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Nonlinear travelling waves in energetic particle phase space

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Abstract
An exact nonlinear solution is found for long-time behaviour of spontaneously formed phase space clumps/holes in dissipative plasmas with a population of energetic particles. This solution represents a Bernstein–Greene–Kruskal mode with slowly varying shape and velocity. It describes a continuous transformation of a plasma eigenmode excited just above the instability threshold into an energetic particle mode with a significantly different frequency. An electrostatic bump-on-tail instability is chosen to illustrate the analysis. However, generality of the resonant particle dynamics makes the described approach applicable to other resonance-dominated instabilities, including rapid frequency-sweeping events for Alfvénic modes in tokamaks.

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1. Introduction

The near-threshold regimes of wave excitation by energetic particles reveal a rich family of nonlinear scenarios ranging from benign mode saturation to spontaneous formation of nonlinear coherent structures (phase space holes and clumps) with time-dependent frequencies [1–3]. In previous work, the build-up of such structures has been demonstrated, but their quantitative description was limited to the case of small frequency deviations from the bulk plasma eigenfrequency [4, 5]. However, there are multiple experimental observations of frequency-sweeping events in which the change in frequency is comparable to the frequency itself [6–9]. The need to interpret such dramatic phenomena requires a non-perturbative theoretical formalism, which is the subject of this paper. The underlying idea is that coherent structures with varying frequencies represent nonlinear travelling waves in fast-particle phase space. Given that the energetic particle density is usually much smaller than the bulk plasma density, one might expect that these particles should not be able to change the eigenmode frequency significantly. From this viewpoint, the strong chirping events look mysterious. The way to resolve this difficulty is to take into account that, regardless of how small is the energetic particle density, a coherent group of these particles can still produce an observable signal with a frequency different from the bulk plasma eigenfrequency. A relevant example is a modulated beam in the plasma. The modulation occurs spontaneously as a result of the initial instability and resonant particle trapping by the excited wave. The initial modulation should then match the frequency of a plasma eigenmode. However, as the coherent structure evolves due to dissipation, the trapped particles slow down without losing coherency, and the resulting frequency shifts considerably from the initial frequency. The corresponding theoretical building block is then a nonlinear Bernstein–Greene–Kruskal (BGK) mode [10], rather than a slowly evolving plasma eigenmode. A rigorous solution of this type is given in this paper for a simple one-dimensional electrostatic bump-on-tail model. This model captures the essential features of trapped resonant particles in more general multidimensional problems, because particle motion is known to be effectively one dimensional in the vicinity of an isolated nonlinear resonance, once expressed in proper action-angle variables [11]. The presented solution, which is based on adiabatic description of the trapped particles, suggests an efficient approach to quantitative modelling of actual experiments. An analytic nonlinear treatment of resonant particles can be combined with a linearized description of the bulk plasma to simplify numerical calculation of the wave fields in realistic geometry.

2. Bump-on-tail problem

As shown in [4, 5], initiation of phase space holes and clumps in the near-threshold regime occurs when collisional relaxation of the resonant particles is negligible. The holes and clumps develop explosively on a time scale that is comparable to the inverse bounce frequency ω_b^{-1} of a resonant particle in the wave field. However, once established, these coherent structures evolve on significantly longer time scales. Namely,

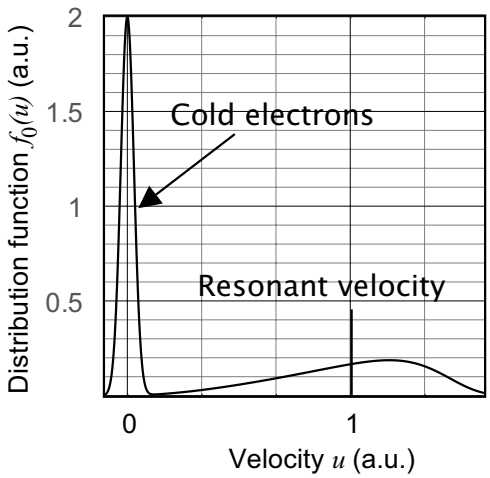

Figure 1. Unstable initial bump-on-tail distribution function.

