Stochasticity in Classical Hamiltonian Systems:
Universal Aspects

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ABSTRACT

For Beginners: This review presents universal aspects of stochasticity of simple (1.5 or 2-degree-of-freedom) Hamiltonian systems. Stochasticity is the seemingly erratic wandering of orbits of non-integrable Hamiltonian systems over some part of phase space, accompanied by exponential divergence of nearby orbits. It is a large-scale phenomenon that spreads over larger and larger regions of phase space by the successive break-ups of barriers called Kolmogorov-Arnold-Moser (KAM) tori when some perturbation to an integrable Hamiltonian is increased. The main emphasis of this review is on the break-up of KAM tori which is described by a renormalization group for Hamiltonians of the KAM type. This paper also reports recent progress in describing chaotic transport which is the large scale manifestation of stochasticity, but this is not the last word to chaos. The central model of this paper is the Hamiltonian of one particle in two longitudinal waves, \( H_p(v,x,t) = \frac{v^2}{2} - Mcosx - Pcosk(x-t), \) which is a paradigm for simple Hamiltonian systems. Simple approximate

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renormalization schemes for KAM tori of $H_p$ are derived, and the way to exactly renormalize a general Hamiltonian of the KAM type is explained as well.

For Experts: The breakup of KAM tori and chaotic transport are the main topics of this review which aims at updating the picture of Hamiltonian stochasticity as described, for instance, in Lichtenberg and Lieberman's textbook. The exact way to renormalize a Hamiltonian of the KAM type for a given torus is explained, and simple computable approximate schemes are described as well. The connection with schemes for area-preserving maps is made. Orbits are characterized by a zoning number whose value is determined by both the spatial periodicity of primary resonances and the rotation number of a KAM torus. Universal classes of KAM tori are related to a periodic continued fraction expansion of the zoning number. When the expansion ends with ones only, the torus is termed noble. The robustness of noble tori is shown to imply a hierarchy in the denumerable set of fixed points of the renormalization group. A simple renormalization scheme for the standard map describes high-order fixed points of the renormalization groups for KAM tori and for period n-tupling. When the perturbation to a given integrable Hamiltonian is increased enough, a KAM torus develops a dense set of gaps and is called a Cantorus. The fact that time is exactly renormalized in schemes for Hamiltonian systems, makes it possible to compute critical exponents for Cantori by simple scaling arguments. Accurate calculations of thresholds of global stochasticity are obtained by the combination of the exact renormalization scheme and of an approximate expression for the noble stable manifold. A
perturbation theory reduces a general Hamiltonian to a many-wave Hamiltonian. The width of stochastic layers is computed by a method different from Melnikov-Chirikov's one, but which agrees with it; it is defined in terms of rotation numbers. Three regimes of chaotic transport are described and the corresponding scaling laws are given: the Cantorus-with-small-hole regime, the quasi-linear regime, and the trapping regime.

For the reader to whom this abstract is too short and the whole article is too long, we recommend sections 1 and 2.
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References
1. Introduction

1.1 What is Hamiltonian Stochasticity?

What is Hamiltonian stochasticity? Paraphrasing Stauffer [1], one may say: Figure 1 gives a first answer: Hamiltonian stochasticity is a very fashionable field. The number of papers published in the field has gone up by a factor of 10 in 10 years. The greater part of the pronounced peak in fig. 1 for the Hamiltonian stochasticity publication rate could not be covered in the earlier reviews of Ford [2], Berry [3], Chirikov [4], Lichtenberg [5], Helleman [6], Tabor [7], Brumer [8], Rice [9], and in the books by Arnold and Avez [10], Percival and Richards [11] (a presentation at an undergraduate level), and by Lichtenberg and Lieberman [12] (a presentation at a graduate level). The estimate of the number of papers on Hamiltonian stochasticity in fig. 1 only yields a tendency since the data base used focusses on physics and does not include all mathematics journals nor chemical physics and celestial mechanics journals which publish a significant amount of articles in the field.

What else is Hamiltonian stochasticity? Besides being fashionable, it is the seemingly erratic wandering of orbits over some part of phase space, accompanied by exponential divergence of nearby orbits from each other (see sec. 2.1). The very nature of Hamiltonian chaos still is a mystery but is ultimately related to the complexity of real numbers as revealed by algorithmic complexity theory [13]. Stochasticity is a general phenomenon as soon as the number of degrees of freedom is larger than one [14], but most of the past studies focussed on systems with 2 or 1.5 degrees of freedom (the last ones are one-degree-of-freedom systems which depend on time explicitly, possibly
in a discontinuous way-like area-preserving maps). Those systems are referred to as simple systems in the following. This review only deals with simple systems, as little progress seems to have been done since the account of ref. [12] for systems with a larger dimensionality.

As already mentioned in previous review papers [2-12]-[15], simple Hamiltonian systems are of interest in various disciplines: celestial mechanics, particle physics, plasma physics, and chemical physics. More recently other fields can be added to that list: solid state physics for models of epitaxial monolayers [16] and of small polaron (electron strongly locally coupled to a lattice deformation) dynamics [17], fluid dynamics with models of vortex flows [18], electrical engineering for the stability of synchronous [19] or reluctance motors [20]. In most cases the time variable of the Hamiltonian corresponds to real time, but it is the number of an atom in a chain in ref. [16] and it is a toroidal angle for the description of toroidal magnetic fields [21]. Celestial mechanics, which played an important role in the development of the field of Hamiltonian stochasticity (see for instance, the famous Hénon-Heiles model [22]), recently evidenced the chaotic behavior of the rotation of Hyperion, a satellite of Saturn [23]. Stochastic heating has been checked experimentally in a plasma [24].

There are two main topics in Hamiltonian stochasticity: one is the statistical description of chaotic motion, and the other is the study of the transition to stochasticity. In simple Hamiltonian systems depending on a parameter s called stochasticity parameter, stochasticity increases by successive jumps as s increases, and becomes more and more global. This is due to the breakup of barriers in phase
space called Kolmogorov–Arnold–Moser (KAM) tori [10]. A large part of the literature on Hamiltonian systems, probably the largest, is devoted to the transition to large scale stochasticity or to the breakup of KAM tori. This breakup appears to be very similar to a second-order phase transition, and renormalization methods have been developed to describe it [25–27].

1.2 Aim of the Review

This review aims at updating the picture of Hamiltonian stochasticity as described, for instance, in Lichtenberg and Lieberman's book [12], and at describing the universal properties of simple Hamiltonian systems, as known in July 1984. Renormalization techniques have induced a percolation of isolated research clusters (see for instance sec. 4.3) that allows us to considerably extend the previous description of Hamiltonian universality by Chirikov [4]. Basically the transition to stochasticity now is understood (secs. 2.3 and 3), even though much work is left to mathematically prove the present physical picture. More universality has been found in the description of stochastic layers, and the Melnikov–Chirikov approach [4] has been connected with KAM theory (see sec. 2.5). Two new chaotic regimes have been identified and described: the Cantorus–with–small–holes regime and the trapping regime (sec. 2.6). However, a complete theory of chaotic transport is not yet available.

The qualitative picture of this review is quite general but the techniques presented here are mostly designed for Hamiltonians of the KAM type.
\[ H(\vec{p}, \vec{q}) = H_0(\vec{p}) + \varepsilon V(\vec{p}, \vec{q}) \]  

where \( H_0 \) is an integrable part and \( \varepsilon V \) is a perturbation. There is not yet any systematic way to apply these techniques to more general cases (see sec. 4.1.4.2).

The existence of Lichtenberg and Lieberman's book [12] allows this review to emphasize only more recent progress. This review also skips a lot of information on regular orbits which is not directly relevant to the appearance of stochasticity (see for instance ref. [28]). Section 2, however, gives a self-contained mundane picture of Hamiltonian stochasticity which should allow the non-specialist to get into the topic without any preliminary study. More information can be obtained by reading the presentation of ref. [3] or chapters 3 to 5 of ref. [12].

A given flow in phase space can be defined by many different differential equations. If this flow is volume-preserving, many different Hamiltonians still may govern it, which are not deduced one from another by canonical transformations. For that reason we often speak in that paper about Hamiltonian systems and represent their dynamics by different Hamiltonians (see for instance sec. 2.1.2.3) whose conjugate variables are connected by rescalings of position, time, etc...

Similarly it now is clear that there is one renormalization group for KAM tori, but some freedom is left in the method of renormalization. The group may be viewed as acting on the pair of a torus and of a Hamiltonian, or on a Hamiltonian for a given torus. When one of those directions is chosen, there still is a lot of freedom
for defining an explicit renormalization scheme, but all of them belong to the same renormalization group.

1.3 A Heuristic Approach

A large part of this paper deals with renormalization schemes that were motivated by the study of plasma turbulence. This section describes the heuristic approach that led to the derivation of the first scheme, and is not necessary for understanding the remaining part of the paper.

Originally the problem of interest was to study the motion of one particle in a 1-dimensional spectrum of longitudinal waves. With only two waves present, the corresponding Hamiltonian is

\[ H(p,q,t) = \frac{p^2}{2m} - \sum_{i=1}^{2} V_i \cos(k_i q - \omega_i t + \phi_i), \quad (1.2) \]

where \( m \) is a mass, \( V_i, k_i, \omega_i \), and \( \phi_i \) respectively are the amplitude, the wave number, the frequency, and the phase of wave \( i \). Appropriate rescalings allow us to define three dimensionless quantities which govern the motion of the system (see appendix A). This yields a new Hamiltonian (1.2) where \( k_1 = m = 1, \omega_1 = \phi_1 = \phi_2 = 0 \). A common choice for the other parameters is \( V_1 = 1 \) so that the dimensionless parameters are \( V_2, k_2 \) and \( \omega_2 \) [4]. This choice amounts to consider wave 2 as a perturbation of wave 1. For the above-mentioned plasma turbulence problem, a priori the two waves are equally important. Therefore, it is better not to fix any amplitude, but to keep the mismatch of phase velocity equal to 1, i.e., \( \omega_2 = k_2 \). The three free parameters then are \( V_1, V_2, \) and \( k_2 \). This choice transforms \( H \) into a new Hamiltonian \( H' \).
When applied to \( H_p \) the resonance-overlap criterion\(^*\) \[4\] does not give an accurate threshold for large scale stochastic motion. Following Chirikov \[4\], it is natural to apply this criterion to higher order resonances. When \( H_p \) is written in the action-angle variables of \( H_p \) for \( V_2 = 0 \), the Fourier expansion of the Fourier expansion of the transformed Hamiltonian displays such resonances and is so reminiscent of the Hamiltonian of one particle in many waves, that one cannot help from writing it this way. By the magic of the two-resonance-approximation \[4\] which retains only two nearby waves, a new Hamiltonian \((1.2)\) is recovered: unwittingly the original system has been renormalized! Iterating the process is tempting, but one needs to choose a pair of nearby resonances at each step. This unpleasant problem is put off for a while, and one runs to the computer in order to see how \((V_1, V_2)\) evolves in the case where the pair of resonances with the same number in the Fourier expansion is taken at each step. Miracle: either \((V_1, V_2)\) goes to \((0,0)\) or to \((\infty, \infty)\)! There obviously is a curve in the initial \((V_1, V_2)\) space that separates both behaviors. This curve is a threshold. Of what? Well, choosing successive pairs of nearby resonances means defining a thinner and thinner object in phase space, which turns out to be a KAM torus: what is computed is a threshold of breakup of a KAM torus. This is disappointing when one looks for a threshold of global stochasticity, but eventually one gets over it. A way to compute stochasticity thresholds is found and a (cryptic) paper \[29\] was written which was made somewhat more readable by referees a

\(^*\)It states that large-scale stochasticity should occur when the trapping domains of the two waves overlap.
few months later [25]. The renormalization scheme is approximate, but in what sense? Even though this question is left open, a lot can be done with the scheme [15, 25, 30–35]. The present review yields an exact version of the renormalization which answers the previous question (sec. 3.2). This scheme differs from exact schemes for area-preserving maps [26–27] (sec. 2.3.1.2) that were developed by analogy with schemes for the period-doubling bifurcations [36].

1.4 Structure of the Paper

This paper deals a lot with renormalization, and is written accordingly! The abstract is a large-scale description, sec. 2 gives a more complete, but yet mundane, picture of the topic, secs. 3 and 4 emphasize aspects of that picture, and finally appendices belong to general literature as most references of that review do.

The central model of this paper is the normalized version $H_p$ of Hamiltonian (1.1), called the paradigm Hamiltonian. Section 3 describes approximate renormalization schemes for $H_p$ and an exact renormalization for Hamiltonian systems. Section 4 describes methods for computing the threshold of global stochasticity which account for the universal properties of Hamiltonian systems. Subsection 4.3 yields a global point of view on the 19121 methods as given in chapter 4 of ref. [12].

For the reader who knows one of refs. [15, 25, 30–35], it should be stressed that this review adopts a different presentation. The basic model still is the Hamiltonian $H(v,x,t) = v^2/2 - M \cos x - P \cos k(x-t)$, but the main renormalization scheme uses the Kolmogorov transformation that kills the $M \cos x$ term at lowest order, instead of the action-angle
transformation. The main scheme is built such that, for $k=1$, each iterate of the renormalization shifts the partial quotients of the continued fraction expansion of the rotation number by one. These two changes, made for pedagogical reasons, yield a much simpler presentation. There is another modification, which is due to the evolution of the theory itself: it is the emphasis on the golden mean (noble) universality class. As a result, thresholds of stochasticity can be computed without using a set of diagrams as originally, but only the simple approximate formula of the noble stable manifold. Finally the present paper explains in which way the original renormalization schemes were approximate and describes a corresponding exact scheme.
2. Picture of Hamiltonian Stochasticity

This section intends to describe the present picture of stochasticity in 1.5 or 2-degree-of-freedom Hamiltonian systems. An important aspect of that picture is the scale invariance of those systems. At a theoretical level this is accounted for by the existence of a renormalization \( R \) that plays the role of a microscope in phase space, and implies the existence of a universal behavior of Hamiltonian systems. This result, though not mathematically proved, is strongly connected with KAM theory. Section 2.1 introduces a Hamiltonian system for the study of stochasticity, sec. 2.2 describes the problems set by large-scale (or global) stochasticity: (threshold, critical behavior nature of chaotic motion), and sec. 2.3 presents the qualitative features of KAM universality.

2.1 A Paradigm Hamiltonian

In order to make the picture simple, it is organized about the Hamiltonian system \( S_p \) defined by the Hamiltonian

\[
H_p(v,x,t) = \frac{1}{2} v^2 - M\cos x - P\cos(x-t), \quad (2.1)
\]

which is the simplest non-integrable, nonlinear Hamiltonian with two resonant terms. It deserves twice over the title of paradigm Hamiltonian: it displays the typical features of Hamiltonian systems, and it is the central object of a simple approximate renormalization scheme that mimics the exact one. Furthermore, sec. 3.1.5-6 and 4.1.2 show that a large class of 1.5 or 2 degree-of-freedom Hamiltonians can locally be reduced to \( H_p \).
2.1.1 Integrable Cases

2.1.1.1 Free Particle

When $M=P=0$, $H_p$ is the Hamiltonian of a free particle. Trajectories are straight lines (fig. 2.1). This phase space displays a trivial scale invariance; any part of it may be blown up by any factor and displays the same structure as the original one. We recover the apparent simplicity of real numbers.

2.1.1.2 Pendulum

When $M \neq 0$ and $P=0$, $H_p$ is the pendulum Hamiltonian (fig. 2.2). Some trajectories keep the same topology as in fig. 2.1, and are called passing or untrapped orbits; they correspond to the rotation of the pendulum. Velocity is no longer a constant along these trajectories, but we can define for each one a mean velocity

\[ u = \frac{1}{T} \lim_{T \to \infty} \int_0^T v(t) dt . \]  

(2.2)

Other trajectories rotate about the origin and are called trapped or libration orbits; they correspond to oscillations of the pendulum about its stable equilibrium point. For such a trajectory a rotation number is defined as

\[ \rho = \lim_{t \to \infty} \frac{N_t}{t} , \]  

(2.3)

where $N_t$ is the number of turns of the orbit around the origin during the time $t$. The set of these trajectories constitute a tear* in phase
space. This tear is bounded by two separatrices anchored at the unstable equilibrium point of the pendulum. The half-width of the separatrix is

$$\Delta v_M = 2\sqrt{M}.$$  \hspace{1cm} (2.4)

The time-averaged velocity is non-zero for passing orbits and zero for trapped ones: the mean velocity of a trapped orbit is the same as the velocity of the stable equilibrium point of the pendulum, i.e., the origin. The $M\cos x$ term is said to be resonant, and the trapping domain to be a resonant domain. The finite extension of the resonant domain is due to the dependence of $u$ on $v$, i.e., to the nonlinearity of $H_p$ for $M=P=0$. This is typical of a nonlinear resonance [12].

2.1.1.3 Particle in a Longitudinal Wave

When $M=0$ and $P\neq 0$, $H_p$ describes the motion of a particle in a longitudinal (electrostatic) wave with phase velocity $1$ and wave-number $k$. From now on, for simplicity, we assume $k$ to be a rational $r/p$. Section 3.1.4 considers the case where $k$ is irrational. Define $v=v-1$ and $y=x-t$. Hamilton's equations make it obvious that the time evolution of $(w,y)$ is governed by Hamiltonian $H(w,y) = \frac{w^2}{2} - P\cos ky$. This is the Hamiltonian of a pendulum, but the periodicity in $y$ (and in $x$) now is $p2\pi/r$. The half-width of the separatrix is

---

*This word refers to the shape of the trapping domain and to the distortion of passing orbits in figs. 2.2 with respect to fig. 2.1. Phase space seems to have been torn.*
\[ \Delta v_p = 2\sqrt{p} \]  \hspace{1cm} (2.5)

A rotation number \( \rho \) for trapped orbits can be defined in that reference frame according to eq. (2.3). Now the resonance domain is moving with a velocity 1 with respect to the \( x \)-axis. A simple way to visualize it in the frame \((x,v)\), is to use a stroboscopic plot (or Poincaré map) that represents the trajectories at time \( nT \) where

\[ T = \frac{2\pi}{k} \]  \hspace{1cm} (2.6)

In order to generate a picture with a finite length, we take advantage of the \( p2\pi \) periodicity in \( k \) for plotting all points between \(-p\pi\) and \(+p\pi\), modulo \( p2\pi \) (in other words the map is wrapped on a cylinder). With that representation (fig. 2.3) trajectories appear as a series of dots. With probability one, \( u \) or \( \rho \) are irrational, and the points densely fill lines on the picture which are the stream lines of \( H(w,y) \).

Consider the set of points \([x,t,v(x,t)]\) such that a trajectory going through that point has a given \( \rho \) or \( u \). The periodicity of \( H \) in both \( x \) and \( t \) yields the same periodicity to \( v(x,t) \). Since the position on surface \( v(x,t) \) is defined by two angles, it is topologically equivalent to a torus, and we call it a torus for short. Such a torus is defined for a continuous set of value of \( \rho \) or \( u \). The Poincaré map of a trajectory characterized by \( \rho \) or \( u \) is the trace of \( v(x,t) \) in plane \( t=0 \).
For the pendulum Hamiltonian, \( v(x,t) \) only is a function of \( x \), but can be considered as a torus with the time period \( T \) as well. For \( u \) or \( \rho \) irrational, the Poincaré map of the orbit looks like fig. 2.2, but is made up of single points.

For \( (M,P) = (M,0) \) or \( (0,P) \), if \( \rho \) or \( u \) is rational, the trace of the orbit in the Poincaré map is periodic, and only a finite number of points appear. This is for instance the case for the stable and unstable equilibria of the pendulum.

At that point the characteristics of the integrable motion have been described. We now turn to the description of non-integrable Hamiltonians.

2.1.2 Non-Integrable Cases

2.1.2.1 Naive Picture

Define the parameter

\[
 s = \Delta v_M + \Delta v_P = 2\sqrt{M} + 2\sqrt{P},
\]

hereafter referred to as the stochasticity parameter. When both \( M \) and \( P \) are non-zero, all trajectories no longer belong to surfaces \( v(x,t) \), and the system is said to be non-integrable. This is proved by showing that the Melnikov-Arnold integral* computed along the unperturbed separatrix of resonance \( M\cos x \) is non-zero (see sec. 7.3b of ref. [12] and sec. 4.4 of ref. [4]). Still it looks intuitively correct that if

*The Melnikov-Arnold integral considered here is \( \int_{-\infty}^{+\infty} \cos k[x(t) - t] \, dt \) where \( x(t) \) corresponds to the motion on the separatrix of resonance \( M\cos x \).
s << 1, i.e. the unperturbed separatrices are far from overlapping (fig. 2.4), the Poincaré map of $H_p$ should look like some kind of superposition of figs. 2.2 and 2.3. We therefore expect the traces of passing tori (those characterized by a mean velocity $u$) to be slightly pinched between the two resonant domains of the primary resonances $M$ and $P$. (short-cuts for $M \cos x$ and $P \cos k(x-t)$) of $H_p$ (fig. 2.4c).

By taking advantage of the $p2\pi$ periodicity of $H_p$ in $x$, we can define a rotation number for passing orbits as the average number of spatial periods $p2\pi$ during a time period $T = \frac{2\pi}{k} = \frac{2\pi\pi}{r}$

$$\rho = \frac{uT}{p2\pi} = \frac{u}{r}.$$  \hspace{1cm} (2.8)

2.1.2.2 KAM Theorem

This naive picture is supported by the KAM Theorem [10] that implies that for each $u$ or $\rho$ irrational enough, the set of trajectories characterized by $u$ or $\rho$ is a torus (a function $v(x,t)$ exists) for $s > 0$ small enough. Such a torus is called a KAM torus. "Irrational enough" corresponds to the Diophantine condition that there is a couple $(A, \varepsilon)$, $A > 0$, such that for all $(m,n)$'s

$$|u - \frac{m}{n}| > \frac{A}{n^{2+\varepsilon}}.$$\hspace{1cm} (2.9)

Furthermore the set of $u$'s $0 < u < 1$, for which a torus with mean velocity $u$ is preserved, has a positive measure that goes to 1 when $s$ goes to 0. In a pictorial way this means that if we draw all preserved
KAM tori in a Poincaré map; dark stripes are going to appear, and that the picture becomes completely black, as expected, for s=0.

Since for \((M,P) = (M,0)\) (resp. \(= (0,P)\)), \(H_p\) is integrable, the same reasoning implies that a trapped torus, with \(\rho\) irrational enough, is preserved for \(P\) (resp. \(M\)) small enough.

So, the KAM theorem proves the existence of both passing and trapped tori. However, Diophantine condition (2.9) suggests that our naive picture of a global structural invariance of the Hamiltonian system \(S_p\) for \(s\) small, overlooks important "details" of the actual picture.

One way to check this feeling is to run numerical calculations of orbits for various values of \(s\). When \(s\) is known, two more parameters, \(k\) and \(M/P\), are needed for \(H_p\) to be defined. A typical case is \(M/P = k=1\), since the two resonances look very similar, and their trapping domains have the same shape. This case is even the central case of Hamiltonian \(H_p\) as we show in the next section.

2.1.2.3 Equivalent Hamiltonian

Define a new position and a new velocity

\[
y = k(x-t),
\]

\[
w = 1-v,
\]

and a new time \(\tau = kt\). Then \([w(\tau),y(\tau)]\) is governed by the equivalent Hamiltonian

\[
H_e(w,y,\tau) = \frac{1}{2} w^2 - P \cos y - M \cos \frac{1}{k} (y-\tau),
\]
i.e. by a Hamiltonian $H_p$ with parameters $(1/k, P, M)$ instead of $(k, M, P)$. For $M/P = k=1$, the two sets of parameters are identical. This is why we term this case the central case.

The Poincaré map of $H_e$ looks upside down with respect to $H_p$'s. The length unit is multiplied by $1/k$.

Hamiltonian system $S_p$ defined by $H_p$ is equivalently described by $H_e$, though the transformation from $(v, x)$ to $(w, y)$ is noncanonical.

2.1.2.4 Numerical Poincaré Maps

2.1.2.4.1 Regular-Like Motion

Figure 2.5 displays the Poincaré map of some orbits numerically integrated for $s = 1/2$. The picture is symmetric with respect to $v = \frac{1}{2}$ and only trajectories of the lower half-plane have been plotted. The continuous lines correspond to the separatrices predicted by a first order perturbation theory [37]. This figure shows that our naive image is both right and wrong. It is right since we see traces of KAM tori similar to those of fig. 2.4c. The closed KAM tori in the trapping domain of resonance $M$ are tori of $H_p$ with $M \neq 0$, $P=0$, perturbed by a small value of $P$.

The picture is wrong since we see new unpredicted islands. They appear as chains of islands. Figure 2.5 displays new chains of $m$ islands $2 \leq m \leq 5$. Their center therefore corresponds to cycles of period $mT$, or to a mean velocity

$$w_m = \frac{1}{m} \tag{2.13}$$
The island chains replace tori with the same mean velocity that exist for $s=0$. The numerical evidence of the disappearance of those tori agrees with by the Poincaré-Birkhoff theorem that only guarantees existence of at least 2 periodic orbits for each rational mean velocity, having opposite indices if isolated [10]. Thus they are stable and unstable for $s>0$ small. The appearance of $m$ islands in fig. 2.5 for $2 \leq m \leq 5$ shows that the cycles characterized by $w_m$ that are stable for $s$ small are still stable for $s = \frac{1}{2}$. The unstable orbits are difficult to see numerically due to their very nature, but must exist to separate successive islands of a chain where the trapped orbits all rotate in the same direction. The Poincaré-Birkhoff theorem tells us that there are many more chains of islands in fig. 2.5 than displayed. They can be seen numerically, but most of them are so small that they would appear as chains of dots at that scale. At that point we anticipate that the chain of islands with mean velocity $w_m$ corresponds to a resonance that we call $R_m$.

For $s = \frac{1}{2}$ the Poincaré map looks more intricate than the naive picture of fig. 2.4c but no chaos is visible as yet. In particular, an orbit started on the "perturbed separatrix" seems to stick to those "separatrices". Perturbation theory still yields good predictions at that scale for $s = \frac{1}{2}$.

2.1.2.4.2 Blowup of Stochastic Layers

Figure 2.6 displays the Poincaré map of $H_p$ for $s = 0.68$. Now an orbit started on the "perturbed separatrix" no longer sticks to it, but rather seems to create a haze of dots about it. The trace of such an orbit is no longer a curve but a layer. This is the
numerical evidence of the breakdown of the invariant of the motion, and of non-integrability [22]. Furthermore the layers are filled in a fairly erratic way with dots: Sometimes the orbit seems to be a passing one with \( u \) small, sometimes to be a trapped one with \( \rho \) small; it switches in an apparently unpredictable way from one behavior to the other. For this reason these layers are called stochastic layers. In fact there are many more stochastic layers than displayed: each tiny chain of islands has its own. Most of them are so thin at the scale of fig. 2.6 that their stochastic nature is not visible. All these stochastic layers already exist for \( s = \frac{1}{2} \), but are too small to be visible in fig. 2.5. As shown in sec. 2.5.1, the width of the stochastic layer of resonance \( M \) scales like \( \mu^{-3} \exp(-1/\mu) \) where \( \mu = s/2\pi \). This function is close to zero for small \( \mu \)'s and blows up for \( \mu = 1/9 \) (the precise definition of this threshold is given in sec. 4.1.2.2). The origin of this layer is mathematically founded on Melnikov's work [38] which shows that the stable and unstable manifolds of the hyperbolic fixed point corresponding to the unstable equilibrium of the pendulum, intersect transversally and oscillate wildly. More information on stochastic layers, especially their width, is given in secs. 2.5 and 4. For the present value of \( s \), the "perturbed separatrix" still corresponds to the skeleton of the stochastic layer, and we call it a virtual separatrix. The trace of a KAM torus still is visible in fig. 2.6. Therefore no trajectory can cross it.

At that point we see that our naive picture of sec. 2.1.2.1 which was made more complete with additional chains of islands, should be made more complete with stochastic layers surrounding the islands as well. Phase space appears as an intricate superposition of KAM tori.
and of chains of islands surrounded with stochastic layers. The trivial scale invariance of the integrable layer is replaced by a structural scale invariance. If we blow up the vicinity of a KAM torus we see the same qualitative structure at all scales, but in general no simple rescaling allows to match the pictures at different scales.

One more feature that appears in fig. 2.6 is the repulsion of neighboring resonances. The virtual separatrices are distorted with respect to those of the pendulum. This phenomenon can be simply understood through fig. 2.7. Figure 2.7(a) displays the torus \( T \) with mean velocity \( 1 \) of the pendulum Hamiltonian \( H(v,x) = v^2/2 - M\cos x \). The velocity is not a constant along \( T \) but oscillates about 1: the x-point pulls it down and the 0-point pushes it up. If we add a small resonant term \( P\cos(k(x-t)) \) to \( H \), the phase space looks like in fig. 2.7(b) (where \( k=1 \)). A tear occurs in the Poincaré map that keeps the shape of \( T \): it is the trapping domain of resonance \( P\cos(x-t) \). Now, if we let \( P \) grow, the resonant domain of resonance \( M \) is affected too (fig. 2.7(c)). A distortion of that kind is visible in fig. 2.6 for the virtual separatrices of resonances \( M,P \) and \( R_2 \) \( (u=w_2 = 1/2) \).

2.1.2.4.3 Global Stochasticity

Figure 2.8 shows the Poincaré map of \( H_p \) for \( s=1 \). A trajectory started at the cross at \( x=0 \) between the virtual separatrices of resonance \( M \) and resonance \( R_2 \), wanders in phase space between the resonant domains of resonances \( M \) and \( P \). It sometimes is apparently trapped in resonance \( M \) or \( P \). This global behavior looks stochastic since the appearance of points does not seem to follow any rule. This regime is termed global (or large scale) stochasticity. In fig. 2.8
virtual separatrices make no sense in the central part of the picture but still correctly indicate the outer boundary of the stochastic domain (this behavior is explained in sec. 2.5.1). The existence of an orbit wandering between $v=0$ and $v=1$ clearly indicates the breaking up of all KAM tori with mean velocity $u$, $0 < u < 1$. Nevertheless a trapped KAM torus still shows up in the trapping domain of resonance $M$. Outside of this torus a "necklace" of nine islands is visible. Such necklaces are also visible in previous maps when looked at with enough resolution. When blown up, a small island of the necklace displays the same structure as the $M$ resonance. This suggests the existence of islands in the islands, in the islands, etc...., a picture firmly founded mathematically and proposed in particular by Arnold (fig. 52 of ref. [3] and 3.5 of ref. [12]). This second type of scale invariance is related to the bifurcations of periodic orbits. It is only shortly reviewed in the present paper (sec. 2.4) since it is less general than the scale invariance related to KAM tori (see sec. 2.3.2.1).

