"Maximum" entropy production in self-organized plasma boundary layer: A thermodynamic discussion about turbulent heat transport

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A thermodynamic model of a plasma boundary layer, characterized by enhanced temperature contrasts and "maximum entropy production," is proposed. The system shows bifurcation if the heat flux entering through the inner boundary exceeds a critical value. The state with a larger temperature contrast (larger entropy production) sustains a self-organized flow. An inverse cascade of energy is proposed as the underlying physical mechanism for the realization of such a heat engine.
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I. INTRODUCTION

Two seemingly opposite "principles" have been formulated to accord "entropy" a controlling role even in the nonequilibrium thermodynamics of macroscopic systems. The first, the principle of *minimum entropy production*, has proven to be a highly successful ansatz, powerful enough to predict a variety of self-organized structures in nonlinear systems.[1] The second, the principle of *maximum entropy production*, first proposed by Paltridge,[2] also seems to work rather well for some fluid systems that may maximize the entropy production by simultaneously enhancing temperature inhomogeneity. Paltridge and co-workers[3–5] invoked this concept to explain the heat transport between warm tropics and cool high latitudes of the Earth; other planets also appear to prefer maximizing the entropy production.[6] Ozawa *et al.*[7] pointed out that the maximum entropy production may be a general consequence of fluid-mechanical instabilities (such as Bénard convection or Kelvin–Helmholtz instabilities) that can work as a heat engine. Dewar[8] developed a statistical-mechanical model of nonequilibrium flux-driven systems, where the maximum entropy production is related to the most probable "paths" of transitions.

It is quite obvious that something so counterintuitive as the principle of maximum entropy production, must be anchored in processes that must necessarily lead to quasiequilibrium states which maximally depart from thermal equilibrium. States that maximize temperature gradients seem tailor made for such processes. Since the standard fluid mechanical instabilities seem to be able to create such states, one wonders if there is a greater generality to the phenomenon: Would, for example, fluidlike instabilities in electromagnetic systems also conspire to create states with sharp gradients? And if yes, will it be reasonable to try to understand such states as maximizing entropy-production?

In this paper we attempt to bring into this general fold one of the most spectacular expressions of self-organization manifested in the high-confinement, high pressure gradient tokamak discharges, the H-mode, or discharges with an internal transport barrier (ITB).[9] It is tempting to enquire if the gross features of phenomena of this genre could be "predicted" and explained by the principle of *maximum entropy production*. We carry out such an enquiry by studying the thermodynamics of a simple generic model of a fluid boundary layer (region with large gradients) in which a specified heat flux enters form the left while the right boundary is kept at a fixed temperature by a heat bath. We will show that the system exhibits bifurcation; two distinct stable states of temperature distribution are possible. The total heat flux is the controlling parameter; when it is greater than a critical value, the system favors the state with a larger temperature contrast. We will also proffer a set of feasibility arguments; in particular, we will discuss how such purely thermodynamic considerations (devoid of electromagnetism) could be relevant to H-modes and ITBs formed in tokamak plasmas.

II. THERMODYNAMIC RELATIONS

We start with the first law of thermodynamics

$$dU = \delta Q - \delta W, \tag{1}$$

relating the change in internal energy dU with δQ, the heat absorbed and δW, the work done by the system. We will distinguish between the changes in state variables and other general variables; the former (latter) will be denoted by dX (δY). In this notation, the second law is written as $\delta Q = T(dS - \delta S_i)$ with T and S as temperature and entropy, respectively. The quantity $\delta S_i (\geq 0)$ denotes internal entropy production. Introducing a positive constant T_{ref} measuring a reference temperature, we may rewrite Eq. (1) as

$$\begin{aligned}\delta W &= \delta Q - T_{\text{ref}} dS - (dU - T_{\text{ref}} dS) \\ &= \left(1 - \frac{T_{\text{ref}}}{T}\right)\delta Q - T_{\text{ref}}\delta S_i - (dU - T_{\text{ref}} dS).\end{aligned} \tag{2}$$

The first term on the right-hand side of Eq. (2) gives the maximum work achievable in a reversible process (Carnot's theorem). The second term $(-T_{\text{ref}}\delta S_i \leq 0)$, proportional to the internal entropy production, diminishes δW in an irreversible process. The third term $(dU - T_{\text{ref}} dS)$, consisting of exact forms, does not contribute to the integral over any closed "cycle."