the rate of their frequency sweeping satisfies the condition $\mathrm{d}\omega/\mathrm{d}t \ll \omega_\mathrm{b}^2$, which enables adiabatic description of resonant particles. The slow evolution of the structures is predominantly due to energy dissipation in the bulk plasma. In the absence of dissipation, the structures would remain stationary. The weakness of dissipation allows a two-step theoretical treatment of the problem. The first step is to construct a single-parameter set of nonlinear BGK modes that carry resonant particles. All modes in this set will have the same spatial period λ (determined by initial conditions) but different phase velocities. In what follows, the phase velocity (rather than frequency) will serve as the mode identification parameter. Because the modes of interest are nonlinear, their Fourier spectrum generally contains multiple frequencies. In contrast, the mode phase velocity is unambiguous. An equivalently convenient frequency parameter would be the frequency of the dominant spectral component. The second step is to use the energy balance condition to determine the time evolution of the mode phase velocity.

To proceed with formalism, we consider a spontaneously generated electron clump in a simple one-dimensional electrostatic bump-on-tail problem. The unperturbed electron population consists of a cold core and a low-density high-energy tail with a broad velocity distribution $f_0(u)$ (shown schematically in figure 1). The tail provides an instability drive γ_L via its positive slope $\mathrm{d}f_0(u)/\mathrm{d}u$. The tail electrons are collisionless whereas the core is slightly dissipative due to collisions with ions. The ion background is assumed to be uniform and immobile. The core electron collision frequency ν provides a linear damping rate $\gamma_\mathrm{d} = \nu/2$ that determines the instability threshold (the minimum slope of $f_0(u)$ needed to excite a mode).

Similar to previous analysis [1–3], we consider a near-threshold case in which only one wavelength (λ) of the perturbed electric field is initially unstable. The initial perturbation is a sinusoidal plasma wave, and its phase velocity is $\lambda\omega_\mathrm{p}/2\pi$, where ω_p is the cold electron plasma frequency. The instability growth rate is assumed to be much smaller than ω_p, so that the wave evolves slowly on the ω_p time scale. As the wave becomes nonlinear, its amplitude, shape and phase velocity change, but the spatial periodicity of the wave is still preserved, with the period being equal to λ. Consequently, we seek the wave electrostatic potential φ in the form

$$\varphi \equiv -\frac{1}{|e|} U\left[x - s(t); t\right], \tag{1}$$

where e is the electron charge, and the electron potential energy U is a periodic function of its first argument $[x - s(t)]$ and a slowly varying function of the second argument t. Also, the wave phase velocity $\dot{s} \equiv \mathrm{d}s(t)/\mathrm{d}t$ is a slowly varying function of time with a sweeping rate $\ddot{s}$. In what follows, we imply that the wave profile $U[x - s(t); t]$ has zero spatial average, which is just a matter of convenience without any additional constraint.

The cold electron response to the perturbed potential is governed by the linear fluid equations:

$$\frac{\partial V}{\partial t} = -\frac{1}{m}\frac{\partial U}{\partial x} - \nu V, \tag{2}$$

$$\frac{\partial \delta n}{\partial t} = -n_0 \frac{\partial V}{\partial x}, \tag{3}$$

where V is the oscillatory flow velocity of the cold electrons and n_0 and δn are the unperturbed and perturbed densities of the cold electrons, respectively.

To lowest order (neglecting ν and $\ddot{s}$), these equations give the following expressions for the perturbed density and velocity:

$$\delta n = n_0 \frac{U}{m\dot{s}^2}, \tag{4}$$

$$V = \frac{U}{m\dot{s}}. \tag{5}$$

The power Q dissipated via weak collisions is due to the work of the friction force, which gives

$$Q = \nu m n_0 \int_0^\lambda V^2 \mathrm{d}x = \nu n_0 \lambda \left\langle U^2 \right\rangle / \left(m\dot{s}^2\right), \tag{6}$$

where angular brackets denote averaging over the wavelength.

