Effect of sheared flow on magnetic islands

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The effect of sheared flow on a magnetic island is examined. In contrast to the density and temperature gradients which are flattened for sufficiently wide islands, it is found that the velocity gradient persists inside the separatrix whenever the constant-ψ approximation is satisfied. It follows that velocity shear has a negligible effect on island amplitude in that approximation. The effect of the violation of the constant-ψ approximation is explored by using the Kelvin-Stuart family of islands, and it is found that flattening is modest even when the separatrix encloses virtually all the current. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2434251]

I. INTRODUCTION

The interaction between magnetic islands and plasma flows is of interest in several contexts ranging from laboratory experiments[1–5] to the investigation of plasmoid ejection in the magnetotail,[6] flux-transfer events at the day-side magnetopause,[7] and reconnection in the solar corona.[8] Previous investigations have focused on the velocity of the island or plasmoid with respect to the background plasma.[3,6,9–14] Theoretical investigations, in particular, have generally assumed that the velocity profile was symmetric (even) about the island, and that the velocity reached a constant value away from the island.[10,11,13,14] In general, however, the transport of momentum across the background plasma is associated with a velocity gradient that is present even at large distances from the island. The velocity profile thus has a component that is antisymmetric (odd) with respect to reflection about the island. Furthermore, antisymmetric flows can be generated *spontaneously* as the result of an equilibrium bifurcation.[9,15] The conditions under which this bifurcation occurs are of considerable interest given the importance of flow for transport and stability.

In the linear regime, Chen and Morrison have shown that a sheared velocity has a stabilizing effect on the tearing mode.[16,17] In the nonlinear regime, by contrast, Waelbroeck and Fitzpatrick showed that the odd part of the velocity profile has no effect on the evolution of the island in the limit where the perturbation of the current is small compared to the background current (the so-called constant-ψ approximation).[18] Smolyakov[19] subsequently revisited the question of the effect of the odd part of the velocity profile and concluded that a stabilizing effect could result for velocity profiles such that the gradient exhibits an unbalanced pair of jumps on the positive and negative branches of the separatrix. Ofman *et al.* and Chen *et al.* have also shown that velocity shear could have a stabilizing influence in the nonlinear regime when it is localized inside the current layer according to a $\tanh x$ profile.[7,8]

In the present paper, we extend the previous work by foregoing most of the simplifying assumptions used in previous analytic investigations. In particular, (i) we allow the island to have finite aspect ratio ($k_y W \sim 1$, where k_y is the azimuthal wave vector and W is the island half-width), (ii) we allow the variation of the background current to be finite (i.e., we allow the constant-ψ approximation to be violated), (iii) we include the effect of the Reynolds stress, and (iv) we allow the transverse Alfvénic Mach number $M = V/B_\perp$ to be finite. Here V is the flow velocity and $B_\perp$ is the magnetic field perpendicular to the direction of symmetry. When solving the equilibrium and transport equations, however, we will restrict consideration to flows that modify only weakly the geometry of the island. We will show that the influence of the flow on the island geometry is measured by MM', so that the geometry will be preserved even for Alfvénic velocity $M \sim 1$ provided that $M' \ll 1$.

The motivations for relaxing the above simplifying assumptions are as follows. The constant-ψ approximation may be violated in two cases. The first case consists of strongly unstable tearing modes ($\Delta' \gg k_y$, where Δ' is the tearing mode stability index), where the perturbed current may be large even when the island is narrow compared to the background current-density distribution. An example of this case is the $m=1$ island that is thought to be responsible for the observation of the "snake" mode in tokamaks.[5] The second case is when the width of the island is comparable to that of the background current-density. In the latter case the often-used thin-island approximation, $k_y W \ll 1$, fails. Examples of this second case include plasmoids in the magnetotail current sheet and magnetic islands at the boundary of fusion experiments. Regarding the magnetic Mach number, finite values of M may be reached in reversed shear discharges where islands resulting from double tearing modes grow near the reversal surface where the magnetic shear (and thus $B_\perp$) is very weak. Lastly, plasmoids and islands with finite aspect-ratio are commonly encountered in the magnetotail.