For an open fluid (plasma) system, one could define the thermodynamic variables (δW, δQ, U, S, etc.) for each mass element; these will be called "specific energy," "specific entropy," and so on. Their variations (dX or δY) are calculated along the streamline. Writing the time derivative of an exact (nonexact) variable as dX/dt ($\dot{Y}$), the rate of work done is found from Eq. (2),

$$\dot{W} = \left(1 - \frac{T_{\rm ref}}{T}\right)\dot{Q} - T_{\rm ref}\dot{S}_i - \frac{d}{dt}(U - T_{\rm ref}S). \tag{3}$$

Integrating Eq. (3) over a domain Ω (fixed), we obtain a macroscopic energy balance relation. Denoting by dM the mass element, the total amount of some state variable X (evaluated for a unit mass) is given by $\bar{X}=\int X dM$. Note that this representation uses the Lagrangian frame (dM moves with the fluid). To evaluate the time derivative of $\bar{X}$, it is convenient to use the Eulerian frame. With defining the mass density ρ, we may write $dM=\rho d^3x$, where d^3x is the volume element (Lebesgue measure) of the laboratory frame. We observe, using the mass conservation law $\partial\rho/\partial t+\nabla\cdot(\boldsymbol{v}\rho)=0$ ($\boldsymbol{v}$ is the flow velocity),

$$\begin{aligned}\frac{d}{dt}\bar{X} &= \int_\Omega \frac{\partial}{\partial t}(X\rho)d^3x \\ &= \int_\Omega \left[\frac{\partial}{\partial t}(X\rho) + \nabla\cdot(\boldsymbol{v}X\rho)\right]d^3x - \int_\Omega \nabla\cdot(\boldsymbol{v}X\rho)d^3x \\ &= \int_\Omega \left(\frac{\partial}{\partial t}X + \boldsymbol{v}\cdot\nabla X\right)\rho d^3x - \int_{\partial\Omega}(\boldsymbol{n}\cdot\boldsymbol{v})\rho X d^2x \\ &= \int_\Omega \left(\frac{d}{dt}X\right)dM - \int_{\partial\Omega}(\boldsymbol{n}\cdot\boldsymbol{v})\rho X d^2x,\end{aligned} \tag{4}$$

where $\boldsymbol{n}$ is the unit normal vector, directed outward, on the boundary $\partial\Omega$, and $dX/dt=\partial X/\partial t+\boldsymbol{v}\cdot\nabla X$ is the convective (Lagrangian) derivative. If we assume that the mass flow is confined in the domain, $(\boldsymbol{n}\cdot\boldsymbol{v})\rho X$ must vanish on the boundary. In what follows, we omit the mass flow through the boundary.

In a "quasistationary state" (could be far from thermal equilibrium), a sufficiently long-term average of a state variable must be constant. Hence, we may assume that the volume integral of the state variables (U and S) are constant. Integrating Eq. (3) over all fluid elements, then, yields

$$\int \dot{W}dM = \int \left(1 - \frac{T_{\rm ref}}{T}\right)\dot{Q}dM - T_{\rm ref}\int \dot{S}_i dM. \tag{5}$$

Generally, the variations of nonexact variables may take finite values even in a quasistationary state. Indeed, Eq. (5) gives the estimate of the long-term average work (power) of a quasistationary thermodynamic engine.

III. QUASISTATIC LAYER SYSTEM

We will now study the thermodynamics of an idealized plasma "layer" bounded from the inside by an internal core plasma, and from the outside by a cold heat bath. We will specify the total heat flux F_1 entering the layer through the inner boundary Γ_1 in contact with the core plasma. The temperature of the outer boundary Γ_0 is fixed by the temperature T_0 of the heat bath. The inner-boundary temperature T_1 (whose value measures the layer temperature gradients), however, is the essential parameter that needs to be determined. The outer-boundary heat flux F_0 must balance F_1 in a quasisteady state (then, we write $F_1=F_0=F$).