Next, we calculate the perturbed density of fast electrons. Their motion is governed by a time-dependent Hamiltonian,

$$H = \frac{p^2}{2m} + U\left[x - s(t); t\right], \tag{7}$$

where the first argument of U describes rapid oscillations with a time scale on the order of ω_p^{-1}. This fast time scale can be eliminated from the Hamiltonian via canonical transformation to new coordinate $z \equiv x - s$ with the same momentum p. The new (wave-frame) Hamiltonian H_w contains only slow time dependence,

$$H_\mathrm{w} = \frac{(p - m\dot{s})^2}{2m} + U\left[z; t\right]. \tag{8}$$

This Hamiltonian preserves adiabatic invariants for co-passing (+), counter-passing (−) and trapped particles:

$$J_\pm = \int_0^\lambda \left(m\dot{s} \pm \sqrt{2m\left(H_\mathrm{w} - U\right)}\right) \mathrm{d}z, \tag{9}$$

$$J_\mathrm{trapped} = \oint \sqrt{2m\left(H_\mathrm{w} - U\right)}\, \mathrm{d}z. \tag{10}$$

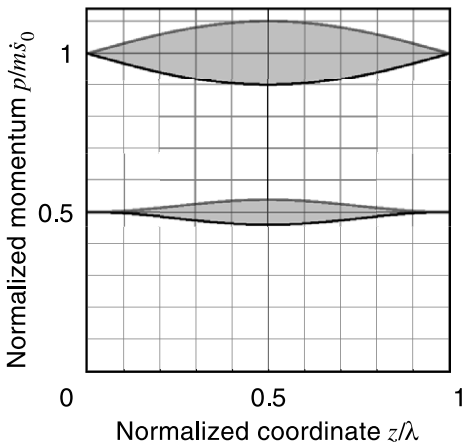

Figure 2. Evolution of the phase space bucket during sweeping event. The plot shows the initial separatrix (upper area) and the shrinked separatrix at half of the initial mode phase velocity (lower area).

The boundary (separatrix) between the passing and trapped particles is given by the condition $H_w = U_{max}$. The separatrix moves in phase space $(p; z)$ due to the change in the wave phase velocity $\dot{s}$, as shown in figure 2.

It should be pointed out that all particles inside the separatrix have nearly the same value of the distribution function f, because the separatrix is assumed to be narrow compared with the characteristic width of the unperturbed distribution function f_0. Conservation of the trapped particle adiabatic invariant means that the phase space density inside the separatrix is preserved during the motion. It is therefore allowable to set

$$f_{\text{trapped}} = f_0(\dot{s}_0), \tag{11}$$

where $\dot{s}_0$ is the initial phase velocity of the wave. As the separatrix travels in phase space, it enters the area where the unperturbed ambient phase space density $f_0(\dot{s})$ is different from $f_0(\dot{s}_0)$. As a result, the distribution function develops a discontinuity at the separatrix: the trapped particles form a phase space clump if $f_0(\dot{s}_0)$ is greater than $f_0(\dot{s})$ or a phase space hole if $f_0(\dot{s}_0)$ is smaller than $f_0(\dot{s})$. Conservation of the adiabatic invariant and phase space volume prevents passing particles from becoming trapped unless the potential well changes its shape.

For this reason, a downward moving separatrix ($\ddot{s} < 0$) simply converts counter-passing particles into co-passing particles. The dominant perturbation of the fast electron density comes from the narrow protrusion or depletion inside the separatrix, associated with the difference between $f_0(\dot{s}_0)$ and $f_0(\dot{s})$, which gives

$$\delta n_b = [f_0(\dot{s}_0) - f_0(\dot{s})]\, 2\sqrt{2/m} \times \left\{\sqrt{[U_{max} - U(z)]} - \left\langle\sqrt{[U_{max} - U(z)]}\right\rangle\right\}, \tag{12}$$

where $2\sqrt{2/m}\sqrt{[U_{max} - U(z)]}$ is the velocity interval occupied by the trapped particles at a given location z. The term $\langle\sqrt{[U_{max} - U(z)]}\rangle$ in this expression accounts for the neutralizing contribution of the ion background, which ensures that $\langle\delta n_b\rangle = 0$.

