k\,\omega})$ and the term in the wavey brackets asymptotes to a constant in both limits $(k\to 0,\infty)$. The nonlinear decorrelation rate, $N_{k\,\omega}$ goes like $k_\vartheta^2 D$ and $\bar{A}(k_\vartheta)$ goes like k_ϑ^2. Thus

$$<\delta n^2>_{k_\vartheta\,\omega} \sim \begin{cases} 1/\omega^2 & (\omega\to\infty) \\ 1/k_\vartheta^4 & (k\to\infty) \end{cases} \tag{109}$$

integrating over ω, we obtain

$$<\delta n^2>_{k_\vartheta} \approx \bar{A}(k_\vartheta) \; |\varphi_{k_\vartheta}(\eta)|^2 \sim \begin{cases} 1/k_\vartheta^2 & (k\to\infty) \\ k_\vartheta & (k\to 0) \end{cases} . \tag{110}$$

With these estimates, we may derive particle transport scaling relationships in the same way we did for the low Reynold's number regime. Summarizing these results we have

$$\Gamma_N \sim \rho_s{}^2 c_s / L^2{}_n \tag{111}$$

and

$$D \sim \rho^2{}_s c_s / L_n . \tag{112}$$

V. CONCLUSIONS

The thrust of this paper has been two-fold. In working with this fluid model a theoretical framework for the analytical treatment of mode coupling equations such as the Hasegawa-Mima equation has been developed. In this theoretical framework the physics of the broad linewidth of the frequency spectrum has been retained in such a way as to illuminate the underlying causes and its relationship to other observable properties of the turbulence. Based on the euqation for two-point density correlation, this theory is the extension of Dupree's theory of two-point phase space density correlation to fluid systems. Here, incoherent fluctuations in the density

arising from the mode coupling contributions to the density at wavenumber k proportional to the potential at some other wavenumber k′ are related to density granulations. The density granulations result from the fact that the relative turbulent diffusion vanishes as the spatial separation of two points decrease to zero. Driving granulations in the electron correlation is a source which is proportional to the ion dissipation. This proportionality is exact in the fluid theory and arises as a consequence of the fact that the total electron density (coherent and incoherent) is related to the ion dynamics by quasineutrality.

In addition, the effect of particle collisions on the density granulations has been included in this dissipative fluid model. Particle collisions are incorporated in the model as a parallel collisional viscosity which results in a relative diffusion inversely proportional to the collision frequency. The magnitude of the parallel collisional viscosity relative to the E×B mixing introduces two limits and a parameter to differentiate them. These two limits are identified by a high Reynold's number regime (inertial range) in which the turbulent E×B diffusion dominates the linear parallel diffusion of the collisional viscosity and a low Reynold's number regime (dissipation range) in which the linear parallel collisional viscosity is dominant.

The Reynold's number is defined as $R_e = k^2 D(Rq)^2 \Delta\eta_c^2 \nu_{ei}/v_{T_e}^2 = \tau_{\parallel}/\tau_{E\times B}$ and is order unity for experiment. In the high Reynold's number regime the separation of trajectories of neighboring fluid elements is exponential. The lifetime of the small scale correlation is logrithmic but not singular since the linear collisional diffusion eventually separates fluid elements.

The incoherent correlation at scales below the clump scale is large, approaching a value of $\tau_c \ell n(R_e)S$, as the separation vanishes, where τ_c is the one-point relaxation time, R_e is the Reynold's number and S is the source. The linewidth in the high Reynold's number regime is intrinsically broad.

In the low Reynold's number regime, trajectories of neighboring fluid elements separate according to a power law. The incoherent correlation is non-zero below the clump scale but small. In this regime it is not possible to estimate the incoherent correlation from the evolution under the dominant diffusion process as the correlation involves a complicate combination of both dominant diffusion (parallel collisional viscosity) and the smaller E×B diffusion. In a Reynold's number expansion of the incoherent correlation, the lowest order term is first order in R_e, indicating that the incoherent correlation vanishes entirely as $R_e \to 0$.