We neglect the mass flow across both boundaries. Consequently the boundary terms in Eq. (4) go to zero. We assume that $\dot{W}$ works only internally to drive a flow in Ω (the energy transformation between the thermal energy and the mechanical energy of collective motion may be represented by $\dot{W}$ or its dual $\dot{Q}=\dot{W}$). The entropy production $\dot{S}_i$ is, by definition, internal in the domain. However, the layer may exchange the heat $\dot{Q}$ with the exterior. In terms of the heat flow vector $\boldsymbol{f}$ ($\dot{Q}\rho=-\nabla\cdot\boldsymbol{f}$), $1/T_{\rm ref}$ times the first term on the right-hand side of Eq. (5) may be manipulated as ($dM=\rho d^3x$),

$$\begin{aligned}\int_\Omega\left(\frac{1}{T_{\rm ref}}-\frac{1}{T}\right)\dot{Q}\rho d^3x &= -\int_{\partial\Omega}\left(\frac{1}{T_{\rm ref}}-\frac{1}{T}\right)\boldsymbol{n}\cdot\boldsymbol{f}d^2x \\ &\quad - \int_\Omega \boldsymbol{f}\cdot\nabla\left(\frac{1}{T}\right)d^3x \\ &= \left[\left(\frac{1}{T_{\rm ref}}-\frac{1}{T_1}\right)F_1 - \left(\frac{1}{T_{\rm ref}}-\frac{1}{T_0}\right)F_0\right] \\ &\quad - \int_\Omega \dot{S}_D dM,\end{aligned} \tag{6}$$

where we have denoted $[\boldsymbol{f}\cdot\nabla(1/T)]=\dot{S}_D\rho$ (T and $\boldsymbol{n}\cdot\boldsymbol{f}$ are assumed to be constant on both boundaries). The first term of Eq. (6) represents the "entropy emission rate" through the boundaries. The second term in Eq. (6) is the "entropy production rate" due to the (irreversible) energy flow $\boldsymbol{f}$.

Hereafter, we set the reference temperature $T_{\rm ref}=T_0$ (the heat bath temperature). Using Eq. (6) transforms Eq. (5) to

$$\int \dot{W}dM = \left(1 - \frac{T_0}{T_1}\right)F_1 - T_0\int(\dot{S}_D+\dot{S}_i)dM. \tag{7}$$

If the heat were to transport only by diffusion in a stationary medium (viz., $\dot{W}=0$), the entropy production (and, thus, the entropy emission) is minimized[10] for the "harmonic heat flow" ($\nabla\cdot\boldsymbol{f}=0$), i.e., Eq. (7) holds with $\dot{W}=0$ and $\dot{S}_i=0$.

In a general quasisteady state, the mechanical energy (plasma flow energy; in the rest of the paper, the work done, and the excitation-dissipation of the flow will be used exchangeably) must saturate, and thus, $\int\dot{W}dM=0$ (by the first law of thermodynamics, $\int\dot{W}dM=F_1-F_0$, and, in a quasisteady state, $F_1=F_0=F$). However, local $\dot{W}$ may remain nonzero. Excitation ($\dot{W}>0$) and dissipation ($\dot{W}<0$) of flow may occur at different space-time locations, and they might have different scales associated with them; the scale separation between excitation and dissipation processes will be discussed in Sec. VI. In the macroscopic energy balance equation (7), the influence of the flow on heat transport may be accounted by the entropy production term $\int(\dot{S}_D+\dot{S}_i)dM$. We note that, in an initial transient phase, F_0 may be smaller than

F_1 and there is power available to generate flows ($\int \dot{W} dM = F_1 - F_0$). Then, entropy production will increase to balance the "heat-engine drive" as a quasisteady state is approached.