The primary objective of this paper is to examine the nature of momentum transport across magnetic islands and its effect on island amplitude. In particular, we will examine the effect of viscous momentum transport on the solutions

proposed by Smolyakov that are characterized by an asymmetric jump in the velocity gradient at the separatrix. A secondary motivation for the work reported here is to lay the foundations and provide the analytical tools needed to identify the conditions for the spontaneous generation of flow around a magnetic island observed by Parker and Dewar[9] and Grasso *et al.*[15]

II. FORMULATION

We consider a plasma described by the reduced MHD model. The magnetic field is given by $\mathbf{B}=B_z\hat{\mathbf{z}}+\mathbf{B}_\perp$ where $\mathbf{B}_\perp=-\hat{\mathbf{z}}\times\nabla\psi$ and the plasma velocity is $\mathbf{v}_\perp=\hat{\mathbf{z}}\times\nabla\phi$, where the $\perp$ subscript indicates the plane perpendicular to the direction of symmetry $\hat{\mathbf{z}}$. The flux ψ and the electrostatic potential φ are governed by Ohm's law and the vorticity equations,

$$\frac{\partial\psi}{\partial t}+[\varphi,\psi]=E_0-\eta J; \tag{1}$$

$$\frac{\partial U}{\partial t}+[\varphi,U]-[J,\psi]=\mu\nabla^2 U, \tag{2}$$

respectively, where $J=-\nabla^2\psi$ is the plasma current, $U=\nabla^2\varphi$ is the vorticity, E_0=constant is the reference electromagnetic induction, η is the resistivity, and μ is the viscosity. Here $[f,g]=\hat{\mathbf{z}}\cdot(\nabla f\times\nabla g)$ denotes the Poisson bracket.

For small transport coefficients, η, $\mu\ll 1$, the RMHD equations admit solutions that describe slowly varying islands in quasiequilibrium. These islands can be investigated using equilibrium and transport theory. Specifically, Eqs. (1) and (2) are solved by expansion in the transport coefficients η and μ, assuming that the time derivatives are of first order in these transport coefficients. This is an extension of the approach followed by Rutherford in his theory of the nonlinear evolution of tearing modes.[20] For simplicity, we will adopt here the more restrictive assumption that the configuration is fully relaxed so that the time derivatives can be neglected ($\partial/\partial t=0$). That is, we consider only states that are in transport as well as dynamic equilibrium. An important feature of Rutherford's theory is that the transport ordering fails in a narrow layer surrounding the separatrix.[21] This is a consequence of the vanishing of the shear-Alfvén frequency at the X-point, as a result of which the dynamical time scales are comparable to transport time scales close to the X-point. Edery *et al.*[22] have shown, however, that for static plasma the singular layer does not contribute to the current balance in the island nor to the island's evolution. We will assume that this remains true in the presence of plasma flow. Investigations of the singular layer in the presence of flow can be found in Refs. 23 and 24.

III. EQUILIBRIUM

To lowest-order, Ohm's law in the stationary state is

$$[\varphi,\psi]=0. \tag{3}$$

This implies that the electrostatic potential, which is the stream function for plasma flows, must be constant on surfaces of constant flux,

$$\varphi=\Phi_\sigma(\psi),$$

where $\sigma=\pm$ is a label that distinguishes between the two flux surfaces corresponding to the same value of ψ but lying on opposite sides of the separatrix [for surfaces lying inside the separatrix, $\Phi_+(\psi)=\Phi_-(\psi)$]. Inserting the above form of the electrostatic potential in the vorticity equation and using the identity $[\Phi_\sigma(\psi),U]=[\psi,\Phi'_\sigma(\psi)U]$, we find

$$[\psi,\Phi'_\sigma(\psi)U+J]=0. \tag{4}$$

This is readily integrated to find the current,

$$J=\hat{I}_\sigma(\psi)-\Phi'_\sigma(\psi)U, \tag{5}$$

where $\hat{I}_\sigma(\psi)$ is an arbitrary function describing the current profile. The presence of $\hat{I}_\sigma(\psi)$ is a manifestation of the freedom to choose the equilibrium current density. Its steady-state (relaxed) form is determined by the solution of the transport equation in the next section (Sec. IV).