Based on the theory of two-point density correlation, we present a study of tokamak edge turbulence providing formulas for wavenumber and frequency spectra, fluctuation level, and particle transport for steady state turbulence. For calculating spectrum properties in the low Reynold's number limit we adopt ion Compton scattering as the saturation mechanism. We find a linewidth which is small for most $R_e \lesssim 0.3$. As R_e approaches 1 it is possible for the linewidth to be as large as ω_{*e}. The wavenumber spectrum is computed by balancing the ion Compton scattering-damping rate with the linear growth rate enhanced by the incoherent emission which is weak for $R_e<1$. The spectrum is strongly peaked at the longest wagelengths and has a cutoff at $k_m = \rho_s^{-1}(1+\hat{s}^2\pi^2/4)^{-1/2}$. Modes with wavenumbers above

this cutoff are only weakly excited due to efficient scattering to longwavelengths which overcomes the linear growth. The steady state fluctuation level scales as $(\rho_s/L_n)[\nu_{ei}c_s/(\omega_{T_e}^2 L_n)]^{1/12}$. The ρ_s/L_n scaling has been observed on many experiments. The scaling with other parameters is extremely weak, going as the 1/12 power. The amplitude is given approximately as $e\varphi/T_e \sim 2\rho_s/L_n$. The particle diffusion is outward and scales as $T_e^{1/6}n^{2/3}$.

For the large Reynold's number regime strong turbulence estimates apply due to the strong excitation relative to linear relaxation. Properties of the spectrum are calculated from general mode coupling saturation in the mixing length approximation. In this regime the linewidth is of the order of or exceeds the diamagnetic drift frequency ω_{*e}. The fluctuation level is of the order of $e\varphi/T_e \sim 7\rho_s/L_n$. The particle diffusion scales identically as it does for $R_e<1$.

In general terms, we find that linewidths on the order of ω_{*e} and fluctuation levels in the neighborhood of those observed experimentally for the tokamak edge are possible for dissipative drift waves. The scaling of the fluctuation level is consistent with experiment both in its dependence on ρ_s/L_n and its lack of dependence on other parameters, most notably ν_{ei}. The particle diffusion has very weak temperature dependence and moderate density dependence. This scaling is decidedly non-Bohm-like and seems to be suggestive of the type of diffusion scaling which is measured experimentally.

Recently, new experimental results[16] indicate that a large, rapidly varying radial electric field is present at the plasma edge. Inclusion of the effects associated with this electric field can modify the density gradient driven turbulence model proposed in this paper or introduce a new source of instability associated with $E_o \times B_o$ velocity shear. Furthermore, the scale size associated with velocity shear may replace the radial mode width as the radial correlation length. These considerations will be discussed in a future publication.

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APPENDIX I

We provide here the details of the derivation of Eq. (63), starting with Eqs. (60), (61) and (62). Substituting Eqs. (61) and (62) into Eq. (60) we perform the η_3 and η_4 integrations and take the derivatives in y_ to yield a factor of $(k^2/\kappa^2)e^{i\kappa\eta''}$. Completing the square in η'' permits the integrations in η'' and η'. The integration constants are chosen so that

$\langle\tilde{H}^2\rangle$ remains finite. This means that η' and η'' are integrated from $|\eta_+|$ to ∞ and $|\eta'|$ to ∞ respectively. The integrations yield

$$\tilde{H}(y_-,\eta_-) = -\frac{(Rq)^4}{D_\parallel^2} D \frac{\sqrt{\pi}}{2} \int d\kappa \int dk\ e^{iky_-} e^{-y_-^2/2\alpha^2}$$

$$\cos(k_o y_-)\Delta\eta^2\ i\ \mathrm{ERFC}\left(\frac{|\eta_-|}{\Delta\eta} - \frac{i\kappa\Delta\eta}{2}\right)\ e^{-\kappa^2\Delta\eta^2/4}$$

$$\int dw\ (\omega-\omega_{*e})\ \varepsilon_{Im}^{Ion}(\kappa\omega)\ \frac{|\hat{\varphi}_k(\kappa_1)\hat{\varphi}_{-k}(\kappa_2)|}{|\varepsilon(k,\omega)|^2}\ \underset{\omega}{\langle\bar{H}^2\rangle_k} \qquad (A1)$$

where

$$i\mathrm{ERFC}(Z) = \int_Z^\infty \mathrm{ERFC}(t)\ dt = -Z\ \mathrm{ERFC}(Z) + \frac{1}{\sqrt{\pi}} e^{-Z^2} \qquad (A2)$$

and ERFC(Z) is the complimentary error function.[17] We take the Fourier transform in η_-, η_+ and y_- of both sides of Eq. (A1). On the left hand side we have

$$\int d\eta_+\ e^{-i\kappa_+\eta_+} \int d\eta_-\ e^{-i\kappa\eta_-} \int dy_-\ e^{-iky_-}\ \tilde{H}(y_-,\eta_+) = \langle\tilde{H}^2(\kappa)\rangle_k.$$