If one were to invoke the ansatz of "maximum entropy production" the internal entropy production term must acquire the largest accessible value. Although in a transient phase, the entropy production rate (energy dissipation rate) may assume any arbitrary value, in a quasistationary state, it is bound by the total entropy emission [that is proportional to the first term on the right-hand side of Eq. (7)]; the factor $(1-T_0/T_1)$ insures that the entropy emission increases as the difference between T_1 and T_0 increases. Maximum entropy production is, therefore, fundamentally tied to maximum temperature inhomogeneity. At the same time, the factor $(1-T_0/T_1)$ scales the maximum work that can be done by the heat engine. Since work done is synonymous with changes in the flow energy, and the maximum entropy production and the maximum work done are controlled by the same temperature difference factor, it follows that entropy production will be maximized, if in the layer, a large temperature inhomogeneity is excited/maintained/accompanied by large plasma flows. Such a quasistate, if found to be stable, will be surely far from thermal equilibrium, and will need to be sustained by an external input, for example, the heat flux entering the layer from the core plasma.

IV. MODEL OF HEAT TRANSPORT

In order to maximize temperature inhomogeneity, the system must involve a mechanism that can fight and overcome processes like heat diffusion that tend to minimize the temperature inhomogeneity (diminishing the entropy production). To work out some details of such a general "mechanism," it is helpful to dwell on an example drawn from the tokamak H-mode experiments where it is found that the transition to a high gradient state is always accompanied with the generation of a strong sheared flow. We understand that this qualitative thermodynamic model cannot even pretend to capture the complicated physics of the H-mode transition, but we believe that this transition does share some of the defining characteristics of the model layer problem that we are investigating. Consequently, this mode of enquiry may shed some conceptual light on this very important phenomenon.

Let us imagine a scenario in which some generic *flow* (collective motion of particles) acts to sustain the temperature inhomogeneity (bringing about an excess of entropy production). If the flow can enhance the temperature contrasts, the power $\dot{W}$ available for driving the flow increases; this positive feedback can, then, become the cause of a *transport barrier*.

For a transparent formulation of the problem, let us invoke a simple transport model in which the temperature difference between the inner and the outer boundary is controlled by a flow dependent heat diffusivity (and the entering heat flux F) via

$$T_1(P) - T_0 = \eta(P) F, \tag{8}$$

where P is the power to drive the flow (to be determined later as a function of F), $T_1(P)$ is the inner boundary temperature that is a function of P (and, thus, of F), and $\eta(P)$ is the impedance (inverse diffusivity). Although we have highlighted the P dependence of η, it could be a function of other system parameters.

Determination of the function $\eta(P)$ will require rather involved analysis and computational studies. But that is not the aim of this effort. Instead, we proceed by constructing an explicitly solvable but reasonable model to extract the necessary conditions on $\eta(P)$ that might produce a new state—a flow-dominated transport barrier, for example. For this purpose, we assume a simple parameterization of $\eta(P)$,

$$\eta(P) = \eta_0 + \eta_1(P) = \eta_0 + aP, \tag{9}$$

where a is a constant. We shall see below that a positive a (increasing the effective inverse diffusivity, and thus, decreasing the prevailing diffusivity) can be the harbinger of a phase transition.

The baseline impedance $\eta_0 = \eta(0)$ characterizes the ambient state in the absence of flows. Naturally this coefficient varies from system to system and is, in general, complicated and often unknown. The "diffusion" in a tokamak, for example, is known to be anomalous (driven by ambient turbulent fluctuations) yielding much higher heat-transfer rates as compared to the purely collisional transport rates. When we apply this model to tokamaks, the turbulent diffusive heat transport will define the reference or the "ground state." To develop the main features of our model, however, we do not need to know much about η_0; it is fully equivalent to specifying the reference inner boundary temperature

$$T_D \equiv T_1(0) = T_0 + \eta_0 F, \tag{10}$$

attained by the flowless ambient state. We assume that the inter-relationships are defined by Fick's law, $F = D\Delta T/\Delta x$, where Δx is the layer thickness and D ($=\Delta x/\eta_0$) is the heat diffusion coefficient (assuming a slab geometry and constant D, the heat flux of the diffusion is $\boldsymbol{f} = -D\nabla T$ that is a constant vector). The entropy production associated with this diffusion process is $\int \dot{S}_D dM = (T_0^{-1} - T_D^{-1})F$.

