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Relativistic generation of vortex and magnetic field[a)]

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The implications of the recently demonstrated relativistic mechanism for generating generalized vorticity in purely ideal dynamics [Mahajan and Yoshida, Phys. Rev. Lett. **105**, 095005 (2010)] are worked out. The said mechanism has its origin in the space-time distortion caused by the demands of special relativity; these distortions break the topological constraint (conservation of generalized helicity) forbidding the emergence of magnetic field (a generalized vorticity) in an ideal nonrelativistic dynamics. After delineating the steps in the "evolution" of vortex dynamics, as the physical system goes from a nonrelativistic to a relativistically fast and hot plasma, a simple theory is developed to disentangle the two distinct components comprising the generalized vorticity—the magnetic field and the thermal-kinetic vorticity. The "strength" of the new universal mechanism is, then, estimated for a few representative cases; in particular, the level of seed fields, created in the cosmic setting of the early hot universe filled with relativistic particle–antiparticle pairs (up to the end of the electron–positron era), are computed. Possible applications of the mechanism in intense laser produced plasmas are also explored. It is suggested that highly relativistic laser plasma could provide a laboratory for testing the essence of the relativistic drive. [doi:10.1063/1.3566081]

I. INTRODUCTION

The notion of fluid vorticity may be generalized, for a charged fluid (plasma), by adding an electromagnetic (EM) part to the standard kinetic part; the *generalized vorticity* is the curl of the *canonical momentum* that combines the fluid (kinetic-thermal) vorticity and the magnetic field.[1] In this paper, the words vorticity, generalized vorticity, and generalized magnetic field will be used synonymously.

A topological constraint forbids the creation (destruction) of vorticity in what is called an ideal fluid (see Sec. II). The *helicity* of generalized vorticity (a *topological charge*)[2] remains unchanged in ideal dynamics; its conservation has serious implications on the matter of the origin of magnetic fields (the EM vorticity). One must, generally, resort to "nonideal effects" to affect a change in helicity. To explain the existence of cosmological magnetic fields, for example, the nonideal baroclinic mechanism [characteristic of nonideal thermodynamics, in which the gradients of p (pressure) and T (temperature) have different directions] has been extensively invoked.[3–8] An anisotropic pressure tensor may also generate vorticity. A velocity-space nonequilibrium (distortion from the Gibbs distribution) could, of course, provide a stronger source of magnetic field via the so-called Weibel instability.[9]

This paper is devoted to working out some of the implications of the recently discovered new ideal but *relativistic* vorticity generating mechanism,[1] which, in a clear-cut departure from the standard practice, shows that vorticity can be generated in strictly ideal dynamics through a combination of inhomogeneous entropy and special-relativistic effects (space-time distortion).[10]

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II. IDEAL VORTEX DYNAMICS AND TOPOLOGICAL CONSTRAINTS (NONRELATIVISTIC)

To set the stage for a proper relativistic calculation we begin with some nonrelativistic preliminaries. We will deal with an ideal one-component charged fluid (the uncharged fluid is a particular case) obeying the well-known equation of motion

$$\partial_t \boldsymbol{P} - \boldsymbol{V} \times \Omega = -n^{-1}\nabla p - \nabla\varphi, \tag{1}$$

where p (n) is the fluid pressure (density), $\boldsymbol{V}$ is the fluid velocity, $\boldsymbol{P} = m\boldsymbol{V} + (q/c)\boldsymbol{A}$ is the canonical momentum [$\boldsymbol{A}$ is the vector potential, m (q) is the mass (charge) of the fluid constituent], $\varphi = mV^2/2 + q\phi$ (ϕ is the scalar potential), and

$$\Omega = \nabla \times \boldsymbol{P} = \nabla \times (m\boldsymbol{V}) + (q/c)\boldsymbol{B} \tag{2}$$

is the generalized vorticity (GV). When the fluid is uncharged ($q = 0$), GV reverts to m times the standard vorticity $\nabla \times \boldsymbol{V}$. In the other limit of minimal magnetohydrodynamics (MHD), the mass is allowed to go to zero, and GV represents just the normalized magnetic field.

We have deliberately split the right-hand side of Eq. (1) into two parts; the latter ($\nabla\varphi$) is a full gradient (exact differential form of a potential) while the former, depending on the thermodynamics of the fluid, may or may not be.