A simple electro-mechanical system that displays the kind of behavior of the central $H_p$ (with $M/P = k = 1$) is the synchronous dipole motor [19]. For that system $v=0$ (resp. 1) corresponds to the locking of the rotor in one direction of rotation (resp. the other), and $v = \frac{1}{2}$ corresponds to the motor at rest. In the global stochasticity regime, the rotor makes a few turns in one direction, stops, hesitates about the zero velocity, then rotates in the other direction or the same as before, etc.... The motion looks intermittent, and the "laminar" periods correspond to the dots of fig. 2.8 inside the virtual separatrices of resonances $M$ and $P$. The chaotic periods corresponds to the dots in between.
The central $H_p$ also corresponds to the motion of plasma particles in a standing electron plasma wave [24]. Here $v = \frac{1}{2}$ is the velocity of the bulk plasma and the thermal velocity is much smaller than $1/2$ in order for the counter-streaming waves not to be damped. When $s$ is small, the plasma is little affected by the standing wave, but when global stochasticity sets in, the distribution function of the plasma gets a width of order 1, and a heating of the plasma is observed.

2.1.2.4.4 Connection with the Standard Map

A system very often studied in the literature is the standard map [4]. It is defined by

$$x' = x + 2\pi v', \hspace{2cm} v' = v - \frac{K}{2\pi} \sin x.$$  \hspace{1cm} (2.14)

This area-preserving map is often written with the variable $I = 2\pi v$ instead of $v$ or $\Phi = x/2\pi$ instead of $x$. The present normalization is made for comparison with $H_p$. This map can be viewed [4,39] as obtained from the canonical equations of a Hamiltonian $H_s$ that depends on time through a $\delta$-function with period $2\pi$. By using the Fourier expansion of that function one gets

$$H_s(v,x,t) = \frac{v^2}{2} - M \sum_{n=-\infty}^{+\infty} \cos(x-nt), \hspace{2cm} (2.15)$$

where $M = K/4\pi^2$. If the sum is restricted to $n = 0$ and 1, $H_s$ is the central $H_p$. $H_s$ therefore can be viewed as the periodization in $v$ of
the central $H_p$. $H_s$ is more complex than $H_p$ and has non-generic properties (existence of accelerating modes [4], periodicity in $v$), yet each iteration of the map yields one point of the Poincaré map of $H_s$, whereas several tens of integration steps of the equations of motion of $H_p$ are necessary to get one point of the Poincaré map of $H_p$. From a computational point of view, the standard map is easier to handle. Nevertheless its theoretical analysis is more intricate than $H_p$'s.

Fortunately the universal properties of Hamiltonian systems allow us to combine the advantages of both $H_p$ and $H_s$: one can think with $H_p$ and compute with $H_s$! Anyway they differ in several ways: there is a periodicity in $v$ and the existence of accelerating modes for $H_s$; the destabilization of cycles occurs for a unique value of $s$ for $H_s$, but, there is a sequence of destabilizations–restabilizations for $H_p$. À la Mathieu equation (see sec. 3.4); for large values of $s$, the stochastic orbits of $H_p$ enter a trapping regime where quasilinear theory breaks down, whereas quasilinear theory becomes excellent for $H_s$ because of the periodicity in $v$ (sec. 2.6.1).

After this introduction to Hamiltonian stochasticity, we now describe the problems posed by this phenomenon.

2.2 Presentation of the Problems of Global Stochasticity

The appearance of large-scale chaos in Hamiltonian systems poses many problems for physicists. Three of them have received a lot of attention: the calculation of thresholds of global stochasticity, the structure of critical KAM tori, and the description of chaotic motion. A renormalization group for KAM tori proves to be all-important for solving the first two problems. This section introduces the general
questions related to global stochasticity that will be addressed to in
the remaining part of the paper, and recalls some important steps of
their solution.

2.2.1 Threshold of Global Stochasticity

Estimating the threshold of global stochasticity is a
non-trivial problem. In 1959 Chirikov [40] proposed the simple overlap
criterion which states that the threshold should occur for \( s = s_c = 1 \),
i.e., when the sum of the half-widths in velocity of the unperturbed
resonances is equal to their velocity mismatch. For \( M \) and \( P \) of the
same order of magnitude, this prediction is correct within a 30% error
in \( s \) [25]. If \( M \) and \( P \) have different orders of magnitude, this
criterion makes no sense. For instance, for \( M = 1/4 \), \( 2\sqrt{M} = 1 \) and the
overlap criterion would suggest that \( H_p \) is stochastic for
\( P = 0 \). Furthermore, the use of this criterion for Hamiltonian systems
that have more than two primary resonances requires first the
approximation of the given Hamiltonian by one with two resonances.

In general, the amplitude of the resonances depend on the
unperturbed action of the system. For instance, consider the
Hamiltonian

\[
H(v,x,t) = \frac{1}{2} \sigma v^2 + \frac{1}{3} \lambda v^3 - M(v) \cos x - P(v) \cos k(x-t) . \tag{2.16}
\]

Chirikov's prescription for applying the overlap criterion to \( H \) is to
compute the amplitudes at the resonant values of \( v \). Thereafter, this
approximation is called the centered-resonance approximation. This
yields for \( \lambda \) small
\[ s = 2\left[\sigma M(0)\right]^{1/2} + 2\left[\sigma(1+2\lambda/\sigma^2)P(1)\right]^{1/2} . \] (2.17)

$s$ depends neither on the details of $M(v)$ and $P(v)$, nor on $k$, whereas the actual threshold does. Furthermore, the overlap criterion is anterior to KAM theory and has little connection with it. Anyway, it is very intuitive, quite easy to memorize, and gives right orders of magnitude of the thresholds in most cases of practical importance. Furthermore, sec. 3.1.5 shows that the centered-resonance approximation is the correct thing to do if $M(v)$ and $P(v)$ vary on a scale larger than 1 and if $\lambda <\ll \sigma^2/2$.

KAM theory motivated people to look for more precise methods accounting for it. An important step in that respect is Greene's work [41] that focuses in the standard map on the golden mean KAM torus defined by $u = 1/g = g-1$ where $g$ is the golden mean

\[ g = \frac{\sqrt{5} + 1}{2} , \] (2.18)

His work clearly recognizes KAM tori as the barriers to global stochasticity, links their existence to the stability of nearby cycles, and yields the threshold of global stochasticity with a high accuracy. However, his method has some drawbacks: it is numerical, finding long cycles close to a given KAM torus in a general Hamiltonian system is not an easy numerical task, and one does not know a priori what is the most robust torus (the last barrier to global stochasticity): the initial guess [41] of $u = 1/g$ is not always the right one [42]. Nevertheless sec. 4.3.3 shows that Greene's method already is a renormalization method.
Upper bounds for the threshold of global stochasticity have been obtained by Lieberman and Lichtenberg [39] by requiring the function \( v(x,t) \) of sec. 2.1.1.3 to be single valued. This shows that the threshold for the standard map is less or equal to 2. By using a theorem of Birkhoff, Mather rigorously shows that the threshold for the standard map is less or equal to 4/3 [43]. An extension of this method allows MacKay and Percival [44] to decrease this upper bound to 63/64 \( \approx 0.9844 \) which is very close to the exact value \( K_c \approx 0.9716 \) [41], but a lot of computer calculations are necessary.

2.2.2 Renormalization Approach and Criticality

The importance of higher order islands and of rescaling for stochasticity was originally recognized by Jaeger and Lichtenberg [5,45,120], but did not lead immediately to a renormalization approach. As said in the introduction (sec. 1.3), the first renormalization scheme for Hamiltonian systems was obtained in 1980 when trying to apply the overlap criterion to subsystems of \( S_p \) [25]. A KAM torus showed up as the limit of infinitely nested small strips in a Poincaré map. Though approximate, this scheme strongly suggested the existence of an exact renormalization group that got further confirmation from renormalization schemes for maps [26–27]. It was later recognized [33,46] that Greene's method is perfectly consistent with the renormalization approach and that his main numerical findings could be understood in that frame. The renormalization approach tells the threshold for a given KAM torus but also allows us to find out where the last KAM torus is (sec. 4.1), and therefore solves the problem of the computation of the threshold of global stochasticity. Rapid
estimates can be obtained with no more calculation than required for applying the resonance overlap criterion, but they are more accurate: for $H_p$, $s_c$ is given within 4% instead of 30% and it has a correct dependence on both $k$ and $M/P$ (sec. 2.3.2.5). Those estimates can be refined with an arbitrary precision by computing successive exact renormalizations of the system under study. One exact renormalization already yields the critical value of $s$ for $H_s$ within 1% (sec. 3.2.4). Naturally the gain in precision is done at the expense of simplicity and rapidity, but the use of symbolic manipulators should improve the feasibility of high-order calculations. In general if high accuracy on the threshold is required it is easier to use numerical methods that rely upon universal features of Hamiltonian systems (sec. 4.2).

These universal features are a mere consequence of the existence of a renormalization group for KAM tori. They are typical of critical KAM tori, i.e., tori at their threshold of breakup. Universal parameters can be computed from renormalization schemes [15,26,27], but were first estimated through direct numerical calculations on specific Hamiltonian systems [27,47].

2.2.3 Description of Chaotic Motion

When deterministic Hamiltonian motion becomes seemingly chaotic it is tempting to approximate it by a stochastic process. Right above $s_c$, this is not quite correct since the last KAM torus develops a dense set of gaps and breaks up into the product of a Cantor set (along the $x$-axis) and of a circle (along the $t$-axis). This set, called Cantorus [48,49], has small holes which govern the flux of
set (along the x-axis) and of a circle (along the t-axis). This set, called Cantorus [48,49], has small holes which govern the flux of chaotic orbit between the parts of the phase space which were disconnected for \( s < s_c \). Therefore just above \( s_c \), the global chaotic motion is still governed locally in phase space, and the renormalization group yields the critical exponent for the flux [50,51] (sec. 2.3.2.4.2).

Further increase of \( s \) yields an apparent global chaos, provided the effect of possible chains of islands is neglected. In that regime, quasi-linear approximation applies (sec. 2.6.1) and allows one to compute a diffusion coefficient \( D_{QL} \) for the action of the unperturbed integrable Hamiltonian system (for instance \( v \) for Hamiltonian (2.15)) [4,12,52]. This estimate can be improved by taking into account some finite time correlations [53,54].

For \( s \) quite large two different behaviors can appear: either the interacting primary resonances are spread quite uniformly in phase space, and the quasi-linear picture stays correct (case of \( H_S \)), or they lie in a bounded region of phase space (case of \( H'_P \)), and the quasi-linear picture breaks down. A trajectory makes loops in phase space with many turns, and an adiabatic invariant appears to be almost preserved, yet it experiences series of small kicks. This allows the estimate of a trapping diffusion coefficient \( D_{tr} \) [55] that scales quite differently from \( D_{QL} \) (sec. 2.6.1). Therefore the chaotic transport that occurs in the crossover domain of these two regimes is unlikely to be a diffusion process and has not yet been seriously studied.
We now describe the presently available answers to the problems of Hamiltonian-stochasticity and begin with KAM universality.

2.3 KAM Universality: Qualitative Picture

Though no mathematical result be presently available to prove the renormalization group picture, all numerical calculations and renormalization schemes suggest the same qualitative picture this section describes. We first begin with simple ideas about what renormalization is for Hamiltonian systems.

2.3.1 Renormalization Schemes

2.3.1.1 Renormalization for Hamiltonians

Here we present the basic ideas of an approximate renormalization scheme for $H_p$ which is in the spirit of ref. [25]. This scheme is shown in sec. 3.2.2 to be a computable example of a more general theory. It is interesting to first deal with a simple scheme since it can be described in a very graphical way.

2.3.1.1.1 Pictorial Approach

Figure 2.9(a) displays a schematic of the Poincaré map of the central $H_p$ ($k = M/P = 1$). In the language of sec. 2.1.2.4.1, we see resonances $R_1$, $R_2$ and $R_\infty$, i.e. resonance $P$ with mean velocity $w_1 = 1$, the resonance with $w_2 = \frac{1}{2}$, and resonance $M$ with $w_\infty = 0$. A torus $T$ with mean velocity $u$ is displayed as well. As we did for fig. 2.5, at that scale we forget about other chains of islands between $P = R_1$ and $R_2$. Basically the renormalization process consists of defining the small box in fig. 2.9(a) and blowing it up so that it
becomes the large box in fig. 2.9(b). During this magnifying process
the picture is put upside down as indicated by arrows. In fig. 2.9(b)
a new chain of 3 islands shows up which corresponds to the symmetric to
$R_3$ with respect to $v = 1/2$ in fig. 2.5. The magnifying process comes
up with a change in origin of the mean velocities too: $w_1, w_2, \text{ and } u$
become respectively $0, 1$, and $u' = (u-w_1)/(w_2-w_1)$; the chain of 3
islands has a mean velocity $w_2$. The important point is that the
dynamics in the new coordinates $(w', x')$ is governed by a new paradigm
Hamiltonian $H_p'$ with parameters $(k', M', P')$ explicitly computable from $u$
and $(k, M, P)$: obviously $k' = 2$ in fig. 2.9(b).

The renormalization process can be iterated by defining a new
small box in fig. 2.9(b) (notice that a rescaling of the $x$ axis is also
involved in that process so that the periodicity $\pi$ of resonance $E$
becomes $2\pi$ for resonance $E'$). This defines a sequence of
$(u(n), H_p(n)) = (u(n), k(n), M(n), p(n))$. Therefore the renormalization $\mathcal{R}$
acts on the couple $(\mathcal{F}, H)$ of a KAM torus and of a Hamiltonian, or more
precisely on the couple of a mean velocity and of a Hamiltonian. We
defer to sec. 3.1.1 the calculation of $M'$ and $P'$, and we give now the
qualitative results.

When the blow-up process is iterated, basically two evolutions can
occur for a given value of $u(0)$ and $k(0)$, the initial value of $u$ and $k$
(fig. 2.10): either $(M(n), p(n))$ goes to $(0, 0)$ or it goes to $(\infty, \infty)$. In
the first case, after a finite number $N$ of iterates of $\mathcal{R}$, $(M(N), p(N))$
is close enough to $(0, 0)$ for the KAM theorem to apply. Therefore $\mathcal{F}$
is preserved in the $N^{th}$ subsystem ($N^{th}$ little box). Since this system
is nothing but a part of the original one (at least within the
approximations of the theory), $\mathcal{F}$ is preserved in the original system.
Notice that the convergence to \((0,0)\) for \((M,P)\) small enough is a mere consequence of KAM theorem. Renormalization just extends the region where this is shown to occur. We term this region the KAM basin.

As shown by Percival [48] and Aubry [49], and later on proved by Aubry et al. [56] Mather [57], and Katok [58], a KAM torus develops a dense set of gaps and breaks up for \(s\) large enough into a Cantorus [48]. Just above threshold, each hole of the Cantor set is very small and the trace of a Cantorus in the Poincaré map looks similar to the trace of a KAM torus. The computation of a positive Lyapunov exponent on the Cantorus* [59], the existence of a stochastic trajectory that crosses the Cantorus [4], or the use of a criterion for the non-existence of tori [39, 43, 44], makes it possible to numerically prove the breaking up of the torus into a Cantorus. In fig. (2.10), the basin of attraction of \((\infty, \infty)\) is the domain of existence of a Cantor set with a mean velocity \(u\). We call this basin the Cantorus basin.

There is a curve \(C\) that separates the KAM and the Cantorus basin. Without further assumption on \(u^{(0)}\), when \((M^{(0)}, P^{(0)})\) belongs to \(C\), \((M^{(n)}, P^{(n)})\) may have a weird motion in \((M, P)\) space, but never falls on \((0,0)\) or \((\infty, \infty)\). When \((M^{(0)}, P^{(0)})\) is on \(C\), there is a torus and it is said to be critical. In order to further discuss critical tori we need to define the concept of zoning number \(z\).

*This Lyapunov exponent is \(h = \lim_{t \to \infty} \frac{1}{t} \ln |x_c(t) - x_n(t)|\) when \(t \to \infty\), where \(x_c(t)\) is one orbit of the Cantorus and \(x_n(t)\) is an orbit started very close to \(x_c\).
2.3.1.1.2 Zoning Number

For $M$ and $P$ small of order $\varepsilon$, we know from the KAM theorem that some orbits are only slightly distorted with respect to the uniform motion of the free particle. We can compute such orbits perturbatively in powers of $\varepsilon$. The calculation shows that the uniform motion of the free particle is perturbed, in particular, by higher order resonances of the type $\alpha_k^P \cos[(k+\ell)x - kt]$ where $\alpha_k$ is a constant (the case $\ell=1$ corresponds to the ponderomotive resonance). The phase of such a resonance is stationary for $\dot{x} = k/(k+\ell)$. For $k=1$, this is $\dot{x} = \omega_{k+1}$ where $\omega_m$ is defined by eq. (2.13), and the corresponding resonance is resonance $R_{k+1}$ of sec. 2.1.2.4.1. This explains the nature of the chains of islands we pointed out in fig. 2.5. For any value of $k = r/p$, we call resonance $R_m$ the resonance with phase velocity

$$\omega_m = \frac{k}{k+m-1} = \frac{r}{r+(m-1)p}. \quad (2.19)$$

For the value of $k$, the period of the Poincaré map is $T = 2\pi/k$. The mean distance between successive points of an orbit with mean velocity $\omega_m$ in this map therefore is $p2\pi/[r+(m-1)p]$. Since the map has a spatial periodicity $p2\pi$ (periodicity of $H_p$ in $x$), the period of the two cycles corresponding to resonance $R_m$ is

$$T_m = r + (m-1)p \quad (2.20)$$

which also is the number of islands of $R_m$. For $k=1$, $m$ is the number of islands in the chain of $R_m$ in fig. 2.5.
Owing to the obvious importance of the $R_m$'s for building our renormalization scheme, we characterize each orbit of $H_p$ that exists for $M=P=0$, by a zoning number $z$ which generalizes $m$ to non-integer values. The mean velocity $u$ of an orbit is expressed in terms of $z$ through

$$u = \frac{k}{k+z-1}, \tag{2.21}$$

which is a mere generalization of eq. (2.19). Accordingly the zoning number is

$$z = k\left(\frac{1}{u} - 1\right) + 1. \tag{2.22}$$

Its value is fixed by both the mean velocity of the orbit and by the ratio $k$ of the spatial periodicity of resonances $M$ and $P$.

Even if the initial value of $k$, $k^{(0)}$, is an integer, the value of $k$ after $n$ iterations of $\mathcal{R}$, $k^{(n)}$, for $n>1$ is not. Figure 2.11 displays a graphical description of the second iterate of $\mathcal{R}$ similar to that of fig. 2.9. The small box in fig. 2.11(a) is similar to the small box of fig. 2.9(b). It encompasses a chain of 2 and a chain of 3 islands. In fig. 2.11(a) they correspond to $R_2$ and $R_3$. In fig. 2.11(b), the torus $\mathcal{F}$ of interest appears to be between a chain of 5 and a chain of 6 islands. Since $M'$ and $P'$ respectively have 3 and 2 islands, $k^{(1)} = k'$ is 2/3, and $r=2$, $p=3$. According to eq. (2.20), $R_m'$ has $2+(m-1)\times 3$ islands, and $\mathcal{F}$ is between $R_2'$ and $R_3'$. According to eq. (2.19) the zoning number $z$ of $\mathcal{F}$ verifies $2 < z < 3$. Here the
rescaling in \( x \) allows the new resonance \( M \) to have a periodicity \( 2\pi \), whereas it had a period \( 2\pi/3 \) originally.

According to eq. (2.11) the mean velocity \( u_e \) of \( \mathcal{F} \) in the equivalent coordinates \((w,z)\) is \( u_e = 1-u \), and according to eq. (2.22) the equivalent zoning number verifies

\[
z_e-1 = \frac{1}{k} \left( \frac{1}{u_e} - 1 \right) = \frac{1}{z-1}.
\]  

Thus \( z=2 \) is an invariant zoning number when one goes from \( H_p \) to \( H_e \). \( R_2 \) is the same resonance in \( H_p \) and \( H_e \). Equation (2.23) tells that orbits of \( H_p \) with \( z>2 \) are orbits of \( H_e \) with \( z_e < 2 \); orbits of \( H_p \) between \( R_1 = P \) and \( R_2 \) are orbits of \( H_e \) between \( R_\infty = M_e \) and \( R_2 \). For the central \( H_p \) \((k = M/P=1)\), \( H_p \) and \( H_e \) are identical, therefore figs. (2.9) and (2.11) focus on the same KAM torus.

Let \( m \) be the zone number containing \( \mathcal{F} \) in the original system. We generalize the picture of fig. 2.9 by requiring that resonances \( M' \) and \( P' \) be resonances \( R_m \) and \( R_{m+1} \) and

\[
u' = \frac{u - w_m}{w_{m+1} - w_m}.
\]  

As a result \( k' \) is the ratio of the number of islands of \( R_{m+1} \) by the corresponding number for \( R_m \). According to eq. (2.20)

\[
k' = \frac{T_{m+1}}{T_m} = 1 + \frac{1}{m+k-1}.
\]
From eqs. (2.22), (2.21), (2.24), and (2.25) it results that

\[ z' = \frac{1}{z-m} \quad (2.26) \]

Define the continued fraction expansion of \( z \) as

\[ z = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} = [a_0, a_1, a_2, \ldots] \quad (2.27) \]

where the \( a_i \)'s are integer. The fact that \( \mathcal{F} \) is between \( R_m \) and \( R_{m+1} \) means that \( m < z < m+1 \). Therefore \( m = a_0 \). According to relation (2.26), the coefficients \( a'_i \) of \( z' \) are given by

\[ a'_i = a_{i+1}, \quad i = 0, 1, 2, \ldots \quad (2.28) \]

Therefore \( \mathcal{R} \) acts like a shift on the continued fraction expansion of the zoning number. After \( n \) iterations of the renormalization, \( \mathcal{F} \) is between \( R_m^{(n)} \) and \( R_{m+1}^{(n)} \). According to (2.28) \( m_n = a_n \) for any \( n \). Therefore the \( a_n \)'s are the successive zone numbers corresponding to \( \mathcal{F} \) in the successive nested subsystems. In fig. 2.9 \( \mathcal{F} \) has a zoning number with \( a_0 = a_1 = 1 \). In fig. 2.11, \( a_0 = 2 \) and \( a_1 = a_2 = 1 \). According to (2.23), in this case \( z_e \) has a continued fraction expansion with \( a_0 = 1 \) and \( a_1 = a_2 = a_3 = 1 \).

2.3.1.1.3 Universal Classes

If the \( a_n \)'s are periodic with a periodicity \( L \) (equivalently \( z \) is a quadratic irrational \([60]\)), eq. (2.26) shows that
$z(n)$ is periodic with period $L$. Through $R^L$, $z$ maps into itself. According to eqs. (2.25) (2.28),

$$k(n) = [b_0, b_1, \ldots, b_n] \tag{2.29}$$

with $b_0 = 1$, $b_i = a_{n-i}$ for $0 < i < n$, and $b_n = a_0 + k(0) - 1$. We slightly generalize the definition of the continued fraction expansion to possibly accommodate a non-integer $b_n$. For $k(0) = 1$, the continued fraction expansion of $k(n)$, is like $z$'s truncated at order $n-1$ and written in descending order [61]. It is a property of the continued fractions that if the $a_i$'s are periodic with period $L$, $k(mL)$ converges toward a value $k_*$ that does not depend on $k(0)$.

$(M(n), p(n))$ has a non-trivial evolution. For simplicity take $k(0) = k_*$. For $n = mL$, it can be proved that the mapping $\mathcal{M} : (M(n), p(n)) \rightarrow (M(n+L), p(n+L))$ has a single hyperbolic fixed point $F$ and two sinks $(0,0)$ and $(\omega, \omega)$, as shown in fig. 2.12. In that case the curve $\mathcal{C}$ of fig. 2.10 is the stable manifold of $F$ and the successive values of $(M(mL), p(mL))$ lie on branches of hyperbolic-like curves that go through $(0,0)$, the KAM fixed point, or $(\omega, \omega)$. If $(M_0, p_0)$ belongs to $\mathcal{C}$, $(M(mL), p(mL))$ converges toward $F$. Therefore $H_{\mathcal{P}}^{(mL)}(mL)$ converges toward a Hamiltonian $H_*$. This means that when looked at with high enough a resolution, all successive scales look alike modulo a rescaling. At $H_*$ the left part of fig. 2.9 (resp. 2.11) looks identical to the right part of that figure for $L=1$ (resp. $L=1$ or 2). At small scale, a more general $H_\mathcal{P}^{(0)}$ with $(M(0), p(0))$ on $\mathcal{C}$, also looks like $H_*$. $\mathcal{C}$ is the stable manifold of an extended $\mathcal{M}$ that acts on $(k, M, P)$. At small scales, the system $S_\mathcal{P}$ therefore displays in the
neighborhood of $\mathcal{C}$ a nontrivial scale invariance, as for critical phenomena. This motivates to term $\mathcal{I}$ critical at the threshold of breakup. In contrast to critical phenomena, however, the series of scales are discrete. Therefore $\mathcal{R}$ is a map, not a flow.

For any $H_p$, $\mathcal{I}'$ critical has the same structure at small scale. The law (2.28) on the renormalization of $z$ shows that a torus $\mathcal{I}'$ with a zoning number $z'$ which has the same continued expansion as $z$ from rank $n \geq N$, is renormalized as $\mathcal{I}$ from the $N$-th step on. We say that $z$ and $z'$ are equivalent zoning numbers and that $\mathcal{I}'$ is similar to $\mathcal{I}$. If $\mathcal{I}'$ is critical, it looks like $\mathcal{I}$ critical at small scale. This shows that $F$ describes the universal behavior of critical $\mathcal{I}'$'s (characterized by $z$) in any $H_p$, but also for similar tori. The set of KAM tori similar to $\mathcal{I}$ is called the universality class of $F$. The universality of $F$ is twofold: with respect to the $H_p$'s and with respect to equivalent zoning numbers.

An important universality class is that of the golden mean $g$ defined by eq. (2.18). In that case all the $a_i$'s are equal to 1. After Percival [62] the KAM tori of this class are said to be noble.

2.3.1.2 Renormalization for Maps

Very often area-preserving maps can be described by Hamiltonians (see [12] and sec. 2.1.2.4.4) and their behavior can be understood through renormalization schemes for Hamiltonians. One also can directly develop renormalization schemes for describing them [26–27]. They can be used for describing the Poincaré map of a given Hamiltonian.
2.3.1.2.1 Intuitive Approach

Consider a one-parameter family of area-preserving maps $\mathcal{P}: (v,x) \rightarrow (v',x')$. For instance $\mathcal{P}$ is the standard map or $\mathcal{P}_p$, the Poincaré map of $S_p$ for a given ratio $M/P$ and a given $k$. We assume

$$\mathcal{P}(v,x+2\pi) = \mathcal{P}(v,x) + (0,2\pi). \quad (2.30)$$

In the case of $\mathcal{P}_p$, this occurs for $k$ integer. We focus on a given KAM circle $\mathcal{C}$ with rotation number $\rho$. For $\mathcal{P}_p$, $\mathcal{C}$ is the intersection of a KAM torus with the plane $t=0 \pmod{2\pi}$, and $\rho$ is given by eq. (2.8). Different points on $\mathcal{C}$ can be distinguished by an angle parameter $\psi$, which is advanced by the same amount at each iterate of the map. $\mathcal{C}$ may be written as $Z(\psi) = [v(\psi),x(\psi)]$ with $x(0) = 0$. The KAM theorem [10] implies that the effect of $\mathcal{P}$ is to advance $\psi$ by $\Delta \psi = \rho 2\pi$

$$\mathcal{P}[Z(\psi)] = Z(\psi + \Delta \psi). \quad (2.31)$$

Let $K$ be the map parameter and $K_c$ be the threshold of breakup of $\mathcal{C}$. For definiteness we focus upon a particular value of $\rho$, $\rho=g^{-1}g^{1}$. The Fibonacci numbers $f_n$ are defined by

$$f_{n+1} = f_n + f_{n-1}, \quad (2.32)$$

and $f_0=f_1=1$. Let $\mathcal{Q}$ be the map defined by

$$\mathcal{Q}(v,x) = (v,x-2\pi). \quad (2.34)$$

According to eq. (2.30)
\[ \mathcal{Z}(\psi) = Z(\psi - 2\pi) \quad (2.35) \]

The application to \( Z(\psi) \) of \( \mathcal{P}^n \) times and of \( \mathcal{Z}^n \) times leads to a new point on \( \mathcal{C} \) very close to the old one for \( n \) large

\[ \mathcal{P}^n \mathcal{Z}^n-1[Z(\psi)] = Z(\psi + \Delta \psi_n) \quad (2.36) \]

since

\[ \frac{\Delta \psi_n}{2\pi} = \rho \frac{f_n - f_{n-1}}{1} = (-1)^n \rho^{n+1} \quad (2.33) \]

is a geometrically decreasing series \([60]\). Let

\[ y = v - \alpha_1 x - \alpha_2 x^2 - \alpha_3 x^3 \quad (2.37) \]

where \( \alpha_n = \frac{1}{n!} \frac{\partial^n v}{\partial x^n} (x=0, K=K_c) \) computed on \( \mathcal{C} \); for any \( i \alpha_{2i+1} = 0 \) for \( \mathcal{P} \) because \( x=0 \) is a symmetry axis, but \( \alpha_{2i+1} \) is nonzero for the standard map. Define components \( \mathcal{P}_n^i, i=x,y \) by

\[ [\mathcal{P}_n^y(y,x), \mathcal{P}_n^x(y,x)] = \mathcal{Z}^{n-1} \mathcal{P}_n^y(y,x) \quad (2.38) \]

For \( K=K_c, n \) large and small \( y \) and \( x \), results established mainly on the standard map \([47]\) suggest the scaling laws \([63]\)

\[ \mathcal{P}_n^y(y,x) = \beta_0^{-n} \mathcal{P}_y^* (\beta_0^n y, \alpha_0^n x) , \]

\[ \mathcal{P}_n^x(y,x) = \alpha_0^{-n} \mathcal{P}_x^* (\beta_0^n y, \alpha_0^n x) \quad (2.39) \]
where $P_y^*$ and $P_x^*$ are universal functions, and $\alpha_0$ and $\beta_0$ are universal scaling constants

$$\alpha_0 = -1.41485 \pm 0.00003,$$

$$\beta_0 = -3.06866 \pm 0.00003.$$ (2.40)

This formulation is strongly reminiscent of the period-doubling universality [36]. Similar relations hold close to $x=\pi$ with other scaling parameters $\alpha_\pi$ and $\beta_\pi$, but the following relation holds

$$\alpha_0 \beta_0 = \alpha_\pi^\prime \beta_\pi^\prime = \xi.$$ (2.41)

The parameter $\xi$ defined by this equation plays the role of an area multiplier later in this paper. Kadanoff derived a renormalization scheme for maps bearing out scaling law (2.38) [26]. Later on, MacKay proposed another scheme with a better convergence of the approximate fixed point solutions [27].

2.3.1.2.2 A Renormalization Scheme

We now follow MacKay's approach, and interpret eq. (2.39) as the hint that there exists a series of coordinate changes $B_n$ such that

$$P_n = B_n^{-1} P f_n (B_n)^{-1} 2^n B_n.$$ (2.42)

converges toward a universal map $P_\star$ that characterizes the noble universality class.
In order to make tractable calculations we want to define $\mathcal{P}$ through its action generating function. Such a function can be defined for $\mathcal{P}$, but not for $\mathcal{L}$. However, we notice that $\mathcal{P}$ and $\mathcal{L}$ commute, and we generalize the problem by looking for $\mathcal{P}$, as a limit of $\mathcal{P}_n$'s defined from eq. (2.42) for commuting pairs of area-preserving maps $(\mathcal{P},\mathcal{M})$. We define a renormalization operator $N$ acting on $(\mathcal{P},\mathcal{M})$ by

$$
N: \mathcal{P}' = B \mathcal{P} MB^{-1},
$$
$$
\mathcal{M}' = B \mathcal{M} B^{-1},
$$

(2.43)

where $B$ is a coordinate change to be defined later. It follows from eq. (2.32) that

$$
N^n(\mathcal{P},\mathcal{M}) = (\mathcal{P}_n, \mathcal{B}_n \mathcal{P}^{n-1} \mathcal{M}^{n-2} \mathcal{B}_n^{-1}),
$$

(2.44)

where $\mathcal{B}_n$ is the composition of the successive coordinate changes. This renormalization scheme is reminiscent of those for the period doubling [36].