It is important to point out that equation (12) is applicable not just to the case of fixed wave profile U but also to the case of slowly evolving U, provided that the time evolution of U does not lead to trapping of ambient particles into the potential well. Contrary to trapping, a leak from the well is permissible. The reason why leaking is easier than trapping to account for is that the trapped particle distribution function remains flat in a leaking well, whereas any trapping of ambient particles would not allow this distribution to remain flat, because of the discontinuity at the separatrix. However, for equation (12) to remain valid, the leak has to be sufficiently slow, so that the escaped particles do not change the ambient distribution significantly. The leaked particles form a wake behind the moving separatrix. If the velocity width of the separatrix decreases by δu during a time interval δt, and the separatrix shifts down by $\delta\dot{s}$ during the same time interval, then the relative correction to the ambient distribution can be roughly estimated as $\delta f/f_0 = \delta u/\delta\dot{s}$. For the solution that will be constructed, the ratio of $\delta u/\delta\dot{s}$ is on the order of $\gamma_L/\omega \ll 1$, where γ_L (defined by equation (22) below) is the fast electron contribution to the mode growth rate. This ordering can be verified *a posteriori* by using the explicit form of the solution.

We now substitute expressions (5) and (12) into the Poisson equation to obtain a nonlinear wave equation for the BGK mode

$$\frac{\partial^2 U}{\partial z^2} = -U\omega_p^2/\dot{s}^2 - A(\dot{s}) \times \left[\sqrt{(U_{max} - U)} - \left\langle\sqrt{(U_{max} - U)}\right\rangle\right], \tag{13}$$

$$A(\dot{s}) \equiv 8\pi e^2 [f_0(\dot{s}_0) - f_0(\dot{s})]\sqrt{2/m}.$$

This equation has a first integral:

$$\frac{1}{2}\left(\frac{\partial U}{\partial z}\right)^2 + \frac{\omega_p^2}{\dot{s}^2}\frac{(U^2 - U_{max}^2)}{2} - \frac{2}{3}A(U_{max} - U)^{3/2} - A(U - U_{max})\left\langle\sqrt{U_{max} - U}\right\rangle = 0. \tag{14}$$

The integration constant here is chosen to satisfy the condition that $\partial U/\partial z = 0$ when $U = U_{max}$.

Introduction of a new unknown function

$$g \equiv \sqrt{U_{max} - U} \tag{15}$$

transforms equation (14) to

$$\frac{2\dot{s}^2}{\omega_p^2}\left(\frac{\partial g}{\partial z}\right)^2 + \frac{\left(g - 2A\dot{s}^2/3\omega_p^2\right)^2}{2} - \frac{\left(A\dot{s}^2/3\omega_p^2\right)^2}{2} + \frac{A\dot{s}^2}{\omega_p^2}\langle g\rangle - U_{max} = 0. \tag{16}$$

As seen from its definition, the function g must vanish at the points where $U = U_{max}$. The translational invariance of equation (16) allows us to choose one of these points as $z = 0$. Because of the spatial periodicity of the wave, g must also vanish at $z = \lambda$. Next, we differentiate equation (16) with respect to z and solve the resulting linear differential equations with the boundary conditions $g(0) = g(\lambda) = 0$ to find

$$g = \left[1 - \cos\left(\omega_p z/2\dot{s}\right) - \tan\left(\omega_p\lambda/4\dot{s}\right)\sin\left(\omega_p z/2\dot{s}\right)\right] \times 2A\dot{s}^2/3\omega_p^2. \tag{17}$$

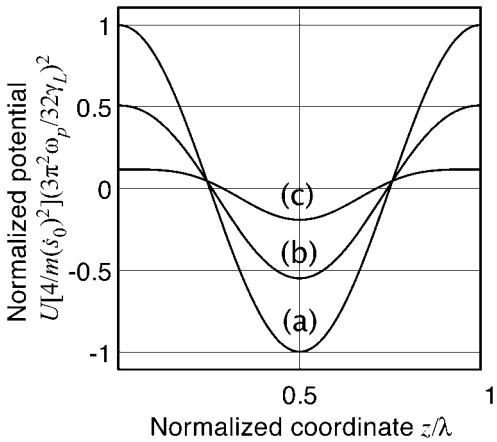

Figure 3. Evolution of the wave profile during frequency sweeping. The plots illustrate equation (20) for three values of the mode phase velocity ((a)—$\dot{s} = \dot{s}_0$; (b)—$\dot{s} = 0.8\dot{s}_0$; (c)—$\dot{s} = \dot{s}_0/2$). The unperturbed distribution of fast electrons $f_0(u)$ is assumed to have a constant slope within the sweeping range.