The curl of the equation of motion (1) gives us the equation for the evolution of GV

$$\begin{aligned}\partial_t\Omega - \nabla \times (\boldsymbol{V} \times \Omega) &= -\nabla \times (n^{-1}\nabla p)\\ &= -\nabla n^{-1} \times \nabla p,\end{aligned} \tag{3}$$

which must be analyzed under two separate headings:

I. The fluid is barotropic and obeys some kind of an equation of state $p = p(n)$ making the right-hand side of (3) vanish. The resulting ideal vortex transport equation (the subscript "id" denotes ideal)

$$\partial_t \boldsymbol{\Omega}_{\rm id} - \nabla \times (\boldsymbol{V} \times \boldsymbol{\Omega}_{\rm id}) = 0 \tag{4}$$

has been investigated in great depth and detail. Here we will content ourselves with simply calling on the results germane to the present work. In distinction to particle dynamics, the generalized vorticity[2] is a hallmark of the ideal fluid dynamics; the ideal fluid dynamics, however, does not allow vorticity to emerge from a zero initial value—if at any time $\boldsymbol{\Omega}_{\rm id} = 0$, it remains zero at all times. This fundamental property makes its appearance in many different forms in literature. Both Kelvin's circulation theorem in hydrodynamics and the frozen-in flux condition of MHD, for example, are a simple consequence of (4) and lend themselves to the same mathematical statement: For any loop $L(t)$ [or the surface $\Sigma(t)$ whose periphery is $L(t)$] transported by the fluid, the *circulation* (or the *flux*)

$$\Psi = \oint_{L(t)} \boldsymbol{P}_{\rm id} \cdot dx = \int_{\Sigma(t)} \boldsymbol{\Omega}_{\rm id} \cdot \boldsymbol{n}\, d^2x$$

is conserved ($\boldsymbol{n}$ is the unit normal vector onto Σ, and d^2x is the surface element); see Fig. 1. It is equally straightforward to demonstrate, by a direct calculation form (4) and its uncurled partner (1), that the *generalized helicity*

$$C = \int_{\Upsilon} \boldsymbol{\Omega}_{\rm id} \cdot \boldsymbol{P}_{\rm id}\, d^3x \tag{5}$$

is a constant of fluid motion ($dC/dt = 0$) when the right-hand side of (1) is a perfect gradient.[11] Here Υ is a "vortex tube" (of the generalized vortex or generalized magnetic field) moving with the fluid such that $\boldsymbol{\Omega}_{\rm id} \cdot \boldsymbol{n} = 0$ for all normal vectors onto the boundary $\partial\Upsilon$. The helicity is a basal quantity characterizing the topology of $\boldsymbol{\Omega}_{\rm id}$.[12]

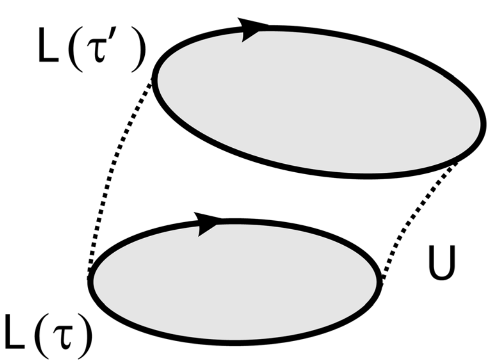

FIG. 1. Transport of a loop and circulation. Given a loop L in space, the circulation of a vector field $\boldsymbol{P}$ is the integral $\oint_L \boldsymbol{P} \cdot d\boldsymbol{x}$. Two loops $L(\tau)$ and $L(\tau')$, connected by the "flow" $dx/d\tau = \boldsymbol{U}$ (the parameter τ may be regarded as time), are shown in the figure. A circulation theorem pertains to a "movement" of loops; the rate of change of circulation is calculated as $\frac{d}{d\tau}\oint_{L(\tau)} \boldsymbol{P} \cdot d\boldsymbol{x} = \oint_{L(\tau)} [\partial_\tau \boldsymbol{P} + (\nabla \times \boldsymbol{P}) \times \boldsymbol{U}] \cdot d\boldsymbol{x}$. To generalize the argument to the relativistic regime, we have to immerse the loop in the four-dimensional space-time and transport it by the four-velocity $dx_\mu/ds = U_\mu$ ($\tau = s$ is the proper time) (Ref. 1). The space-time circulation obeys $\frac{d}{ds}\oint_{L(s)} P^\mu dx_\mu = \oint_{L(s)} (\partial^\mu P^\nu - \partial^\nu P^\mu) U_\nu dx_\mu$; compare the right-hand side with (10).