An approximate fixed point of $N$, and the corresponding tangent mapping are found by approximating the action generating function of $\mathcal{P}$, by a polynomial of some degree. This yields the values of the universal parameters. They agree with the values given in table 2.1, originally found by looking at specific maps.
This renormalization scheme can be generalized to more general rotation numbers than $\rho = g - 1$ [27]. The continued fraction expansion of $\rho$ plays the same role as played by the continued fraction expansion of $z$ for the renormalization scheme for Hamiltonian systems. We come back later to this. There is also another fixed point of $N$, for the same value of $\rho$, which corresponds to the KAM fixed point (the origin of fig. 2.12) [27].

It is easy to make the link between the scheme for maps and the scheme for Hamiltonians as described by fig. 2.9. In the $L$-th subsystem, $S_p^{(L)}$, resonance $p^{(L)}$ has a mean velocity 1, and a zoning number 1. Let $\mathcal{T}$ be the golden mean torus related to $F_1$. According to eqs. (2.27-28) and (2.32), in the original system $S_p^{(0)}$ resonances $p^{(L)}$ corresponds to a zoning number $z^{(0)} = f_{L+1}/f_L$ and, from eqs. (2.8) and (2.19) to a rotation number

$$\rho_L = \frac{1}{r + (z^{(0)} - 1)p}, \quad (2.45)$$

where $r$ and $p$ are given by $k^{(0)} = r/p$. For $k^{(0)} = 1$ as in fig. 2.9(a), $\rho_L = (f_L/f_{L+1})$. The stable (resp. unstable) cycle with zoning number $z^{(0)}$ corresponds in fig. 2.9 to the center (resp. the intersection) of the virtual separatrices of the corresponding chain of islands. Let $L = n - 1$. Then a point of the cycles with zoning number $z^{(0)}$ is a fixed point of $\mathcal{D}_p^{(n)} \mathcal{D}_n^{(n-1)}$, where $\mathcal{D}$ is defined by eq. (2.34). The coordinate change $\mathcal{B}_n$ in eq. (2.42) allows for the picture of the cycles with rotation number $\rho_n$ to look the same for any large $n$ if $\mathcal{T}$ is critical. $\mathcal{B}$ in eq. (2.43) corresponds to the passage from the small box of fig. 2.9(a) to the large box of fig. 2.9(b). Therefore
eq. (2.32) is the recursion law for the period of the cycles that approximate the KAM circle at successive scales. This law is the same for the golden mean rotation number of the circle map [64,65]. It replaces the law \( f_{n+1} = 2f_n f_0 = 1 \), of the period doubling problem, which is a one-point recursion formula. Equation (2.32) is a two-point recursion formula. Therefore both \( f_0 \) and \( f_1 \) must be specified. Similarly the renormalization operator acts on a pair of maps [26–27] and no longer on a single map as for the period-doubling case [36].

If \( k^{(0)} \) is not an integer \((p \neq 1)\), eq. (2.45) shows that the truncation of the continued fraction expansions of \( \rho \) and of \( z \) are different. In this case, apparently the renormalization schemes for Hamiltonians and for maps do not act identically on mean velocities. The reason for this difference is hard to elucidate presently, since the renormalization scheme for maps has been defined explicitly (through the definition of \( \mathcal{B} \)) only at the noble fixed point where the zoning number and the rotation number for commuting pairs of maps [27] are identical (since \( k \) is irrational, a rotation number cannot be defined through eq. (2.8)).

Physically a universality class in a given Hamiltonian system is not defined by a property of the rotation number \( \rho \) alone, but the ratio of spatial periodicities of primary resonances come in as well (for \( S_p \) this is the parameter \( k \)). The universal parameter defining the classes is the zoning number which is a compromise between this ratio and the rotation number. This point has been missed in numerical studies [27,41,47] because they focussed on the standard map for which \( k=1 \). From the point of view of KAM theory the distinction between \( \rho \) and \( z \) is irrelevant, since the attractiveness of the KAM fixed point only
depends on Diophantine condition (2.9) which is equally verified by $\rho$ and $z$.

2.3.2 Common Picture of Renormalization

The schemes for Hamiltonian [25,35] and for maps [26,27], and numerical calculations [27,41,47] support the same picture of the renormalization group, but it is not mathematically proven. For a general Hamiltonian of the KAM type

$$H(\dot{p},\dot{x}) = H_0(\dot{p}) + \epsilon V(\dot{p},\dot{x}),$$

(2.46)

where $H_0$ is nonlinear and $V$ is of class $C^{4+\eta}$, for any $\eta > 0$ in $(\dot{p},\dot{x})$, i.e. four times continuously differentiable, they predict a behavior of the kind shown in fig. 2.10 for any KAM torus with a zoning number of the constant type. Now the space $(M,P)$ is the space of Hamiltonians and the origin corresponds to $H_0$; $C$ is a manifold of co-dimension 1: it separates the KAM and Cantorus basins.

2.3.2.1 Universal Classes

If the zoning number $z$ has a periodic continued fraction expansion, the picture is like fig. 2.12 with the same extensions of dimensionality. Therefore the concept of universality defined for $H_0$ is applicable to all 1.5 or 2 degree-of-freedom Hamiltonian systems. The KAM point corresponds to the trivial scale invariance of the free-particle Hamiltonian (fig. 2.1). A fixed point of renormalization $R$ corresponds to a nontrivial discrete scale invariance. This discreteness is reflected in the fact that $R$ is a
mapping and not a flow as for critical phenomena. Even though the analytical calculation of the characteristics of the fixed points be tedious (secs. 2.3.1.2 and 3.2.2), they can be measured by numerical calculations of orbits in one specific Hamiltonian system. Historically, this is what was first done [41,47].

As yet no evidence has been brought against the generality of that picture. Section 3.2.5 shows how to build a counterexample, but it obviously is of codimension 1 and thus of probability 0. KAM universality therefore has to be understood in a quite strong sense. This is to be contrasted with period n-tupling universality. Since cycles have several generic bifurcations [66], other scenarios of bifurcation than the period-doubling sequence can be seen [67,68]. Therefore period-doubling "universality" is not universal in a strong sense.

Putting together KAM theory and renormalization group analysis for KAM tori, yields a picture like fig. 2.13. It is a sketch of the functional space of Hamiltonians in the vicinity of \( H_0 \) which is integrable (for instance \( H_p \) with \( M=P=0 \)). This functional space is the space of Hamiltonians of class \( C^4 \) (KAM theorem is proved in class \( C^4 \) and there are counterexamples of class \( C^4-\eta \), for any \( \eta>0 \) [69]; the proofs in fact are for maps with one order of differentiability below Hamiltonians). The shaded area in fig. 2.13 corresponds to the Cantorus domain for the universality class described at criticality by the Hamiltonian \( H_* \). In certain directions the corresponding torus \( \mathcal{T} \) is never destroyed (integrable directions for instance \( H_p \) with \( P=0 \)), even infinitely far from \( H_0 \). The KAM theorem states that there is an \( \varepsilon \) such that \( \mathcal{T} \) is preserved in a bowl of radius \( \varepsilon \) about \( H_0 \) (in any
Hamiltonians fairly close to \( H_0 \) can be found such that \( \mathcal{T} \) is destroyed, for instance \( H_c \) in fig. 2.13 [69]. This gives low upper bounds to the maximum value \( \varepsilon_m \) of \( \varepsilon \) (in fig. 2.13 circle \( \Sigma \) has a radius \( \varepsilon_m \)), which are close to present lower bounds on \( \varepsilon \) (the dashed circle in fig. 2.13 is of radius \( \varepsilon \)). Therefore the domain of existence of \( \mathcal{T} \) looks very anisotropic. In some directions the boundary is quite far from the KAM bowl. There, renormalization allows the physicist (not yet the mathematician!) to map the subcritical domain of \( \mathcal{T} \) into the KAM bowl (in that figure \( T \) is the renormalization transformation noted \( \mathcal{R} \) elsewhere). The critical boundary is the stable manifold of the fixed point \( H_* \). The reason why the KAM theorem yields low bounds on the break-up of tori is because it is very general: it works for a bowl imbedded in quite an anisotropic space and for quite general rotation numbers that obey the weak Diophantine condition (2.9). From Arnold's and Moser's proofs of the KAM theorem, Hénon estimated the maximum perturbation to \( H_0 \) for the theorem to apply, to be \( 10^{-333} \) and \( 10^{-48} \) respectively [70]. If one makes the Diophantine condition more stringent by dealing with numbers of constant type as defined by relation (2.9) with \( \varepsilon = 0 \), one finds much better lower bounds [69]. For the golden mean torus (\( z = g \)) in the standard map, Herman predicts [71] \( K_c \geq 1/34.5 \) when numerical calculations indicate \( K_c \approx 0.9716 \) [41].

When renormalizing on the universal torus \( \mathcal{T} \) related to fixed point \( F \), successive iterates converge to the unstable manifold of \( F \) which is of dimension 1 and corresponds to a one-parameter family of systems called the universal one-parameter family of \( F \).
Section 2.3.1.1.3 has shown that any periodic continued fraction expansion is related to a universality class. Thus there is an infinity of such classes. Each of them describes KAM tori, i.e. infinitely thin objects in phase space, but a physicist very often needs to know what is the large scale (in mean velocities or rotation numbers) behavior of a given Hamiltonian system. The existence of infinitely many classes is of little help if none plays a leading role. Fortunately, there is a leader.

2.3.2.2 Robustness of noble tori

Indeed it appears that the universality class of \( z=g \) plays a very special role. This was originally suspected by Greene [41,42] and further confirmed by Mac Kay's numerical calculations on the standard and quadratic maps [27,72]. These calculations indicate that noble tori are locally the most robust: (i) a critical noble has a neighborhood (in rotation number) containing no other invariant circle; when a noble torus breaks up into a Cantorus, then no torus is left in a neighborhood. (ii) A critical non-noble always has a non-critical noble in any neighborhood. This last statement was not explicitly made in ref. [27] but follows from the same data as the first one (see sec. 3.4.2).

This implies that, on a large scale, only the noble universality can show up: in a given domain of phase space, the last torus is noble and breaks up into a Cantorus with small holes that governs the flux of stochastic trajectories up and down the surface of section (see 2.3.2.4.3).
2.3.2.3 Hierarchy of Fixed Points

The robustness of noble tori immediately implies a hierarchy of the fixed points of $\mathcal{R}$. Let us say that a point is below (resp. above) the stable manifold $\mathcal{P}$ of a fixed point $F$ if it is in the KAM (resp. Cantorus) basin of $F$. The hierarchy can be stated as follows: (i) The noble fixed point $F_n$ is above all non-noble manifolds. (ii) All non-noble fixed points are below the noble stable manifold $\mathcal{P}_1$.

This statement is borne out by the renormalization scheme of sec. 3.1.7 which indicates that the picture looks like in fig. 2.14 where $F_2$ is a non-noble fixed point. In fact, $F_2$ and $F_1$ correspond to different values of $k$, but are shown in the same plane for simplicity of the picture. More generally $\mathcal{P}_1$ and $\mathcal{P}_2$ are codimension 1 manifolds in the space of Hamiltonians.

Section 3.3 shows that the hierarchy of fixed points is necessary but not sufficient for the robustness of noble tori. One can expect that, if this robustness proves to be correct, a stronger version of the above hierarchy should allow to prove it from $\mathcal{R}$. This strong version would include metric statements like "sufficiently above or below".

Table 2.1 gives the main characteristics of the golden mean fixed point.
2.3.2.4 Computation of Critical Exponents

One useful application of the fixed point properties is the calculation of critical exponents for noble Cantori. We outline this calculation in the present section, but defer to more technical sections the derivation of some intermediate steps of the reasoning.

2.3.2.4.1 Critical Exponents and Scaling

Consider a torus or a Cantor us \( \mathcal{F} \) that belongs to the universal class of a fixed point \( F \). Let \( Q \) be a quantity depending on both \( H \) and \( T \) that is rescaled by a factor \( \chi \) (for definiteness we call it a multiplier) when renormalized for \( H \) in the vicinity of \( F \)

\[
Q' = \mathcal{R}(Q) = \chi Q .
\]  
(2.47)

Let \( \delta \) be the unstable eigenvalue of \( F \). Consider a one-parameter-family of Hamiltonians \( H \) that depend on the stochasticity parameter \( s \). Assume \( \mathcal{F} \) to be critical for \( s=s_c \). Then we will show later (sec. 3.2.3) that \( Q \) as measured in \( H_s \) scales like

\[
Q \propto |s-s_c|^{\nu} ,
\]  
(2.48)

with

\[
\nu = \frac{\ln|\chi|}{\ln \delta} .
\]  
(2.49)
2.3.2.4.2 Lyapunov Exponents for Cantori

Let $h$ be the Lyapunov exponent \[12\] of a Cantorus.

Since the exponential divergence of trajectories is the same at all scales, $ht = h't'$ where the primes indicate renormalized quantities. Through $R$, $t$ is renormalized according to

$$t' = \tau t,$$ \hspace{1cm} (2.50)

Thus $h' = h/\tau$ and $\chi = \tau^{-1}$ for $Q=h$. Thus the critical exponent for $h$ is \[15\]

$$\nu_h = \frac{\ln|\tau^{-1}|}{\ln\delta}. \hspace{1cm} (2.51)$$

According to the values of $\delta$ and $\tau$ given in table 2.1, the noble value of $\nu_h$ is

$$\nu_h \approx 0.98746. \hspace{1cm} (2.52)$$

$h$ has a physical interpretation in models for solid state physics (epitaxy, defects). It was numerically computed in refs. [59,72]. For instance, it appears as the inverse of the correlation length along an epitaxial monolayer when the potential troughs of the substratum are deep enough to forbid a free slipping of the monolayer. If the potential of the substratum is $k \sin x$, it turns out that the position $x_{n+1}$ of atom $n+1$ of the layer is given as a function of the position of atoms $n$ and $n-1$ by $x_{n+1} - 2x_n + x_{n-1} = k \sin x_n$ which is the standard map. A given number of atoms per unit length corresponds to a given mean
velocity for the map. Typically if $k$ is large enough, the $x_n$'s belong to a Cantorus.

2.3.2.4.3 Flux through Cantori

As was recognized by MacKay, Meiss, and Percival [50], the motion close to a Cantorus has regular features. If one draws a continuous line $L$ that goes through each point of the Cantorus in a Poincaré map of the system (fig. 2.15), one period $T$ of the map later, $L'$ maps into a similar line $L'$ that makes ripples with respect to $L$. The total area $A$ of the ripples above $L$ is equal to $fT$, where $f$ is the upward flux of area through the Cantorus. Conservation of area imposes the downward flux to be equal. Depending on the choice of $L'$, many fluxes may be defined, but the actual flux is defined by $L'$'s that minimize $f$ [50]. Through $R$ acting at a fixed point a unit area is multiplied by $\xi$ and a unit length by $d$. Only $1/d$ of the total length of the system is kept (notice that the small boxes in figs. 2.9(b) and 2.11 are narrower than the large ones). Therefore $A$ is renormalized according to $A' = RA = \xi A/d$. Combining this with eq. (2.50) yields $f' = f/\xi d\tau$, and

$$\chi = \frac{\xi}{d\tau},$$  \hspace{1cm} (2.53)

for $Q=f$. According to eq. (2.43) the critical exponent for the flux is

$$\nu_f = \frac{\ln(\xi/d\tau)}{\ln \delta}. \hspace{1cm} (2.54)$$

According to table 2.1, for noble tori $d\tau = 1$ and
\[ \nu_f = \frac{\ln(\xi)}{\ln \delta} \approx 3.011722. \] (2.55)

This result was first numerically computed in ref. [59,72]. Reference [50] shows that \( fT \) is exactly equal to the difference \( \Delta W \) of the actions of the Cantorus and of an orbit homoclinic to it. The specific choice for \( \mathcal{L} \) of ref. [50] yields a curve \( \mathcal{L}' \) which coincides with \( \mathcal{L} \) everywhere, except in one gap between two nearby points of the Cantorus, where they create a "turnstile" whose area is \( \Delta W \). It should be pointed out that \( \Delta W \) may be small even though the gap looks large.

The renormalization schemes for maps tell how \( \Delta W \) is rescaled and do not affect \( T \) which is not renormalized. Therefore the exponent for \( f \) is again got through eq. (2.49). Using the renormalization for Hamiltonian systems directly yields the scaling of \( f \) through dimensional analysis, because time is a quantity explicitly renormalized in that case.

Since noble tori are the most robust, the last torus to break up in a given region of phase space is noble, and the flux through the Cantorus with small holes that replaces it, scales according to eqs. (2.48) and (2.55). When \( f \) is small it governs the chaotic transport of stochastic orbits in that region, and in particular the critical exponent for the diffusion coefficient. The agreement of this prediction with numerical data is excellent. Figure 2.16 plots the diffusion coefficient \( D \) in the standard map, with some noise of amplitude \( \sigma \) added, versus \( \Delta k = K - K_c \) (this is fig. 6 of ref. [54] with the new abscissa \( \log \Delta k \)). The lower is the amount \( \sigma \) of noise, the
better is the agreement of \(D\) with the law (2.48) where \(\nu\) is given by (2.55).

Figure 2.17 checks the same scaling law through the transition time \(N\) of a stochastic orbit from the vicinity of one primary resonance of the standard map to the next one, as measured by Chirikov (fig. 5.4 of ref. [4]). There is excellent agreement of the present scaling (continuous line) with the minimal transition time, for each value of \(K\).

Bensimon and Kadanoff [51] defined an escape rate through a broken torus and found it to be a critical quantity with an exponent given by eq. (2.55) too. In models for solid state physics, \(\Delta W = \int T\) appears as the energy per unit length (Peierls-Nabarro barrier) necessary for making an epitaxial monolayer to slip above a given substrate when free slipping is forbidden [28].

2.3.2.5 Threshold of Global Stochasticity for \(H_p\)

For the paradigm Hamiltonian \(H_p\), sec. 3.1.7 shows that the golden mean fixed point has a stable manifold approximately given by [35, 73]

\[
M P^{g-1} [1+c(k)P^2] = R(k)
\]

(2.56)

where \(R(k)\) and \(c(k)\) are functions of \(k\) plotted in fig. 2.18. This equation yields the threshold of breakup of the golden mean torus of \(H_p\) and of \(H_e\) too (\(H_e\) is defined by eq. (2.12)). According to eq. (2.23), in this second case, the torus has a zoning number \(z = g+1\) in \(H_p\). For \(1/25 < M/P < 25\) and \(k = 1\) or \(M=P\) and \(1/4 \leq k \leq 4\), the zoning number of
the last torus between resonances $M$ and $P$ is $z=g$ or $g+1$. In these
domains of parameter, eq. (2.56) directly yields the threshold of
global stochasticity. Section 4.1.1 shows that outside this domain,
KAM tori with $z = g+n$ or $z = 1 + (g+n-1)^{-1}$, $n \geq 0$, are the most robust or
are close to be. This means that after one iterate of the
renormalization either on $H_p$ or $H_e$, $z' = g$, and that the threshold of
breakup of one torus is obtained by setting the renormalized values of
$(k, M, P)$ into eq. (2.56). When $n$ increases, the threshold drops down
very fast. The computation of the maximum threshold is easy and gives
a good estimate of the thresholds of global stochasticity. Figure
2.19(a) yields $s_c(dots)$ versus $\rho = \Delta v_M / \Delta v_P = (M/P)^{1/2}$ for $k=1$ and
fig. 2.19(b) yields $s_c(dots)$ versus $k$ for $M=P$. The lines $s=1$
correspond to the resonance-overlap criterion. The agreement with
numerical results (crosses) is excellent for $\partial s_c / \partial p$ or $\partial s_c / \partial k$ and is
within 4% for $s_c$. [73].

When $M/P$ and $k$ are not too far from 1, $z=g$ or $g+1$ is the zoning
number of the last torus, and the computation of the threshold through
eq. (2.56) is more precise than the overlap criterion, yet simple.

Section 3.1.5–6 and 4.1.4.1 show how to locally (in rotation
number) reduce more general Hamiltonians to the paradigm Hamiltonian.
Calculations involved in that process are quite similar to those
necessary to apply the resonance overlap criterion, and also involve,
in particular, the "two-resonance-approximation". In that frame
eq (2.56) should yield better results than the overlap criterion.

Section 3.2 shows how to perform an exact renormalization on
Hamiltonians, and this is the key to high analytical accuracy on
thresholds. Section 4.1 gives a complete set of methods that allow the
analytical calculation of global stochasticity thresholds, in the frame of renormalization theory.

Section 4.2 indicates numerical methods for computing threshold that take advantage of the universal features of critical noble KAM tori.

Section 4.3 reviews the methods of chapter 4 of ref. [12] and shows how renormalization theory allows us to see them from a global point of view. After reviewing the present picture of KAM universality, we now deal with non-KAM scale invariance.

2.4 Non-KAM Scale Invariance

Much before the structural scale invariance related to KAM has been discussed, Arnold pointed out another such invariance for chains of islands in the islands of a given Hamiltonian system [10]. Graphically, this structural scale invariance looks as indicated in fig. 2.20(a) which is a sketch of the Poincaré map of Hamiltonian

$$H(v,x,t) = \frac{1}{2} v^2 - M \cos x - P \cos k(x-t) - R \cos (qx+kt), \quad (2.57)$$

where $M=P=R$, and $k=q=1$. Inside resonance $M$, chains with 2, 3, and 4 islands show up (they are similar to the chain of 9 islands in fig. 2.8). Figure 2.20(b) repeats the same chains of islands in the action-angle coordinates of the pendulum Hamiltonian (2.57) with $P=R=0$. The previous "necklaces" now are open and their "beads" are aligned. They look like the three chains of islands of fig. 2.20(a). The magnification procedure included in building fig. 2.20(b) allows us to discover new small chains of islands trapped in the chain of 3 islands.
Those chains were already present in fig. 2.20(a), but not visible. The fact that we again see chains of 2, 3, and 4 islands is only due to the convenience of the drawing. Three chains of islands possibly could not show up simultaneously in a Poincaré map of Hamiltonian (2.57) but the following argument would be the same, since it bears on the stable cycle of the central chain of islands.

More generally one would see chains of n-1, n, and n+1 islands in fig. 2.20(a) and of n', n', and n'+1 islands in fig. 2.20(b), where n,n'>=2 (in fact a chain of 1 island would not be visible, but, as said before this does not matter). The chain of n islands bifurcates out of the x=v=0 point for M,P, and R large enough, and there is a manifold B_n in the space of parameters (k,q,M,P,R) that corresponds to this bifurcation. The process of blowup of islands chains can be repeated indefinitely, revealing the existence of islands in the islands, in the islands, etc... Similar nested chains of islands also exist in resonances P and R.

As shown in appendix B an approximate renormalization transformation can be built on Hamiltonian (2.57) [74]. It also has a denumerable set of fixed points F_n' which correspond to chains of n islands in chains of n islands, etc... i.e. to a period n-tupling of the original period T = 2π/k. A picture similar to fig. 2.12 can be drawn with the exception that the equivalent of KAM point M=P=R=0 is a point where there is no chain of n islands, since the rotation number about the center of resonance M is O. The picture must be made more complete with the trace of the bifurcation manifold B_n which is under the stable manifold of F_n'. Sec. 2.5.2 yields a simpler renormalization scheme for high values of n.
Combination of the renormalization for KAM tori and for cycles allows one to reach many regions of phase space but not all of it: for instance chaotic and homoclinic orbits cannot be reached.

Historically, the period-doubling bifurcation in Hamiltonian systems has received a great deal of attention [6,75-85]. Since the account given in appendix B of ref. [12], the universal parameters of the Hamiltonian period-doubling fixed-point have been measured with a high accuracy [77-85], and efficient exact renormalization schemes have been developed [79,82]; the characteristics of the period-tripling fixed point have been investigated too [78,80]. However, as already explained in sec. 2.3.2.1, periodic orbits can experience a lot of other bifurcations than the period-doubling sequence. This has been pointed out by numerical calculations which show that the period-doubling bifurcation may be interrupted by tangent bifurcations or exchange of stability; one can even see remerging trees of bifurcations [67-68].

After this general description of scale invariance, we now focus on stochastic layers where this invariance shows up in a quite simple way.

2.5 Stochastic Layers

We first show how to compute the width of stochastic layers before describing two simple renormalization schemes for high-order fixed points of the renormalization groups for KAM tori and period n-tupling.
2.5.1 Width of Stochastic Layers

For Hamiltonian $H_p$, if $P$ is small, the motion close to the virtual separatrix of resonance $M$ may be envisioned as a perturbation of the motion close to the separatrix of the pendulum Hamiltonian

$$H_0(v,x) = \frac{1}{2} v^2 - M\cos x .$$ (2.58)

Through this equation an energy $E = H_0(v,x)$ can be defined at any point $(v,x)$ in phase space. It is convenient to introduce the normalized energy

$$W = \frac{E-M}{2M} .$$ (2.59)

The time necessary for a complete turn in the stochastic layer of resonance $M$ is approximately equal to the period of an orbit of Hamiltonian (2.58) starting at the same point. By using this approximation and Melnikov's technique for computing the change of energy $E$ during one turn in the stochastic layer, Chirikov [4] showed that for motion near the energy $W$ and for $W \ll 1$, the motion is governed by the standard map with a parameter

$$K = \frac{\beta P_0^{2\lambda+1}}{M^\lambda \Gamma(\lambda) \mu^{\lambda+1} W} \exp(-\alpha/\mu) ,$$ (2.60)

where $\lambda = 2k$ ($k$ is one of the parameters of $H_p$), $\Gamma$ is the gamma function (for $\lambda$ integer $\Gamma(\lambda) = (\lambda-1)!$),
\[ \mu = \frac{2\sqrt{M}}{\pi k}, \quad (2.61) \]

with

\[ \mu \ll 1, \quad (2.62) \]

\( \beta = 1 \) for outer stochastic layers and \( \beta = 2 \) for inner stochastic layers (inner and outer refer to the position with respect to the virtual separatrix of resonance \( M \)). \( \alpha = 1 \) for \( \beta = 2 \) and for \( \beta = 1 \) and the outer layer with a positive mean velocity, \( \alpha = 3 \) for \( \beta = 1 \) and the outer layer with a negative velocity. Originally Chirikov only gave formula (2.60) with \( \lambda = \alpha = \beta = M/P = 1 \) because he dealt with the standard mapping instead of Hamiltonian \( H_p \), but eq. (2.60) is a trivial extension of his result.

The motion is stochastic for energy \( W \) provided \( K > K_c \approx 0.9716 \) [41]. Therefore the motion should be stochastic till an energy

\[ W_c = \beta \frac{P}{M} \frac{2^{2\lambda+1}}{K_c n^\lambda \Gamma(\lambda) \mu^{\lambda+1}} \exp(-\alpha/\mu). \quad (2.63) \]

The same scaling was obtained by a resonance-overlap argument on resonances \( R_m \) in ref. [86].

Chirikov's standard mapping for stochastic layers can be recovered for describing orbits with a mean velocity \( u \) (or rotation number \( \rho = u \)) by going to the action-angle variables of Hamiltonian (2.58) and by recognizing that, close to the separatrix, the transformed Hamiltonian is quite close to the standard map Hamiltonian (2.15) (see appendix E and ref [31]). In a surface of section, the chains of islands
corresponding to resonances $R_m$ with $m$ large look very similar to each
other and their spacing changes very slowly too. This is the basic
reason why a standard map can locally describe the motion. The
parameter $K$ of the standard map still is given by eq. (2.60), where now

$$W = 16 \exp \left( -\frac{\pi^2 k \mu}{\beta u} \right).$$  \hspace{1cm} (2.64)

When numerically checking eq. (2.63), Chirikov found the actual
width in energy to be 2.15 larger than predicted by the equation
[4,87]. This has not yet found a satisfactory explanation.

We can take advantage of eq. (2.64) to define the width of the
stochastic layer in terms of a maximum mean velocity (or rotation
number) $u_c$, which is an estimate of the mean velocity (resp. rotation
number) of the last KAM torus bounding the outer (resp. inner)
stochastic layer.

For $\mu \gg 1$, the width of the layer in $u$ is given by eq. (2.64) with

$$W = \frac{\lambda^4 \mu}{2K_c M^{k+1}}.$$  \hspace{1cm} (2.65)

for both the inner and outer layers, and $k=1$ and $1/2$ [31]. The same
scaling was originally found by Smith [88]. Since $u \ll 1$, eq. (2.64)
shows that eq. (2.65) only is correct for very small $W$'s.

Expressions similar to eq. (2.63) can be found for the width of
stochastic layers of perturbed integrable systems more general than the
pendulum Hamiltonian [89–90].
The coefficient $\alpha = 3$ for the outer layer of resonance $M$ with a negative mean velocity explains why this layer blows up later than the outer layer, and why the chaotic domain in Fig. 2.8 still is bounded by the lower (resp. upper) half of the virtual separatrix of resonance $M$ (resp. $P$).

The center of the primary resonances of the standard map destabilizes for $K = 4$ [91]. Setting this value in Eq. (2.60) and using Eq. (2.64) shows that the center of resonance $R_m$ of $H_p$ is destabilized for $m$ large at

$$M = \frac{k^2}{4(k+m-1)} \tag{2.66}$$

if $P/M$ is bounded away from 0 and $\infty$.

2.5.2 Simple Renormalization Schemes

Making the two-resonance approximation locally reduces the standard map Hamiltonian (2.15) to the central paradigm Hamiltonian ($k = M/P = 1$) with $M = K/4\pi^2$.

We now have two transformations: one maps a standard mapping into a central $H_p$, the other maps $H_p$ into a standard mapping through Eqs. (2.60) and (2.64) provided a mean velocity $u$ is given. The alternate combination of these two transformations, already found by Chirikov [4] but not used by him, allows us to define a sequence of nested standard maps with parameters $K_n$ given by the mapping [31].
\[ K_{n+1} = \frac{2\pi^4}{K_n^{3/2}} \exp(m_n \sqrt{K_n} - \pi^2/K_n) \] (2.67)

obtained from eqs. (2.60–61) by setting \( M/P=k=\alpha=\beta=1 \), \( M=K_n/4\pi^2 \), and \( m_n = 1/u \).

For untrapped orbits, the two–resonance approximation makes sense for the standard map, since the neglected resonance \( \text{Mcos}(x+t) \) of eq. (2.15) corresponds to a factor \( \alpha=3 \) in eqs. (2.60) and (2.63). This approximation no longer holds for an orbit trapped in resonance \( \text{Mcos}x \) since both resonances \( \text{Mcos}(x-t) \) and \( \text{Mcos}(x+t) \) contribute equally to the amplitude of the trapped resonances. As shown in appendix E, this implies that \( \beta \) should be taken equal to 1 in eqs. (2.60) and (2.63) for trapped orbits too. Therefore the mapping (2.67) applies in both the trapped and untrapped domains of the primary resonances of the standard map Hamiltonian (2.15).