All we are interested in, from now on, is to demonstrate that additional processes, like self-generated flows (possibly through an instability) allow us to reach solutions for which $T_1(P) \geq T_D > T_0$. Using Eqs. (8)–(10), we eliminate η_0 to arrive at

$$T_1(P) = T_D + aPF. \tag{11}$$

The model (8) and (9) can be represented by an equivalent circuit shown in Fig. 1, where T_D may be considered as an intermediate temperature (voltage) between two different impedances, η_0 and $\eta_1(P)$. However, we are not considering separate "regions" in the layer for both impedances.

The next step is to estimate the power P available to generate the flow. We must subtract the power wasted through the ubiquitous entropy production in a diffusive

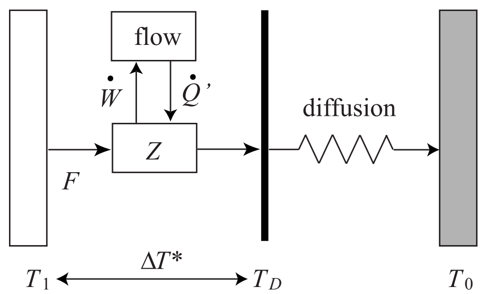

FIG. 1. Equivalent diagram of a heat engine in a boundary layer. If $\Delta T^* > 0$, the heat engine can work to drive flow (P is the power driving the flow; $\dot{Q}'$ is the dissipated power; in steady state, $P = \dot{Q}'$). The flow produces an additional "nonlinear impedance" $Z = \eta_1(P)$ that sustains the temperature contrast ΔT^* yielding a free energy to drive the flow itself.

process, $T_0[T_0^{-1} - T_D^{-1}]F$, from the maximum power $P_{\max} = [1 - T_0/T_1(P)]F$ available in an ideal Carnot process (6),

$$P = P_{\max} - T_0\left(\frac{1}{T_0} - \frac{1}{T_D}\right)F = T_0\left(\frac{1}{T_D} - \frac{1}{T_1(P)}\right)F. \tag{12}$$

However, not all of this P can appear as the flow energy because of inherent damping mechanisms (additional entropy production). The coefficient a multiplying P in Eq. (9) can be viewed as some sort of an efficiency factor, and scales the overall influence of the flow on the thermal transport.

Equations (11) and (12) are simultaneous in T_1 and P, we can solve them for either. We first solve for T_1 determined by

$$T_1 = T_D + aF^2\left(\frac{T_0}{T_D} - \frac{T_0}{T_1}\right), \tag{13}$$

or equivalently,

$$\Delta T^* = \frac{aF^2 T_0 \Delta T^*}{(T_D + \Delta T^*)T_D} \equiv g(\Delta T^*), \tag{14}$$

where $\Delta T^* \equiv T_1 - T_D$ measures the temperature increase (caused by the flows) at the inner boundary from its diffusive reference value. There are two solutions of Eq. (13):

$$T_1 = \begin{cases} T_D, \\ aF^2 T_0/T_D. \end{cases} \tag{15}$$

The first is simply the reference diffusive one and occurs when $P = 0$.

It is the second solution, capable of supporting a higher temperature contrast ($T_1 > T_D$) as shown below, that is the primary object of our search; it is accompanied by, in fact, is driven by a finite P, and its very definition $T_1 > T_D$ turns out to be exactly the "bifurcation" condition. The graphical solution of Eq. (14) for ΔT^*, shown in Fig. 2, illustrates how the second solution can emerge in $\Delta T^* > 0$. When the graph of $y = g(\Delta T^*)$ goes over that of $y = \Delta T^*$ near the origin $\Delta T^* = 0$, they will intersect at some point in $\Delta T^* > 0$, yielding a solution of Eq. (14). This condition reads as