II. Since the conclusions stated in the preceding paragraphs are contingent upon the vanishing of the right-hand side of (3), many authors have attempted to escape from or bypass the topological constraint by invoking a baroclinic pressure state with $\nabla \times (n^{-1}\nabla p) = \nabla n^{-1} \times \nabla p \neq 0$. If the density and pressure gradients are not parallel, then the GV equation has a source and vorticity could, in principle, be generated. In a locally thermal-equilibrium fluid (i.e., internal entropy production = 0 in every fluid element), we may write

$$\nabla h = T\nabla\sigma + n^{-1}\nabla p, \tag{6}$$

where h is the enthalpy, σ is the entropy, and T is the temperature. Like pressure and density all these quantities are, in general, inhomogeneous. Using (6), the baroclinic term could also be expressed as

$$-\nabla \times (n^{-1}\nabla p) = \nabla T \times \nabla\sigma, \tag{7}$$

the form that it would take for a relativistically hot fluid. When expressed in this form, it is easier to notice that the baroclinic term may have an appreciable magnitude when a system is driven by an external energy input causing strong inhomogeneity in temperature and entropy. For example, in atmospheric flows, inhomogeneous solar irradiation and density distribution may cause vortexes.[3] Similarly, strong laser light irradiating a pellet of nuclear fusion fuel produces rather strong magnetic fields.[4–8]

This is the end of nonrelativistic preliminaries. We now proceed to the relativistic formulation which will yield to us a brand new source of vorticity within an ideal fluid setting. We will show that for a variety of plasmas (cosmological and astrophysical, in particular), this ubiquitous mechanism associated with inhomogeneous entropy and relativistic velocity fields may be dominant.

III. RELATIVISTIC VORTEX DYNAMICS

The relativistic theory of vorticity generation[1] is based on the recent, rather perspectively formulated, unified theory of relativistic, hot magnetofluids.[13] The essence of the formulation lies in the creation of the unified tensor

$$M^{\nu\mu} = F^{\nu\mu} + \frac{m}{q} S^{\nu\mu} \tag{8}$$

from the EM field tensor (A^μ is the four-vector potential)

$$F^{\nu\mu} = \partial^\nu A^\mu - \partial^\mu A^\nu$$

and the *flow field tensor* (representing both inertia and thermal forces)

$$S^{\nu\mu} = \partial^\nu (fU^\mu) - \partial^\mu (fU^\nu), \tag{9}$$

where $U^\mu = (\gamma, \gamma\boldsymbol{V}/c)$ is the fluid four-velocity, $\boldsymbol{V}$ is the ordinary three velocity, $\gamma = 1/\sqrt{1-(V/c)^2}$, and the statistical factor f is related to the enthalpy through $h = mn_R f$, where

n_R is the fluid density in the rest frame. For a Maxwellian fluid with relativistic temperatures, $f = f(T) = K_3(mc^2/T)/K_2(mc^2/T)$; K_j is the modified Bessel function,[14] and reduces to $f \approx 1 + (5/2)(T/mc^2)$ in the nonrelativistic limit.

In terms of the fully antisymmetric $M^{\nu\mu}$, the equation of motion of a hot relativistic fluid takes on the succinct form

$$qU_\mu M^{\nu\mu} = T\partial^\nu \sigma, \tag{10}$$

from which the isentropic condition $U_\mu \partial^\mu \sigma = 0$ follows trivially.

For the subject matter of this paper, we need only the vector part (the scalar part is not independent; it is a consequence of the vector part plus the isentropic condition)

$$q\left(\tilde{\boldsymbol{E}} + \frac{\boldsymbol{V}}{c} \times \tilde{\boldsymbol{B}}\right) = \frac{cT\nabla\sigma}{\gamma}, \tag{11}$$

where

$$\tilde{\boldsymbol{E}} = \boldsymbol{E} - \frac{m}{q}\left[\partial_t(f\gamma \boldsymbol{V}) + \nabla(f\gamma c^2)\right],$$
$$\tilde{\boldsymbol{B}} = \boldsymbol{B} + \frac{mc}{q}\nabla \times (f\gamma \boldsymbol{V}),$$

may be viewed as generalized electric and magnetic fields representing the electromagnetic and fluid forces. Apart from a multiplicative constant (q/c), $\tilde{\boldsymbol{B}}$ is simply the relativistic generalization of the GV $= \Omega$ defined in Eq. (2). Relativity makes its appearance not only through the standard kinematic γ but also through the thermal factor f (reflecting large random motions); the fluid particle acquires an effective mass $m_{\rm eff} = \gamma f m$. *The nonmagnetic part of GV is now kinetic-thermal; quite different from its nonrelativistic limit that is purely kinetic*. The generalized electric field $\tilde{\boldsymbol{E}}$, in addition to the electric field and the relativistic inertial term, has $\nabla(f\gamma)$ representing the gradient of the sum of thermal and hydrostatic pressures; in the nonrelativistic limit, $\nabla(f\gamma) \approx c^{-2}\nabla[V^2/2 + (5/2)(T/m)]$.

Notice the appearance of the factor γ in Eq. (11); this will prove to be the origin of the new vorticity generating mechanism.