For the untrapped domains, \( m_n \) is the number of the zone of interest at the \( n \)-th step of the renormalization for KAM tori. It is the \( n \)-th coefficient of the continued fraction expansion of the zoning number of the KAM torus under study. Naturally condition (2.62) implies that this scheme is only correct if \( m_n >> 1 \) for all \( n \)'s. In particular fixed point \( F_m \) with \( m \) large is described by map (2.67) with \( m_n = m \) for all \( n \)'s. The coordinates \( M_m \) and \( P_m \) of \( F_m \) are equal to \([1+0(\frac{1}{m})]/4m \). The unstable eigenvalue of \( F_m \) is computed by linearization of the map (2.67) at its fixed point. It is

\[ \delta_m = \pi \sqrt{m} \left[ 1 + O\left(\frac{1}{m}\right) \right] \] (2.68)
For the trapped domains, \( m_n \) is the number of islands in the chain of interest at the \( n \)-th step of the renormalization described in sec. 2.4. The fixed point \( F^*_m \) with \( m \) large corresponds to \( m_n = m \) for all \( n \)'s. The corresponding values of \((M,P)\) and of the unstable eigenvalue are the same as for \( F^*_m \).

These schemes are the simplest approximations of the corresponding renormalization groups: only the unstable manifold of a fixed point is kept.

2.6 Description of Chaos

Apart from the calculation of a critical exponent for the flux of chaotic orbits through Cantori, very little has been said till now about stochasticity. We now deal with this topic and emphasize two aspects: chaotic transport and correlation times.

2.6.1 Chaotic Transport

When global stochasticity sets in, the divergence of nearby trajectories makes physically irrelevant the concept of trajectory since any error on the initial position becomes large during the time of observation of the system \([92]\). Therefore a more global description of the motion is suitable, and one rather thinks in terms of a chaotic transport in phase space. The paradigm of such a transport is the diffusion connected to quasi-linear theory \([4,12,52]\), but there are other kinds of transport.

For a stochasticity parameter \( s \) sufficiently above the threshold of global stochasticity, \( s_c \), many chaotic systems can accurately be described by a random-walk description of the unperturbed action (see
refs. [4,39,93]. This occurs, for instance, for the standard map (2.14) [12]. For large values of parameter $K$, $I = 2\pi \nu$ diffuses with a diffusion coefficient

$$D_{QL} = \frac{K^2}{4},$$  \hspace{1cm} (2.69)

where the subscript QL refers to quasi-linear for reasons given later. For values of $K$ not far from $K_c$, the assumption of decorrelation of successive steps done when computing $D_{QL}$ is incorrect. Taking into account finite-correlation-time effects adds a multiplication factor to (2.69) that oscillates with $K$ about 1 with an amplitude that scales like $K^{-1/2}$ [53,54]. Chapter 5 of ref. [12] and ref. [52] review this diffusion regime.

For values of $K$ close to $K_c$, as explained in sec. 2.3.2.4.3, the diffusion coefficient scales like $(K-K_c)^{\nu_f}$, $\nu_f \approx 3$ [43,44].

The standard map Hamiltonian (2.15) is a special case of Hamiltonians of the type

$$H(p,x,t) = \frac{p^2}{2} - \sum_{n=-N}^{N} \phi_n \cos(x-n\delta\omega t+\alpha_n),$$  \hspace{1cm} (2.70)

which describes the motion of one particle in a spectrum of longitudinal waves. The standard map corresponds to $\delta\omega = 1$, $\alpha_n = 0$, $\phi_n = K/4\pi^2$, and $N = \infty$. Now consider a case where $N$ is finite and where the $\phi_n$'s slowly vary with $n$, and have a typical amplitude $M$. Equation (2.56) shows that global stochasticity occurs in the domain of phase velocities $[-N,N]$ for
\[ \frac{M}{\delta \omega^2} \geq 3 \times 10^{-2} \]  \hspace{1cm} (2.71)

The correlation time for chaotic motion is related to the width

\[ \Delta \omega = 2N \delta \omega \]  \hspace{1cm} (2.72)

of the spectrum in eq. (2.70) and is given by

\[ \tau_c = \frac{1}{\Delta \omega} = \frac{1}{2N \delta \omega} \]  \hspace{1cm} (2.73)

Hamiltonian (2.70) may be rewritten as

\[ H(p,x,t) = \frac{p^2}{2} - A(t) \cos[x - \varphi(t)] \]  \hspace{1cm} (2.74)

where \( A(t) \) has a typical amplitude

\[ \varphi = M(2N+1)^{1/2} \]  \hspace{1cm} (2.75)

Equation (2.74) allows us to define a trapping time for the motion in the wave-spectrum as

\[ \tau_{tr} = \frac{1}{\sqrt{\varphi}} = M^{-1/2}(2N+1)^{-1/4} \]  \hspace{1cm} (2.76)
If

\[ \tau_c \ll \tau_{tr}, \quad (2.77) \]

then \( A(t) \) changes too fast for an orbit to be appreciably bent by the wave potential before this potential changes. The momentum varies very little during a time \( \tau_c \), and the effect of the wave-spectrum is to give small decorrelated kicks to \( p \). Consequently, \( p \) makes a random walk. The particle may be viewed as only interacting with the wave having the closest phase-velocity, and as having an unperturbed momentum during the interaction. This is the quasi-linear approximation \([4,12,52]\).

One finds

\[ D_{QL} \propto \rho^2 \tau_c = \frac{\tau_c}{\tau_{tr}^4} \propto M^2. \quad (2.78) \]

The scaling of \( D_{QL} \) like \( M^2 \) agrees with eq. (2.69), since for the standard map \( K = 4\pi^2 M \).

For any finite \( M \), if \( N \) increases, \( \tau_c/\tau_{tr} \) goes to 0. Therefore the standard map \((N=\infty)\) always verifies eq. (2.77). For \( N \) finite and \( M \) large enough, the opposite condition holds

\[ \tau_c \gg \tau_{tr}. \quad (2.79) \]

Then the trapping of an orbit in the many-wave potential becomes effective. This regime was originally investigated by Dupree who proposed a new diffusion coefficient for \( p \) based on resonance broadening \([94]\). Nevertheless, numerical calculations disagree with
this prediction [95,96,97]. More recently, a different approach of that regime has been developed that takes advantage of relation (2.79) for defining an adiabatic invariant \( J \) of the motion, from the action of the pendulum Hamiltonian (2.74) frozen at a given time [55]. This adiabatic invariant gets a kick \( \Delta J \) each time the orbit crosses the separatrix of the pendulum, and the \( \Delta J \)'s are decorrelated owing to the long time elapsed between two separatrix crossings. The physical picture again is a diffusion, not of the momentum, but of \( J \) which is something like an action. A diffusion coefficient can be computed for \( J \) which scales like

\[
D_{tr} \propto \frac{1}{\tau_c^3} = \Delta \omega^3.
\]  
(2.80)

It is independent of the amplitude of the many-wave potential. As expected the diffusion goes to zero when \( \Delta \omega \to 0 \) since the spectrum becomes monochromatic. Equations (2.78) and (2.80) imply

\[
\frac{D_{tr}}{D_{QL}} \propto \left( \frac{\tau_{tr}}{\tau_c} \right)^4.
\]  
(2.81)

This shows that the chaotic transport strongly diminishes in the trapping regime. This agrees with numerical calculations of refs. [95,96]. This slow transport can lead to misleading conclusions when looking at Poincaré maps obtained from numerical integration of orbits. During the computation time a stochastic orbit can seem to be confined in phase space, suggesting the reappearance of KAM tori. In fact, no such torus exists but there is a good adiabatic invariant
which needs a lot of time to be destroyed! For $\Delta \omega$ small Hamiltonian (2.70) is close to an integrable Hamiltonian but not in a KAM sense [98].

Figure 2.21 summarizes the present picture of chaotic transport in Hamiltonian (2.70). In fig. 2.21(a) $\Delta \omega$ and $N$ are kept fixed and $\phi$ varies. Global stochasticity sets in at $\phi = \phi_c$ [according to eqs. (2.71-72) and (2.75), $\phi_c \propto \Delta \omega N^{-3/2}$]. Then the last noble KAM torus breaks up into a Cantorus with small holes that governs the chaotic transport, leading to the scaling $D_\alpha(\phi-\phi_c)^{\nu_f}, \nu_f = 3$ (see sec. 2.3.2.4.3). When $\phi$ increases there is a transition to the quasilinear regime where $D$ scales like $\phi^2$. For large $\phi$ the transport is constant. For $\phi = \Delta \omega$ there is a crossover regime between a diffusion in $p$ and in $J$. The work of Thomson and Benford [99] suggests that a distribution function of momenta would evolve according to a non-Markovian master equation including derivatives in $p$ of order higher than 2.

Reference [50] suggests considering adiabatic invariants which instead of being real valued, diffuse on a "tree" [61]. This breaks new grounds for future investigation.

Figure 2.21(b) describes the transport with $\phi$ and $N$ fixed and $\Delta \omega$ varying. Global stochasticity sets in for $\Delta \omega \leq \Delta \omega_c \propto \phi^{1/2}N^{3/4}$. The Cantorus-with-small-holes regime yields a scaling $D \propto (\Delta \omega_c - \Delta \omega)^3$. The quasi-linear regime follows with $D_{QL} \propto 1/\Delta \omega$. For $\Delta \omega$ of order $\phi^{1/2}$, the trapping regime appears and $D_{tr} \propto \Delta \omega^3$. 
2.6.2 Correlations

A complete theory of the correlations in the chaotic domains of Hamiltonian systems is not yet available. A measure of these correlations is given by autocorrelation functions: \( \langle x(t)x(0) \rangle \) for \( H_p \) and \( H_s \) (defined by eq. (2.15)) or \( \langle v(t)v(0) \rangle \), or \( \langle \sin x(t) \sin x(0) \rangle \) for \( H_s \), where the average is taken over initial conditions. Historically it was first believed that autocorrelation functions should decay exponentially, till the work of Alder and Wainwright [100] showed numerically the existence of long-time tails in the hard-sphere gas. More recently Vivaldi, Casati and Guarneri [101] demonstrated, both analytically and numerically, the existence of long-time tails in the stadium billiard [102] and connected it with the presence of arbitrarily long segments of regular motion in the time evolution of the chaotic orbits. However, the bulk of the autocorrelation function exhibits an exponential decay which suggests the existence of two distinct time-scales controlling the approach of the microcanonical distribution of a phase space density [101]. It is interesting to notice that for a system where no segment of regular motion is present, only the exponential decay exists [103]. This is consistent with a theoretical result [106] which predicts an exponential decay for maps similar to the standard map, by neglecting terms which are numerically checked to be large whenever sizeable islands are present in the stochastic sea. When such islands are present, numerical calculations [104,105] show the existence of long-time tails, with a seemingly algebraic decay like \( t^{-1} \) [105]. As was noticed in ref. [101], this feature is germane to what occurs in
the stadium billiard: when a stochastic orbit comes close to an island, it experiences an almost-regular motion.

Provided islands in the stochastic sea are not too big, the exponentially-decaying part of the correlation function may be the most relevant for physical applications. The Lyapunov exponent \( \lambda \) which measures the mean rate of divergence of nearby orbits in the chaotic domain, gives a rough estimate of the decay rate of the exponential-part of the correlation functions [106]. It is therefore an interesting quantity to know from that respect. Several methods have been derived for the numerical calculation of \( \lambda \) (see sec. 4.3 of ref. [12]), which was analytically computed in ref. [107] by writing a quasi-linear equation corresponding to the two-point distribution function of orbits for the Hamiltonian

\[
H(p,x,t) = \frac{1}{2} p^2 - \frac{e}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{M} \frac{1}{m} \cos(m2\pi x - n2\pi t + m\psi_{mp}), \tag{2.82}
\]

where the \( \psi_{mp} \)'s are random phases, \( p=n \pmod{N} \), and \( M \) and \( N \) are fixed numbers. The diffusion coefficient is found in ref. [107] to be

\[
D_{QL} = \pi^2 e^2 M^3 / 3, \tag{2.83}
\]

for \( M >> 1 \), and the Lyapunov exponent is derived as

\[
\lambda = 0.54 M e^{2/3}, \tag{2.84}
\]

provided \( \lambda < N \). Equation (2.84) agrees very well with numerical calculations. It is useful to point out that in the literature \( \lambda \) is
often called Kolmogorov (−Sinai) entropy (a simple definition of this quantity is given in sec. 4.2c of ref. [12]) because of a theorem of Pesin [108] which shows that both quantities are equal for 2-degree-of-freedom Hamiltonian systems.

2.7 Islands of Stability

When a stochastic domain appears in a region of phase space at a given scale, stable islands are often visible at that scale which are embedded in the "stochastic sea". With a further increase of chaos, these islands may disappear. As explained in sec. 3.4, a critical noble torus is surrounded by island chains with a universal rotation number \( \rho = \frac{1}{6} \) (the trace T, of the tangent map at their center is given in table 2.1) [41]. Therefore global stochasticity sets in before the disappearance of islands nearby to the torus. One could dream about a threshold of complete chaos in a given region of phase space where all islands would vanish: it would, for instance, correspond to the higher period-doubling accumulation point of the stable cycles related to resonances M and P. Unfortunately a given cycle may destabilize and restabilize later though chaos become more global (case of Hamiltonian \( H_p \), see sec. 3.4.1), or new chains of islands may appear (case of the accelerator modes of the standard map [4]). In any case it seems as if typically islands are dense [61].

After this mundane picture of Hamiltonian stochasticity, we now turn to the technical aspects of renormalization for KAM tori in Hamiltonian systems.
3. Renormalization for KAM Tori: technical aspects

This section is devoted to the presentation of explicit renormalization schemes for Hamiltonian systems that make concrete the picture of the renormalization group given in sec. 2.3.1.1. These schemes apply to mappings as well, since these can be written in a Hamiltonian way (see ref. [39], sec. 2.1.2.4.4 and sec. 3.1(c) of ref. [12]). Section 3.1 deals with an approximate renormalization scheme \( T_r \) for the paradigm Hamiltonian \( H_p \). Then sec. 3.2 shows that \( T_r \) is nothing but a computable example of a more general and exact theory. Section 3.3 justifies the hierarchy presented in sec. 2.3.2. Section 3.4 introduces fractal diagrams built on cycles.

3.1 Approximate Approach

This section presents the derivation of explicit renormalization schemes corresponding to figs. 2.9 and 2.11. The first scheme gives simple formulae whose lowest degree part yields an even simpler scheme, and allows us to compute the rescaling multipliers for area and flux. We also justify the two approximations involved in the derivation of the simple schemes.

3.1.1 A Simple Scheme

We now describe a simple renormalization scheme on the paradigm Hamiltonian system \( S_p \) which is defined by eq. (2.1). We first assume, as in sec. 2, that \( k \) is a rational \( r/p \), but the final renormalization formulae work for \( k \) irrational as well. In order to build the small box of fig. 2.9(a), we need to isolate the elements of that figure, in particular the chains of islands, which correspond to
resonances $R_m$ with a zoning number $z=m$ and a mean velocity given by eq. (2.21)

\[ w_m = \frac{k}{k-1+m} \tag{3.1} \]

For $k=1$ we recover the definition of $w_m$ given by eq. (2.13). Those elements are not explicitly present in $H_p$ but are generated when considering $S_p$ in other canonical coordinates, for instance, the coordinates $(I,\phi)$ obtained from $(v,x)$ by Kolmogorov's superconvergent transformation [4,12] with generating function

\[ F(I,x) =Ix + \frac{M\sin x}{I} \tag{3.2} \]

This relation implies

\[ v = \frac{\partial F}{\partial x} = I + \frac{M\cos x}{I} \tag{3.3} \]

The new Hamiltonian is

\[ H'(I,\phi,t) = H_0(I) - \varepsilon \sum_{n=-\infty}^{+\infty} \nu_n(I)\cos[(k+n)\phi-kt] + W\cos 2x(I,\phi) \tag{3.4} \]

with $\varepsilon=1$,

\[ H_0(I) = \frac{I^2}{2} + \frac{\nu^2}{4I^2} \tag{3.5} \]
\( x(I, \vartheta) \) is defined from \( \vartheta = \partial \Phi(I, x)/\partial I \), and the \( V_n \)'s and \( W \) are computed in appendix C. For \( n \geq 0 \) one obtains

\[
V_n(I) = A_n(k, I) M^n P[1 + O(M^2)] , \quad (3.6)
\]

where

\[
A_n(k, I) = \frac{k(k+n)^{n-1}}{n!(2I^2)^n} . \quad (3.7)
\]

The phase

\[
\varphi_n(\vartheta, t) = (k+n)\vartheta - kt , \quad (3.8)
\]

of the \( n \)-th resonant term in the sum of eq. (3.4) is stationary for \( \vartheta = \omega_{n+1} \), and therefore corresponds to resonance \( R_{n+1} \).

Now consider a torus \( \mathcal{T} \) with a zoning number \( z \). Let

\[
m = \text{Int}(z) \quad (3.9)
\]

\[
\lambda = m-1 .
\]

Thus \( \mathcal{T} \) is between \( R_m \) and \( R_{m+1} \). Then building the little box of fig. 2.9(a) is done through the two-resonance approximation: in eq. (3.4) we only keep \( H_0(I) \) and the \( n=\lambda \) and \( m \) terms of the sum. This yields a new Hamiltonian

\[
H'(I, \vartheta, t) = H_0(I) - \varepsilon \sum_{n=\lambda}^{m} V_n(I) \cos \varphi_n(\vartheta, t) . \quad (3.10)
\]
A rationale for this approximation is: that along $\mathcal{F}$, the secular behavior of $\varphi_n$ is $\varphi_n(ut,t)$ with $u=k/(k+z-1)$, and

$$\varphi_n(ut,t) = (n+1-z) \frac{kt}{k+z-1}, \quad (3.11)$$

is the most slowly varying for $n=1$ and $m$; the slowest perturbations are known to be the most efficient to destabilize a motion: see for instance the behavior of Hill's equation. This approximation is also done in the resonance-overlap approach. The correct justification of this kind of approximation is given in sec. 3.1.6.

We therefore expect $H''$ still to correctly tell whether $\mathcal{F}$ exists or not. We emphasize that this approximation differs strongly from perturbation theory if $m>1$: we do not keep the largest perturbations but the slowest. Our purpose is not to describe the shape of $\mathcal{F}$ in a given Poincaré map, for instance, but to assess the existence of $\mathcal{F}$.

At that point we remark that $H$ has a structure similar to $H_p$: there are two resonant terms and one angle-independent nonlinear part $H_0$. There are differences too: $H_0$ is not quadratic in $I$ and the $V_n$'s depend on $I$. In order to map $H''$ into a new $H_p$ corresponding to the large box of fig. 2.9(b), two more steps must be done.

The first one is the local-quadratic approximation: we Taylor-expand $H_0(I)$ to second order about $I_m$, and leave out higher order terms, where $I_\zeta$ is defined for any $\zeta$ by

$$\omega(I_\zeta) = u_\zeta = \frac{k}{k+\zeta-1}, \quad (3.12)$$

with
\[
\omega(I) = \frac{dH_0}{dl} = I - \frac{M^2}{2l^3}.
\] (3.13)

A rationale for this approximation is that \(I(t) - I_z\) remains small if \(M\) and \(P\) are not too large and that \(I_z - I_m\) is small, so that the first terms in the expansion should correctly approximate \(H_0\). Nonetheless we must go to second order to keep the nonlinearity of \(H_0\) which is a basic ingredient of KAM theory. The choice of \(I_m\) for the expansion is somewhat arbitrary, \(I_\lambda\) would do as well.

The second step is the centered-resonance approximation: we approximate \(V_n(I)\) by the constant \(V_n(I_{n+1})\). \(I_m\) is the resonant value of \(I\) for \(R_m\) when \(\epsilon = 0\) in eq. (3.4). A perturbation approach justifies this approximation in sec. 3.1.5. As yet we notice that it is a classical approximation since it is made when applying the resonance-overlap criterion.

These last two approximations yield the new Hamiltonian

\[
H''''(I, \psi, t) = H_1(I) - \sum_{n=\lambda}^{m} V_n(I_{n+1}) \cos \varphi_n(\psi, t).
\] (3.14)

with

\[
H_1(I) = \frac{1}{2} \sigma (I - I_m)^2 + \Omega (I - I_m) + H_0(I_m),
\] (3.15)

where
\[
\sigma = \frac{d^2H_0}{dI^2} (I_m) = 1 + \frac{3}{2} \frac{M^2}{I_m}, \tag{3.16}
\]

and

\[
\Omega = \omega(I_m). \tag{3.17}
\]

If we consider \( H''' \) as describing the motion of one particle in two waves, \( 1/\sigma \) plays the role of a mass. We call it the renormalized mass.

The last thing left before mapping the small box of fig. 2.9(a), as described by \( H''' \), into the large box of fig. 2.9(b), is to recognize in fig. 2.9 that \( M' = R_m \) and \( P' = R_{m+1} \). This means that we describe \( S_p' \), the part of \( S_p \) in the small box of fig. 2.9(a), in a new frame of reference that moves with \( R_m \), and we define a new position

\[
x' = \varphi_k(\psi, t). \tag{3.18}
\]

Define a new time

\[
t' = \tau t \tag{3.19}
\]

where

\[
\tau = \frac{-k}{k+m}. \tag{3.20}
\]

One obtains
\[ \varphi_m(t) = k'(x' - t') \quad (3.21) \]

where

\[ k' = \frac{k+m}{k+l} \quad (3.22) \]

Combining the canonical equation for \( H''' \) yields

\[ \frac{d^2x'}{dt'^2} = -M'sin x' - P'k'sin k'(x' - t') \quad (3.23) \]

where

\[ M' = F(k) \]
\[ P' = F(m) \quad (3.24) \]

with

\[ F(n) = \frac{\sigma}{\Delta w_n^2} V_n(l_{n+1}) \quad (3.25) \]

and

\[ \Delta w_m = w_m - w_{m+1} \quad (3.26) \]

Equation (3.23) also can be obtained from the canonical equations of Hamiltonian
\[ H'_p = \frac{1}{2} v'^2 - M'\cos x' - P'\cos k'(x'-t') \, \] 

(3.27)

that therefore describes the motion in \( S'_p \).

Equations (3.24)-(3.25) can be simply understood through dimensional analysis: the unit of mass and velocity in eq. (3.27) is chosen such that a mass \( 1/\sigma \) and a mismatch of velocity \( \Delta w'_m \) in eq. (3.14) become both 1 in eq. (3.27). Therefore \( V_n \) that scales like an energy is rescaled according to eqs. (3.24)-(3.25).

The new mean velocity in \( S'_p \) is a linear function of \( u \) such that, for \( u = w'_m \), \( u' = 0 \) and for \( u = w'_m + 1 \), \( u' = 1 \). Thus

\[ u' = \frac{k+m}{k} [k - (k+1)u] \, \] 

(3.28)

This formula can be recovered by requiring that for \( M = P = M' = P' = 0 \), \( \Delta x = u\Delta t \) implies \( \Delta x' = u'\Delta t' \). According to eqs. (2.22), (2.21), (3.20), and (3.28)

\[ z' = \frac{1}{z-m} \, \] 

(3.29)

Notice that eqs. (3.20), (3.28), and (3.29) already are given in sec. 2.3.1 as eqs. (2.25), (2.24) and (2.26). Their derivation is reproduced here for completeness, but does not rely upon the approximate treatment of this section. Define

\[ \delta z = z-m = z-\text{Int}(z) \, \] 

(3.30)
Figure 3.1 displays $z'(\delta z)$ and $\delta z'(\delta z)$, and makes it obvious that $\delta z'(\delta z)$ has a denumerable set of unstable fixed points.

$$\delta z_m = \frac{1}{2} \left[ (m^2 + 4)^{1/2} - m \right], \quad m=1,2,\ldots \quad (3.31)$$

The corresponding value of $z$ is

$$z_m = m + \delta z_m. \quad (3.32)$$

Notice that $z_1 = g = (\sqrt{5} + 1)/2$. In fig. 3.1, the vertical branches of $\delta z'(\delta z)$ can be numbered from right to left from 1 to $\infty$, and $\delta z_m$ is on branch number $m$. As a result of eq. (3.29)

$$z_m = [m,m,\ldots,m,\ldots] \quad (3.33)$$

In order to designate a quantity $Q$ obtained after $i$ renormalizations, we denote it $Q^{(i)}$. In particular, $Q=Q^{(0)}$ and $Q'=Q^{(1)}$. If $z^{(0)} = z^*$, then $m^{(i)} = m$ for all $i$'s and eq. (3.22) shows that $k^{(i)}$ converges to a fixed point

$$k_m = 1 + \delta z_m \quad (3.34)$$

According to eqs. (3.6-7), (3.12), (3.16) and (3.24-26)

$$M' = B_m(k)M\hat{q}P[1 + o(M^2)] \quad ,$$

$$P' = C_m(k)M\hat{r}P[1 + o(M^2)] \quad .$$
with \( \lambda = m-1 \),

\[
B_m(k) = \frac{A_{\lambda}(k, I_m)}{\Delta w_m^2},
\]

(3.36)

\[
C_m(k) = \frac{A_m(k, I_{m+1})}{\Delta w_m^2}.
\]

Equations (3.22), (3.29), and (3.35) define a renormalization transformation \( T_r \) on the the couple \((\mathcal{F}, H_p)\) as defined by parameters \((z, k, M, P)\). It is interesting to split \( z \) into an integer and a fractional part according to eq. (3.30) and to consider that \( T_r \) acts on \((\delta z, m, k, M, P)\). Figure 3.2 shows how the renormalized quantities depend on the original ones. It shows that the mapping of the interval \( \delta z' (\delta z) \) plays a leading role in \( T_r \). This map accounts for the irrationality of zoning numbers. \( H_p' \) is a function of \( H_p \) and \( m \) only: renormalizing is like peeling off an onion; each iterate corresponds to one coefficient of the continued fraction expansion of \( z(0) \).

3.1.2 A Simpler Scheme

Let \( T_\rho \) be the renormalization scheme obtained by dropping the \( O(M^2) \) terms in eq. (3.35). The fixed point \( F_m \) of \( T_\rho \) corresponding to \( z_m \) is defined by \( k = k_m \) as given by eq. (3.34) and

\[
M_m = (C_m^*)^{-1}.
\]
\[ P_m = \left( c_m^* \right)^{m-2} / b_m^* , \]  

(3.37)

with

\[ b_m^* = b_m(k_m) , \]  

(3.38)

\[ c_m^* = c_m(k_m) . \]  

Linearization of the lowest order of equations (3.35) about \( F_m \) yields

\[
\begin{pmatrix}
\delta M' \\
\delta P'
\end{pmatrix}
= \mathcal{M}_m
\begin{pmatrix}
\delta M \\
\delta P
\end{pmatrix}
\]  

(3.39)

with

\[
\mathcal{M}_m =
\begin{pmatrix}
\ell & b_m^* (c_m^*)^{-\ell/m} \\
\frac{m (c_m^*)^{\ell/m}}{b_m^*} & 1
\end{pmatrix}
\]  

(3.40)

As a result \( T_r(\mathcal{M}_m) = m \) and \( \text{Det}(\mathcal{M}_m) = -1 \). \( \mathcal{M}_m \) has an unstable eigenvalue \( z_m \) and a stable eigenvalue \(-z_m\). For the golden mean fixed point \( F_1 \), these are \( g \) and \( 1-g \), very close to the numerical values for \( \delta \) and \( \delta' \) as given in table 2.1. This is quite natural as, the value of \( M_m \) being small, the correction \( O(M^2) \) in the scheme \( T_r \) only yields a small modification to the eigenvalues for \( T_p \). The eigenvalue corresponding to the \( k \) direction is \(-z_m^{-2} \).
For a given fixed point \( F_m \), \( T_p \) has two stable directions: one in the plane \((M,P)\) and one parallel to the \( k \)-axis, and one unstable direction in the plane \((M,P)\). This corresponds exactly to the picture given in secs. 2.3.1.1.3 and 2.3.2 (fig. 2.12). Here the space of Hamiltonians is artificially kept of dimension 3 when renormalizing \( H_p \); the unstable manifold is of dimension 1 as it should, but the stable manifold is only of dimension 2, yet of codimension 1 as it should.

The lowest order part of eq. (3.35), when written for \( \dot{X} = (\dot{\ln M}, \dot{\ln P}) \) is a linear transformation \([33, 46]\)

\[
\dot{X}' = \dot{C} + L \dot{X},
\]

where \( \dot{C} = [\dot{\ln B_m}(k), \dot{\ln C_m}(k)] \) and

\[
L = \begin{pmatrix}
\ell & 1 \\
m & 1
\end{pmatrix},
\]

with \( \ell = m-1 \). Let

\[
\eta(n) = \begin{pmatrix}
\gamma(n) \\
\delta(n)
\end{pmatrix} = \prod_{i=1}^{n} L_i,
\]

where \( L_i \) is \( L \) where \( m \) is computed through eq. (3.9) with \( z = z^{(i-1)} \). Then \((M^{(n)}, P^{(n)})\) scale like

\[
M^{(n)} \propto (M^{(0)})^{\delta(n)} (P^{(0)})^{\gamma(n)},
\]

\[
P^{(n)} \propto (M^{(0)})^{\beta(n)} (P^{(0)})^{\alpha(n)}.
\]
If \( f^{(i)} = 0 \) (and therefore \( m^{(i)} = 1 \)) for all \( i \)'s, then \( \alpha^{(n)} = f_n \), \( \beta^{(n)} = \gamma^{(n)} = f_{n-1} \), and \( \delta^{(n)} = f_{n-2} \), where \( f_n \) is defined by eq. (2.32). Equation (3.44) shows that the convergence to \((0,0)\) or \((\infty,\infty)\) is like exponential of exponential \( n \) since \( f_n \) scales like \( g^n \). The stable manifold of \( F_m \) can be computed explicitly for \( T_\rho \) by using eq. (3.4), and is given by \[ MP^{\alpha_m} = R_m(k) , \] (3.45)

where \( \alpha_m = (z_m - 1)^{-1} \) and \( R_m(k) \) is analytically computable.

For \( T_\rho \) a result similar to the KAM theorem can be proved: for almost all zoning numbers (in the sense of Lebesgue measure), if \( M^{(0)} \) and \( P^{(0)} \) are small enough, then \( [M^{(i)}, P^{(i)}] \) converges toward \((0,0)\) when \( i \) goes to infinity \([33,46]\).

3.1.3 Rescaling of Area and Flux

Here we go back to the more complete scheme \( T_\rho \). An important universal parameter is the multiplier \( \xi_m = |dv'dx'/dvdx| \) that tells how the area is rescaled through \( \mathcal{R} \). Since transformation \((v,x) \rightarrow (1,\phi)\) is canonical, \( dv \, dx = dI \, d\phi \). According to eq. (3.18)

\[ dx' = (k+\ell)d\phi - kdt , \] (3.46)

and from eq. (3.27), \( dx'/dt' = v' \). From eqs. (3.19), (3.20), and the canonical equation for \( \phi \) of eq. (3.14), we get

\[ dv' = \frac{1}{\tau} (k+\ell) \, s \, dI . \] (3.47)
From (3.46), for a change $d\psi$ for $t$ fixed,

$$dx' = (k+i)d\psi .$$  \hspace{1cm} (3.48)

Equations (3.47) and (3.48) computed at fixed point $F_m$ yield

$$\xi_m = \sigma_m (k_m+i) \frac{2(k_m+m)}{k_m} ,$$  \hspace{1cm} (3.49)

where $\sigma_m$ (called the inverse mass multiplier) is computed from eqs. (3.12) and (3.16) with $k=k_m$ and $M=M_m$.