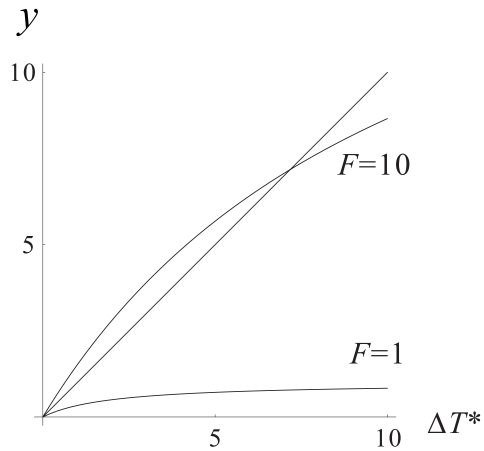

FIG. 2. Bifurcation of solutions [intersections of the graphs $y = g(\Delta T^*)$ and $y = \Delta T^*$]. If $g'(0) > 1$, we have the second branch of solutions with $\Delta T^* > 0$. In this graph, parameters are $T_0 = 1$, $a = 2$, $\eta_0 = 1$, and $F = 1$, 10.

$$g'(0) = \frac{aF^2 T_0}{T_D^2} > 1, \tag{16}$$

where the $'$ denotes differentiation with respect to ΔT^*. Remembering that $T_D = T_0 + \eta_0 F$ contains F, the condition (16) translates as (for $a > 0$)

$$F > F_{\min} \equiv \frac{T_0}{\sqrt{T_0 a} - \eta_0}. \tag{17}$$

For positive heat flow into the layer from the inner boundary ($F > 0$), the threshold condition (17) is meaningful only if the edge temperature T_0 is sufficiently high so that

$$T_0 > \frac{\eta_0^2}{a} \tag{18}$$

is satisfied. It is interesting that in this model, a very cold outer edge could prevent the transition even for arbitrary amounts of heat flux input into the layer.

Notice that the existence or the essential nature of the bifurcation of solutions does not depend on the detailed shape of the function $\eta(P)$, as long as $\eta'(P) > 0$. For an $\eta(P)$ more general than Eq. (9), the bifurcation condition (16) becomes

$$g'(0) = \frac{\eta'(0)F^2 T_0}{(T_0 + \eta_0 F)^2} = \frac{\eta'(0)F^2 T_0}{T_D^2} > 1, \tag{19}$$

and the minimum heat flux and the minimum temperature are given by replacing a by $\eta'(0)$ in Eqs. (17) and (18), respectively.

V. STABILITY OF THE BIFURCATED STATE

The stability of each equilibrium point is determined by evaluating the response of T_1 to a perturbation δT. We can imagine the following chain of events:

$$\delta T \to \delta P = F\frac{T_0}{T_1^2}\delta T$$
$$\to \delta\eta = aF\frac{T_0}{T_1^2}\delta T$$
$$\to \delta T_1 = aF^2\frac{T_0}{T_1^2}\delta T \equiv \alpha\delta T.$$

If $\alpha > 1$ ($\alpha < 1$), the equilibrium temperature T_1 is unstable (stable), because the perturbation is amplified (diminished). If this cycle of processes takes a period of time τ, the evolution of the perturbation may be written as $\delta T(t) = e^{\gamma t}\delta T(0)$ with $\gamma = (\log \alpha)/\tau$.

Let us first examine the stability of the trivial state $T_1 = T_D = T_0 + \eta_0 F$. Below the bifurcation point, i.e., $aF^2 T_0/T_D^2 \leq 1$ [see Eq. (16)],

$$\alpha \equiv aF^2\frac{T_0}{T_1^2} \leq \frac{T_D^2}{T_1^2} = 1$$

(equality holds at the bifurcation point), the trivial state is stable. For fluxes above the bifurcation threshold, however, the trivial state becomes unstable. On the other hand, the high $T_1 (> T_D)$ state ($T_1 = aF^2 T_0/T_D$), if it exists, is always stable, because we have

$$\alpha \equiv aF^2\frac{T_0}{T_1^2} = \frac{T_D}{T_1} < 1.$$

VI. MULTISCALE FLOW-TURBULENCE SYSTEM

The working of the model “heat engine” we constructed to sharpen temperature gradients in a plasma layer, depends on the concurrent expression of two contradictory processes: the processes that create disorder (maximum entropy production) and processes that create order (generation of flows). The sharper temperature gradients result because the coherent flow suppresses the ambient turbulence-caused diffusive heat transport, which, in the language used in this paper is equivalent to an increased impedance [$\eta(P) > \eta_0$].