To exploit our nonrelativistic knowledge to understand the relativistic system, we proceed to cast Eq. (11) into a form analogous to what the vorticity Eq. (3) takes after invoking Eq. (7) to convert the right-hand side to $-\nabla T \times \nabla\sigma$. Dividing both sides by $q\,\gamma$, taking the curl of both equations, and using the generalized Faraday law $\nabla \times \tilde{\boldsymbol{E}} = -c^{-1}\partial_t \tilde{\boldsymbol{B}}$,[13] we derive

$$\partial_t \tilde{\boldsymbol{B}} - \nabla \times (\boldsymbol{V} \times \tilde{\boldsymbol{B}}) = -\nabla\left(\frac{cT}{q\gamma}\right) \times \nabla\sigma = \mathcal{S}. \tag{12}$$

But for the factor γ on the right-hand side, Eq. (12), describing the evolution of the relativistic generalized vorticity (RGV), is structurally identical to the systems (3) and (7). Needless to say, Eqs. (3)–(7) is the nonrelativistic limit of Eq. (12) as Ω is of $\tilde{\boldsymbol{B}}$.

The source ($\mathcal{S} = \mathcal{S}_b + \mathcal{S}_r$) is readily broken into the familiar baroclinic term

$$\mathcal{S}_b = -\frac{c}{q\gamma}\nabla T \times \nabla\sigma, \tag{13}$$

and the relativistically induced new vorticity drive

$$\mathcal{S}_r = -\frac{cT}{q}\nabla\gamma^{-1} \times \nabla\sigma = \frac{cT\gamma}{2q}\nabla\left(\frac{V}{c}\right)^2 \times \nabla\sigma. \tag{14}$$

The rest of this paper is an investigation of $\mathcal{S}_r$, first reported in Ref. 1. Following conclusions are readily deducible from Eqs. (12)–(14):

(1) For homogeneous entropy σ, there is no vorticity drive. Naturally for this case, the topological invariants are "impossible" to break (within this formalism).
(2) Even in the ideal fluid thermodynamics, when baroclinic drive is zero, special-relativistic effects provide a vorticity generating mechanism. As long as the fluid velocity field is inhomogeneous, its interaction with inhomogeneous entropy keeps S_r nonzero.
(3) When baroclinic drive is nonzero, and the kinematic and thermal gradients are comparable, then the ratio of their strengths is

$$\frac{|\mathcal{S}_r|}{|\mathcal{S}_b|} \approx \gamma^2 (V/c)^2 = \frac{(V/c)^2}{1-(V/c)^2}. \tag{15}$$

For highly relativistic flows (cosmic particle–antiparticle plasmas, electron–positron plasmas in the magnetosphere of neutron stars, relativistic jets, laser produced plasmas, etc.) for which $(V/c)^2$ approaches 1, this ratio can be very large, and $\mathcal{S}_r$ will be the dominant drive. In moderately relativistic flows the drives can be comparable.
(4) One must, however, remember that the baroclinic drives are rather difficult to create; most long lived plasma systems will tend to have $\nabla T \times \nabla\sigma = 0$ because of the thermodynamic coupling of temperature and entropy. In these large majority of physical situations, the relativistically induced $\mathcal{S}_r$ may be the only vorticity generation mechanism, because there is no constraint that will make the gradients of γ (kinematic quantity) align with gradients of σ (statistical quantity). Thus, whether small or large, the relativistic drive will always be there.

Because of their complete structural equivalence, all general results proved for Eq. (3), including the deep connection between the creation (noncreation) of vorticity with nonconservation (conservation) of helicity, remain valid for Eq. (12). By somewhat tedious but straightforward manipulations, one finds that the rate of change of helicity ($C = \int \tilde{\boldsymbol{P}} \cdot \tilde{\boldsymbol{B}}\, d^3x$ with $\nabla \times \tilde{\boldsymbol{P}} = \tilde{\boldsymbol{B}}$),[15] caused by the relativistic drive, is given by

$$\frac{dC}{dt} = \int \frac{2cT}{q}\sigma \tilde{\boldsymbol{B}} \cdot \nabla\gamma^{-1} d^3x, \tag{16}$$

where the baroclinic term is neglected and it is assumed that $\tilde{\boldsymbol{B}} \cdot \boldsymbol{n} = 0$ at the surface surrounding the volume of integration. To change helicity, it seems, that the gradients in the kinetic energy (measured by γ) of the fluid elements must have a component along the $RGV = \tilde{\boldsymbol{B}}$.

IV. ESTIMATES OF THE SEED VORTEX/MAGNETIC FIELD

After having shown that the new drive $\mathcal{S}_r$ will always dominate the traditional baroclinic drive $\mathcal{S}_b$ for relativistic plasmas, we will now attempt to estimate its strength in a few representative cases. Since the basic theory pertains to the generation of the GV, the eventual apportioning of GV into the magnetic part and the thermal-kinetic part will be a difficult system-dependent exercise—for example, if the plasma consists of relativistic electrons in a neutralizing ion background or it is an electron–positron pair plasma where both species are dynamic. The present paper is basically a theoretical-conceptual paper and will be limited to general considerations; no detailed models will be built.