The renormalized mass $1/\sigma_m$ enters the expansion for $\xi_m$. The ultimate reason for that lies in relation (3.3) which shows that going to $(1,\psi)$ coordinates straightens the Poincaré section by suppressing resonance $M$ at order $M$. Setting eq. (3.3) in $H_p$ [eq. (2.1)] yields the $M^2/4l^2$ term in $H_0$ [eq. (3.5)] that renormalizes the mass in $H_4$ [eq. (3.14)]. This also shows that passing from the small box of fig. 2.9(a) to the large box of fig. 2.9(b) also corresponds to a straightening of the picture.

Equation (3.48) shows that the new large box of width $\Delta x' = 2\pi$ corresponds to a width $\Delta\psi$ such that

$$\Delta x' = d\Delta\psi .$$  \hspace{1cm} (3.50)

where $d = k+i$ in general and $d = z_m$ at $F_m$. At $F_m$, the small boxes in figs. 2.9(a) and 2.9(b) only take a fraction $1/d = 1/z_m$ of the width of the large box. We call $d$ the width multiplier.
At \( F_m \), according to eq. (3.20), time is rescaled with a coefficient

\[
\tau_m = \frac{-k_m}{k_m + m}.
\]  

(3.51)

The flux through a near-critical Cantorus of the universal class governed by \( F_m \) is rescaled according to eq. (2.53) with the coefficient

\[
\chi_m = \sigma_m \tau_m^2 \left( \frac{\tau_m + 1}{\delta \tau_m + 1} \right)^2.
\]  

(3.52)

For \( T_\rho \), \( \sigma_m = 1 \) and \( \chi_1 = 2 + g = g^3 \). Since for \( T_\rho \), \( \delta = \tau_m \), the simplest scheme predicts \( \nu_g = 3 \) which agrees with the exact one (2.55) within \( 3 \cdot 10^{-3} \).

3.1.4 Remarks

Since the equations defining \( T_r \) are continuous with respect to \( k \), we can relax our original assumption that \( k \) is irrational and deal with more general barriers in phase space than KAM tori.

\( T_r \) can be modified by replacing \((1,\psi)\), as given by the generating function of eq. (3.2), by the action-angle variables of \( H_p \) for \( P=0 \) [25]. The action is defined by

\[
I = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(x,E)dx
\]  

(3.53)

where \( v(x,E) \) is obtained from
\begin{equation}
E = \frac{1}{2} v^2 - M \cos x .
\end{equation}

(3.54)

The generating function these new \((I, \phi)\) variables is

\begin{equation}
P(x, I) = \int \lambda \left. \nu[y, E(I)] \right| dy ,
\end{equation}

(3.55)

When written in the \((I, \phi)\) variables \(H_p\) becomes a new Hamiltonian that has the same structure as \(H'\) [eq. (3.4)], but with \(W=0\): resonance \(M\) is completely suppressed (see appendix E). \(H_0\) and the \(\lambda_n\)'s are given in appendix D. At lowest order the \(\lambda_n\)'s are given by eq. (3.6) with

\begin{equation}
\lambda_n(k, I) = \frac{1}{(4l^2)^n} \sum^{k}_{n} L_n ,
\end{equation}

(3.56)

where \(\sum^{k}_{n}\) is given by eq. (D.17).

This technique is more involved than the one of sec. 3.1.1, but yields a more accurate scheme. It is the only correct one for large values of \(z\) (tori close to the separatrix of resonance \(M\)) because of the small denominator present in eq. (3.2).

Both schemes, however, suffer from the fact that only resonance \(M\) is killed, and not resonance \(P\): the distortion of \(I(t)\) due to \(P\) makes the local-quadratic and centered-resonance approximations only correct marginally (see secs. 3.1.5-6). In particular the golden mean fixed point \(F_1\) is not correctly described, and those schemes do not verify the hierarchy of the fixed points introduced in sec. 2.3.2.3.
3.1.5 Reduction to a Many-wave Hamiltonian

Consider a Hamiltonian

$$H(p,x,t) = h(p) - \varepsilon \sum_{n \in \mathcal{N}} V_n(p) \cos(k_n x - \omega_n t + \alpha_n), \quad (3.57)$$

where $\alpha_n$ is a phase, $\varepsilon$ is a small parameter, $\mathcal{N}$ is a set of integers, and $h$ and the $V_n$'s are analytic functions. We assume

$$\sigma(p) = g(\eta p), \quad (3.58)$$

with

$$\sigma(p) = \frac{d^2 h(p)}{dp^2}, \quad (3.59)$$

and

$$\frac{dV_n}{dp} (p) = g_n(\eta p), \quad (3.60)$$

where $\eta$ is a small parameter. Thus the nonlinearity of $h$ and the derivative of the amplitudes of the angle-dependent terms are slowly varying, and they change on a scale $\Delta p \eta \propto 1/\eta$. Let

$$v_n = \frac{\omega_n}{k_n}, \quad (3.61)$$

and

$$\Delta v_n = v_n - v_{n+1}. \quad (3.62)$$
We assume that
\[ \Delta v_n > 0 \text{ for all } n \text{'s}. \]  
(3.63)

Let \( p_n \) be the solution, if it exists, of
\[ \Omega(p_n) = v_n, \]  
(3.64)
with
\[ \Omega(p) = \frac{dh}{dp}(p). \]  
(3.65)

Consider an orbit \( \mathcal{O} \) with mean velocity \( \bar{w} \),
\[ v_\mathcal{O} > \bar{w} > v_{\mathcal{O}+1}. \]  
(3.66)

We assume
\[ \sigma(p) \neq 0 \text{ for } v_\mathcal{O} > \Omega(p) > v_{\mathcal{O}+1}. \]  
(3.67)

Let \( p_* \) and \( \delta p \) be defined by
\[ \Omega(p_*) = w, \]  
(3.68)
\[ \delta p = p - p_* . \]  
(3.69)

The canonical equation for \( p \) yields
\[ \delta \dot{p} = \epsilon \sum_{n \in \mathcal{N}} V_n(p) k_n \sin \phi_n(x,t), \]  
(3.70)

where
\( \phi_n(x,t) = k_n x - \omega_n t + \alpha_n \). \hfill (3.71)

Thus \( \delta \dot{p} \) scales like \( \varepsilon \). According to eq. (3.68), along \( \mathcal{E} \), \( \delta p(t) \) does too. We therefore Taylor-expand the canonical equations of \( H \) about \( p=p_* \). This yields

\[
\dot{x} = w + \sigma_* \delta p + \varepsilon \sum_{n \in \mathcal{N}} V_n^* \cos \varphi_n(x,t) + O(\eta \varepsilon^2), \hfill (3.72)
\]

\[
\delta \dot{p} = \varepsilon \sum_{n \in \mathcal{N}} (V_n^* + V_n^* \delta p) k_n \sin \varphi_n(x,t) + O(\eta \varepsilon^2). \hfill (3.73)
\]

where all starred quantities are computed at \( p=p_* \). Deriving eq. (3.72) with respect to time, combining it with eqs. (3.72-73), allows us to eliminate both \( \delta p \) and \( \delta \dot{p} \). This yields

\[
\ddot{x} = \varepsilon \sum_{n \in \mathcal{N}} \phi_n k_n \sin \varphi_n(x,t) - \frac{\varepsilon^2}{2} \sum_{l, n \in \mathcal{N}} V_l^* V_n^* k_n \left[ \sin(\varphi_n + \varphi_l) + \sin(\varphi_n - \varphi_l) \right] + O(\eta \varepsilon^2) \hfill (3.74)
\]

with

\[
\phi_n = \sigma_* V_n^* + V_n^* (v_n - w) = \sigma_* (V_n^* + \frac{v_n}{\sigma_*} - w) \hfill (3.75)
\]

Equation (3.74) is derived, through elimination of \( \dot{P} \), from the canonical equations of Hamiltonian

\[
H'(P, x, t) = \frac{P^2}{2} + \varepsilon \sum_{n \in \mathcal{N}} \phi_n \cos \varphi_n(x,t)
\]

\[
- \frac{\varepsilon^2}{2} \sum_{l, n \in \mathcal{N}} V_l^* V_n^* \frac{k_n}{k_n + k_l} \cos(\varphi_n + \varphi_l) - \frac{\varepsilon^2}{2} \sum_{l, n \in \mathcal{N}} V_l^* V_n^* \frac{k_n}{k_n - k_l} \cos(\varphi_n - \varphi_l). \hfill (3.76)
\]
This is a Hamiltonian describing the motion of a particle in several longitudinal waves. It is a generalization of the paradigm Hamiltonian given by eq. (2.1). If \( V_n' \ll O(\eta^{1/2}) \), only the first sum in (3.76) makes sense.

Figure 3.3 graphically shows how \( \phi_n \) is computed from \( V_n \). For simplicity we have assumed \( V_{n_1} = V_{n_2} = V_{n_3} = V \). We choose \( n = n_1 \) such that the \( n_1 \)-th term is resonant, and \( v_{n_1}^{-w} \) is small enough for \( \Delta p_1 = |p_+ - p_{n_1}| \ll \Delta p_\eta \). Geometrically, the line \( \Omega = v_{n_1}^{-w} \) intersects \( \Omega(p) \) at \( p_{n_1} \), according to eq. (3.64), and intersects \( p_{n_1} \) the tangent \( T_\Omega \) to \( \Omega(p) \) at \( p_+ \); \( p_{n_1} \) is very close to \( p_{n_1} \) which has not been represented.

According to eq. (3.75), the line \( p = p_{n_1}^{-w} \) intersects \( T_V \) (the tangent to \( V(p) \) at \( p_+ \)) at \( \phi_{n_1} \), which is very close to \( V(p_{n_1}) \). This implies

\[
v_{n_1}^{-w} = \sigma_v(p_{n_1}^{-w} - p_+) + O(\eta \Delta p_1)
\]

(3.77)

and

\[
V_n(p_{n_1}) = V_{n_1} + V_{n_1}' \frac{v_{n_1}^{-w}}{\sigma_+} + O(\eta \Delta p_1) = \frac{\phi_n}{\sigma_+} + O(\eta \Delta p_1).
\]

(3.78)

It so appears that \( \phi_{n_1} \) is computed by applying Chirikov's prescription when using the overlap criterion, namely \( V_n(p) \) is computed at the corresponding resonant value \( p_n \) of the action \( p \). His prescription simply stemmed from the approximation of the \( n \)-th isolated resonance by a pendulum. Here the same approximation stems from a perturbation approach to the complete equations of motion. This result is not intuitive at all, and intuitive approaches can lead to completely wrong
results (see sec. 4.3.1 and refs. [35,73]). The centered-resonance approximation was independently found appropriate in the work by Reichl and Zheng [109].

For a resonant term like \( n=n_2 \) in fig. 3.3, such that \( v_{n_2} \) is too far from \( w \) for eq. (3.77) to hold, \( \phi_n \) can differ a lot from \( \sigma \cdot V_n(p_n) \). The line \( \Omega=v_{n_2} \) intersects \( \Omega(p) \) at \( p_{n_2} \) and \( T_\Omega \) at \( p_{n_2}' \), but \( p_{n_2} \) and \( p_{n_2}' \) are no longer close to each other. The line \( p=p_{n_2}' \) intersects \( T_V \) at \( \xi_{n_2} \), which is far from \( V(p_{n_2}) \).

For a non-resonant term like \( n=n_3 \) in fig. 3.3, \( \phi_n \) is defined, even so \( p_n \) is not. The line \( \Omega=v_{n_3} \) intersects \( T_\Omega \) but not \( \Omega(p) \). Thus \( \xi_{n_2} \) is constructed as for \( \xi_{n_1} \) and \( \phi_{n_2} \), but \( V(p_{n_3}) \) does not exist.

In order to apply the overlap criterion to Hamiltonian (3.58) between \( v_k \) and \( v_{k+1} \) in a reliable way one needs the reduction to Hamiltonian (3.76) to be possible, and eq. (3.78) to hold. Therefore relation

\[
\Delta p_{\eta} \gg \Delta p_k = |p_k - p_{k+1}| ,
\]

is necessary. Then

\[
\Delta p_k = \frac{\Delta v_k}{\sigma(p_k)} + O(\eta \Delta p_k) .
\]

Equation (3.79) can be made explicit through

\[
\frac{d\sigma}{dp}(p_k) \Delta v_k \ll \sigma^2(p_k)
\]
\[
\frac{d^2 \nu_n}{dp^2} (p_k) \Delta \nu_k \ll \sigma(p_k) \frac{d\nu_n}{dp} (p_k), \quad n=\ell, \ell+1.
\] (3.81)

These equations can be viewed as bounding \( \sigma(p_k) \) from below. In contrast to a similar bound (condition of moderate nonlinearity) for the local reduction to a pendulum Hamiltonian \([4,12]\), eq. (3.81) involves neither \( \Omega(p_k) \) nor \( \varepsilon \).

If Hamiltonian (3.57) is truncated with only resonances \( \ell \) and \( \ell+1 \), the present perturbation theory simultaneously justifies the local–quadratic–approximation, and the centered–resonance–approximation introduced in sec. 3.1.1. In that section the expansion of \( H_0(1) \) was done at \( I_m \) instead of \( I_z \). This induces an error \( \delta M^2 |I_m - I_z|/I_z^5 \ll 1 \) on \( \sigma \). This is done in order to avoid a dependence of \( H'_p \) on \( z \) that does not exist when renormalizing exactly (sec. 3.2.1).

3.1.6 Two–resonance Approximation

Transition to global stochasticity can be viewed as the breakup of the last KAM torus in a domain \( \mathcal{D} \) of phase space of interest. It corresponds to the merging of outer stochastic layers of primary resonances bounding \( \mathcal{D} \) as well. Therefore we motivate the two–resonance approximation by showing that the neglected terms have a much smaller contribution than the kept terms to the outer width of stochastic layers of primary resonances bounding \( \mathcal{D} \).

Consider the many–wave Hamiltonian

\[
H(p,x,t) = \frac{1}{2} p^2 - \varepsilon \sum_{n \in \mathcal{N}} U_n \cos(k_n x - \omega_n t + \alpha_n), \quad (3.82)
\]
where $\epsilon$ is a small parameter and $\mathcal{N}$ is a set of integers. We assume the phase velocities $v_n = \omega_n / k_n$ to be monotonically changing with $n$. The width of the stochastic layer of resonance $k$ induced by resonance $n$ is computed from Hamiltonian (3.82) where only resonances $k$ and $n$ are kept. According to Chirikov's results [4] the total width is the sum of the individual contributions. Using the result of appendix A allows us to characterize the Hamiltonian system (3.82) restricted to resonances $k$ and $n$ by a paradigm Hamiltonian $H_p$ with parameters

$$M = \frac{\epsilon U_k}{\Delta v^2}$$

$$P = \frac{\epsilon U_n}{\Delta v^2}$$

$$k = \frac{k_n}{k_k}$$

(3.83)

where

$$\Delta v = |v_n - v_k|.$$  

The standard map that governs the domain close to resonance $M$ has a parameter $K$ given by eq. (2.60). Here

$$\mu = \frac{2(\epsilon U_k)^{1/2}}{\pi |\omega_n - k_n| k_n}$$

$$\mu = \frac{2(\epsilon U_k)^{1/2}}{\pi |\omega_n - \omega_k| k_n k_k}$$

(3.85)
The exponential dependence of \( K \) on \( \alpha/\mu \) makes it quite sensitive to a small change of this parameter. The largest \( \mu/\alpha \) gives the largest contribution to the width of the layer.

For the standard map Hamiltonian [eq. (2.15)] \( U_n = M = K/4\pi^2 \) for all \( n \)'s and \( \mu = 2\sqrt{M}/\pi|n-\ell| \). Thus, the \( n=\ell+1 \) primary resonance gives the main contribution to the width of the outer stochastic layer of primary resonance \( \ell \) in the domain of mean velocities [\( v_\ell, v_{\ell+1} \)]. The \( n=\ell+2 \) term yields a factor \( \mu/\alpha \) two times smaller; the \( n=\ell-1 \) term gives a factor \( \mu/\alpha \) three times smaller than the \( \mu/\alpha \) of the \( n=\ell+1 \) term, since resonance \( \ell-1 \) is not on the side of the outer stochastic layer of interest (\( \alpha = 3 \)).

We now can justify the two-resonance approximation for building the scheme \( T_r \) of sec. 3.1.1. For Hamiltonian (3.4) the width of the stochastic layer of resonance \( R_i \), \( i=m, m+1 \) in the domain \([w_i, w_{i+1}]\) is mainly due to resonance \( R_{i+1} \). The coefficient \( \mu/\alpha \) of resonance \( R_{i+2} \) is two times smaller, and of resonance \( R_{i-1} \) is three times smaller, than the \( \mu/\alpha \) of resonance \( R_{i+1} \). The Fourier expansion of the \( W \cos2x(1,\theta) \) is made up of harmonics \( \cos2N\theta \) which yield a \( \mu/\alpha \) \( 2N \) times smaller than the \( \mu/\alpha \) of \( R_{i+1} \).

The preceding argument relies upon eq. (2.60) which is only correct for small \( \mu \)'s. This implies

\[
\mu = \frac{2\sqrt{M}}{nk} \ll 1 ,
\]

which bounds through eq. (3.85), the maximum amplitude of resonance \( \ell \) of Hamiltonian (3.82) for which the two-resonance approximation holds.
between \( v_1 \) and \( v_1 \pm 1 \). By using the equivalent Hamiltonian (2.12), the maximum amplitude of resonance \( \pm 1 \) is bounded by

\[
\mu_k = \frac{2\sqrt{F}}{\pi} \, k \ll 1.
\]

(3.87)

The rationale about the frequencies \( \dot{\varphi}_n(\text{ut}, t) \) given in sec. 3.1.1 is more intuitive, but less powerful since the ratio of the frequencies of the deleted terms to those of the kept ones is not really large.

3.1.7 A More Precise Scheme

The derivation of the scheme \( T_r \) of sec. 3.1.1 accounts for the distortion of the KAM torus \( \mathcal{T} \) by resonance \( \mathcal{M} \) and not by resonance \( \mathcal{P} \). For \( \mathcal{T} \) critical and \( \mathcal{M}/\mathcal{P} \) close to 1, the parameter \( \epsilon \) of sec. 3.1.5 is not small when computed in Hamiltonian (3.10), and the reduction to a many-wave Hamiltonian only is marginally correct.

In order to account for the distortion of \( \mathcal{T} \) induced by resonance \( \mathcal{P} \), this resonance also must be killed at lowest order. This is done by applying to \( H_\mathcal{P} \) the Kolmogorov transformation \([4, 12]\) with generating function

\[
F(I, x, t) = I x + M \frac{\sin x}{I} + P \frac{\sin k(x-t)}{k(I-1)},
\]

(3.88)

and one gets the new Hamiltonian

\[
\mathcal{H}'(I, \varphi, t) = H_\mathcal{P}(I) + \epsilon \sum_{i=0}^{+\infty} \sum_{j=\infty}^{\infty} K_{ij}(I) \cos \varphi_{ij}(\varphi, t),
\]

(3.89)

where \( \epsilon = 1 \).
\[ H_2(I) = \frac{I^2}{2} + \frac{M^2}{4I^2} + \frac{P^2}{4(I-1)^2}, \quad (3.90) \]

\[ \varphi_{ij}(\delta,t) = (jk+1)\delta - jkt, \quad (3.91) \]

and the \( K_{ij} \)'s, given in appendix C, are at lowest order

\[ K_{ij}(I) = A_{ij}(k,l)M^iP^j[1+O(M^2) + O(P^2)], \quad (3.92) \]

with

\[ A_{ij}(k,l) = \frac{(i+j+1)^{i+j-3}}{i!j!k!l!(i-1)^2(1-1)^2} \left[ \frac{1(i-1)}{1(i-1)} + \frac{1(i)}{(i-1)^2} + \frac{k^3 j(j-1)}{i^2} \right]. \quad (3.93) \]

for \( i,j \geq 1 \). Hamiltonian \( \mathcal{H}' \) is both similar to and different from Hamiltonian \( \mathcal{H} \) given by eq. (3.4): it also has terms resonant for \( \delta = \omega_m \) \((i=m-1, j=1)\), and a nonlinear angle-independent part \( H_2 \), but the \( i=0 \) terms in eq. (3.89) now are non-resonant since there is a new small denominator \( (I-1)^2 \) in \( H_2 \): the \( P=R_1 \) resonance no longer exists. Obviously, the small box no longer can be built as in fig. 2.9(a).

For the golden mean fixed point, resonance \( P \) becomes resonance \( M' \) which is killed when \( T_r \) is applied a second time. Since the Kolmogorov transformation (3.89) kills both resonances \( M \) and \( P \) at the same time, we can try to build something like \( T_r^2 \) right away. This is done by following the recipe of fig. 2.11: we build the small box on the \( R_m \)'s \( m \geq 2 \). We take advantage of the two possible descriptions of \( S_p \) by \( H_p \) and \( H_e \) [eq. (2.12)] for only considering KAM tori with zoning numbers \( z \geq 2 \) (see sec. 2.3.1.1.2). In order to keep that feature at all
steps of the renormalization (i.e. \( z^{(n)} > 2 \) for all \( n \)'s), resonance \( M' \) is chosen to be resonance \( R_{m+\lambda} \) where \( m \) is defined by relation (3.9) and \( \lambda \) by

\[
\lambda = 0 \quad \text{for} \quad \delta z < \frac{1}{2},
\]
\[
\lambda = 1 \quad \text{for} \quad \delta z > \frac{1}{2},
\] (3.94)

where \( \delta z \) is given by eq. (3.29). Let

\[
\mu = 1 - \lambda .
\] (3.95)

The new mean velocity in the renormalized system \( S'_P \) is a linear function of \( u \) such that for \( u = w_{m+\lambda}, \ u' = 0 \), and for \( u = w_{m+\mu}, \ u' = 1 \). One obtains

\[
u' = (1-2\lambda) \frac{k+\ell+\mu}{k} \ [k-(k+\ell+\lambda)u],
\] (3.96)

and

\[
k' = \frac{k+\ell+\mu}{k+\ell+\lambda},
\] (3.97)

where \( \ell \) is defined by eq. (3.9). According to eqs. (2.22-21) and (3.96-97)

\[
z' = \frac{1-2\lambda}{z-m-\lambda}.
\] (3.98)
For \( \lambda = 0 \) we naturally recover the results of sec. 3.1.1, with the exception that now \( m \geq 2 \) and the previous golden mean fixed point \( F_1 \) (\( z_1 = g \)) vanishes.

We denote with a superscript \( \lambda \) the universal quantities corresponding to \( \lambda \). For instance \( z^0 \) is \( z_m \) of sec. 3.1.1. Relation (3.98) also gives a denumerable set of unstable fixed points to \( z'(z) \) for \( \lambda = 1 \).

\[
z^1_m = m + \delta z^1_m
\]  
(3.99)

with

\[
\delta z^1_m = \frac{1}{2} (m^2 + 2m - 3)^{1/2} + 1 - m,
\]  
(3.100)

where \( m = 2, 3, \ldots \). Figure 3.4 displays \( z'(\delta z) \) and \( \delta z'(\delta z) \) for the new renormalization scheme noted by \( T_R \). Owing to the definition of \( \lambda(\delta z) \), these functions are symmetric with respect to \( \delta z = \frac{1}{2} \). Their left part, corresponding to \( \lambda = 0 \), looks like the \( \delta z < \frac{1}{2} \) part of fig. 3.1. Let \( z = [a_0, a_1, a_2, \ldots] \). Then \( \frac{1}{2} < \delta z < 1 \) means \( a_1 = 1 \). From eq. (3.98) we get

\[
z' = [a_2 + 1, a_3, a_4, \ldots].
\]  
(3.101)

The continued fraction expansion of \( z' \) is that of \( z \) shifted by 2, except that \( m' = a'_0 = a_2 + 1 \). As a consequence

\[
z^1_m = [m, m - 1, m - 1, m - 1, \ldots].
\]  
(3.102)
In particular \( z_2^1 = 1+g \), and \( \delta z_2^1 = g-1 \) is the \( \delta z_1 \) of sec. 3.1.1. Therefore \( T_R \) has a new golden mean fixed point \( F_2^1 \) corresponding to \( z_2^1 \). The golden mean torus now lies between \( R_2 \) and \( R_3 \) (it is the golden mean torus with \( z=z_1 \), of the equivalent Hamiltonian \( H_e \)). For \( k(0) = 1 \), fig. 2.11 displays \( R_2 \) and \( R_3 \) with respectively 2 and 3 islands and \( R_2' \) and \( R_3' \) with respectively 5 and 8 islands (these two last numbers come from \( k' = 2/3 \), see sec. 2.3.1.1.2).

The renormalized values of \( M \) and \( P \) through \( T_R \) are got by dealing with \( \mathcal{H}' \) as we dealt with \( H' \) when building \( T_R \), but in a more rigorous way that takes account of the results of secs. 3.1.5 and 3.1.6.

Because of the existence of small denominators in \( \mathcal{H}' \), we cannot expect \( T_R \) to describe the \( F_m^s \)'s for \( m \) large better than \( T_R \) does. We expect \( T_R \) to be better than \( T_R \) for \( m \) small, in particular for \( m=2 \). In this case the small box encompasses the mean velocities between \( w_2 \) and \( w_3 \), i.e. a range \( \Delta \omega_2 = w_2 - w_3 \) of mean velocities. For a given torus \( \mathcal{F} \) with zoning number \( z \), \( I \) oscillates about \( I_z \), the constant value of \( I \) for \( \varepsilon = 0 \) with an amplitude \( \delta I = O(\varepsilon) \ll \Delta \omega_2 \). The \( K_{ij} \)'s and \( \sigma = \frac{d^2 H_e}{dI^2} \) vary on the larger scale \( \Delta v = 1 \).

We now can use the results of sec. 3.1.5, where the existence of the three scales \( \Delta v = 1 \), \( \Delta \omega_2 \), and \( \delta I \) are accounted for by the two small parameters

\[
\eta = \frac{\Delta \omega_2}{\Delta v}
\]

and
\[ \varepsilon = \frac{\delta I}{\Delta w_2} \quad (3.103) \]

Therefore \( \mathcal{H}' \) can be reduced to a many-wave Hamiltonian of the type given by eq. (3.76). Here the structure is even simpler since the \( V_n' \) also are of order \( \eta \). Application of formula (3.73) yields the \( \phi_{ij} \)'s from the \( K_{ij} \)'s, and a new Hamiltonian

\[ \mathcal{H}''(\vartheta, t) = \frac{1}{2} J^2 + \varepsilon \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} \phi_{ij} \cos \phi_{ij}(\vartheta, t) \quad (3.104) \]

By analogy with the derivation of \( T_r \), we now only retain the \((i,j) = (l,1)\) and \((m,1)\) terms in the sum of eq. (3.104). This is justified by the results of sec. 3.1.6: the \((i,j)\) term yields a parameter \( \mu_{ij} \) in eq. (2.60) for parameter \( K \) of the map ruling the stochastic layer of resonance \((l,1)\); this parameter \( \mu_{ij} \) scales like \((i-lj)^{-1}\). In the domain \([w_m, w_{m+1}]\) the largest \( \mu_{ij}/\alpha \) is thus given by \((i,j) = (m,1)\).

After rescaling \( \Delta w_m \) into 1 and \( \Delta \vartheta = (k+m+\lambda)^{-1}2\pi \) into \( 2\pi \), one gets a new paradigm Hamiltonian \( \mathcal{H}' \) with parameters given by eq. (3.97) and

\[ M' = F(l+\lambda) \quad (3.105) \]
\[ P' = F(l+\mu) \quad (3.106) \]

with

\[ F(n) = \frac{\sigma_+}{\Delta w^2_m} K_{ln}(1_{n+1}) \quad (3.107) \]
where

$$\sigma_* = \frac{d^2 H_2}{d l^2} (I_m) ,$$

(3.108)

and $I_\xi$ is defined for any $\xi$ by

$$\frac{\partial H_2}{\partial I} (I_\xi) = \frac{k}{k+\xi-1} .$$

(3.109)

A simpler scheme analogous to $T_{\rho}$ (sec. 3.1.2) can be built from $T_R$ by keeping only the lowest order terms in eq. (3.92). In particular an equation similar to eq. (3.45) can be defined for the stable manifold of the fixed point $F_m^\lambda$

$$MP^\lambda_{\sigma_m} = R_{m}^\lambda (k) ,$$

(3.110)

where $\sigma_m^\lambda = (z_m^\lambda-1)^{-1}$ and $R_m^\lambda (k)$ is analytically computable [73]. The O(P^2) term in eq. (3.92) yields the more accurate expression

$$MP^\lambda_{\sigma_m} [1+c(k)P^2] = R_m^\lambda (k) ,$$

(3.111)

where $c_m^\lambda (k)$ is analytically computable [73]. Figure 2.18 yields the coefficients $R = R^1_2$ and $c=c^1_2$ for the golden mean fixed point.

An important feature of $T_R$ is that it verifies the hierarchy of the fixed points defined in sec. 2.3.2.3 [35,73]. This and the numerical calculations of ref. [27] bear out the existence of such a hierarchy for $R$, and the robustness of noble KAM tori.
3.1.8 Conditions of Validity

The domain of validity of Kolmogorov transformation (3.88) puts a natural boundary on the domain of validity of $T_R$ [37]. In the KAM domain, M and P are small enough for the transformation $(v, x) \to (I, \phi)$ to be close to identity. When M and P grow, this property no longer holds, but we may require a special feature of this property to stay, namely

$$\frac{\partial I}{\partial v} > 0,$$  \hspace{1cm} (3.112)

$$\frac{\partial \phi}{\partial x} > 0.$$  \hspace{1cm} (3.113)

This is equivalent to condition $\frac{\partial^2 F(I, x, t)}{\partial I \partial x} > 0$ for all x's and t's in some domain of I's. This condition is the most difficult to satisfy when both cosines are equal to 1 in $\frac{\partial^2 F}{\partial I \partial x}$. It is therefore equivalent to

$$g(I) = \frac{\partial^2 F}{\partial I \partial x} (I, 0, 0) > 0.$$  \hspace{1cm} (3.114)

This condition is satisfied for an interval of I's between 0 and 1 if the maximum of $g(I)$ is positive. This last condition is

$$M^{1/3} + P^{1/3} < 1,$$  \hspace{1cm} (3.115)

and limits the domain where transformation (3.88) makes sense. In particular if $M/P = 1$ then $M=P < 1/8$. 
In order to compute a stable manifold for large values of \( M \) or \( P \), it is necessary first to renormalize \( \mathcal{H}_p \) with the scheme using the action–angle transformation defined in sec. 3.1.4, and to then apply eq. (3.111) to the renormalized Hamiltonian.