One is, of course, more familiar with the “principle” of “minimum entropy production” widely applied to explain self-organization of “ordered structures;” the flow being the ordered structure. In this narrative, large entropy production is believed to destroy coherent structures and the system is pushed toward a disordered state.

One is forced to ask, then: What is it that we have done differently? What is the essential ingredient that enables the simultaneous enhancement of entropy production, and the channeling of energy into ordered motion (flow)?

Going back to the essentials of the model, we note that the only possible new element that could impart this nonstandard behavior to the engine is our choice of the inner boundary condition; instead of specifying the temperature T_1 at the inner boundary, we have chosen to specify the amount of heat flux F entering the layer. In fact it is F that brings in the energy that would be eventually channeled into an ordered flow; unless F is large enough (larger than a threshold value), the engine does not work, i.e., the high temperature-contrast state is not accessible.

Since the high temperature-contrast state is the final product of the heat engine, the factor measuring the temperature-contrast, $(1 - T_0/T_1)$ scales the strength of the two seemingly contradictory constituent processes: the Carnot efficiency for generating mechanical energy (flow), and the entropy production (emission). Such a state of affairs could pertain if, for instance, the dissipation mechanisms that create the total entropy (including the “damping” of the flow $\dot{Q}' = P$) were independent of the mechanisms that convert the “free energy” into an ordered flow.

The two opposing mechanisms could, indeed, act independently and simultaneously if the “domains” of their efficient operation were nonoverlapping. We propose that a recourse to scale-separation does precisely what is needed: (1) the total entropy production is dominated by small scale perturbations with a large damping rate ($\propto L^{-2}$; L: eddy size) keeping the eddy amplitudes (sacrifice for the dissipation) to be very small. (2) The flow, being a coherent macroscopic structure, is created in the large scale, perhaps, from an instability driven by the entering heat flux F; its creation/characteristics are not affected by the short scale dissipation responsible for entropy production.

Thus the existence of a scale-hierarchy allows the system to transition to a state that can maintain order while maximizing disorder. This transition is an expression of self-organization of the “heat engine;” the self-organization, most likely, taking place through the so-called “dual-cascade” process investigated in two-dimensional (2D) turbulence. Using this approach, Hasegawa and Wakatani[11] predicted self-organization of “zonal flows” in electrostatic turbulence of plasmas.

The canonical example of two-dimensional turbulence, however, is provided by the 2D Navier–Stokes system[12] where the dual cascade is facilitated by the existence of two different ideal constants of motion, the energy, and the enstrophy. When fluctuations are excited (by an instability) at an intermediate range of wave numbers, the energy and the enstrophy move in opposite directions in the wave number: the energy transfers, through the inverse cascade route, toward larger scales (gets ordered), while the enstrophy goes to the small-scale dissipation range (gets disordered). At the large-scale, a flow self-organizes, and the stretching effect suppresses turbulent transport. It should be noted that these processes have been shown to function only in a 2D fluid; in three dimensions (3D) the vortex-tube stretching effect violates the ideal conservation of the enstrophy. A 3D tokamak plasma, however, has an advantage over the 3D neutral fluid; the strong axial magnetic field imparts an effective 2D behavior to the confined plasma so that simpler 2D fluid like considerations could be relevant to H-mode layers and ITBs.

In the standard approach for exploring the large-scale structure of the flow, one sets up a variational principle; a constrained minimization of enstrophy while keeping the energy constant (as well as the total angular momentum).[11,12] The minimum enstrophy principles, naturally, implies “minimum entropy production" because the dissipation is proportional to the enstrophy.