We now work out an essential theoretical addition to Ref. 1 that is crucial to the understanding of the pair plasma systems; the pair plasmas of all kinds fill the early hot universe, for example.

A. Separation of kinetic vorticity and magnetic field in pair plasmas

We consider a typical pair plasma, neutral in its rest frame, with density $n_+ = n_- = n =$ constant. The suffix $+$ $(-)$ labels the positive (negative) particles. We also assume that the particles have the same homogeneous temperature so that their temperature modified effective masses $m^*_\pm = f_\pm m$ (m: rest mass) are also the same ($m^*_+ = m^*_- = m^* =$ constant). The generalized canonical momenta are

$$P^j_\pm = m^* c U^j_\pm \pm (e/c) A^j,$$

and the associated *generalized vorticities* are

$$\begin{aligned}\frac{1}{m^*}\nabla\times\boldsymbol{P}_\pm &= \nabla\times(c\boldsymbol{U}_\pm) \pm \frac{\boldsymbol{e}}{m^* c}\boldsymbol{B}\\ &\equiv \boldsymbol{\omega}_\pm \pm \boldsymbol{\omega}_c,\end{aligned} \tag{17}$$

in terms of which, the induction Eq. (12) takes the form

$$\begin{aligned}&\partial_t(\boldsymbol{\omega}_\pm \pm \boldsymbol{\omega}_c) - \nabla\times[\boldsymbol{V}_\pm \times (\boldsymbol{\omega}_\pm \pm \boldsymbol{\omega}_c)]\\ &= -\nabla\times\left(\frac{cT\nabla\sigma_\pm}{\gamma_\pm m^*}\right).\end{aligned} \tag{18}$$

To close the system, we need a determining equation for $\boldsymbol{\omega}_c = e\boldsymbol{B}/(m^* c)$ (the normalized magnetic field). When the large-scale slowly evolving EM is decoupled from the photons, the displacement current may be neglected[16] and the resulting Ampere's law

$$\nabla\times\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} = 4\pi e n(\boldsymbol{U}_+ - \boldsymbol{U}_-) \tag{19}$$

may be written as

$$\delta^2 \nabla\times\boldsymbol{\omega}_c = c\hat{n}(\boldsymbol{U}_+ - \boldsymbol{U}_-). \tag{20}$$

Here $\delta = c/\omega_{pe}$ (electron inertia length) with $\omega_{pe}^2 = 4\pi e^2 \bar{n}/m^*$ (plasma frequency), $\bar{n}$ is the average density, and $\hat{n} = n/\bar{n}$ is the normalized density. The curl of (20)

$$\delta^2 \nabla\times\left(\hat{n}^{-1}\nabla\times\boldsymbol{\omega}_c\right) = \boldsymbol{\omega}_+ - \boldsymbol{\omega}_- \tag{21}$$

shows that the magnetic field is related to the difference in the normal vorticities of the two fluids.

We denote the generation drives as

$$\boldsymbol{G}_\pm = -\nabla\times\left[\frac{cT\nabla\sigma_\pm}{\gamma_\pm m^*}\right].$$

Assuming $\boldsymbol{V}_+ \approx \boldsymbol{V}_- \approx \overline{\boldsymbol{V}}$, and defining $\overline{\boldsymbol{\varpi}} = (\boldsymbol{\omega}_+ + \boldsymbol{\omega}_-)/2$, we may rewrite Eq. (18) as

$$\partial_t\overline{\boldsymbol{\varpi}} - \nabla\times\left(\overline{\boldsymbol{V}}\times\overline{\varpi}\right) = \frac{\boldsymbol{G}_+ + \boldsymbol{G}_-}{2}, \tag{22}$$

$$\partial_t\boldsymbol{\omega}_c - \nabla\times\left(\overline{\boldsymbol{V}}\times\omega_c\right) = \frac{\boldsymbol{G}_+ - \boldsymbol{G}_-}{2}. \tag{23}$$

Here we have approximated, using Eq. (21) and assuming a large scale ($\gg \delta$),

$$\boldsymbol{\omega}_c + (\boldsymbol{\omega}_+ - \boldsymbol{\omega}_-)/2 = \boldsymbol{\omega}_c + (\delta^2/2)\nabla\times(\hat{n}^{-1}\nabla\times\boldsymbol{\omega}_c) \approx \boldsymbol{\omega}_c.$$

B. Cosmological application

One of the primary motivations to look for an ideal drive was to investigate if such a drive could generate a magnetic field in early universe (when the plasma is in strict thermal equilibrium) that is strong enough to leave its mark in an expanding universe. We present here a possible scenario that could emerge in the light of the current relativistic drive. The scenario is intertwined with the thermal history of the universe. Although there are earlier hotter eras, let us begin our considerations around 100 MeV.