It is convenient to introduce the Alfvénic Mach number $M_\sigma(\psi)=d\Phi_\sigma/d\psi$. In terms of $M_\sigma(\psi)$ the current takes the form

$$J=\hat{I}_\sigma(\psi)-UM_\sigma(\psi).$$

The vorticity U may likewise be expressed in terms of $M_\sigma(\psi)$ as

$$U=-JM_\sigma(\psi)+B_\perp^2M'_\sigma(\psi), \tag{6}$$

where $B_\perp=|\mathbf{B}_\perp|$ and the prime denotes derivation with respect to the argument, $M'_\sigma(\psi)=dM_\sigma/d\psi$. We see that the vorticity and the current are linearly interrelated. Solving for J and U in terms of M, we find

$$J=\frac{\hat{I}_\sigma-B_\perp^2M_\sigma M'_\sigma}{1-M_\sigma^2}; \tag{7}$$

$$U=\frac{-\hat{I}_\sigma M_\sigma+B_\perp^2M'_\sigma}{1-M_\sigma^2}. \tag{8}$$

We see that the equilibrium configuration is completely determined by the two pairs of profile functions $\hat{I}_\sigma(\psi)$ and $M_\sigma(\psi)$. These profile functions can in principle be controlled by the experimentalist, but in the absence of external action they will eventually relax to a form that is determined by the transport equations. We next derive the transport equations. To simplify notations we will henceforth omit the σ subscript whenever possible.

IV. TRANSPORT

In order to obtain transport equations we regard the steady-state version of the RMHD equations as magnetic differential equations for φ and J respectively:

$$\mathbf{B}\cdot\nabla\varphi = E_0 - \eta J; \tag{9}$$

$$\mathbf{B}\cdot\nabla J = [\varphi, U] - \mu\nabla^2 U. \tag{10}$$

The transport equations are the solubility conditions for these magnetic differential equations.

We will show that substituting the equilibrium results into the solubility conditions yields a differential equation for the profile M. As mentioned in Sec. II, however, the equilibrium assumption is violated in a narrow singular layer enclosing the separatrix. In order to deal with this singular layer we separate the analysis of the transport equations into two parts. In the first part, we will obtain an exact transport equation that is valid inside as well as outside the singular layer. We will use this transport equation to obtain a connection formula across the singular layer. In the second part, we will combine the exact transport equation with the results of the equilibrium analysis of Sec. III to obtain a differential equation governing the profile of M outside the singular layer.

A. Nonperturbative transport

We assume that the resistivity is constant along flux surfaces, $\eta = \eta(\psi)$. This is the case when the island is sufficiently wide that the parallel transport dominates the perpendicular transport. In order for Ohm's law to be soluble the integral of the right-hand side of Eq. (9) along a closed flux-surface must vanish. Thus,

$$\langle J\rangle = \frac{E_0}{\eta(\psi)}\langle 1\rangle, \tag{11}$$

where we have used the fact that E_0 is spatially constant. Here the flux-surface integral operator is defined by

$$\langle f(x,y)\rangle = \begin{cases} \oint \dfrac{dy}{B_y} f[x(\psi,y),y], & \psi < \psi_{\text{sep}}; \\ \displaystyle\int_{-y_{\text{tp}}}^{+y_{\text{tp}}} \frac{dy}{B_y}\frac{1}{2}\{f[x_+(\psi,y),y] + f[x_-(\psi,y),y]\}, & \psi > \psi_{\text{sep}}, \end{cases} \tag{12}$$

where ψ_{sep} is the flux label for the separatrix, $\psi > \psi_{\text{sep}}$ ($\psi < \psi_{\text{sep}}$) labels flux surfaces lying inside (outside) the separatrix, $x_\pm(\psi,y)$ is the transverse position of the flux surfaces, and y_{tp} is the azimuthal coordinate of the turning point for the flux surfaces lying inside the separatrix. We will find it convenient to use the flux-surface integral defined above to separate the fields into their average along the field line and a remainder according to $f = \bar{f} + \tilde{f}$, where $\bar{f} = \langle f\rangle/\langle 1\rangle$. Using this notation, Eq. (11) takes the form $E_0 = \eta\bar{J}$ and Ohm's law becomes $[\varphi,\psi] = \eta\tilde{J}$.