On the right hand side we write $\cos(k_o y_-) = \left(\exp(ik_o y_-) + \exp(-ik_o y_-)\right)/2$ and complete the square in y_-. Integrating in y_- we obtain

$$\langle\tilde{H}^2(\kappa)\rangle\rangle_k = \frac{-(Rq)^4}{D_\parallel^2} D \frac{\sqrt{\pi}}{2} \int d\eta_-\ e^{-i\kappa\eta} \int d\kappa'\ \int dk'\ \Delta\eta^2\ \frac{\alpha}{\sqrt{2}}$$

$$[e^{-(k-k'-k_o)^2\alpha^2/2} + e^{-k-k'+k_o)^2\alpha^2/2}]\ \text{iERFC}[\frac{|\eta_-|}{\Delta\eta} - \frac{i\kappa'\Delta\eta}{2}]$$

$$(\frac{k'^2}{\kappa'^2})e^{-\kappa'^2\Delta\eta^2/4} \int d\omega'(\omega'-\omega'_{*e})\ \frac{\varepsilon^{Ion}_{Im}|\hat{\varphi}^2|_{k',\omega'}\langle\bar{\tilde{H}}^2\rangle_{k',\omega'}}{|\varepsilon|^2} \quad . \tag{A3}$$

The integration over η_- may be carried out next using the identity

$$\int d\eta'\ e^{-i\eta'\kappa}\ \text{iERFC}[\frac{\Delta}{2}|\eta'| - \frac{iu}{\Delta}]$$

$$= -\frac{\Delta}{2}\ e^{u^2/\Delta^2}\ \frac{1}{\kappa^2}\{\text{ERFC}[-\frac{i}{\Delta}(u+\kappa)]e^{-(u-\kappa)^2/\Delta^2}\}\ (\kappa \neq 0) \quad . \tag{A4}$$

Upon taking the real part we have

$$\langle\tilde{H}^2\rangle_k = \frac{(Rq)^4}{D_\parallel^2} D\alpha\Delta\eta \sqrt{\frac{\pi}{2}} \int d\kappa' \int dk'\ e^{-(\kappa-\kappa')^2\Delta\eta^2/4}$$

$$\frac{k'^2}{\kappa^2\kappa'^2}\ [e^{-(k-k'-k_o)^2\alpha^2/2} + e^{-(k-k'+k_o)^2\alpha^2/2}]$$

$$\int d\omega'(\omega'-\omega'_{*e})\ \frac{\varepsilon^{Ion}_{Im}|\hat{\varphi}|^2_{k',\omega'}}{|\varepsilon|^2}\ \langle\bar{\tilde{H}}^2\rangle_{k',\omega'} \quad . \tag{A5}$$

The variable κ' is the "wavenumber" corresponding to the relative parallel variation η_-. The squared wave function $|\hat{\varphi}|^2_{k',\omega'}$ depends on both κ' and κ_+, where κ_+ corresponds to η_+. We assume that the wavefuction is Gaussian in η with width Δ_k^{-1}. Then

$$|\varphi_k(\eta_1)\varphi_{-k}(\eta_2)| \cong e^{-\eta_1^2\Delta_k^2} \; e^{-\eta_2^2\Delta_k^2} = e^{-2\eta_+^2\Delta_k^2} \; e^{-2\eta_-^2\Delta_k^2}$$

and

$$|\varphi^2(\kappa+\kappa')| = e^{-\kappa_+^2/8\Delta_k^2} \; e^{-\kappa^{1/2}/8\Delta_k^2}.$$

With this assumption we may perform the integration over κ' by completing the square and using the identity

$$\int d\kappa \; \frac{1}{\kappa^2} \; e^{-(\kappa-u)^2/\Delta^2} = -\left(\frac{2}{\Delta}\right) \; e^{-u^2/\Delta^2} \; \text{iERFC}\left(-\frac{iu}{\Delta}\right) . \qquad \text{(A6)}$$

upon taking the real part we arrive at Eq. (63) of the text.