Substitution of this expression into equation (16) yields

$$U_{\max} = \left(2A/3k^2\cos\alpha\right)^2 \times \frac{1}{2}\left\{\sin^2\alpha + 3\cos^2\alpha\left[1 - \frac{\sin\alpha}{\alpha\cos\alpha}\right]\right\}, \tag{18}$$

where

$$A(k) \equiv 8\pi e^2\left[f_0(\dot{s}_0) - f_0(\dot{s})\right]\sqrt{2/m} \equiv 2k^2\dot{s}^2\frac{1}{n_0}\left[f_0(\dot{s}_0) - f_0(\dot{s})\right]\sqrt{2m}, \tag{19}$$

$\alpha \equiv k\lambda/4 = \omega_p\lambda/4\dot{s}$.

We finally use equations (15), (17), (18) to obtain

$$U = \frac{m\dot{s}}{2}\left\{\frac{8\dot{s}\left[f_0(\dot{s}_0) - f_0(\dot{s})\right]}{3n_0\cos\alpha}\right\}^2\left\{\frac{1+2\cos^2\alpha}{2} - \frac{3\sin 2\alpha}{4\alpha} - \left[\cos\alpha - \cos\left(\alpha\frac{2z}{\lambda} - \alpha\right)\right]^2\right\}. \tag{20}$$

It is noteworthy that this expression applies both to clumps and holes; the distinction between the two is in the sign of $f_0(\dot{s}_0) - f_0(\dot{s})$, which does not affect equation (20).

For small deviations of $\dot{s}$ from $\dot{s}_0$ (early phase of frequency sweeping), equation (20) simplifies to

$$U = \frac{m\dot{s}_0^2}{4}\left\{\frac{32}{3\pi^2}\frac{\gamma_L}{\omega_p}\right\}^2\cos\left(\frac{2\pi z}{\lambda}\right), \tag{21}$$

where

$$\gamma_L \equiv \omega_p\frac{\pi}{2n_0}\dot{s}_0^2\frac{\partial f_0(\dot{s}_0)}{\partial\dot{s}_0} \tag{22}$$

is the fast electron contribution to the mode growth rate.

Equation (21) reproduces the result of [4], i.e. a sinusoidal mode with constant amplitude at the beginning of frequency sweeping. On the other hand, the more general expression (20) shows that the amplitude and the mode structure change significantly for larger variations of $\dot{s}$. These changes are illustrated in figure 3. To conclude this section, we note that the presented derivation involves a tacit assumption that the wave profile U has no additional maxima within the wavelength. A straightforward analysis of equation (20) shows that this assumption is justified as long as $\dot{s}$ remains greater than $\dot{s}_0/2$, which defines the range of sweeping tractable by equation (20).

3. Power balance and sweeping rate

Based on the mode structure given by equation (20), we now calculate how much power is released by the phase space clump when the clump velocity $\dot{s}$ decreases in time. This calculation relies on the knowledge that the distribution function of passing particles is smooth around the clump. One can therefore treat the ambient distribution as uniform and consider a small displacement of the separatrix (phase space bucket) in the sea of passing particles with an unperturbed phase space density $f_0(\dot{s})$. It is important to keep in mind that a completely uniform distribution (including the interior of the separatrix) would not be affected by any variation of the electrostatic potential. Consequently, the power release is only due to the difference between $f_0(\dot{s}_0)$ and $f_0(\dot{s})$, the values of the distribution function inside and outside the separatrix. This difference represents a narrow flat-top peak within the separatrix. The height of the peak is $f_0(\dot{s}_0) - f_0(\dot{s})$ and the area involved is the phase space area within the separatrix. For one wavelength, λ, this area is equal to the value of J_{trapped} at the separatrix (see equation (10)). The number of particles N in the peak is then $N = (1/m)[f_0(\dot{s}_0) - f_0(\dot{s})]\oint\sqrt{2m(U_{\max} - U)}\mathrm{d}z$ and the kinetic energy of each particle is $m\dot{s}^2/2$, except for an insignificant small correction associated with the finite size of the separatrix. Let $\delta\dot{s}$ be a small reduction in the wave phase velocity. As the peak shifts together with the separatrix, the corresponding energy release is