If conditions (3.112–113) are imposed on transformation (3.2) used for deriving \( T_L \), one gets the condition \( 1 > \sqrt{M} \), and, according to eq. (3.3)

\[
v > \sqrt{M}(1+\cos x) .
\]  

(3.116)

At \( x=0 \) this implies

\[
v > 2\sqrt{M} .
\]  

(3.117)

If \( M/P \) and \( k \) one close to 1, a similar condition should apply, by symmetry, on the equivalent Hamiltonian (2.12). This yields

\[
1-v > 2\sqrt{P} .
\]  

(3.118)

For conditions (3.117) and (3.118) to hold simultaneously one must require \( s<1 \) where \( s \) is defined by eq. (2.7). This is another way to motivate the resonance overlap criterion for \( \mathcal{H}_p \)'s not too far from the central one (\( k=M/P=1 \)).

Among the terms deleted when making the two–resonance approximation in Hamiltonian (3.89), some have a mean velocity that corresponds to the mean velocity in \( S_p \) of the primary resonances of \( S_p^{(L)} \), \( L>1 \). In order to check this, for instance consider the golden
mean torus \((z = z_2^{1/2} = 1+g, \; u=k/(k+g))\). From eq. (3.98) we get for \(\ell=1\) (i.e., \(m=2\))

\[
z = 2 + \frac{1}{1 + \frac{1}{z' - 1}} \tag{3.119}
\]

According to the definition of sec. 2.3.1.1.2, for any value of \(L\), resonance \(R_n^{(L)}, \; n=2,3\), of the \(L\)-th subsystem \(S_p^{(L)}\), has a zoning number \(z_n^{(L)} = n\). It is easily shown recurrently from eq. (3.119) that in the original system \((S_p^{(0)} = S_p)\) \(R_n^{(L)}\) has a zoning number

\[
z_n^{(0)} = \frac{f_2L+n}{f_2L+n-2} \tag{3.120}
\]

Now we recognize that the \((i,j)\) term in eq. (3.89) has a mean velocity \(k/(k + i/j)\) and a zoning number \(z_1^{(0)} = i/(j+1)\). For \((i,j) = (f_2L+n-1, f_2L+n-2)\), where \(f_\nu\) is defined by eq. (2.32), the corresponding zoning number is \(z_n^{(0)}\) defined by eq. (3.120). This apparently makes inconsistent deleting terms with zoning numbers \(z_n^{(0)}\) at step 0, when such resonant terms are reconstructed at step \(L\) of the renormalization process. In fact the amplitudes of the deleted terms can be renormalized as explained in sec. 3.2.1 and compared to the amplitudes \(M^{(L)}\) and \(P^{(L)}\). It is found that those neglected terms bring a correction decreasing with \(L\), which already is \(310^{-2}\) smaller than \(M^{(L)}\) and \(P^{(L)}\) for \(L=2\) [73].
3.2 Exact Approach

We have just described approximate renormalization schemes. The present section shows in what sense these schemes are approximate, and how to build an exact scheme.

3.2.1 Principle

Section 3.1 suggests that renormalizing is similar to following the KAM procedure of elimination of resonances order by order, combined with some rescalings and a change of reference frame. The first element of the method is a canonical transformation; the second one depends on the torus of interest, and is non-canonical.

Let us make this idea more precise for the golden mean fixed point, as approximately described by $T_r$ of sec. 3.1.1. Let $Q_n$ be the resonance corresponding to the truncation at order n of the continued fraction expansion of the zoning number of the golden mean torus. When adopting the slow renormalization pace of fig. 2.9, $Q_0$ is resonance $M$ of $S_p$, $Q_1$ is resonance $P$ of $S_p$ and resonance $M'$ of $S_p'$, and for any $n \geq 1$, $Q_n$ is resonance $P^{(n-1)}$ of $S_p^{(n-1)}$ and resonance $M^{(n)}$ of $S_p^{(n)}$. Define $\alpha$ as the ratio of the amplitudes of resonances $M$ and $P$ at the golden mean fixed point. According to eqs. (3.31) and (3.34), $k = g$. Equations (3.1) and (3.28) show that $\Delta w_1 = g^{-2}$ at the noble fixed point $F_1$. Equations (3.7) and (3.36) yield $B_1(k) = \Delta w_1^{-2} = g^4$. As a result, at $F_1$, the first of eqs. (3.35) yields $M_1 = g^4 P_1$, and

$$\alpha = \frac{P_1}{M_1} \approx g^{-4}.$$  

(3.121)
As we did for resonances $M$ and $P$, let us also call $Q_n$ the amplitude of resonance $Q_n$ as measured in the original system $S_P$. Since resonance $p(n-1)$ of $S_P^{(n-1)}$ is resonance $M^{(n)}$ of $S_P^{(n)}$, at fixed point $F_1$ $p(n-1) = P_1 = \alpha M_1 = \alpha M^{(n)}$. Therefore, at each step of the renormalization, amplitudes are multiplied by a factor $1/\alpha$. $Q_n$ corresponds to renormalizing $M^{(n)}$ back $n$ times. Therefore

$$Q_n = \alpha^n M^{(n)} = \alpha^n M_1.$$  \hfill (3.122)

At the KAM fixed point (i.e. for $M$ and $P$ small), according to eq. (3.44), $M^{(n)}$ scales like

$$M^{(n)} \sim (M^{(0)})^n (P^{(0)})^{n-2}.$$  \hfill (3.123)

If we assume

$$M^{(0)} = 0(\varepsilon)$$  \hfill (3.124)

$$P^{(0)} = 0(\varepsilon)$$  \hfill (3.125)

we get

$$Q_n \sim \varepsilon^n.$$  \hfill (3.126)

Since $f_n = O(\varepsilon^n)$, eq. (3.126) shows that $Q_n$ decreases like exponential of exponential $n$, whereas eq. (3.122) shows that $Q_n$ only decreases like exponential $n$ at the golden mean critical fixed point. However, the correct behavior of approximate renormalization schemes which only
retain the $Q_n$'s is a hint that some kind of KAM convergence is kept for the neglected terms.

Therefore the recipe for renormalizing $H_p$ exactly with scaling (3.124), (3.125) consists in carrying the KAM procedure to successive orders $\epsilon^{n-1}$, $n=1,2,3...$ with

$$q_n = \gamma(n) + \delta(n),$$

(3.127)

when $\gamma(n)$ and $\delta(n)$ are given by eq. (3.43), and in making some changes of coordinates. The first renormalization step kills all terms up to order $\epsilon^k$ where $k$ is defined by eq. (3.9). For the golden mean fixed point, $k=0$ since resonance $P$, of order $\epsilon$, is kept in $H_p'$. In order to kill all $O(\epsilon)$ terms, two iterates of the renormalization must be performed, or these terms can be suppressed by Kolmogorov transformation (3.88) used for defining $T_R$. This second solution is retained in the following. An alternate solution would be to consider $P$ as being of order $O(\epsilon^2)$, as suggested by scaling (3.122), and to take $q_n = \gamma(n) + 2\delta(n)$.

The present way of removing resonances requires a slight modification of Kolmogorov's superconvergent method which rather proceeds by steps $\epsilon^{2n}$ instead of $\epsilon^n$. For fixed point $P_m$, the unstable eigenvalue of matrix (3.42) is $z_m$. Coefficients $\gamma(n)$ and $\delta(n)$ of matrix (3.43) scale like $z_m^n$, and $q_n$ too, according to eq. (3.127). For the golden mean fixed point, the $q_n$'s are the $f_n$'s of eq. (2.32) which scale like $g^n$. Since $g<2$, the renormalization pace for killing resonances is slower than the KAM pace. For other fixed points $z_m>2$ and the pace is faster. This is no problem since, as shown in
sec. 3.2.2, one can define the generating function of a canonical transformation that kills all terms up to any a priori order.

For a more general Hamiltonian of the type (3.57) the same renormalization procedure can be followed provided the set $\mathcal{N}$ only has two elements. If more than two primary resonances are present, there is no a priori reason to choose one couple of primary resonances rather than another in order to define the equivalent of resonances $M$ and $P$. Fortunately eqs. (3.42), (3.43), and (3.127) show that $I_n$ depends only on the zoning number, and the canonical transformation killing all resonances up to order $\varepsilon^{q_n-1}$ is the same whatever be the chosen couple. In a region of phase space of interest, it is possible to identify the largest $O(\varepsilon^{q_n})$ term when $n$ grows, and to find out what is the couple of primary resonances that mainly contributes to it. This is the couple to be kept in order to define the zoning number of a given KAM torus. We call it the basic couple. The non–canonical transformation which allows us, when combined with the canonical removal of resonances, to renormalize, commutes with the canonical transformation. Therefore resonances can be first removed to order $\varepsilon^{q_n-1}$, and, after that, the non–canonical transformation corresponding to $n$ steps of the renormalization can be done. This procedure is made more explicit in the next section.

3.2.2 Converging to the Exact Fixed Point

Now that we have an exact renormalization scheme, it can be used in several ways. One is improving the accuracy of threshold estimates, and is done in sec. 3.2.4. Another is to get a highly accurate description of the fixed points.
A priori this second task seems to be formidable: even if we begin with the paradigm Hamiltonian $H_p$, after two iterates of the renormalization we are left with a very general Hamiltonian with a dense set of resonant terms with action-dependent amplitudes. Fortunately, we can take advantage of the simplicity of $H_p$ for directly computing $n$ steps of the renormalization procedure.

Let us again concentrate on the golden mean fixed point. With scaling (3.124) and (3.125), a term of order $\epsilon^L$ corresponds to a product $M^k P^n$ with $l+n = L$, and therefore to a product $(\cos x)^l [\cosh(x-t)]^n$ which yields a phase

$$\varphi_{L \ell \eta}(x,t) = (nk+\eta \ell) x - nk t$$, \hspace{1cm} (3.128)

where $\eta = \pm 1$, and $n = L-\ell$. As explained later, there is a set of functions $A_{L \ell \eta}(I)$ such that we can kill all resonances up to order $\epsilon^N$ by making a canonical transformation with generating function

$$F(I,x,t) = Ix + \sum_{L=1}^{N} \sum_{\ell=0}^{L} A_{L \ell \eta}(I) \frac{\cos \varphi_{L \ell \eta}(x,t)}{1-u_{L \ell}}$$, \hspace{1cm} (3.129)

where

$$u_{L \ell} = \frac{(L-\ell)k}{(L-\ell)k+\eta \ell}$$ \hspace{1cm} (3.130)

This yields a new Hamiltonian $H_N(I,\psi,t)$ which may be Fourier-expanded with terms $\cos \varphi_{L \ell \eta}(\psi,t)$ where $L>N$. Following a method analogous to that of appendix C, all Fourier coefficients can be exactly computed in
terms of Bessel functions, and the correct expression for the $A_{l\ell \eta}$'s of eq. (3.129) can be recursively found in $L$.

For $N = f_n - 1$ we can define a new position

$$x^{(n)} = \varphi_{f_n f_{n-1}^{-1}}(\psi, t), \quad (3.131)$$

and a new time $t^{(n)}$ such that

$$\varphi_{f_n f_{n-1}^{-1}}(\psi, t) = k^{(n)}(x^{(n)} - t^{(n)}), \quad (3.132)$$

where

$$k^{(n)} = \frac{f_n f_{n-1}^{k+f_{n-2}}}{f_{n-1}^{k+f_{n-2}}}. \quad (3.133)$$

For large $n$'s $k^{(n)}$ converges to $g$, as it did for $T_r$ (eq. (3.34)). In order to make that convergence trivial, we can take $k^{(0)} = g$; then $k^{(n)} = g$ for all $n$'s.

Let $p^{(n)}$ be the conjugate variable of $x^{(n)}$. We can choose the origin of $p^{(n)}$ such that $p^{(n)} = 0$ is the resonant value of $p^{(n)}$ corresponding to $dx^{(n)}/dt^{(n)} = 0$. Let $\sigma^{(n)}$ be the second derivative of the angle-independent part of $H_{f_n^{-1}}$ computed at $p^{(n)} = 0$, let $\nu^{(n)} = \sigma^{(n)} p^{(n)}$, and $H(n)(\nu^{(n)}, x^{(n)}, t^{(n)})$ be Hamiltonian $H_{f_n^{-1}}$ rewritten with the $(\nu^{(n)}, x^{(n)}, t^{(n)})$ coordinates. Notice that, if one applies to $H(n)$ the reduction to a many-wave Hamiltonian of sec. 3.1.5 and the two-resonance approximation that keeps resonances $Q_n$ and $Q_{n+1}$, one finds a Hamiltonian quite close to $H_p^{(n)}$ as obtained from $T_r^n$. 
If the golden mean torus is critical for \( n \) large, \( H^{(n)} \) converges toward a universal Hamiltonian. In particular resonance \( Q_n \) (with \( \dot{x}^{(n)} = 0 \)) and resonance \( Q_{n+1} \) (with \( \dot{x}^{(n)} = 1 \)) have amplitudes \( M^{(n)} \) and \( p^{(n)} \) computed at the corresponding resonant values of \( v^{(n)} \) (by definition it is 0 for \( Q_n \)). \([M^{(n)},p^{(n)}]\) converges toward a universal value \((M_1,P_1)\) typical of the golden mean fixed point. Successive converging approximate values \([M_1^{(n)},P_1^{(n)}]\) of \((M_1,P_1)\) can be found by requiring

\[
M^{(n)} = M^{(n+1)}
\]
\[
p^{(n)} = p^{(n+1)} .
\]

(3.134)

This defines corresponding initial values \([M^{(0)}_{\text{in}},p^{(0)}_{\text{in}}]\) of parameters \([M^{(0)},p^{(0)}]\). The tangent mapping to \([M^{(0)},p^{(0)}] \to [M^{(n)},p^{(n)}]\) computed at \([M^{(0)}_{\text{in}},p^{(0)}_{\text{in}}]\) has eigenvalues \( d_n \) and \( d'_n \) which approximate \( \delta^n \) and \( \delta'^n \). Relation (3.134) is a property of the fixed point as well. According to sec. 3.2.3, \([M^{(0)}_{\text{in}},k^{(0)}_{\text{in}}], [M^{(n)}_{\text{in}},P^{(n)}_{\text{in}}]\), \( d_n^{1/n} \), and \( d'_n^{1/n} \) converge to their asymptotic values at a rate \((\delta/\delta')^n\). The renormalization scheme \( T_r \) of sec. 3.1.1 amounts to take \( n=0 \) in eq. (3.134), and therefore yields the lowest order approximation to the exact fixed point.

This exact technique of renormalization should be directly transposable into a renormalization for area-preserving maps. We can begin with the standard map, and perform the KAM procedure up to order \( K^{f_n^{-1}} \). This yields a new map acting on \([I^{(n)},\phi^{(n)}]\) with a term \( K_n(I^{(n+1)})\sin(f_n\phi) \) where \( K_n \) is of order \( K^{f_n} \). Then define a new position \( \dot{x}^{(n)} = f_n\phi^{(n)} \) and a new action \( v^{(n)} = \sigma_n f_n I^{(n)} \) (this is a
non-canonical transformation), where \( \sigma_n \) again is given by the derivative of the frequency at the resonant value \( v^{(n)}_0 \) of \( v^{(n)} \) for resonance \( K_n \). When rewritten in the \( (v^{(n)}, x^{(n)}) \) coordinates, the map should converge to a fixed map when the golden mean torus is critical. The fixed point can then be computed by a method similar to eq. (3.134).

3.2.3 Hyperbolicity Arguments

Figure 3.5 is a schematic description of the space of Hamiltonians close to an integrable one located at the origin. A fixed point \( F \) of the renormalization group is shown along with its stable and unstable manifold \( S \) and \( U \). Circle \( C \) encloses the region where the renormalization operator may be approximated by its linearized version at the fixed point. Let \( N \) be a codimension 1 manifold that intersects \( U \) transversally at \( V \), and let \( \Delta \) represent a family of Hamiltonian systems depending on one parameter \( s \). \( \Delta \) intersects \( N \) at \( A_0 \) and \( S \) at \( A_\infty \). Let \( A_i \) correspond to the member of the family such that its \( i \)-th iterate through the renormalization belongs to \( N \). Since the least stable eigenvalue \( \delta' \) of \( F \) is negative, only one out of two \( A_i \)'s is displayed for clarity in fig. 3.5, but the argument is the same for all \( i \)'s. Let \( s_i \) be the value of \( s \) corresponding to \( A_i \). By following a reasoning similar to one in ref. [110] we show that

\[
(s_i - s_\infty) \delta^i \to a \quad \text{for} \quad i \to \infty, \tag{3.135}
\]

where \( a \) is a constant and \( \delta \) is the unstable eigenvalue of \( F \).
Obviously $A_i$ may not remain at a finite distance from $A\infty$; otherwise the $i$-th iterate $R^i(A_i)$ would go to $(\infty, \infty)$. Therefore $A_i$ converges to $A\infty$. Moreover for large values of $i$, only a small number of the $R^n(A_i), 1 \leq n \leq i$ are outside C. Thus the tangent mapping $DR$ at $F$ governs the evolution and implies property (3.135).

Figure 3.6 is similar to fig. 3.5, but now $F$ belongs to $N$, and $B_i = R^i(A_i)$. By again using $DR$ at $F$, the projection of $FB_i$ on the least stable direction scales like $\delta^{-i}$ and the projection on the unstable direction scales like $(s_i-s\infty)\delta^i$. The trace of $N$ in the plane of these two directions locally is a straight line close to $F$.

Therefore both projections are proportional and

$$(s_i-s\infty) \left( \frac{\delta}{\delta_s} \right)^i \rightarrow b \text{ for } i \rightarrow \infty,$$

(3.136)

where $b$ is a constant. This result is used in sec. 4.2.

We now go back to fig. 3.5 in order to prove eq. (2.48). Let $\mathcal{T}$ be the torus or the Cantorus corresponding to $F$, and $H_v$ be the Hamiltonian corresponding to point $V$ of fig. 3.5. The quantity $Q$ introduced in sec. 2.3.2.4.1 has a value $Q_v$ for $\mathcal{T}$ at $H_v$. Since $R^i(A_i)$ converges to $V$, the corresponding value of $Q$ converges to $Q_v$. As $Q$ is renormalized according to eq. (2.47), $A_i$ corresponds to a value

$$Q_i \approx \chi^{-i} Q_v.$$

(3.137)

This combined with eq. (3.135) implies eq. (2.48).
3.2.4 Improvement of Threshold Estimates

According to sec. 3.2.3, when renormalizing a Hamiltonian L times and applying to the renormalized Hamiltonian a criterion N which is not verified by the fixed point, the improvement $\eta_L$ on the estimate scales like $\delta^L$. Naturally the better is the criterion, the better is the L-th approximation. When the two-resonance approximation is made on the standard map Hamiltonian of eq. (2.15), and criterion (2.56) is applied one gets $K_0 = 1.107$. When performing the canonical transformation that kills all resonant terms of eq. (2.15) at order M, one directly computes the second iterate of the renormalization. Plugging the renormalized amplitudes $P^{(2)}$ and $M^{(2)}$ of resonances $R_2$ and $R_3$ into criterion (2.56) computed with $k = 2/3$, yields the estimate $K_0 = 0.991$ [35, 73]. This corresponds to an improvement $\delta_2 = (K_0 - K_c)/(K_2 - K_c) \approx 7$, larger than $\delta^2 \approx 2$. The reason this improvement is better than the asymptotic one is the exponential of exponential convergence of $(M,P)$ to $(0,0)$ and $(\infty, \infty)$ away from the critical fixed point (see eq. (3.43)).

The standard map Hamiltonian has a lower threshold than the central paradigm Hamiltonian. This is due to the special phases of resonant terms in eq. (2.15), and can be checked [73] on Hamiltonian

$$H(v,x,t) = \frac{1}{2} v^2 - \frac{K}{4\pi^2} \sum_{n=-1}^{2} \cos(x - nt + \varphi_n).$$

(3.138)

The case $\varphi_n = 0$ for all $n$'s corresponds to a 4-resonance approximation of the standard map. The preceding method yields $K_2 = 1.100$ for the threshold between resonances $n=0$ and $n=1$. The case $\varphi_0 = \varphi_1 = 0$ and $\varphi_{-1} = \varphi_2 = \pi$ yields $K_2'' = 1.114$. This bears out the concept of repulsion
of resonances and of attraction of hyperbolic fixed points explained in fig. 2.7. Pictorially speaking, resonances \( n=-1 \) and \( 2 \), in phase with resonances \( n=0 \) and \( 1 \), increase the compression of the KAM torus and allow it to break up more rapidly. Out-of-phase resonances \( n=-1 \) and \( 2 \) have the opposite effect.

If we have to deal with a Hamiltonian of type (3.57) which does not verify the necessary conditions for the reduction to a many-wave Hamiltonian, renormalizing it exactly several times (possibly by a unique calculation as in sec. 3.2.2) allows us to create a Hamiltonian where the previous rapid variation of parameters now is a slow one, since the mismatch in phase-velocity of primary resonances is on a smaller scale. After a finite number of renormalizations, parameter \( \eta \) of eqs. (3.58) and (3.60) corresponds to the ratio of the mismatch in phase velocity of the primary resonances at the final scale to the same mismatch at the scale immediately larger.

### 3.2.5 Counterexample to Universality

Consider the Hamiltonian

\[
H(v,x,t) = \frac{1}{2} v^2 - Mcosx - Q_1cos(x-t) - Q_2cos2(x-t). \tag{3.139}
\]

For \( Q_n = 0, n=1 \) or \( 2 \), \( H \) is a paradigm Hamiltonian with parameters \( (k,M,P) = (3-n,M, Q_{3-n}) \). Resonances \( R_m \) have a mean velocity

\[
w_m = \frac{3-n}{2-n+m}. \tag{3.140}
\]

For \( n=2, w_2 = 1/2 \) and \( w_3 = 1/3 \), and for \( n=2, w_2 = 2/3 \) and \( w_3 = 1/2 \).
This shows that the golden mean torus lies in different regions of phase space in each case. More generally, universal classes do not correspond in both cases. If \( Q_1 \) is of order \( M \) and \( Q_{3-1} \) is small, when renormalizing, the behavior is governed by \( M \) and \( Q_1 \). Obviously, for a given value of \( M \) there is a special ratio \( Q_1/Q_2 \) for which there would be a coexistence of two distinct noble families, which is absurd. This corresponds to the breakdown of the universal picture of sec. 2, but non-universal Hamiltonians are of codimension 1. When \( Q_1/Q_2 \) is close to the special ratio, it is necessary to renormalize many times before finding out which of \((M,Q_1)\) and \((M,Q_2)\) is the basic couple defined in sec. 3.2.1. Another counterexample to universality can be constructed by taking a map with a generating function quadratic in \((x'+x)\) [61].

3.2.6 Importance of Differentiability

Consider the Hamiltonian

\[
H(p,x,t) = h(p) + \sum_{\ell=-\infty}^{+\infty} \sum_{m=0}^{\infty} V_{\ell m}(p) \cos(\ell x - m t),
\]

(3.141)

where the \( V_{\ell m} \)'s are assumed to scale with \( \ell \) like

\[
V_{\ell m} = \frac{1}{\ell^\alpha},
\]

(3.142)

where \( \alpha \) is a constant. A priori nothing tells us which are the leading resonances and how to define a zoning number. When picking up the pair \((V_{\ell m}, V_{\ell+1m})\) and using normalizations similar to those of appendix A, the new amplitudes scale like \( \ell^4 V_{\ell m} \) and \( \ell^4 V_{\ell+1 m} \). For these quantities not to diverge when \( \ell \) goes to infinity, one must require \( \alpha \geq 4 \) in
eq. (3.142), i.e. H to be four times continuously differentiable in x. A similar argument has already been given by Chirikov [87] although he did not get the right connection between differentiability and decay of Fourier coefficients. For this property to hold for the renormalized Hamiltonian, one must require the same differentiability of H with respect to p. This is in perfect agreement with the condition for the KAM theorem to hold (see sec. 2.3.2.1). For renormalizing at the critical point \( \alpha > 4 \) is necessary in order to find the universal behavior. Otherwise at each scale, a different pair of resonances might govern the dynamics.

3.3 Hierarchy of Fixed Points

Here we show that if the robustness of noble KAM tori is true, then the hierarchy of fixed points defined in sec. 2.3.2.3 is necessarily true as well. This is proved geometrically by using fig. 2.14. Line \( \Delta \) corresponds to a one-parameter family of Hamiltonian systems and intersects \( \mathcal{P}_2 \) at A. Let \( \mathcal{T}_1 \) be the KAM torus with zoning number \( z_1 \) defined by eq. (3.32).

We first show how the hierarchy of fixed points is related to the robustness of KAM tori. Since A is above \( \mathcal{T}_1 \), the stable manifold of \( F_1 \), \( \mathcal{T}_2 \) is stabler than \( \mathcal{T}_1 \) for the \( \Delta \) family. When renormalizing from A for \( \mathcal{T}_2 \), the successive iterates converge to \( F_2 \) along \( \mathcal{P}_2 \), and lie below \( \mathcal{T}_1 \), after say, \( n \) iterates. Choose a number \( m \gg n \), and let \( z_m \) be the zoning number with a continued fraction expansion \([a_0, a_1, a_2, \ldots]\) and \( a_j = 2 \) for \( 0 \leq j < m \) and \( a_j = 1 \) for \( j \geq m \). Then, renormalizing from A for the torus \( \mathcal{T} \) with zoning number \( z_m \) yields \( m \) first steps identical to those for \( \mathcal{T}_2 \), but, after that, the
renormalization mapping is governed by $F_1$. Since the $m$-th iterate lies below $\mathcal{P}_1$, the forthcoming iterates converge to the origin. Therefore $\mathcal{I}$ is under-critical at $A$ when $\mathcal{I}_2$ is critical. Thus $\mathcal{I}$ is a noble torus more robust than $\mathcal{I}_2$ which can be chosen arbitrarily close to $\mathcal{I}_2$ by taking $m$ large enough. This rationale can be extended to noble tori close to any non-noble torus.

Imagine that $F_2$ is above $\mathcal{P}_1$ (for instance, $F_2$ is shifted to the left along $\mathcal{P}_2$ in fig. 2.14). A reasoning similar to the preceding one shows that all tori with zoning number $Z_m$ are less stable than $\mathcal{I}_2$. This would contradict the robustness of noble tori.

Now imagine that $F_1$ is below $\mathcal{P}_2$ (for instance, $F_1$ is shifted to the left along $\mathcal{P}_1$ in fig. 2.14). The same reasoning as before, shows that, arbitrarily close to a noble torus, there is a torus of the universal class of $F_2$ which is more robust. This again contradicts the robustness of noble tori.

3.4 Application of KAM Renormalization to Cycles

3.4.1 A Mathieu Equation for Elliptic Cycles

Hamiltonian $H_p$ has two cycles with a mean velocity $w_m$ given by eq. (3.1). One of them is stable for small values of $M$ and $P$, and is termed an elliptic cycle. This cycle $\mathcal{O}_m$ is destabilized for larger values of $M$ and $P$. Let $N_m$ be the manifold corresponding to this destabilization in space $(k,M,P)$. $N_m$ can approximately be computed by characterizing the stability of $\mathcal{O}_m$ by a Mathieu equation [30]. A rigorous Mathieu equation can be obtained for $\mathcal{O}_2$ when $M/P=k=1$ [30].
For describing $\Theta_m$, we no longer can make the two-resonance approximation, since resonances $R_{m-1}$ and $R_{m+1}$ give perturbations with the same frequency. Therefore, we make in Hamiltonian (3.4) the three-resonance approximation that retains resonances $R_{m-1}$, $R_m$, and $R_{m+1}$. Then the reduction to a many-wave Hamiltonian of sec. 3.1.5 yields a new Hamiltonian

$$H''(J, \phi) = \frac{1}{2} J^2 - \sum_{n=m-2}^{m} U_n \cos \phi_n(\phi, t), \quad (3.143)$$

where

$$U_n = \sigma V_n(I_{n+1}),$$

with $I_{n+1}$ defined by eq. (3.12) and $\sigma$ by eq. (3.16). Let cycle $\Theta_m$ have coordinates $[J(t), \phi(t)]$, and $[J(t)+\delta J(t), \phi(t)+\delta \phi(t)]$ be a nearby orbit. The equations for the tangent flow of $\Theta_m$ are obtained from the canonical equations of $H''$

$$\delta \dot{\phi} = \delta J, \quad (3.144)$$
$$\delta J = -\delta \phi \sum_{n=m-2}^{m} U_n(k+n) \cos \phi_n[\phi(t), t]. \quad (3.145)$$

Eliminating $\delta J$ from these equations and approximating $\phi(t)$ by $w_m t$

yields the Mathieu equation

$$\frac{d^2 \delta \phi}{dt^2} + (a-2q \cos 2t') \delta \phi = 0, \quad (3.146)$$
with

\[ t' = \frac{1}{2} w_m t, \quad (3.147) \]

\[ a = \frac{4}{k^2} (k+m-1)^4 U_{m-1}, \quad (3.148) \]

\[ q = 2 \left( \frac{k+m-1}{k} \right)^2 \left[ (k+m)^2 U_m + (k+m-2)^2 U_{m-2} \right]. \quad (3.149) \]

The stability diagram of the Mathieu equation [111] shows that, when \( s = 2\sqrt{M} + 2\sqrt{P} \) grows for given values of \( M/P \) and \( k \), cycle \( \mathcal{E}_m \) experiences a series of destabilization and restabilizations. However, restabilization is not a general feature of Hamiltonian systems. It does not exist for the standard map, nor for the noble one parameter family [61]. The Mathieu equation also implies that the Lyapunov exponent of \( \mathcal{E}_m \) scales like \( (s-s_m)^{1/2} \) above the first destabilization threshold, \( s_m \), of \( \mathcal{E}_m \). This threshold can be computed through formula

\[ a = 1 - q - \frac{1}{8} q^2 + O(q^3), \quad (3.150) \]

which yields the first threshold of destabilization of the solutions of eq. (3.146) [111]. Equations (3.148–150) define \( N_m \) through an implicit equation

\[ G(m,k,M,P) = 0. \quad (3.151) \]

Similar functions can be obtained by deriving a Hamiltonian of the type (3.143) from Hamiltonians (3.89) or (E.1) instead of (3.4).
3.4.2 Fractal Diagrams

These diagrams were originally introduced for the standard map by G. Schmidt [91,112]. In the case of Hamiltonian $H_p$ they look like in fig. 3.7 [32]. Figure 3.7 is made up of several sheets of discrete points. The first one only has two points at $u=0$ and $u=1$ which are the thresholds of $\theta_\infty$ and $\theta_0$ (the centers of resonances $M$ and $P$). The second one corresponds to the thresholds of resonances $R_m$, $m>0$ finite of $H_p$ and $H_e$. It has a parabolic shape and converges to $s=0$ for $u=0$ and $u=1$, in agreement with eq. (2.66). The third sheet is made up of infinitely many parabolic arches obtained by renormalizing two first sheets, i.e. by considering $R_m$ and $R_{m+1}$ of $H_p$ and $H_e$, $m=2,3,\ldots$, as being resonances $M'$ and $P'$ (or $P'$ and $M'$) of a new $H_p'$, and by constructing a new second arch in between.