Figure Captions

1. Toroidal correlation time as a function of relative separation in the toroidal angle. For several values of the Reynold's number.
2. Width of the frequency spectrum for a fixed wavenumber $(k_\vartheta \rho_s)^2=0.1$ as a function of Reynold's number.
3. Wavenumber spectrum for moderate Reynold's numbers.

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FIG. 1

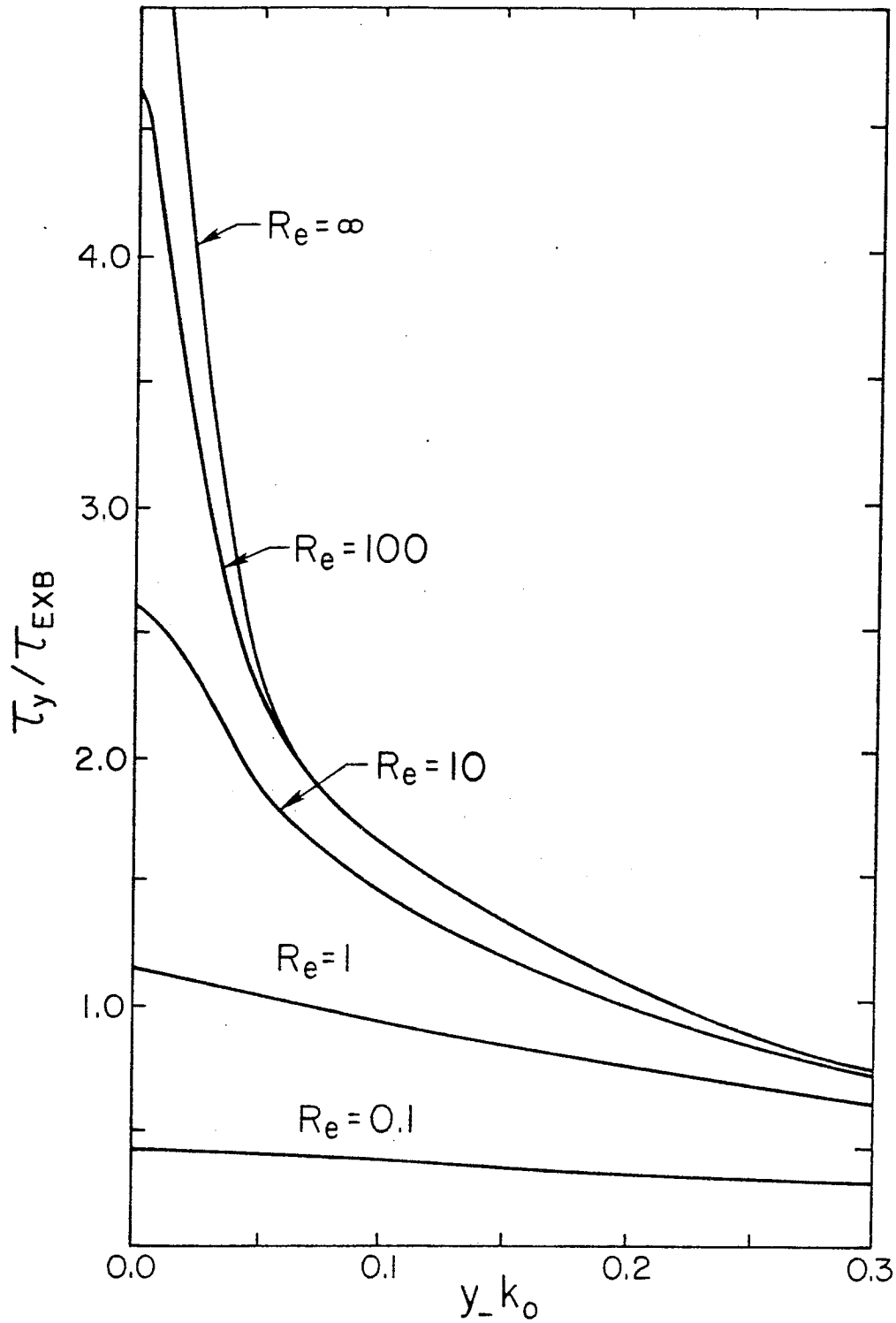

$R_e = \infty$
$R_e = 100$
$R_e = 10$
$R_e = 1$
$R_e = 0.1$
4.0
3.0
2.0
1.0
0.0
0.1
0.2
0.3
τ_y / τ_{EXB}
y_k_o