(i) 100 MeV (10^{12} K) age ($\sim 10^{-4}$ s): At this time the muon–antimuon are beginning to annihilate, and primary constituents of the universe are electron–positron pairs, neutrionos–antineutrinos and photons, all in thermal equilibrium, with a very small amount of nucleons (protons and neutrons).

(ii) 10 MeV age ($\sim 10^{-2}$ s): The main constituents are electron–positron pairs, neutrinos and photons, and a very small amount of nucleons (protons getting considerably more than neutrons). At this stage neutrinos are decoupled and are freely expanding. The electron–positron pairs and photons are coupled and in thermal equilibrium.

(iii) 0.5 MeV age (~ 4 s): The neutrinos are in free expansion. The electron–positron pairs are beginning to annihilate.

(iv) 0.1 MeV age (~ 180 s): Almost all pairs are gone, and what we have now is an electron–proton plasma contaminated by lots of neutrons and gammas (gammas are still electromagnetically coupled).

(v) This plasma continues for a long time, but it is barely relativistic (protons are not). Nucleosynthesis converts some protons and neutrons to He (about 25% of the mass). But we continue with an electron-ion mildly relativistic and then essentially nonrelativistic electrons till 4000 K when atomic hydrogen forms by the absorption of electrons in the protons. At this time the universe is about 400 000 yr old.

Notice that till there is plasma whether electron–positron or electron–proton (He), the radiation keeps the particles in thermal equilibrium, so there is no baroclinic term. Hence the only thing that could generate seed vorticity is the relativistic source. Now we could envisage the "magnetic field" generation in several stages:

(1) The universal "ideal" relativistic drive creates seed vorticity in the MeV era of the early universe when the electron–positron (e_-, e_+) plasma is the dominant component decoupled from neutrinos. In the context of this paper, this is the crucial element of the total scenario—the rest is cobbling together pieces of highly investigated phenomena.
(2) Below 1 MeV, as the (e_-, e_+) pairs begin to annihilate, the electron–proton plasma tends to be the dominant component. This era lasts for 400 000 yr till the temperature falls to 4000 K when the plasma disappears and the radiation decouples from matter. During this relatively long era, the seed field, created in the MeV era, is vastly magnified by what we call the early universe dynamo (EUD).
(3) At the hydrogen formation time, this magnetic field (however big it is) is decoupled from matter—there is no plasma left, and like the photons, the macroscopic magnetic field becomes a relic and red shifts (goes down in intensity) conserving flux. The relic field manifests in later eras appropriately diluted (conserving flux) by the cosmic expansion.
(4) This diluted field would, then, provide the seed for an intergalactic or a galactic dynamo.

In this paper we will simply estimate the level of the seed magnetic field that is fed into the EUD. Amplification during the EUD constitutes work under progress.

C. Order estimate of the seed magnetic field

Let us suppose that, in the era of seed generation, L is the scale factor of the universe, and ℓ denotes the scale length (during seed generation) of perturbations of $\gamma_\pm$ and T (or σ) that would grow into 1 Mpc of the current era. A qualitative estimate by Eq. (23) for the magnetic field at this scale length is

$$B \approx a\frac{\tau c^2}{\ell^2}\frac{mc}{2e}\gamma\left(\frac{V}{c}\right)^2\left(\frac{T}{mc^2}\right), \tag{24}$$

where a is the asymmetry factor (measuring the difference between $\boldsymbol{G}_+$ and $\boldsymbol{G}_-$), and τ is the period of time during which the magnetic field is generated (the relativistic drive works). In an expanding universe this field is diluted while conserving the total flux $\Phi = BL^2$ as far as the dumping mechanism do not diminish the flux.

Let us take the average energy (both thermal and kinetic) during the first second ($\tau \approx 1$ s) to be of order 10 MeV, implying $T_0/mc^2 \approx 10$ and $\gamma_0 \approx 10$ ($V_0/c \approx 1$). The relevant scale length ℓ is approximated to be $\ell = (3\times 10^{-4}/10^7)\times 1$ Mpc $\approx 10^{14}$ cm, because the scale factor L is inversely proportional to the temperature T. Thus the seed field that is being fed into the EUD may be estimated as $B \approx a \times 10^{-13}$ G. We expect (calculations underway) this seed field to be very strongly amplified, via the EUD, operating in a very high density cosmic electron–proton plasma of the early universe (lasting $\approx 400\,000$ yr). In order to create a seed field for the galactic dynamo of the order 10^{-16} G, the EUD must boost up the relativistic seed by a factor of $10^3/a$ since the expansion of the universe from 4000 K (end of EUD) to 2.7 K (present state) will dilute the field by a factor of 10^{-6}.

V. APPLICATION FOR RELATIVISTIC HOT-DENSE MATTER EXPERIMENT

Fortunately the functioning of the relativistic drive can be tested in the relativistic laser created plasmas in which electrons are highly relativistic but ions are not.