In our model, however, the conventional variational principle is essentially stood on its head; the relevant new principle will constitute a dual[13] or an antitheses of the old one; one must maximize the energy for some reference value of the enstrophy. As mentioned above, the boundary condition, specifying the heat flux, is the key to our departure from the standard picture and, hence, the cause for the new variational principle. In an open system where the entering heat flux F is given, the entropy production rate is bounded,

$$\int (\dot{S}_D + \dot{S}_i) dM = \left(\frac{1}{T_0} - \frac{1}{T_1}\right) F = \frac{F^2 \eta(P)}{T_0[T_0 + F\eta(P)]} < \frac{F}{T_0}.$$

Hence, the enstrophy (dissipation) is bounded, supplying us a constraint while we maximize the energy. Under the assumption that $\eta'(P) > 0$, maximization of P (energy) raises the entropy production to its maximum.

The maximum entropy production yields a most "disordered" state in the small-scale, while an ordered flow with the maximum energy appears in the large-scale of the hierarchy.

Thus the working of this somewhat peculiar "heat engine" described in this paper can be understood in the backdrop of processes and ideas that have been invoked to study a variety of self-organizing systems. Special and distinguishing feature of this system is that when the entering heat flux F exceeds a well-defined threshold, a transition to a stable state with enhanced temperature gradients occurs. Simple thermodynamics can capture the essential qualitative features of the transition as well as of the new state.

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[1] G. Nicolis and I. Prigogine, *Self-Organization in Nonequilibrium Systems—From Dissipative Structures to Order through Fluctuations* (Wiley, New York, 1977).
[2] G. W. Paltridge, Q. J. R. Meteorol. Soc. **101**, 475 (1975); G. W. Paltridge, Nature (London) **279**, 630 (1979).
[3] P. H. Wyant, A. Mongroo, and S. Hameed, J. Atmos. Sci. **45**, 189 (1988).
[4] H. Grassl, Q. J. R. Meteorol. Soc. **107**, 153 (1990).
[5] S. Shimozawa and H. Ozawa, Q. J. R. Meteorol. Soc. **128**, 2115 (2002).
[6] R. D. Lorenz, J. I. Lunine, and P. G. Withers, Geophys. Res. Lett. **28**, 415, DOI: 10.1029/2000GL012336 (2001).
[7] H. Ozawa, S. Shimokawa, and H. Sakuma, Phys. Rev. E **64**, 026303 (2001).
[8] R. Dewar, J. Phys. A **36**, 631 (2003).
[9] F. Wagner, G. Becker, K. Behringer *et al.*, Phys. Rev. Lett. **49**, 1408 (1982); ASDEX Team, Nucl. Fusion **29**, 1959 (1989); R. J. Taylor, M. L. Brown, B. D. Fried, H. Grote, J. R. Liberati, G. J. Morales, P. Pribyl, D. Darrow, and M. Ono, Phys. Rev. Lett. **63**, 2365 (1989); R. J. Groebner, K. H. Burrell, and R. P. Seraydarian, *ibid.* **64**, 3015 (1990).
[10] Here, the "minimum" means that $\dot{S}_i=0$ (so that $\dot{W}=0$), but not minimizing $\dot{S}_D$ with examining the relation between the distributions of the temperature T, heat flux $\boldsymbol{f}$, and the diffusion coefficient. For a critical analysis of "minimum entropy production" in a fluid system, see E. Barbera, Continuum Mech. Thermodyn. **11**, 327 (1999).
[11] A. Hasegawa and M. Wakatani, Phys. Rev. Lett. **59**, 1581 (1987).
[12] A. Hasegawa, Adv. Phys. **34**, 1 (1985).
[13] Examples of a pair of dual variational principles are: (1) for a fixed perimeter, find a curve that maximizes the area of its enclosed region (isoperimetric problem), (2) for a fixed area, find a curve that minimize the perimeter. Note that the opposite problems are "ill-posed": (1') for a fixed perimeter, find a curve that minimizes the area of its enclosed region, (2') for a fixed area, find a curve that maximized the perimeter. To formulate a well-posed variational principle, the target functional to be maximized (minimized) must be "higher-order" ("lower-order") in comparison to the functional representing the constraint. In a variational principle of field theory, "higher-order" means that the functional includes higher-order derivatives. The enstrophy is higher-order with respect to the energy; see Z. Yoshida and S. M. Mahajan, Phys. Rev. Lett. **88**, 095001 (2002).