FIG. 2

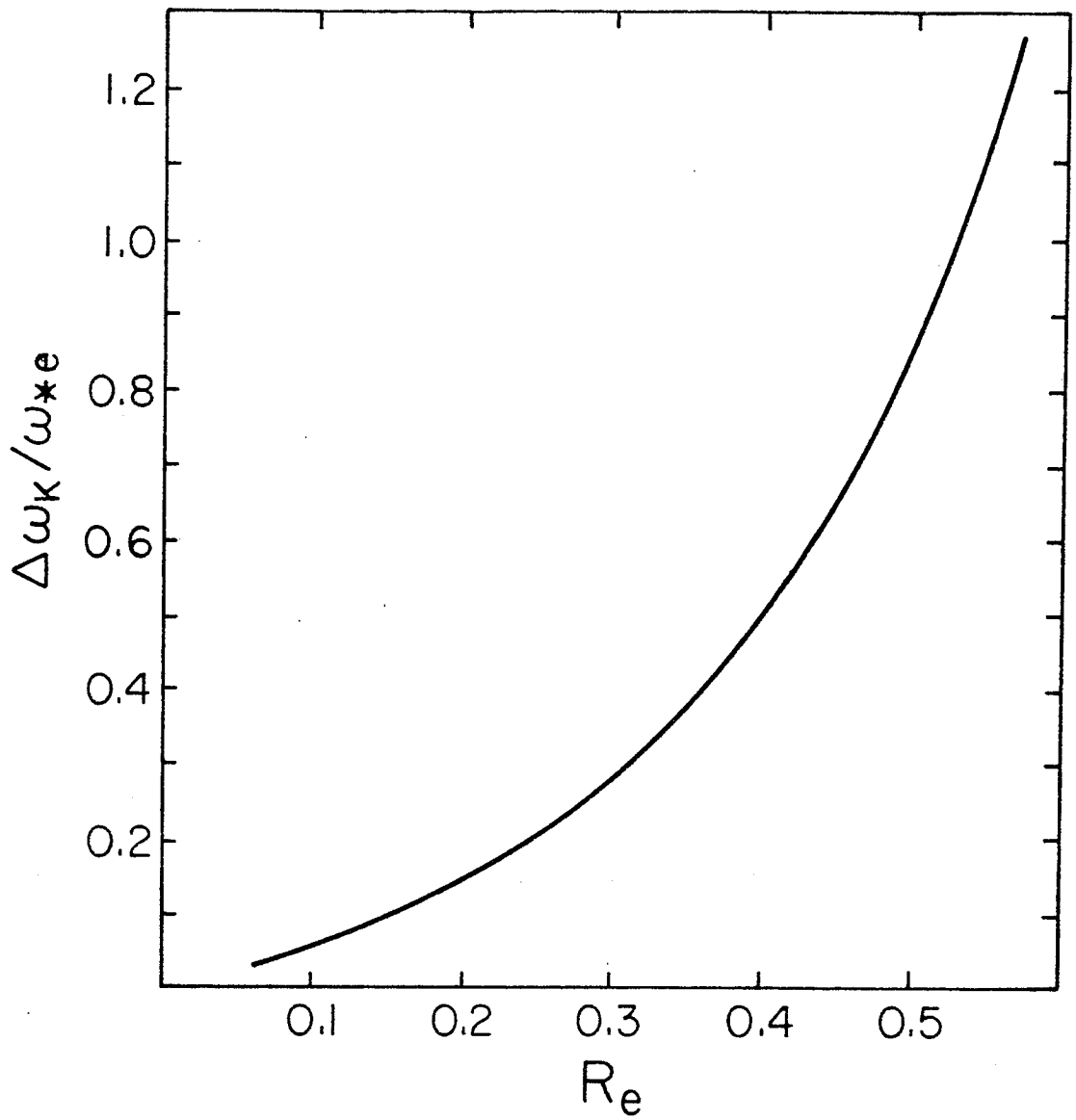
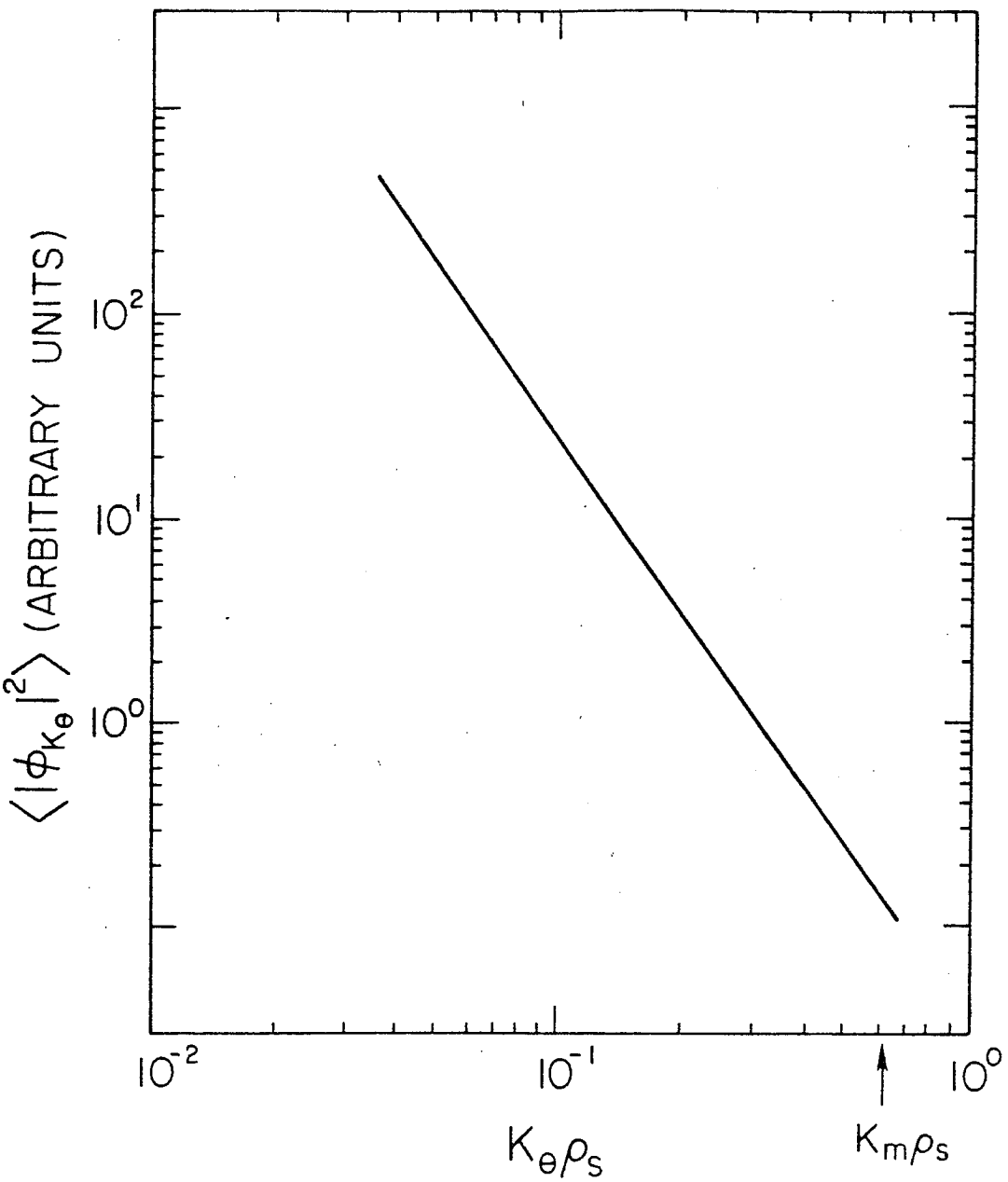


FIG. 3


$\langle|\phi_{K_\theta}|^2\rangle$ (ARBITRARY UNITS)
10^2
10^1
10^0
10^{-2}
10^{-1}
10^0
$K_\theta\rho_s$
$K_m\rho_s$