$$\delta E = \dot{s}\delta\dot{s}\left[f_0(\dot{s}_0) - f_0(\dot{s})\right]\oint\sqrt{2m\left(U_{\max} - U\right)}\mathrm{d}z, \tag{23}$$

which, together with equations (18), (19) and (20), gives the following expression for the power release:

$$P = -\left[f_0(\dot{s}_0) - f_0(\dot{s})\right]2m\dot{s}^2\lambda \times\left|\frac{8\dot{s}\left[f_0(\dot{s}_0) - f_0(\dot{s})\right]}{3n_0\cos\alpha}\right|\left[\frac{\sin\alpha}{\alpha} - \cos\alpha\right]\frac{\mathrm{d}\dot{s}}{\mathrm{d}t}. \tag{24}$$

Equation (24) applies directly to the case of externally controlled sweeping rate, which complements the analysis performed in [12, 13]. On the other hand, this equation, together with the power balance condition, governs self-sustained sweeping in dissipative plasmas.

To balance collisonal dissipation of the BGK mode, the released power must compensate the power Q absorbed by the bulk electrons (see equation (6)). This condition determines the slowing-down rate of the phase space clump, namely

$$\frac{\mathrm{d}\dot{s}}{\mathrm{d}t} = -\frac{\nu\dot{s}}{3\left|\cos\alpha\right|}\frac{\alpha}{\left[\sin\alpha - \alpha\cos\alpha\right]}\left|\frac{4\dot{s}\left[f_0(\dot{s}_0) - f_0(\dot{s})\right]}{3n_0\cos\alpha}\right|^2 \times\left\langle\left[\cos\left(\frac{4\alpha z}{\lambda} - 2\alpha\right) + \frac{3\sin 2\alpha}{2\alpha} - 4\cos\alpha\cos\left(\frac{2\alpha z}{\lambda} - \alpha\right)\right]^2\right\rangle. \tag{25}$$

Same as in equation (6), the angular brackets here denote averaging over the wavelength λ. More explicitly,

$$\left\langle\left[\cos\left(\frac{4\alpha z}{\lambda} - 2\alpha\right) + \frac{3\sin 2\alpha}{2\alpha} - 4\cos\alpha\cos\left(\frac{2\alpha z}{\lambda} - \alpha\right)\right]^2\right\rangle = \frac{1}{2} + \frac{11\sin 4\alpha}{24\alpha} + 8\cos^2\alpha - \left(\frac{3\sin 2\alpha}{2\alpha}\right)^2 - \frac{2\sin 2\alpha}{3\alpha}. \tag{26}$$

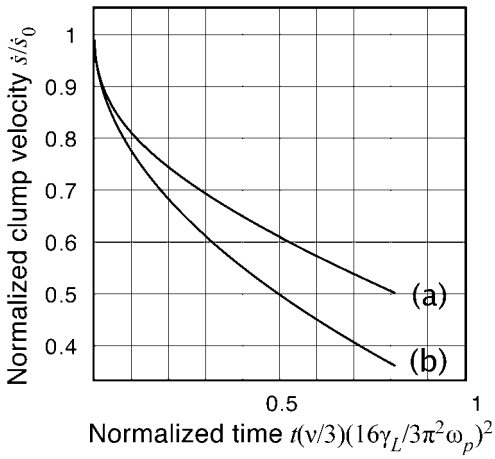

Figure 4. Evolution of the clump velocity for a constant-slope unperturbed distribution function of the fast electrons $f_0(u)$. The upper curve (a) shows the solution of equation (25). The lower curve (b) shows the square root dependence (equation (27)) obtained in [4] for the initial phase of frequency sweeping.