An approximate threshold for any elliptic cycle of $H_p$ which exists for $s=0$ can be computed by combining the renormalization for KAM tori and eq. (3.151). Let $Z_{mL}$ be the truncation at order $L$ of the continued fraction expansion of $z_m$ defined by eq. (3.33), and let $s_{mL}$ be the threshold of first destabilization of the elliptic cycle with zoning number $Z_{mL}$. According to the hyperbolicity argument of sec. 3.2.3, $s_{mL}$ converges toward the threshold of breakup of the universal KAM torus with zoning number $z_m$ at a rate $\delta_m^L$ [32]. This was first seen numerically in ref. [112]. More generally the thresholds of breakup of KAM tori correspond to the accumulation points of the fractal diagram with an irrational zoning number.

The accumulation points with a noble zoning number constitute also a fractal diagram originally proposed by Percival [62]. Schmidt's fractal diagrams plot the value of $s$ for which a given elliptic cycle
destabilizes. This corresponds to a value $T = -2$ of the trace of the tangent map. MacKay proposed modified fractal diagrams that correspond to other values of $T$ [27]. In particular, if the trace is taken equal to its noble critical value given by table 2.1, $T_* = 1$, sec. 4.2 shows that one gets a diagram that converges about twice as quickly to its noble accumulation points. Figure 10 of ref. [27] plots fractal diagrams with $T = -2, -1, 0$, and 1. The more 1's has the continued fraction expansion of the zoning number the higher is the value $s(T)$ of the perturbation corresponding to a given value of $T$. This shows the first aspect of the robustness of noble circles quoted in sec. 2.3.2.2. For a zoning number whose all partial quotients are 2, if the last 2's are changed into 1's, $s(T)$ increases. This shows the second aspect of the robustness of noble tori quoted in sec. 2.3.2.2 for the fixed point $F_2$.

This other aspect is not yet checked for the other fixed points $F_m$, $m \geq 3$, but we can expect it to hold too, for the following reason. The $F_m$'s come closer and closer to the KAM fixed point when $m$ increases.

Thus, if $F_2$ is below the noble stable manifold, all $F_m$'s with $m \geq 3$ should be as well. Section 4.3.3 comes back to the link between KAM tori and nearby cycles.
4. Methods for Computing the Threshold of Global Stochasticity

This section is devoted to the computation of the threshold of global stochasticity. Its first part presents the methods that take advantage of the knowledge of the widths of stochastic layers and of an approximate noble stable manifold for the paradigm Hamiltonian. The second part shows how to compute a threshold by requiring a critical feature of the golden mean fixed point. The third one shows how renormalization allows us to globally understand the methods existing in 1981 as presented in chapter 4 of ref. [12].

4.1 Computation by reduction to the paradigm Hamiltonian

4.1.1 Case of $H_p$

Equations (2.56) and (3.111) yield a quite accurate expression for the noble stable manifold in the domain of parameters $(M,P)$ indicated by eq. (3.115). This is checked through comparison [73] with numerical upper bounds for the threshold of global stochasticity as shown in 2.3.2.5 (fig. 2.19). These results suggest that for $1/25 < M/P < 25$ and $1/k_m < k < k_m'$ with $k_m \approx 2.2$ one can expect one of the golden mean tori ($z=g$ or $z=1+g$) to be the most robust. Equation (2.56) yields the threshold for $z=1+g$ (resp. $g$) when computed with parameter $(k,M,P)$ (resp. $(1/k,P,M)$); the result for $z=g$ is obtained by using eq. (2.23) and the equivalent Hamiltonian $H_e$ of eq. (2.12).

Using eq. (2.56) might seem tedious, since $M(P,k)$ is not a simple function. Anyway, since coefficient $c$ is small, one can first use it by neglecting the $O(P^2)$ term and compute the threshold by perturbation. If $M/P$ and $k$ are not too far from 1, the threshold can be computed from
\[ M_{P}^{g-1} \simeq 3 \cdot 10^{-3} \frac{R(1)}{(1+c(1)M_{C}^2)} \approx 3 \cdot 10^{-3} \quad . \tag{4.1} \]

where \( M_{C} \) is the value of \( M \) obtained from eq. (2.56) for \( M=P, \ k=1 \).
Equation (4.1) is more accurate than the overlap criterion and is as easy to use. At that point we still have two tori as candidates to be the most robust \((z=g \text{ and } g+1)\). Equation (2.56) however, confirms the intuitive argument of resonance overlap that for \( k=1 \) and \( M<P \) (fig. 4.1(a)), and for \( M=P \) and \( k>1 \) (fig. 4.1(b)), the most robust torus is between \( R_{2} \) and \( M \), so that formulas (2.56) and (4.1) need only to be applied to \( z=1+g \), i.e. with parameters \((k, M, P)\).

If \( M/P \) or \( k \) are far from 1, then the last torus can be in other zones than 1 and 2. We know it is a noble torus, so that the continued fraction expansion of the zoning number should be like \( g \)'s after some rank. Now, the attraction toward the fixed point \( F_{m}^{\lambda} \) is so strong (the least stable eigenvalue of \( F_{m}^{\lambda} \) is of order \( m \)) that a few iterates of the renormalization about other fixed points than \( F_{2}^{1} \) should quickly bring \([k(n), M(n), P(n)]\) under \( \mathcal{R}_{2}^{1} \) (see sec. 3.3). This should make very small the number of \( a_{i} \)'s different from 1 in \( z = [a_{0}, a_{1}, a_{2}, \ldots] \).
Practically the threshold can be estimated by considering only tori with a zoning number

\[ z = l+g, \ l \geq 1, \tag{4.2} \]

for both \( H_{p} \) and \( H_{e} \). The thresholds quickly drop down when \( l \) increases since those tori become close to resonance \( M \). Therefore only few values of \( l \) need to be considered.
Computation of the breakup threshold for those tori makes it necessary, for \( \epsilon \geq 2 \), to first apply eqs. (3.97) and (3.105–106) with \( \lambda = 1 \) in order to compute parameters \( (k', M', P') \) and to set them into eq. (2.56), since their zoning number is \( z' = 1 + g \) in \( S'_p \). The theoretical prediction of fig. 2.19 for \( k > 2.2 \) and \( M/P > 25 \) is obtained this way.

4.1.2 Reduction to \( H_p \) for time-dependent Hamiltonians

Consider a Hamiltonian of type (3.57) that has a nonlinearity and resonance amplitudes which vary slowly according to conditions (3.58), (3.60), and

\[
\frac{dV_n(p)}{dp} = O(\eta^\alpha)
\]

(4.3)

with \( \alpha \geq \frac{1}{2} \).

4.1.2.1 Global Stochasticity Between Two Resonances

For computing the threshold of global stochasticity between two resonances \( n = \ell \) and \( \ell + 1 \) of Hamiltonian (3.57), one first can make the two-resonance approximation retaining the \( n = \ell \) and \( \ell + 1 \) terms. For a KAM torus \( \mathcal{T} \) with a mean velocity \( w \), such that \( v_\ell > w > v_{\ell+1} \), reduction to Hamiltonian (3.76) can be performed, and the second and third sums can be deleted since they are of order \( \eta^2 \) too, as a consequence of eq. (4.3). Then resonances \( \phi_n, n = \ell, \ell + 1 \) can be viewed as resonances \( M \) and \( P \) of a paradigm Hamiltonian obtained by the normalization procedure of appendix A. This yields \( M = \phi_\ell / \Delta v_\ell^2 \), \( P = \phi_{\ell+1} / \Delta v_{\ell+1}^2 \), and \( k = k_{\ell+1} / k_\ell \). The KAM torus \( \mathcal{T} \) now has a mean velocity
\[ u = \frac{v_k - w}{\Delta v_k} \]  

(4.4)

KAM tori with a zoning number given by eq. (4.2) have a mean velocity \( u = k/(k+z-1) \) in \( S_p \) and a mean velocity \( w = v_k - u \Delta v_k \) in coordinates of Hamiltonian (3.57). Amplitudes \( M \) and \( P \) should be computed for those tori, and the values of \( k \) in eq. (4.2) should be chosen as for \( H_p \) depending on the values of \( M/P \) and \( k \). If more accuracy is desired on the final result, one should first perform one or more exact renormalizations about the tori of interest before making the reduction to a paradigm Hamiltonian (see sec. 3.2.4).

4.1.2.2 Large-scale Stochasticity Without Primary Resonance Overlap

Now we consider the case where the \( n=k \) term of eq. (3.57) is resonant but the \( n=k+1 \) is not. If widespread stochasticity appears in the Poincaré map of Hamiltonian (3.57) in the region \( w < v_k \), this is due to the interaction between higher-order resonances that are close to the stochastic layer of resonance \( k \). This stochasticity corresponds to the blowup of the stochastic layer of resonance \( k \) in direction \( w < v_k \) [113]. Reduction of Hamiltonian (3.57) to a many-wave Hamiltonian can be done for \( w = v_k \). A paradigm Hamiltonian can then be defined as in the preceding section, and the width of the stochastic layer of resonance \( M \) increases according to eq. (2.63). This width has a sudden blowup in a small range of parameter \( \mu \). A threshold \( \mu_t \) can be defined as the intersection of the \( \mu \)-axis with the tangent at the
inflection point of function $W(\mu)$ [113]. For Hamiltonian (3.57) all resonances scale like $\varepsilon$ and

$$\mu_{\pm} = \left[ \frac{\lambda^2 - 2(\lambda + 1)^2}{\lambda^2} \right] \lambda$$

with $\lambda = 2k_{l+1}/k_l$. The blowup of a stochastic layer as a tool for predicting thresholds was originally used in refs. [114,115].

If there is a resonance domain of mean velocities $w$ between two non-resonant values of $v_l$ and $v_{l+1}$, then the method for two resonant $v_n$'s can be applied as a first trial. More accuracy can be obtained by (3.57), which eventually creates resonant terms in the domain of interest. Then the previous methods can be used for the renormalized Hamiltonian.

4.1.2.3 Large Derivative of the Resonance Amplitudes

If the $V_{\nu,*}$ are large, the second and third sums in eq. (3.78) can become more important than the first one, and should rather be considered. If all sums are equally important, then the two-resonance approximation no longer makes sense for Hamiltonian (3.76). This occurs in particular if $\sigma_\nu$ is small. Then renormalizing exactly, at least once, Hamiltonian (3.57) yields, in general, a new Hamiltonian with larger $\sigma_\nu$'s and smaller $V_n$'s.

4.1.3 Case of a Linear Integrable Part

If $\sigma(p)$, as defined by eq. (3.59), is 0, but eq. (4.3) holds with $\alpha < \frac{1}{2}$, then the reduction of Hamiltonian (3.57) to Hamiltonian (3.76) still makes sense. All $\varphi_n$'s are 0 but the second and third sums of eq. (3.76) are not. The methods of the previous section can be applied to Hamiltonian (3.76) by retaining two resonant terms with phase velocities close to the unique frequency $w = dh/dp$. 
Otherwise specific methods must be derived. First consider the case where one velocity \( v_\lambda = w \) is resonant and \( V_\lambda(p) \) is not constant. Then, one can go to the action-angle variables of Hamiltonian (3.57) where only the \( \lambda \)-th term is kept \([114, 115]\). The new Hamiltonian has a nonlinear angle independent part and one can use the methods of sec. 4.1.2.

If no \( v_\lambda \) is resonant, then one, or more, exact renormalization (see sec. 3.2.4) of Hamiltonian (3.57) allows us both to build a nonlinear angle-independent part and to increase the number of possibly resonant frequencies. Then the methods of sec. 4.1.2 may be used.

4.1.4 Two-degree-of-freedom Hamiltonians

4.1.4.1 Reduction to a many-wave Hamiltonian

Till now we only considered time-dependent Hamiltonian systems and area-preserving maps. Here we deal with general two-degree-of-freedom Hamiltonians of the type

\[
\mathcal{H}(\vec{I}, \vec{J}) = \mathcal{H}_0(\vec{I}) + \varepsilon \sum_{n \in \mathcal{N}} V_n(\vec{I}) \cos \varphi_n(\vec{J}),
\]

(4.5)

with

\[
\varphi_n(\vec{J}) = \vec{\alpha}_n \cdot \vec{J} + \gamma_n,
\]

(4.6)

where \( \varepsilon \) is a small parameter, \( \mathcal{N} \) is a set of integers, \( \gamma_n \) is a phase, and all vectors are two-dimensional. We assume \( d^2 \mathcal{H}_0/d\vec{I}^2 \) and \( V_n(\vec{I}) \) to depend on \( \vec{I} \) through \( \gamma \vec{I} \) where \( \gamma \) is a small parameter. By using a method similar to the one of sec. 3.1.5 we show how Hamiltonian (4.5) can be
locally reduced to a 1.5-degree-of-freedom Hamiltonian, namely a many-wave Hamiltonian.

This reduction is trivial to do if

$$\mathcal{H}(\dot{I}, \dot{\varphi}) = H(I_1, \varphi_1, \varphi_2) + I_2 \quad (4.7)$$

The canonical equations show that \( \dot{\varphi}_2 = 1 \), and

$$\dot{\varphi}_1 = \frac{\partial H}{\partial I_1}(I_1, \varphi_1, t), \quad (4.8)$$

$$\dot{I}_1 = -\frac{\partial H}{\partial \varphi_1}(I_1, \varphi_1, t), \quad (4.9)$$

where \( \varphi_2 \) is replaced by \( t \). Equations (4.8–9) are the canonical equations of Hamiltonian \( H(I_1, \varphi_1, t) \) which describes the non-trivial motion of the first degree of freedom.

We now return to the more general case. Consider an orbit \( \mathcal{O} \) of the system (4.5) with energy \( E \), characterized by a frequency vector \( \omega_r \), such that \( \dot{\omega}_r^2 = 1 \). Let \( \hat{r} \) be a unit vector such that \( \hat{r} \cdot \omega_r = 0 \). Let \( \hat{I}_r \) be the corresponding action for \( \varepsilon=0 \) defined by

$$\mathcal{H}_0(\hat{I}_r) = E, \quad (4.10)$$

$$\hat{r} \cdot \frac{d\mathcal{H}_0}{d\hat{I}}(\hat{I}_r) = 0. \quad (4.11)$$

Generically, in the vicinity of \( \hat{I}_r \) the energy curve \( E = \mathcal{H}_0(\hat{I}) \) is nonlinear (fig. 4.2) and can locally be approximated by a parabola. By going to a reference frame corresponding to the principal axis of this
parabola (those parallel to \( \hat{\omega}_r \) and \( \hat{r} \)), \( \mathcal{H} \) can be approximated by a Hamiltonian of type (4.7) for which the reduction to a 1.5-degree-of-freedom Hamiltonian is trivial.

We thus perform on Hamiltonian (4.5) the canonical transformation \((\hat{I}, \hat{\psi}) \rightarrow (J, L, \varphi, \psi)\) with generating function

\[
F(J, L, \hat{\psi}) = \hat{\psi} \cdot (\hat{I}_r + J \hat{r} + L \hat{\omega}_r) .
\]  

This yields

\[
\hat{I} = \frac{\partial F}{\partial \hat{\psi}} = \hat{I}_r + J \hat{r} + L \hat{\omega}_r , \tag{4.13}
\]

\[
\varphi = \frac{\partial F}{\partial J} = \hat{r} \cdot \hat{\psi} , \tag{4.14}
\]

\[
\psi = \frac{\partial F}{\partial L} = \hat{\omega}_r \cdot \hat{\psi} . \tag{4.15}
\]

Let \((\alpha_n, \beta_n)\) be defined by

\[
\alpha_n = \hat{r} \cdot \hat{q}_n , \tag{4.16}
\]

\[
\beta_n = \hat{\omega}_r \cdot \hat{q}_n . \tag{4.17}
\]

It follows that

\[
\varphi_n(\hat{\psi}) = \psi_n(\varphi, \psi) = \alpha_n \varphi - \beta_n \psi + \gamma_n . \tag{4.18}
\]

In the new coordinates \(\mathcal{H}\) becomes

\[
\mathcal{H}'(J, L, \varphi, \psi) = \mathcal{H}_1(J, L) + \varepsilon \sum_{n \in \mathcal{A}} W_n(J, L) \cos \psi_n(\varphi, \psi) , \tag{4.19}
\]
with

$$\mathcal{H}_1(J, L) = E + \Omega L + \frac{1}{2} \left[ aJ^2 + 2bJL + cL^2 \right] + O(\eta \varepsilon^2) \ , \quad (4.20)$$

$$W_n(J, L) = V_n(\tilde{T}) = V_{n*} + \tilde{v}_{n*} \cdot \left( J\tilde{\omega} + L\tilde{\omega}_r \right) + O(\eta \varepsilon^2) \ , \quad (4.21)$$

where \( V_{n*} = V_n(\tilde{T}_r) \), \( \tilde{v}_{n*} = \frac{\partial V_n(\tilde{T}_r)}{\partial \tilde{T}} \), \( \Omega = d\mathcal{H}_0(\tilde{T}_r)/d\tilde{T} \), \( a = \tilde{r}_{\tilde{T}} \), \( b = \tilde{r}_{\tilde{\omega}_r} \), \( c = \tilde{\omega}_r \cdot \tilde{\omega}_r \), and \( \tilde{\sigma} = \frac{d^2 \mathcal{H}_0(\tilde{T}_r)}{d\tilde{T}^2} \). The canonical equations of \( \mathcal{H} \) yield

$$\dot{\varphi} = aJ + bL + \varepsilon \sum v'_{n*} \cdot \tilde{T} \cos \psi_n(\varphi, \psi) + O(\eta \varepsilon^2) \ , \quad (4.22)$$

$$\dot{\psi} = \Omega + O(\varepsilon) \ , \quad (4.23)$$

$$\dot{J} = \varepsilon \sum V_{n*} \alpha_n \sin \psi_n(\varphi, \psi) + O(\eta \varepsilon^2) \ , \quad (4.24)$$

$$\dot{L} = \varepsilon \sum V_{n*} \beta_n \sin \psi_n(\varphi, \psi) + O(\eta \varepsilon^2) \ . \quad (4.25)$$

Deriving eq. (4.22) with respect to time and combining it with equations (4.23–25) yields

$$\ddot{\varphi} = \varepsilon \sum \alpha_n \beta_n \sin(\alpha_n \varphi - \beta_n \Omega t + \gamma_n) + P(\eta \varepsilon^2) \ , \quad (4.26)$$

$$\dot{\beta}_n = V_{n*}(a-b) + \Omega \frac{\beta_n}{\alpha_n} \tilde{v}_{n*} \cdot \tilde{T} \ . \quad (4.27)$$

Equation (4.26) can be derived from the canonical equations of the Hamiltonian

$$H(P, \varphi, t) = \frac{1}{2} P^2 + \varepsilon \sum \phi_n \cos(\alpha_n \varphi - \beta_n \Omega t + \gamma_n) \ . \quad (4.28)$$
Equations (4.26-28) are reminiscent of eqs. (3.74-76) for time-dependent Hamiltonians. Hamiltonian (4.28) still is a many-wave Hamiltonian. Assume the n-th term of eq. (4.26) to be resonant, and let $\hat{f}_n$ be the corresponding value of $\hat{f}$ for $\epsilon=0$. $\mathcal{H}_1(J,L)=E$ implies

$$\Omega L + \frac{1}{2} a J^2 = O(J^3). \quad (4.29)$$

Thus, $L$ scales like $J^2$. For $a=0$, $\hat{f}_n$ has coordinates $(J,L)$ given by

$$0 = \alpha_n \dot{\phi} - \omega_n \dot{\psi} = \alpha_n (aJ+bL) - \beta_n (\Omega bJ+cL) + O(J^3).$$

The lowest order solution of this equation is

$$(J,L) = \left( \frac{-\beta_n \Omega}{\alpha_n a - \beta_n b}, 0 \right). \quad (4.30)$$

If $\hat{f}_n$ is close enough to $\hat{f}_r$, eq. (4.30) implies

$$\phi_n = \gamma_n(\hat{f}_n), \quad (4.31)$$

which is the centered-resonance approximation.

4.1.4.2 Polynomial Hamiltonians

In order to apply renormalization techniques to polynomial Hamiltonians, it is necessary to write them in the form (4.5). A general rule cannot be given for the definition of action-angle variables since a given non-integrable system can be close to several integrable Hamiltonians. If the lowest degree part of those Hamiltonians is separable, it is natural to use the action-angle
variables of the two separable parts. For instance this can be done for the Hénon–Heiles Hamiltonian [22] whose lowest degree part is the sum of two harmonic oscillators. Using the techniques of secs. 4.1.3 and 4.1.2.2 allows us to analytically estimate a threshold $E=0.08$ [113] to be compared to the numerical value $E=0.11$ [22]. A better estimate $E=0.106$ can be obtained [113] by first using the generalized Birkhoff normal forms [116]. The improvement of the estimate suggests that those normal forms provide an efficient mean to rewrite polynomial Hamiltonians in the form (4.5) where $\varepsilon$ is small, i.e. to find out a close integrable system.

4.2 Computation by Requiring a Critical Noble Feature

When there is some hint about the location of the most robust noble torus $\mathcal{T}$ in phase space (for instance those hints given in sec. 4.1.1), one can find the threshold by requiring a critical noble feature in the vicinity of $\mathcal{T}$. In particular, if $\mathcal{T}$ is critical, neighboring islands have features typical of the critical noble fixed point: normalized widths, ratio of widths, rotation number about the center of the island, etc. Equation (3.136) and fig. 3.6 show that the precision of a criterion $N$ which requires a critical noble feature is improved by $(\delta/\delta')^L$ (and not by $\delta^L$ as for other criteria, cf. sec. 3.2.4) after a large number $L$ of iterates. This yields a twice more rapid convergence than any other criterion. When dealing with islands close to the torus $\mathcal{T}$, renormalizing means considering higher and higher order truncations of the continued fraction expansion of the zoning number of $\mathcal{T}$. 
A feature easy to measure numerically is the rotation number \( \rho \) about the center of the nearby islands [41]. It is simply related to the trace \( \text{Tr} \mathcal{M} \) of the tangent mapping \( \mathcal{M} \) to the Poincaré map of the corresponding stable cycle, by

\[
\text{Tr} \mathcal{M} = 2 \cos 2\rho \pi .
\]  

(4.32)

The value \( T_s \) of \( \text{Tr} \mathcal{M} \) for the noble fixed point given in Table 2.1 yields \( \rho = 1/6 \).

For the standard map, \( \text{Tr} \mathcal{M} \) can be analytically computed for low period cycles. For the primary islands of the standard map \( \text{Tr} \mathcal{M} = 2 - K \). This yields a prediction \( K_1 = 1.0004 \) for the threshold [72], to be compared with the numerical value \( K_C = 0.9716 \). For resonance \( R_2 \), \( \text{Tr} \mathcal{M} = 2 - K^2 \). This yields the estimate \( K_2 = 1.0002 \). The improvement \( (K_1 - K_C)/(K_2 - K_C) \) is very close to 1 and not to \( \delta/\delta' \approx 2.7 \). This shows that the standard map at large scale still is far from the universal 1-parameter family of sec. 2.3.2.1.

A fractal diagram can be computed [27] by replacing the threshold of destabilization of a cycle by the value of \( s \) corresponding to a trace equal to \( T_s \) (table 2.1). The points of this fractal converge more rapidly than in Schmidt's diagrams toward the accumulation points corresponding to the breakup of KAM tori. These fractal diagrams can in principle indicate where the last KAM torus is by inspection of their successive sheets. For a Hamiltonian system more general than \( H_p \) or the standard map, the problem for building such a diagram is that of finding the basic couple defined in sec. 3.2.1. The non-triviality of this problem has been shown in sec. 3.2.5.
4.3 Global Point of View on the 1981 Methods

4.3.1 Resonance Overlap

Section 3.1.8 showed that the resonance overlap criterion \( s_c = 1 \) \[4\], simply motivated in sec. 2.2.1, can be recovered from the condition for the Kolmogorov transformation that kills resonance \( M \) to be simultaneously feasible on both \( H_p \) and \( H_e \). Equations (2.56) and (4.1) show that this criterion gives the right order of magnitude for the threshold provided the amplitudes of the resonances and their periodicity are not too different. The "2/3 rule", \( s_c = 2/3 \) \[12\] is only correct for special pairs of \( k \) and \( M/P \); it generally underestimates the threshold, when fig. 2.19 shows that the resonance-overlap criterion sometimes overestimates and sometimes underestimates the actual threshold. Underestimates, in particular, occur for \( M/P \) far from 1, in agreement with the repulsion of nearby resonances indicated in fig. 2.7.

Applying the overlap criterion to resonances \( R_2 \) and \( P \) of \( H_p \) (or of the standard map Hamiltonian \( H_S \) of eq. (2.15)) consists of applying it to the renormalized system \( S'_p \). According to the argument of sec. 3.2.4 this improves the estimate of the threshold. For the standard map Chirikov \[4\] obtains \( K_1 = 1.46 \) instead of \( K_0 = 2.47 \) when the criterion is applied to resonances \( M \) and \( P \) (or \( n=0 \) and \( 1 \) of eq. (2.15)). This corresponds to an improvement \( K' = (K_0 - K_c)/(K_1 - K_c) \approx 3.1 \) where \( K_c = 0.9716 \) \[41\]. When applied to resonances \( R_2 \) and \( R_3 \), the criterion gives \( K = 1.35 \). In fact, if one renormalizes correctly, there also is a renormalization of the mass parameter \( \sigma \) of eq. (3.107) is larger than 1. This, as already noted by Cary (see the note, page 317 of ref. \[4\]), further improves the estimate obtained by applying the
criterion to $S'_p$ (or resonances $R_2$ and $R_3$) and yields $K_2 = 1.28$ [4].

The improvement is then $\Delta_2 = (K_0-K_c)/(K_2-K_c) \approx 4.85$. As explained in sec. 3.2.4, when the overlap criterion is applied to the L-th subsystem $S_p^{(L)}$ of $S_p$, the improvement $\Delta_L$ scales like $L^L$ for $L$ large. For small values of $L$ the improvement is larger, due to the strong divergence from $F^L_m$ far from $F^L_m$. It therefore appears that, without knowing it, Chirikov took advantage of renormalization to improve his estimates!

Estimate $K_1$ is obtained from resonance-overlap of resonances $R_2$ and $P$ through equation $2K^{1/2} + \frac{1}{2}K = \pi$ [4]. A similar, but simpler to derive, formula is obtained through a trick originally used by Cohen for estimating the threshold of vanishing of regular gaps in a map modeling the motion of particles in a tandem mirror [117]. He estimates the width of the gap, and consider it to vanish when the jump of action induced by the map is larger than the estimated width. The symmetry of the standard map implies that a trajectory going from resonance $M$ to resonance $R_2$, also goes from $R_2$ to resonance $P$. Therefore, the gap is of order $(w_2-w_\infty = w_1-w_2 = \frac{1}{2})$ minus the width of resonance $M$, $\Delta_M = \sqrt{K}/\pi$. The jump in action induced by the map is $K/2\pi$. The criterion therefore yields $\frac{1}{2} - \sqrt{K}/\pi = K/2\pi$, i.e. $2K^{1/2} + K = \pi$. This yields $K = 1.07$, which is not too bad!

In order to further improve his estimates, Chirikov added to the width of a primary resonance the width of its stochastic layer. The formula he uses (eq. (2.63)) is correct for thin layers, where one may approximate by a continuous increase the steplike progression of the width of the layer, induced by the successive breakup of the last noble torus between resonances $R_m$ and $R_{m+1}$, for $m$ decreasing. Global stochasticity corresponds to a layer that typically extends up to
resonance $R_2$ and by symmetry to resonance $P$, i.e. to $m=2$. For so small $m$, eq. (2.63) is no longer correct. A good formula should indicate by itself that resonance $R_2$ is in the layer of resonance $M$. Therefore, adding widths of resonances to widths of stochastic layers is somewhat artificial. The lack of validity of this method is no problem since the present state of renormalization theory now makes it possible to compute quite accurate threshold without ad hoc assumptions (see sec. 4.1).

A natural improvement of the resonance-overlap criterion was proposed by Walker and Ford [118] and later on used by Wisdom [119]. It corresponds to the overlap of the exact separatrices of the resonances taken separately (the pendulum approximation is not made). For Hamiltonian (2.16) where

$$\lambda=0,$$

$$M(v) = 1 + \alpha v \quad \text{and}$$

$$P(v) = 1 + \alpha (1-v) \quad (4.33)$$

this predicts that the threshold is a decreasing function of $\alpha$. Anyway the calculations of sec. 3.1.5 show that the threshold depends only on $\alpha^2$ [35, 73]. This non-intuitive result shows that Chirikov's original prescription for applying the overlap criterion is the right one, namely the amplitudes should be evaluated at the resonant value of the action.
4.3.2 Growth of Second-order Islands

Section 4.3 of ref. [12] makes a connection between the overlap of two primary resonances, i.e. the breakup of the last KAM torus in between, and a similar feature for second-order resonances trapped inside the primaries (like the chain of 9 islands in fig. 2.8). In particular a standard map is derived close to the separatrix whose primary resonances correspond to the second-order resonances. Such a standard map can also be derived by a local linearization of the whisker map [4]. Sec. 4 showed that it is directly obtained by investigation of the paradigm Hamiltonian as written in the action-angle coordinates of resonance M. All methods show that the inner and outer stochastic layers blow up simultaneously.

For deeply trapped islands, the picture of ref. [12] should be somewhat modified, since the rotation number at the center of islands close (in the sense of sec. 4.3.3) to a critical noble torus is very close to 1/6 but slightly larger (sec. 4.2). Furthermore the vanishing of the trapped chains with an odd number of islands is a special feature of the standard map that does not exist for the Hamiltonian corresponding to the noble fixed point (see appendix B).

4.3.3 Stability of High-order Fixed Points

Greene [41] numerically established a link between the breakup of a KAM torus and the destabilization of nearby cycles. These cycles have a zoning number which corresponds to the truncation at some order of the continued fraction expansion of the zoning number of the torus. For a critical noble torus the trace of the tangent mapping of those cycles has a universal value very close to 1 (see table 2.1).
These properties are in full agreement with renormalization theories [33,46], and even were the seed for renormalization schemes for area-preserving maps [26,27]. A link between the existence of a KAM torus and the properties of nearby cycles is even proved mathematically [57,120].

For the standard map, the parameter $k$ of the paradigm Hamiltonian obtained by keeping two successive terms in eq. (2.15), is 1. It is therefore equivalent to truncate the continued fraction expansion of the zoning number or the winding (rotation) number. This equivalence breaks down when $k$ is not an integer (see sec. 2.3.1.2.2).

4.3.4 Renormalization for Two Resonances

Section 4.5 of ref. [12] describes the first renormalization scheme derived for Hamiltonian systems [25]. It is similar to Tr as presented in sec. 3.1.1 with a few differences. At that time the validity of the centered-resonance approximation was not yet recognized, and all resonance amplitudes where computed at the unperturbed action $I_z$ of the torus under interest. When done on Hamiltonian (2.16) with parameters (4.33), this approximation also predicts that the stochasticity threshold is a decreasing function of $\alpha$ [73]. Fortunately, the first scheme used the action-angle variables of resonance $M$ for which coefficients $W_n$ of eq. (E.1) vary very slowly with $I$. Therefore, the wrong choice of approximation had no bearing on the final result.

An other difference is that parameter $\sigma$ defined by eq. (3.16) also was computed at $I_z$. This induced a spurious dependence on $z$ of $H_p$ that does not exist in an exact approach (sec. 3.2.1). Finally this
scheme did not verify the hierarchy of fixed points defined in sec. 2.3.2.3, and this made somewhat cumbersome the computation of thresholds of global stochasticity.