The drive $\mathcal{S}_r$, in order to leave measurable effects, must beat any damping in the system. In fact the damping due to resistive dissipation, being proportional to the magnetic field, can set an upper limit on the magnitude of the generated field.

A systematic and rigorous inclusion of dissipative processes in a relativistic system is nontrivial, and we will not attempt to do it here. We will, instead, attempt at a simple heuristic approach. The resistive dissipative term $\mathcal{D} = \nabla \times (\eta \boldsymbol{J}) = (c/4\pi)\nabla\times(\eta\nabla\times\boldsymbol{B})$, that is pertinent for the nonrelativistic evolution Eq. (3), will be assumed to be valid for Eq. (12). Notice that, in the initial stages when one is looking to create the seed field, the resistive dissipation term will be necessarily negligible because the vorticity is zero in the beginning. Being proportional to the generated vorticity, the resistive dissipation term will become progressively large as the fluid builds up vorticity. At some arbitrary stage in the development, a ratio of the strengths of the source $\mathcal{S}_r$ and the sink $\mathcal{D}$ may be written as

$$\frac{|\mathcal{S}_r|}{|\mathcal{D}|} \approx \frac{(\gamma^2V^2/c^2)(T/mc^2)}{(V_A/c)(\nu/\omega_p)}, \tag{25}$$

where we have introduced the Alfvén speed V_A, the collision frequency ν and the plasma frequency ω_p. Though this ratio could be expressed in many ways, we have chosen the rather transparent form Eq. (25).

Laser created plasmas come in a huge variety. Let us focus our attention on a typical system of interest: A 100 femtosecond (fs) laser at a wavelength of 1 μm focused on a spot measuring about 5 μm, with an intensity that corresponds to the normalized vector potential $\approx$10 (effective particle $\gamma \approx 10$). Critical density for this wavelength is $n_c \approx 10^{21}$ cm^{-3}, and typical plasma temperature is about 10 keV (experiments exist for both under dense and over dense systems).

Let us calculate the drive/dissipation ratio. For electrons, the two factors in the denominator are calculated to be small: $\nu/\omega_p = 10^{-9}n^{1/2}/T^{3/2} \approx 3\times 10^{-5}$, and $V_A/c \approx 10^{-8}B$ (in Gauss). Substituting all the factors in Eq. (24), we find

$$\frac{|\mathcal{S}_r|}{|\mathcal{D}|} \approx \frac{10^{13}}{B}. \tag{26}$$

The drive is way stronger than dissipation for any reasonable fields; the saturation must come from some other mechanism.

It is now time to estimate the magnetic fields that can be generated in the plasma that we just described. Taking the inverse gradients to be of order 5 μm and the generation times to equal the pulse length (100 fs), we estimate $B \approx 10^6$ G (i.e., the Mega gauss range). The magnetic field levels, of course, can rise for larger intensities that would have created even more highly relativistic electrons.

Finally a remark on the relativistic drive for mildly to vestigially relativistic plasmas is in order. Even for such cases, the drive can easily overcome damping. However, in this regime the baroclinic term (due to small departures from thermal equilibrium constraint) may be equally or even more important. With this caveat we will go ahead and evaluate the strength of the drive for a hypothetical electron fluid with $n = 10^{10}\text{cm}^{-3}$, $T = 20$ eV ($T/mc^2 = 4 \times 10^{-5}$), $V/c = 10^{-2}$, one can calculate $|\mathcal{S}_r|/|\mathcal{D}| \simeq B^{-1}$ (in Gauss). Thus the drive remains dominant till one reaches the magnetic fields of 1 G or so. Notice that the ratio is really independent of the density, and is immensely boosted up by higher temperatures. The "relativistic drive" has turned out to be strong even for plasmas that are quite nonrelativistic.

VI. CONCLUDING REMARKS

The relativistic drive $\mathcal{S}_r$ was derived from a Lorentz covariant theory of a hot charged ideal (perfect) fluid. The construction of such an underlying theory revealed that the form as well as the content of standard physical variables like the canonical momentum and vorticity also had to evolve to accommodate the additional physics added to the system. It is worthwhile here to comment on the finer points concerning vorticity, and its evolved counterparts the generalized vorticity (GV $= \Omega$), and the relativistic generalized vorticity (RGV $= \tilde{\boldsymbol{B}}$). As the system becomes more and more complicated (from an uncharged fluid to a charged fluid to a relativistic charged fluid), one was forced to successively invent more and more sophisticated physical variables so that the fundamental dynamical structure (vortical form), epitomized in Eq. (3) or Eq. (12), is maintained. Reducing the more complicated system to this form immediately advances our understanding of the new larger physical system because the very beautiful vortical structure has been so thoroughly studied. For example, we can immediately guess conservation laws that pertain to the larger system.[13]