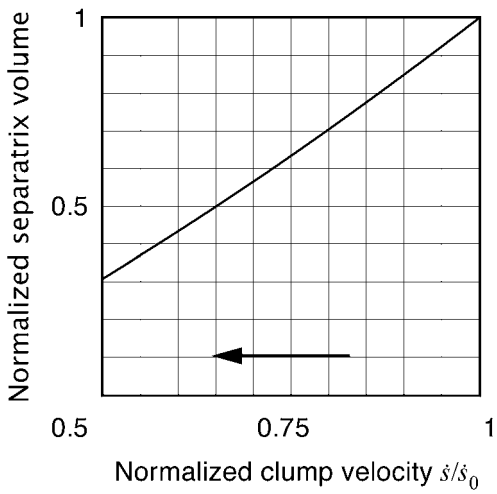

Figure 5. Decay of the phase space clump due to particle leak through the separatrix (decrease in the separatrix volume during sweeping event). The solid line shows the value of $J_{\text{trapped}}(\dot{s})/J_{\text{trapped}}(\dot{s}_0)$ at the separatrix. The arrow in the plot indicates that the clump slows down.

Early in time, equation (25) simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{(\dot{s}-\dot{s}_0)^2}{\dot{s}_0^2} = \frac{\nu}{3}\left|\frac{16\gamma_{\mathrm{L}}}{3\pi^2\omega_{\mathrm{p}}}\right|^2 \tag{27}$$

and reproduces the square root scaling of frequency sweeping found in [4]. Later in time, the mode moves further away from the initial linear eigenmode, and its phase velocity $\dot{s}$ deviates gradually from the simple square root scaling, as shown in figure 4.

The phase space area inside the separatrix shrinks during this process (figure 5), and some of the initially trapped particles leave the clump as a result. This nonlinear evolution can be viewed as spontaneous transformation of the initial plasma wave into an energetic particle mode. It also presents a plausible scenario for energetic particle modes generated by Alfvén wave instabilities [14–16], for which nonlinear modification of the mode structure appears to be essential, especially when the instability is non-perturbative even in the linear regime.

4. Generalization

The presented consideration of nonlinear frequency sweeping in the 1D electrostatic bump-on-tail problem suggests a similar approach to the frequency-sweeping events in tokamaks. Experimentally, such events have been observed, for example, in [6, 9]. They occur in the shear Alfvén frequency range, and their early stage can be attributed to the excitation of toroidal Alfvén eigenmodes. However, the measured frequency quickly moves away from the original eigenmode frequency, and a plausible underlying mechanism is spontaneous formation of coherent phase space structures at the wave–particle resonances. This scenario implies that the resonances are well separated in phase space, so that the energetic particle response can be treated as a sum over several independent resonances. For a linear mode, the resonance condition has the form

$$\omega - n\omega_\varphi\left(P_\varphi; P_\theta; P_\psi\right) - l\omega_\theta\left(P_\varphi; P_\theta; P_\psi\right) = 0, \tag{28}$$

where ω is the mode frequency, $\omega_\varphi(P_\varphi; P_\theta; P_\psi)$ and $\omega_\theta(P_\varphi; P_\theta; P_\psi)$ are the toroidal and poloidal transit frequencies and n and l are integers. The pairs $(P_\varphi; \varphi)$, $(P_\theta; \theta)$ and $(P_\psi; \psi)$ are the canonical action-angle variables for the integrable unperturbed motion. The third pair $(P_\psi; \psi)$ describes fast gyro-motion that does not resonate with shear Alfvenic perturbations. As a result, the resonance condition involves only two frequencies. For an isolated linear resonance, the perturbed particle Hamiltonian is a sinusoidal function of $\omega t - n\varphi - l\theta$. Similarly to the bump-on-tail problem, transition to the nonlinear case generalizes the Hamiltonian to

$$H = H_0 + U\left(\int_0^t \omega(\tau)\,\mathrm{d}\tau - n\varphi - l\theta; t\right), \tag{29}$$