Scheme \( T_R \) of sec. 3.1.7 yields a better threshold \( K = 1.107 \) for the standard map than the original scheme, and one exact renormalization (sec. 3.2.4) yields the quite good prediction \( K = 0.991 \) to be compared to the exact one \( K_C = 0.9716 \).

**4.3.5 Variational Principle for KAM Tori**

Section 4.6 of ref. [12] describes a numerical method used by Percival which consists of the computation of the Fourier coefficients of a given KAM torus and in checking that high-order harmonics are decreasing [48]. This method is as accurate as Greene's method. More recently Greene and Percival [121], and Shenker and Kadanoff [47] found that, at critically the Fourier coefficients for the action (coordinate \( v \) for \( H_p \)) decrease like \( \kappa^{-2} \), where \( \kappa \) is the spatial frequency. This agrees with renormalization predictions [15].
5. Conclusion

What is Hamiltonian stochasticity? A critical point of view could be: something which is not described in that paper! Indeed this review deals a lot with KAM tori, Cantori, cycles, and when it comes to the description of chaos, it only speaks about random walks (quasilinear and trapping regimes) or regular motion (Cantorus-with-small-holes regime). Unfortunately, as of May 1984, it is not possible to say more about the real nature of Hamiltonian stochasticity.

Even if this "essential" problem may be kept aside by an application-oriented physicist, progress would be welcome for a better description of chaotic transport and of the correlations or power spectra in the stochastic domains. More universality than presently recognized seem to exist in stochastic layers [89-90], and its better understanding could help in the description of systems with a higher dimensionality, and of Arnold diffusion [4,12] in particular. A systematic way of dealing with Hamiltonians which are not of the KAM type (eq. (1.1)) still needs to be found. The paradigm Hamiltonian in Chirikov's review [4] is the pendulum Hamiltonian; here it is the particle-in-two-longitudinal-waves Hamiltonian. New paradigms have to be found for higher dimensional systems.

A topic currently under study, but whose theory is not yet completed, deals with the singularities of chaotic orbits in the complex time plane. Natural boundaries with a self-similar structure have been found (see ref. [122] and references therein), and a link has been numerically established between the strength of chaos and the nature of singularities [123,124].
Obviously we can expect a more global understanding of Hamiltonian stochasticity in the forthcoming years. The study of Hamiltonian stochasticity itself can also be expected to make further connections with the theories of other dynamical systems. For instance, one may dream about a theory that would explain why so many renormalization groups for dynamical systems all have fixed points with a dimension one unstable manifold.

Even though obviously this paper is not the last word to chaos, it has reported important recent progress in understanding Hamiltonian stochasticity:

(i) The transition to global stochasticity is now understood and related to KAM theory. This makes it possible to compute analytically the threshold of this transition with a good accuracy.

(ii) More universal properties of stochastic layers have been found, and the Melnikov–Chirikov approach has been connected with KAM theory.

(iii) Two new regimes of chaotic transport have been added to the quasilinear regime: the trapping regime and the Cantorus-with-small-holes regime. The second one indicates more order in chaotic transport than originally thought.

(iv) The origin of long-time tails in correlation functions has been identified.
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Appendix A: Normalizations of a Two-wave Hamiltonian

Consider the Hamiltonian

\[ \mathcal{H}(p, z, \tau) = p^2 / 2m_0 - e \sum_{i=1}^{2} V_i \cos(k_i z - \omega_i \tau + \varphi_i) \]  

(A.1)

which describes the motion of a particle of charge \(-e\), mass \(m_0\) in the potential of two longitudinal waves with potential \(V_i\), wave number \(k_i\) and frequency \(\omega_i\); \(z\) is the position, \(p\) the momentum, and \(\tau\) is the time. A shift in the origin of time and position, allows one to make \(\varphi_i = 0\) and to replace the \(V_i\)'s by the \(|V_i|\)'s. Let \(x = k_1 z - \omega_1 \tau\), \(k = k_2 / k_1\), \(\Delta v = \omega_2 / k_2 - \omega_1 / k_1\), \(M = eV_1 / m_0 \Delta v^2\), \(P = eV_2 / m_0 \Delta v^2\), and \(t = k_1 \Delta \tau\). Then, the evolution of \(x\) as a function of \(t\) is given by the Hamiltonian,

\[ H(v, x, t) = v^2 / 2m0 - M \cos x - P \cos k(x - t) \]  

(A.2)

which is defined by parameters \((M, P, k)\). If the roles of waves labeled 1 and 2 are reversed a similar Hamiltonian \(H_e\) is obtained which is nothing but \(H\) with the parameters \((P, M, 1/k)\).

We may invert the procedure and look for a Hamiltonian of the type (A.1) which reduces to (A.2), with parameters \(e' = m_0' = V_1' = k_1' = 1\), and \(\omega_1' = 0\). One obtains \(k_2' = k\), \(V_2' = V_2 / V_1\), and \(\omega_2' = k / \sqrt{M}\).

Appendix B: Renormalization Scheme for Period n-tupling

We derive a renormalization scheme for period n-tupling [74] already mentioned in ref. [34], but never published since its approximations are too rough for \(n=2\), and it does not accurately describe the period-doubling fixed point which was fashionable at that
time. Anyway, it allows the global calculations of stable manifolds and should be adequate for moderate values of $n$.

This scheme renormalizes Hamiltonian (2.57) into itself. To derive it we use the action-angle variables of trapped orbits of the pendulum Hamiltonian (2.58). The new Hamiltonian is

$$H'(I, \phi, t) = H_0(I) - \sum_{n=-\infty}^{+\infty} V_n(I) \cos \phi_n(\phi, t), \quad (B.1)$$

where

$$\phi_n(\phi, t) = n\phi - kt, \quad (B.2)$$

$$V_n(I) = PW_n^k(I) + RW_n^q(I), \quad (B.3)$$

and $W_m^c$ is coefficient $W_m$ of appendix D computed for $k=c$. When focusing on period $k$-tupling, we deal with Hamiltonian (B.1) as we did in sec. 3.4.1, and we get the three-wave Hamiltonian

$$H''(I, \phi, t) = \frac{1}{2} j^2 - \sum_{n=\lambda-1}^{\lambda+1} U_n \cos \phi_n(\phi, t), \quad (B.4)$$

with

$$U_n = \sigma V_n(I_n), \quad (B.5)$$

where $I_n$ is computed through eq. (E.5), and

$$\sigma = \frac{d^2H_0'(I_j)}{dl^2}. \quad (B.6)$$
At that point we choose to call resonances $M'$, $P'$ and $R'$ respectively resonances $n=\ell$, $\ell-1$, and $\ell+1$ of Hamiltonian (B.4). The normalizations of appendix A yield a new Hamiltonian (2.57) with parameters

$$k' = \frac{\ell-1}{\ell},$$  \hspace{1cm} (B.7)

$$q' = \frac{\ell+1}{\ell},$$  \hspace{1cm} (B.8)

$$M' = \frac{1}{\Delta v^2} |U_\ell|,$$  \hspace{1cm} (B.9)

$$P' = \frac{1}{\Delta v^2} |U_{\ell-1}|,$$  \hspace{1cm} (B.10)

$$R' = \frac{1}{\Delta v^2} |U_{\ell+1}|,$$  \hspace{1cm} (B.11)

with

$$\Delta v = \frac{k}{\ell(\ell-1)}.$$  \hspace{1cm} (B.12)

Notice that for $q=k$ and $P=R$, eq. (B.3) implies $V_{2\ell+1} \equiv 0$ since $W_n = (-1)^n W_n$ (eq. (D.5)). This also occurs when Hamiltonian (2.57) is replaced by the standard map Hamiltonian in the above derivation. The scheme (B.7-11) is easy to implement on a computer. The three largest eigenmodes of any fixed point are simply given as those of a 3x3 matrix.
Appendix C: Computation of the Hamiltonian obtained from $H_p$ through Kolmogorov Transformation

When applied to $H_p$, Kolmogorov transformation with generating function

$$F(I, x, t) = IX + \alpha M\text{sin}x \frac{M\text{sin}x}{I} + \beta \text{Psink}(x-t) \frac{\beta \text{Psink}(x-t)}{k(I-1)},$$  \hfill (C.1)

defines new conjugate variables $(I, \theta)$ related to $(v, x)$ by

$$\theta = \frac{\partial F}{\partial I} = x - \frac{\alpha M\text{sin}x}{I^2} - \frac{\beta \text{Psink}(x-t)}{k(I-1)^2}, \hfill (C.2)$$
$$v = \frac{\partial F}{\partial x} = I + \frac{\alpha M\text{cos}x}{I} + \frac{\beta \text{Pcosk}(x-t)}{I-1}. \hfill (C.3)$$

The new Hamiltonian is

$$H(I, \theta, t) = \frac{I^2}{2} + \frac{\alpha^2 M^2}{4I^2} + \frac{\beta^2 p^2}{4(I-1)^2} - (1-\alpha) M\text{cos}x(I, \theta) - (1-\beta) P\text{cosk}(x-t)$$

$$+ \frac{\alpha^2 M^2}{4I^2} \cos 2x(I, \theta) + \frac{\beta^2 p^2}{4(I-1)^2} \cos 2k[x(I, \theta)-t]$$

$$+ \frac{\alpha \beta M P}{2I(I-1)} [\cos((k+1)x(I, \theta)-kt)+\cos((k-1)x(I, \theta)-kt)]$$

$$= \frac{I^2}{2} + \frac{\alpha^2 M^2}{4I^2} + \frac{\beta^2 p^2}{4(I-1)^2} + \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} K_{km}(I) \cos[(mk+\lambda)x(I, \theta)-mkt]. \hfill (C.4)$$

Let $A_{km}^\mu$ be defined by

$$\cos[(\mu k+\lambda)x(I, \theta)-\mu kt] = \sum_{k \geq 0} \sum_{m=-\infty}^{\infty} A_{km}^\mu \cos[(mk+\lambda)x(I, \theta)-mkt]. \hfill (C.5)$$
According to eq. (C.4) we get

\[ K_{\lambda m}(I) = \frac{\alpha^2 R^2 \alpha^2}{4I^2} + \frac{\beta^2 P^2 \alpha^2}{4(1-I)^2} \]

\[ + (\alpha-I)A_{\lambda m}^{01} + (\beta-I)A_{\lambda m}^{10} + \frac{\alpha\beta M P (A_{\lambda m}^{11} + A_{\lambda m}^{-11})}{2I(I-1)} \]  \hspace{1cm} (C.6)

Equation (C.5) implies

\[ A_{\lambda m}^{\mu} = \frac{1}{\pi T} \int_0^{2\pi} \cos[(\mu k+\lambda)\chi(I,\psi)-\mu k t] \cos[(mk+\lambda)\psi-mk t] d\psi dt, \]

where \( T=2\pi/\mu k \). Equation (C.2) yields

\[ A_{\lambda m}^{\mu\lambda} = \frac{1}{\pi T} \int_0^{2\pi} \cos[(\mu k+\lambda)\chi-x-\mu k t] \cos(mk+\lambda)x-mk t-(mk+\lambda) \left[ \alpha M \sin x/I^2 + \beta P \sin k \right] \]

\[ \times \left( x-t/[[k(1-I)^2]] \right) \frac{d\phi}{dx} dx dt. \]  \hspace{1cm} (C.7)

Deriving eq. (C.2) yields

\[ \frac{d\phi}{dx} = \frac{\partial^2 P}{\partial x \partial I} = 1 - \frac{\alpha M \cos x}{I^2} - \frac{\beta M \cos (x-t)}{(1-I)^2} \]  \hspace{1cm} (C.8)

Therefore

\[ A_{\lambda m}^{\mu\lambda} = C_{\lambda m}^{\mu} - \frac{\alpha M}{2I^2} \left( C_{\lambda m}^{\mu\lambda+1} + C_{\lambda m}^{\mu\lambda-1} \right) \]
\[ -\frac{\beta P}{2(I-1)^2} \left( C_{\lambda m}^{\mu+1} + C_{\lambda m}^{\mu-1} \right), \quad (C.9) \]

where

\[ C_{\lambda m}^{\mu} = \frac{1}{\pi T} \int_0^{2\pi} \int_0^T dx dt \cos[(\mu k + \lambda)\omega_0 x - \mu k t] \times \cos[(mk + l)x - mkt - \omega_0 x + z \sin(x - t)], \quad (C.10) \]

with

\[ y = \frac{(mk + l)\omega_0 M}{I^2}, \]

\[ z = \frac{(mk + l)\beta P}{k(I-1)^2}. \]

By using the expansions [125]

\[ \cos(y \sin x) = J_0(y) + 2 \sum_{\ell \geq 0} J_{2\ell}(y) \cos 2\ell x, \]

\[ \sin(y \sin x) = L \sum_{\ell \geq 0} J_{2\ell + 1}(y) \sin(2\ell + 1)x, \quad (C.12) \]

where \( J_n \) is the Bessel function of first kind, we get

\[ C_{\lambda m}^{\mu} = \sum_{\nu = \pm 1} \eta^{\nu''} \varepsilon^{\nu'''} J_{\ell''}(y) J_{m''}(z), \quad (C.13) \]

where \( \ell' = \ell + \nu \lambda, m' = m + \nu \mu, \ell'' = |\ell'|, m'' = |m'|, \eta = \text{sgn}(\ell'), \) and \( \varepsilon = \text{sgn}(m'), \)
with $\text{sgn}(u)=+1$ (resp. $-1$) for $u \geq 0$ ($u < 0$). Equations (C.6), (C.9), (C.11), and (C.13) give the exact expressions for the $K_{\ell m}$'s.

For $y$ and $z$ small, Bessel functions may be expanded at lowest order and we get much simpler expressions. For Kolmogorov transformation (3.88), $\alpha=\beta=1$, and one obtains [30]

$$K_{\ell m}(I) = \frac{(mk+\ell)^{m+\ell-3}M^2\ell^m}{\ell!m!2^{\ell+m-1}k_m2^\ell-2(1-1)^2m-21(1-1)} \left[ \frac{\ell m(k+1)}{1(1-1)} \right. $$

$$+ \left. \frac{\ell(\ell-1)}{(1-1)^2} + \frac{k^3m(m-1)}{1^2} \right], \quad (C.14)$$

for $\ell \geq 1$, $m \geq 1$.

For Kolmogorov transformation (3.2), $\alpha=1$, $\beta=0$, and it comes

$$K_{01} = -P, \quad (C.15)$$

$$K_{\ell 1} = -\frac{k(k+\ell)^{\ell-1}}{\ell!} P\left(\frac{M}{21^2}\right)^\ell, \quad (C.16)$$

for $\ell \geq 1$.

It is interesting to notice that the definition of transformation $(v,x) \rightarrow (I,\phi)$ by a generating function allows us to compute exactly the $K_{\ell m}$'s in terms of Bessel functions, when a Lie transformation would compute the transformed Hamiltonian order by order in $M$ and $P$. The Bessel functions would be recovered (if one were clever!) as sums of series.
Appendix D: Expansion of $H_p$ in the Action-Angle Variables of the Pendulum

Let $H_0(v,x) = v^2/2 - M \cos x$.

D.1. Trapped Orbits Let $m = \frac{1}{2} [H_0(v,x)/M + 1]$. Then

$$I = \frac{8}{\pi} \sqrt{M} [E(m) - m_1 K(m)]$$

$$\Omega = \frac{dH_0}{dI} = \sqrt{M} \frac{\pi}{2K(m)}$$

$$\sigma = \frac{d\Omega}{dI} = -\frac{\pi^2}{16mK^3(m)} [E(m)/m_1 - K(m)]$$

where $m_1 = 1 - m$ and $K$ and $E$ are the elliptic integrals of first and second kind [126]. $x$ is related to $\phi$ through

$$x = 2 \arcsin \left[ \sqrt{m} \; \text{sn} \left( \frac{2K(m)\phi}{\pi}, m \right) \right]$$

where $\text{sn}$ is the Jacobian elliptic function [126].

For $2k$ integer the $W_n$'s of eq. (E.1) can be computed exactly

$$W_n = \left[ -\frac{\pi}{2K(m)} \right]^{2k} \frac{q^{n/2}}{1 - (-1)^{2k+n} q^n} \sum_{n}$$

where

$$q = \exp - [\pi K(m_1)/K(m)]$$
and

\[ \sum_{n}^{1/2} = 2, \quad \sum_{n}^{1} = 4n, \quad \sum_{n}^{3/2} = (2n)^{2} + 2[2K(m)/\pi]^{2} \]

\[ \sum_{n}^{2} = 8n^{3}/3 + 16[2K(m)/\pi]^{2}/3. \] For small values of q the method given in [25] yields for any k.

\[ w_{n} = \left[ \frac{\pi}{2K(m)} \right]^{2k} n^{2} q^{2n-2k} \sum_{n} \] 

(D.7)

with

\[ \sum_{n}^{k} = \sum_{\ell=1}^{\infty} 2^{\ell} c_{z}^{\ell} z^{n-1} \]  

(D.8)

where

\[ c_{z}^{\ell} = (z-1) \ldots (z-\ell+1)/\ell! \]  

(D.9)

for z real and \( \ell \) integer. For a large \( n/K(m) \) ref. [127] yields eq. (D.7) with

\[ \sum_{n}^{k} = \frac{2(2n)^{2k-1}}{\Gamma(2k)} \]  

(D.10)

D.2. Untrapped Orbits

Let \( m = 2/H_{0}(v,x)/M+1 \), then

\[ I = \frac{4}{\pi} \left[ \frac{M}{m} \right]^{1/2} E(m) \]  

(D.11)
\[ \Omega = \pi \left( \frac{M}{m} \right)^{1/2} \frac{1}{K(m)} \]  \hspace{1cm} (D.12)

\[ \sigma = \frac{\pi^2 E(m)}{4m_1 K(m)^3} \]  \hspace{1cm} (D.13)

\[ x = 2am \left[ \pm \frac{K(m)\psi}{\pi}, m \right] \]  \hspace{1cm} (D.14)

where \( am \) is the amplitude of elliptic functions [126].

For \( 2k \) integer the \( W_n \)'s can be computed exactly in a similar way as for trapped orbits

\[ W_n = \left[ \frac{2\pi}{K(m)} \right]^{2k} \left( \frac{q}{m} \right)^k \sum_{n}^{2k} \frac{q^n}{1-q^n} \sum_{n}^{k} \]  \hspace{1cm} (D.15)

where \( q \) is given by eq. (D.6), \( \sum_{n}^{1/2} = 1, \sum_{n}^{1} = 1+n, \)

\[ \sum_{n}^{3/2} = \left( \frac{3}{2}+n \right)^2 /2-(2-m)[2K(m)/\pi]^2/16 \]

\[ \sum_{n}^{2} = (2+n)^3 /2-(2-m)(2+n)[2K(m)/\pi]^2/12. \]  For small values of \( q \)

\[ W_n = \left[ \frac{2\pi}{K(m)} \right]^{2k} \left( \frac{q}{m} \right)^k \sum_{n}^{k} q^n \]  \hspace{1cm} (D.16)

with

\[ \sum_{n}^{k} = \sum_{\ell=1}^{k} C_{\ell}^{k} C_{n-1}^{\ell-1} \]  \hspace{1cm} (D.17)

where \( C_{\ell}^{k} \) is given by eq. (D.9). For a large \( n/K(m) \) ref. [127] yields eq. (D.16) with
\[ \sum_{n}^{k} \frac{(k+n)^{2k-1}}{\Gamma(2k)} \]  \hspace{1cm} (D.18)

Notice that the definition of \((I, \theta)\) for both trapped and untrapped orbits implies that \(I=\nu\) and \(\theta=\pi\) for \(M=0\).

Appendix E: Stochastic Layers

This appendix shows how to compute a standard map for describing stochastic layers, by a method different from Melnikov–Chirikov’s [4,12].

Rewriting the paradigm Hamiltonian (2.1) into the action–angle variables \((I, \theta)\) of the pendulum Hamiltonian (2.58) yields the new Hamiltonian

\[ H'(I, \theta, t) = H_0(I) - P \sum_{n=-\infty}^{+\infty} W_n(I) \cos \varphi_n(\theta, t), \]  \hspace{1cm} (E.1)

where

\[ \varphi_n(\theta, t) = k_n \theta - kt \]  \hspace{1cm} (E.2)

with

\[ k_n = (2-\beta)k+n \]  \hspace{1cm} (E.3)

\[ \beta = \begin{cases} 1 & \text{for untrapped orbits,} \\ 2 & \text{for untrapped orbits.} \end{cases} \]  \hspace{1cm} (E.4)
and $H'_0$ and $W_n$ are given in appendix D.

Let $I_Q$ be defined by

$$\frac{dH'_0}{dl}(I_Q) = \frac{k}{Q}. \quad (E.5)$$

$I_Q$ is the value of the action for $P=0$ corresponding to a mean velocity (rotation number) $k/Q$. In particular $I_{k_n}$ is the resonant value of the action for the $n$-th term of eq. (E.1).

Now consider an orbit $\mathcal{O}$ with a mean velocity (or rotation number) $u=k/Q_0$ with $Q_0$ large. For $P=0$ this orbit is quite close to the separatrix of resonance $M$. According to eqs. (E.5), (D.3), (D.10), (D.13), and (D.18), $W_n(I_Q)$ and $d^2H'_0(I_Q)/dl^2$ vary with $Q$ on a typical scale $Q_0$ in the neighborhood of $I_{Q_0}$. Therefore their variation is infinitely small for large $Q_0$'s on the scale $\Delta Q = 1$ separating two nearby resonances, and the reduction to a many-wave Hamiltonian is excellent in the limit of large $Q_0$'s. A typical number $Q_0$ of resonant terms have their amplitude computed according to the centered-resonance approximation $W_n(I) \rightarrow W_n(I_{k_n})$. For the other terms, formula (3.75) would give another approximation, but these terms very little contribute to the dynamics of $\mathcal{O}$, and the error made by extending to these terms the centered-resonance approximation is small. As a result we get the many-wave Hamiltonian

$$H''(p,\phi, t) = \frac{1}{2} p^2 - |\sigma| P \sum_{n=-\infty}^{+\infty} U_n \cos n(\phi, t), \quad (E.6)$$

where, according to eqs. (E.5), (D.3), (D.10), (D.13), and (D.18),
\[ U_n = \frac{2^{2\lambda}}{\pi^\lambda k_n \Gamma(\lambda) \mu^\lambda} \exp \left( -\frac{1}{\mu} \right) , \]  
(E.7)

\[ |\sigma| = \frac{g}{4\pi W} \left( -\frac{k}{Q_0 \sqrt{M}} \right)^3 , \]  
(E.8)

with

\[ W = 16 \exp \left( -\mu \pi^2 \frac{Q_0}{\beta} \right) , \]  
(E.9)

\( \lambda = 2k, \Gamma \) is the gamma function (for \( \lambda \) integer \( \Gamma(\lambda) = (\lambda-1)! \)), and \( \mu \) is defined by eq. (2.61). Equation (E.9) is obtained from eqs. (E.5), (2.64) and the asymptotic expression of the elliptic integral of the first kind \( [126] \). Define \( k \) by

\[ k_\lambda \leq Q_0 < k_{\lambda+1} , \]  
(E.10)

a new position

\[ y = \varphi_\lambda(\theta, t) , \]  
(E.11)

and a new time

\[ t' = \frac{kt}{k_\lambda} \]  
(E.12)

Then \( y(t') \) is governed by Hamiltonian
\[ H^{(w,y,t')} = \frac{1}{2} w^2 + A \sum_{n=-\infty}^{\infty} \frac{1}{\kappa_n} \cos[\kappa_n y - (\lambda - n)t'], \]  

(E.13)

where

\[ A = |c| \frac{k^4}{k^2} \frac{P U_k}{\lambda} = \beta \frac{P}{M} \frac{2^{2\lambda-1}}{\pi^{\lambda+2} \Gamma(\lambda) \mu^\lambda + 1} \exp - \frac{1}{\mu} + o\left(\frac{1}{Q_0}\right), \]  

(E.14)

and

\[ \kappa_n = \frac{k_n}{k_\lambda} = \frac{U_k}{U_n}. \]  

(E.15)

Let \( \varepsilon \) be a small number and \( \Delta n(\varepsilon) \) be the range of \( |n-\lambda| \) such that \( |\kappa_n - 1| \leq \varepsilon \). When \( Q_0 \) goes to infinity, \( \Delta n(\varepsilon) \) goes to infinity as well. Therefore \( H^{(w,y,t')} \) becomes closer and closer to Hamiltonian (E.13) computed with \( \kappa_n = 1 \) for all \( n \)'s, i.e. to the standard map Hamiltonian (2.15) with \( M = A \). As a result the dynamics of orbit \( \mathcal{O} \) is governed by a standard map with a parameter \( K \) given by eq. (2.60). This agrees with Chirikov's result \([4]\) for untrapped orbits \( (\beta = 1) \). The difference for trapped orbits \( (\beta = 2) \) is related to the fact that Chirikov's result applies to the standard map: \( \alpha \) and \( \beta \) should be taken equal to 1 in eq. (2.60) for both the outer and the inner layer, for the following reason. For trapped orbits a strong difference appears between \( H_p \) and the standard map Hamiltonian \( H_s \) of eq. (2.15): the \( \cos(k+t) \) resonance which exists in \( H_s \), but not in \( H_p \), is equally important as resonance \( \cos(x-t) \) for computing the width of the layer. As shown in appendix B,
trapped resonances whose rotation number has an odd (resp. even) inverse cancel each other (resp. add together). The non-zero trapped resonances now have an amplitude 4 times that of the untrapped resonances, but their spacing is 2 times larger. According to the scaling of appendix A, this yields the same reduced amplitude. Therefore, for the standard map, $\beta$ should be taken equal to 1, even for the inner stochastic layer.
References


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Figure Captions

Fig. 1.1 Variation with time of the number of articles published each year on Hamiltonian stochasticity as measured from the INSPEC Physics Abstracts with the key words: (resonance overlap) or (KAM) or \{[(stochastic) or (chaos) or (random process)] and [(Hamiltonian) or (classical mechanics) or (area-preserving map) or (nonlinear oscillations)]\}.

Fig. 2.1 Trivial scale invariance.

Fig. 2.2 Phase space of a pendulum.

Fig. 2.3 Phase-space of a particle in a longitudinal wave.

Fig. 2.4 Naive description of the phase space of a particle in two waves with amplitudes $M$ and $P$. (a) $P=0$, (b) $M=0$, (c) $MP \neq 0$, $s = 2\sqrt{M} + 2\sqrt{P} << 1$.

Fig. 2.5 Poincaré map of $H_p$, $M/P=k=1$, $s=0.5$.

Fig. 2.6 Poincaré map of $H_p$, $M/P=k=1$, $s=0.68$.

Fig. 2.7 Graphical proof of the repulsion of resonances for $H_p$, (a) $P=0$, (b) $P$ small, (c) $P$ and $M$ of the same order of magnitude.

Fig. 2.8 Poincaré map of $H_p$, $M/P=k=1$, $s=1$.

Fig. 2.9 Graphical description of the simple renormalization scheme for KAM tori.

Fig. 2.10 Behavior of $(M,P)$ when renormalizing.

Fig. 2.11 Renormalizing by double steps.

Fig. 2.12 Structure of the space of Hamiltonians with respect to a universal class of KAM tori.

Fig. 2.13 Sketch of the functional space of Hamiltonians close to an integrable Hamiltonian $H_0$. The shaded area corresponds to the domain where KAM torus is destroyed.

Fig. 2.14 Hierarchy of fixed points.

Fig. 2.15 Flux through a Cantorus.

Fig. 2.16 Scaling of diffusion in the Cantorus-with-small-hole regime of the standard map with noise. By courtesy of R. White.

Fig. 2.17 Transition time between two primary resonances of the standard map. Reproduced by permission from MacKay, Meiss, and Percival [50].
Fig. 2.18 Coefficients $R(k)$ (continuous line) and $c(k)$ (dashed line) of the formula for the approximate noble stable manifold.

Fig. 2.19 Comparison of the theoretical predictions (dots) of the threshold of global stochasticity of $H_p$ with numerical upper bounds (crosses), (a) $k=1$, $\rho = (M/P)^{1/2}$ varying, (b) $\rho = 1$, $k$ varying.

Fig. 2.20 Graphical description of a renormalization scheme for the bifurcation of period-$n$ cycles; $n=3$.

Fig. 2.21 Regimes of chaotic transport of a many-wave Hamiltonian, (a) spectral width $\Delta \omega$ fixed, amplitude $\phi$ of the amplitude varying, (b) $\phi$ fixed, $\Delta \omega$ varying.

Fig. 3.1 Evolution of the zonning number $z$ when renormalizing, (a) new $z$ vs. fractional part $\delta z$ of the old $z$, (b) new $\delta z$ vs. $\delta z$.

Fig. 3.2 Hierarchy of parameters.

Fig. 3.3 Geometrical construction of the amplitudes of the many-wave Hamiltonian obtained by reduction of a general Hamiltonian. $V(p)$ stands for $V_{n_1}(p)$, $V_{n_2}(p)$, and $V_{n_3}(p)$.

Fig. 3.4 Evolution of the zonning number $z$ when renormalizing with $T_r$.

Fig. 3.5 Hyperbolicity argument; case of a manifold $N$ not containing the fixed point.

Fig. 3.6 Hyperbolicity argument; case of a manifold $N$ going through the fixed point.

Fig. 3.7 Fractal diagram for $H_p$, $k=1$, $M=16P$.

Fig. 4.1 Graphical argument for the appearance of global stochasticity first between resonances $P$ and $R_2$ when (a) $k=1$, $P>M$, (b) $k>1$, $M=P$.

Fig. 4.2 Energy line of a two-degree-of-freedom Hamiltonian with indication of the positions of two primary resonances, $I^2_m$ and $I^2_p$, and of an orbit, $I^2_1$. 
<table>
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<tr>
<th>Symbol</th>
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<td>$\delta$</td>
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<td>[27]</td>
<td>2.3.2.4.1</td>
<td>secs. 3.1.2, 3.1.7</td>
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<td>$\delta'$</td>
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<td>3.2.3</td>
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<tr>
<td>$\tau$</td>
<td>time multiplier</td>
<td>$g^{-1}$</td>
<td>-</td>
<td>2.3.2.4.2</td>
<td>eq. (3.51)</td>
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<td>$\xi$</td>
<td>area multiplier</td>
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<td>[27] and eq. (2.41)</td>
<td>2.3.2.4.3</td>
<td>eq. (3.52)</td>
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<tr>
<td>$d$</td>
<td>width multiplier</td>
<td>$g$</td>
<td>-</td>
<td>3.1.3.</td>
<td>eq. (3.50)</td>
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<tr>
<td>$\sigma$</td>
<td>inverse mass multiplier</td>
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<td>3.1.1,3.1.7</td>
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<td>$T_w$</td>
<td>trace of cycles nearby to a critical noble torus</td>
<td>0.999644</td>
<td>[47,72]</td>
<td>5.2</td>
<td>-</td>
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</table>

Table 2.1 Universal parameters of the noble fixed point. $g = 1.6180340$. 
Figure 1.1: Number of articles over the years.
Fig 2.18
Fig. 2.21
Fig. 3.3
Fig. 3.4
\mathcal{H}_0(I) = E

Fig. 4.2