We next seek a transport equation describing the velocity profile. To do this, we consider the flux-surface average of the vorticity equation:

$$\mu\langle\nabla^2 U\rangle = \langle[\varphi, U]\rangle, \tag{13}$$

where the right-hand side Poisson bracket represents the Reynolds stress. In order to simplify this equation, we write the Reynolds stress as a divergence,

$$[\varphi, U] = \nabla\cdot(U\hat{\mathbf{z}}\times\nabla\varphi).$$

This allows us to write the vorticity equation in conservation form,

$$\langle\nabla\cdot(\mu\nabla U - U\hat{\mathbf{z}}\times\nabla\varphi)\rangle = 0. \tag{14}$$

We may apply Gauss' theorem by expressing the flux-surface average as a volume integral over an infinitesimal flux tube. For a general vector field $\mathbf{\Gamma}$, we write

$$\langle\nabla\cdot\mathbf{\Gamma}\rangle = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\psi}^{\psi+\epsilon} d\psi \oint \frac{dy}{|\partial_x\psi|}\nabla\cdot\mathbf{\Gamma}. \tag{15}$$

Changing the integration variables to (x,y), the flux-surface integral takes the form of a volume integral:

$$\langle\nabla\cdot\mathbf{\Gamma}\rangle = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int\oint \mathrm{d}x\mathrm{d}y\,\nabla\cdot\mathbf{\Gamma}.$$

We first apply Gauss' theorem to a flux tube that does not contain the separatrix,

$$\langle\nabla\cdot\mathbf{\Gamma}\rangle = \frac{d}{d\psi}\langle\mathbf{\Gamma}\cdot\nabla\psi\rangle, \quad \psi \neq \psi_{\text{sep}}.$$

Using this in the vorticity equation (14) leads to

$$\frac{d}{d\psi}\langle\mu\nabla\psi\cdot\nabla U - U[\varphi,\psi]\rangle = 0, \quad \psi \neq \psi_{\text{sep}}.$$

Making use of Ohm's law in the last term, there follows

$$\frac{d}{d\psi}\langle\mu\nabla\psi\cdot\nabla U + \eta\tilde{J}U\rangle = 0, \quad \psi \neq \psi_{\text{sep}}. \tag{16}$$

We may integrate Eq. (16) in each of the three regions bounded by the separatrix. We find

$$\mu\langle\nabla\psi\cdot\nabla U\rangle + \eta\langle\tilde{J}U\rangle = C_j, \quad \psi \neq \psi_{\text{sep}}, \tag{17}$$

where the C_j ($j = 0, +, -$) are integration constants applying to the regions on either side ($\pm$) and inside (0) the separatrix.

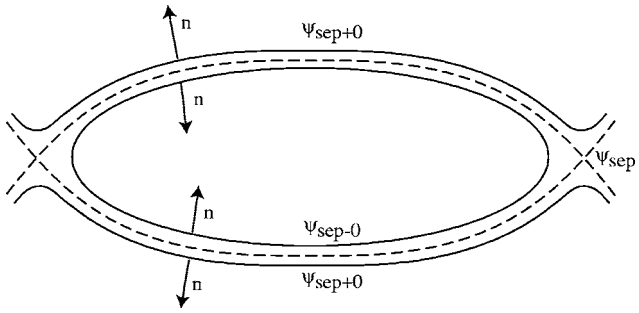

FIG. 1. Sketch of the surfaces (full lines) used to apply Gauss' theorem across the separatrix (dashed line). The arrows marked **n** indicate the direction of unit normals.