where the function U (to be determined numerically) is still periodic (but not necessarily sinusoidal) function of its first argument. The function U represents a projection of the perturbed electromagnetic field onto the wave–particle resonance, which is the dominant part of the total perturbed Hamiltonian. The projection operator establishes a linear functional relation between this unknown function and the perturbed fields. We now note that the quantities P_ψ and $P = lP_\varphi - nP_\varphi$ are constants of motion for such Hamiltonian and that slow evolution of the function U should also preserve an adiabatic invariant for trapped particles. These three conservation laws establish a simple relationship between the flat-top trapped particle distributions at any two locations of the resonance (see figure 6).

The distribution of the ambient passing particles remains virtually unperturbed. We thereby eliminate the need to solve the kinetic equation for energetic particles numerically. Any macroscopic quantity, such as perturbed energetic particle pressure, now becomes a known functional of the unperturbed distribution and the 'potential energy profile' U. What remains to be solved is a set of linear MHD equations for bulk

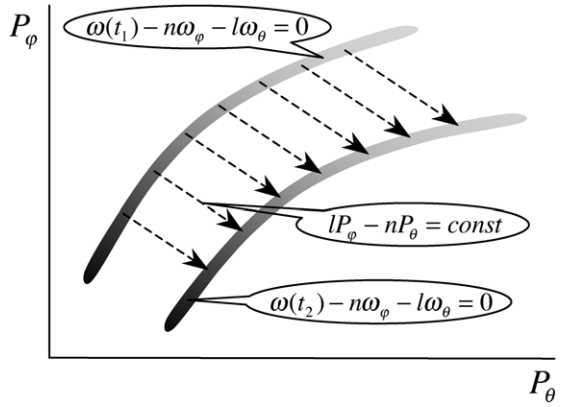

Figure 6. Transport of resonant particles during frequency sweeping. The shaded areas are snapshots of the moving resonant region in the momentum space. The shades of gray mark different values of the particle distribution function. The trapped resonant particles form a locally flat distribution across the resonance and preserve the value of their distribution function when the resonance carries them along the dashed lines.

plasma response with an analytic nonlinear input from the energetic particles. These equations represent an analogue of equation (13), and they need to be solved numerically to determine the wave profile U. After that, the power balance condition should be used to calculate the frequency-sweeping rate. An effort is currently underway to develop a corresponding numerical procedure based on the AEGIS code [17] that provides the required linear description of the bulk plasma in the Alfvénic range.

5. Concluding remarks

The presented new solution for a nonlinear travelling wave with time-dependent phase velocity shows how a weakly driven eigenmode of the bulk plasma evolves continuously into a nonlinear energetic particle mode. It is noteworthy that the spatial structure of the wave changes together with the phase velocity. This idealized example highlights the essential physics of the commonly observed frequency-sweeping events in real systems. The main simplifying element in the analysis is the adiabatic description of the trapped particles, which proves to be relevant to the cases of interest. The problem is particularly simple when the evolving wave does not trap new particles from the ambient phase space, so that the initially flat distribution of the trapped particles remains flat in the process. This auspicious regime is tractable analytically. The more complicated case of possible trapping into a deepening potential well seems to require a numerical procedure to keep track of the particle flux through the separatrix. Yet, such procedure can still benefit a great deal from the adiabatic approximation and Liouville theorem, which eliminates the need to follow the particle dynamics on the fast bounce frequency time scale. The only part of the problem that needs full-scale modelling is the initiation of holes and clumps, because the characteristic time of this process is comparable to the bounce period. Previous simulations show that several holes and clumps can emerge simultaneously from an initially unstable wave. However, these structures appear to be reasonably well separated to make the presented consideration of an isolated clump meaningful. An interesting nonlinear problem of interactions between neighbouring holes and clumps deserves a separate study. Another relevant next step is to analyse the effect of resonant particle collisions on the dynamics and lifetime of holes and clumps. Finally, specific features of the fast-particle resonances with Alfvénic modes need to be accounted for in a quantitative way in order to extend the theoretical model to diagnostic applications.

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