Although in this paper we have concentrated on the generation of vorticity (whatever is relevant, vorticity, GV, or RGV), it is possible to carry out another profitable exercise. If the source terms are zero, then the appropriate helicity is conserved. One could, then, investigate, for example, the transfer of fluid kinetic-thermal helicity ($\int (f\gamma \boldsymbol{V}) \cdot \nabla \times (f\gamma \boldsymbol{V}) d^3x$) to the magnetic helicity ($\int \boldsymbol{A} \cdot \boldsymbol{B} d^3x$) or vice versa. When the two vorticities combine to form the GV, it is the helicity associated with GV that is conserved and not the individual helicities—the fluid or the magnetic.[15]

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[2] So called "Casimir invariants" represent the singularity (topological defect) of the Poisson bracket. Helicity is a typical example of Casimir invariants, which allows existence of vorticity in an ideal Hamiltonian mechanics; for example see V. Arnold and B. Khesin, *Topological Methods in Hydrodynamics* (Springer-Verlag, Berlin, 1998).

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[10] "Ideal mechanism" in this paper denotes a process in which the entropy σ is a function of temperature T so that the heat$= Td\sigma(T)$ becomes an exact differential. A "space-time distortion" that special-relativistic effects create, may, for instance covert an exact differential dF into a "Clebsch form" such as gdf that may have a vorticity $dg \wedge df \neq 0$. We note that the vortical field derived from a Clebsch form is helicity free $[gdf \cdot (dg \wedge df) = 0]$; creation of helicity does need the help of a flow that can produce another Clebsch component $g'df'$; cf. Z. Yoshida, J. Math. Phys. **50**, 113101 (2009).

[11] Let us denote $K = \Omega_{\text{id}} \cdot \boldsymbol{P}_{\text{id}}$. Then, $dC/dt = \int_\Upsilon [\partial_t K + \nabla \cdot (K\boldsymbol{V})] d^3x$. Using Eq. (4) and the boundary condition $\Omega_{\text{id}} \cdot \boldsymbol{n} \equiv 0$ (by the flux conservation, this boundary condition holds constantly on the surface $\partial\Upsilon$ that moves with the fluid), we observe $\int_\Upsilon \partial_t K d^3x = -\int_\Upsilon \nabla \cdot (K\boldsymbol{V}) d^3x$. Hence, $dC/dt = 0$.

[12] The helicity may be generalized to produce, for example, Gauss's linking number of the "vortex line", the curve tangential to Ω; for example, see H. K. Moffatt, *Magnetic field generation in electrically conducting fluids* (Cambridge University Press, Cambridge, 1978).

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[15] The notion of "helicity" parallels the total "charge" $C = \int K^0 d^3x$ of a four-vector; if K^μ is conserved, i.e., $\partial_\mu K^\mu = 0$, we obtain the charge (helicity) conservation $dC/dt = 0$. Defining the dual $\mathcal{M}^{\nu\mu} = (1/2)\phi^{\nu\mu\alpha\beta} M_{\alpha\beta}$, we set $\mathrm{K}^\mu = P_\nu \mathcal{M}^{\nu\mu}$. Then, using $\partial_\mu \mathcal{M}^{\nu\mu} = 0$, the antisymmetry of $\mathcal{M}^{\nu\mu}$, and $M^{\nu\mu}\mathcal{M}_{\nu\mu} = -4\tilde{\boldsymbol{E}} \cdot \tilde{\boldsymbol{B}}$, we observe $\partial_\mu K^\mu = \partial_\mu (P_\nu \mathcal{M}^{\nu\mu}) = (1/2)\big(\partial_\mu P_\nu - \partial_\nu P_\mu\big)\mathcal{M}^{\nu\mu} = (1/2) M_{\nu\mu}\mathcal{M}^{\nu\mu} = -2\tilde{\boldsymbol{E}} \cdot \tilde{\boldsymbol{B}}$. We note that all these relations parallels the well-known facts of EM field theory with replacing A^μ by P^μ and $F^{\nu\mu}$ by $M^{\nu\mu}$; see Ref. 13.

[16] The displacement-current term $\partial_t \boldsymbol{E}/c$ in Ampere's law will appear as $-\omega_{pe}^{-2}\partial_t^2 \boldsymbol{\omega}_c$ on the left-hand side of Eq. (21). In the time scale $\tau \gg \ell/c$ (ℓ is the length scale of the structures), this term may be neglected with respect to $\delta^2 \nabla \times (\hat{n}^{-1} \nabla \times \boldsymbol{\omega}_c)$, and the D'Alembert operator collapses to the elliptic operator, eliminating the EM waves. The displacement-current term may not be neglected when one compares the divergence of Ampere's law with the mass conservation law. But this is not pertinent to the present calculations.