In order to obtain connection formulas across the two branches of the separatrix, we return to Eq. (15) and consider a flux tube enclosing the separatrix (Fig. 1). Applying Gauss' theorem to such a flux tube, we find

$$\int_{\psi_{\rm sep}-\epsilon}^{\psi_{\rm sep}+\epsilon} d\psi\langle\nabla\cdot\mathbf{\Gamma}\rangle = 2\langle\mathbf{\Gamma}\cdot\nabla\psi\rangle_{\psi_{\rm sep}+\epsilon} - (\langle\mathbf{\Gamma}\cdot\nabla\psi\rangle_{\psi_{\rm sep}-\epsilon,+} + \langle\mathbf{\Gamma}\cdot\nabla\psi\rangle_{\psi_{\rm sep}-\epsilon,-}), \tag{18}$$

where $\psi_{\rm sep}\pm\epsilon$ represents flux-surfaces lying immediately inside and outside the separatrix and the $\pm$ subscripts label the flux surfaces lying to either side of the separatrix. Applying this result to Eq. (14) and using Eq. (17), we find

$$C_+ + C_- = 2C_0. \tag{19}$$

In order to determine the values of the integration constants $C_\pm$, we match the left-hand side of Eq. (17) to its asymptotic value away from the island. Note that $\tilde{J}\to 0$ away from the island. We will assume that any volumetric momentum source that may be present is such that $\lim_{x\to\pm\infty}U = V_0/L_V \ll V_0/W$, where V_0 is the background velocity and $L_V \gg W$ is the velocity shear-length. It follows that $C_\pm=0$. By virtue of Eq. (19) C_0 must then also vanish. Thus, momentum transport is governed by

$$\mu\langle\nabla\psi\cdot\nabla U\rangle + \eta\langle\tilde{J}U\rangle = 0 \tag{20}$$

on all flux surfaces both inside and outside the singular layer.

We next integrate the first term of Eq. (20) by parts,

$$\nabla\psi\cdot\nabla U = \nabla\cdot(U\nabla\psi) + UJ.$$

Substituting in the momentum transport Eq. (20), there follows

$$\langle\nabla\cdot(U\nabla\psi)\rangle + \langle U\rangle\bar{J} + (1+\eta/\mu)\langle\tilde{J}\tilde{U}\rangle = 0.$$

We may once again apply Gauss' theorem. First, we exclude the separatrix to find

$$\frac{d}{d\psi}\langle B_\perp^2 U\rangle + \langle U\rangle\bar{J} + (1+\eta/\mu)\langle\tilde{J}\tilde{U}\rangle = 0, \quad \psi\neq\psi_s. \tag{21}$$

Second, we apply Gauss' theorem to the singular layer in order to obtain connection formulas. In this instance we need connection formulas relating quantities on either side of the singular layer on surfaces where the equilibrium approximation applies. We thus take the half-width ϵ of the flux tube to be such that $\delta\ll\epsilon\ll W$. Here δ is the width of the singular layer. We separate the integral into a part containing the divergence term and a part containing the remaining terms

$$\int_{\psi_{\rm sep}-\epsilon}^{\psi_{\rm sep}+\epsilon} d\psi\langle\nabla\cdot(U\nabla\psi)\rangle + \int_{\psi_{\rm sep}-\epsilon}^{\psi_{\rm sep}+\epsilon} d\psi(\langle U\rangle\bar{J} + (1+\eta/\mu)\times\langle\tilde{J}\tilde{U}\rangle) = 0.$$

The second integral is of order δ and may be neglected. To see this, recall that the velocity is continuous across the separatrix whereas the gradient of velocity (proportional to the momentum flux), may experience a jump when the island is subject to an external electromagnetic force. Thus, the vorticity U is discontinuous but finite. The current $\tilde{J}$ must be singular, $\tilde{J}=O(\delta^{-1})$, in order to provide an electromagnetic force balancing the viscous force $\mu\nabla^2 U$. The singular part of $\tilde{J}$ has odd parity in y, however, so that the term $\langle\tilde{J}\tilde{U}\rangle$ is finite. Note that the vorticity convection term $[\varphi,U]=O(\delta)$ in the singular layer.

Applying Eq. (18) to the first term, we finally find the jump condition

$$\langle B_\perp^2 U\rangle_{\psi_{\rm sep}-\epsilon,+} + \langle B_\perp^2 U\rangle_{\psi_{\rm sep}-\epsilon,-} = 2\langle B_\perp^2 U\rangle_{\psi_{\rm sep}+\epsilon}. \tag{22}$$

Equation (22) shows that the even part of the vorticity U and thus the odd part of the velocity must be continuous across the singular layer. It follows that solutions with unbalanced jumps in the vorticity U (i.e., jumps that are not antisymmetric with respect to reflection about the midplane of the island) such as those considered in Ref. 19 are not allowed in a relaxed state. The relaxation of a velocity profile with unbalanced jumps in the vorticity is illustrated in Fig. 2.

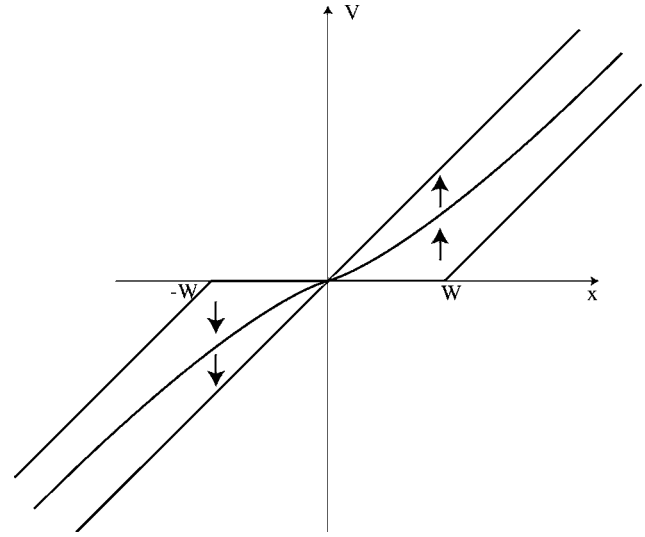

FIG. 2. Sketch of the relaxation of a velocity profile with unbalanced jumps in the vorticity $U=V'$ on the two branches of the separatrix at $x=\pm W$. The profiles shown are for a cross section with constant y containing the O-point of the island.

In the following subsection, we use the results of the equilibrium analysis of Sec. III in order to evaluate the fields appearing in Eqs. (21) and (22) in terms of the profile functions $\hat{I}$ and M.

B. Equilibrium transport

We begin by expressing the equilibrium results in terms of flux-averaged and oscillating quantities. It is convenient to use $I(\psi)$ to denote the average of the current density. That is,

$$I(\psi) \equiv \bar{J} = \frac{\langle J\rangle}{\langle 1\rangle} = \frac{\hat{I}(\psi) - MM'\langle B_\perp^2\rangle/\langle 1\rangle}{1-M^2}.$$

Equation (11) shows that $I(\psi)=E_0/\eta(\psi)$. The part of the current that oscillates along the field lines, $\tilde{J}$, is the polarization current:

$$\tilde{J} = -M\tilde{U}, \tag{23}$$

where $\tilde{U}$ is the part of the vorticity that oscillates along the field lines,

$$\tilde{U} = \frac{M'}{1-M^2}\cdot\left(B_\perp^2 - \frac{\langle B_\perp^2\rangle}{\langle 1\rangle}\right). \tag{24}$$

The field-line averaged part of the vorticity, by contrast, is

$$\bar{U} = -IM + M'\frac{\langle B_\perp^2\rangle}{\langle 1\rangle}. \tag{25}$$

Note that the polarization current varies like MM'. It follows that for $MM' \ll 1$, the effect of the polarization current on the island structure may be neglected and the island geometry is approximately independent of the flows. In this case we may investigate the transport of momentum by assuming that the geometry of the island is fixed.

To express the transport equation (21) as a second order ordinary differential equation for M, we recall that $\tilde{J}=-M\tilde{U}$ and substitute for $\tilde{U}$ and $\bar{U}$ from Eqs. (24) and (25), respectively. We may simplify the resulting equation by using Gauss' theorem to show that

$$\langle J\rangle = -\langle \nabla\cdot(\nabla\psi)\rangle = -\frac{d}{d\psi}\langle B_\perp^2\rangle.$$

It follows from the preceding identity that

$$\frac{d}{d\psi}(MI\langle B_\perp^2\rangle) = M\langle B_\perp^2\rangle\frac{dI}{d\psi} + I\langle U\rangle.$$

Using the above result to simplify Eq. (21), we obtain the following momentum transport equation governing the profiles $M_\sigma(\psi)$:

$$\frac{d}{d\psi}\left(\frac{\langle B_\perp^4\rangle - M_\sigma^2\langle B_\perp^2\rangle^2/\langle 1\rangle}{1-M_\sigma^2}\frac{dM_\sigma}{d\psi}\right) - M_\sigma\langle B_\perp^2\rangle\frac{dI_\sigma}{d\psi} = \left(1+\frac{\eta}{\mu}\right)\frac{M_\sigma M_\sigma'^2}{(1-M_\sigma^2)^2}\left(\langle B_\perp^4\rangle - \frac{\langle B_\perp^2\rangle^2}{\langle 1\rangle}\right), \quad \psi \neq \psi_s. \tag{26}$$

We have restored the σ labels to emphasize that the momentum transport equation applies separately to both branches of the profile function outside the separatrix. The jump condition at the separatrix is given by (22):

$$\frac{dM_+}{d\psi}(\psi_s-\epsilon) + \frac{dM_-}{d\psi}(\psi_s-\epsilon) = 2\frac{dM}{d\psi}(\psi_s+\epsilon). \tag{27}$$

At the magnetic axis of the island, the momentum equation is singular since $\langle \mathbf{B}_\perp^4\rangle$ and $\langle \mathbf{B}_\perp^2\rangle^2$ both vanish at this point. The appropriate boundary condition at the magnetic axis is thus

$$\frac{dM}{d\psi}(\psi_0) = 0. \tag{28}$$

The boundary conditions are completed by specifying the velocity shear at large distances from the island,

$$\lim_{\psi\to\infty}\frac{dM_\pm}{d\psi} = \frac{V_\pm'}{B_{y0}'}, \tag{29}$$

where the primes indicate differentiation with respect to x and where B_{y0}' describes the shear in the reference magnetic field.

Equation (26) together with its boundary conditions, Eqs. (27)–(29), governs the transport of momentum across a finite aspect-ratio, nonconstant-ψ magnetic island. This equation constitutes a central result of the present paper. In the following section we will investigate the effect of the island on momentum transport by solving the transport equation for two particular cases, the case of sub-Alfvénic flow and the case of the Kelvin-Stuart solution of Liouville's problem.

V. SOLUTION FOR SUB-ALFVÉNIC FLOW

In the sub-Alfvénic approximation ($M \ll 1$), for large aspect-ratio islands ($kW \ll 1$), and for constant-ψ islands ($I'(\psi)=O(kW)$), Eq. (26) reduces to that used in Refs. 18 and 19:

$$\frac{d}{d\psi}\left(\langle B_y^4\rangle\frac{dM}{d\psi}\right) = 0. \tag{30}$$

The general solution of this equation is

$$M(\psi) = A_j + C_j\int_{\psi_{\rm sep}}^{\psi}\frac{d\psi}{\langle B_y^4\rangle}$$

where the A's and C's are integration constants and $j=0,\pm$ labels the regions inside the separatrix and on either side of it, respectively. Since $B_y\to 0$ on the island's magnetic axis, we must clearly take $C_0=0$. Furthermore, continuity of the velocity across the separatrix imposes $A_\pm=A_0$. Lastly, Eq. (19) imposes that $C_+ + C_- = 0$. We thus conclude that the solutions with *unbalanced* localized jumps in M' at the separatrix proposed by Smolyakov *et al.*[19] are not allowed in the relaxed state. These solutions represent states with *unbalanced localized viscous torques* and they will rapidly relax to the solutions examined in Ref. 18. The relaxed solutions are

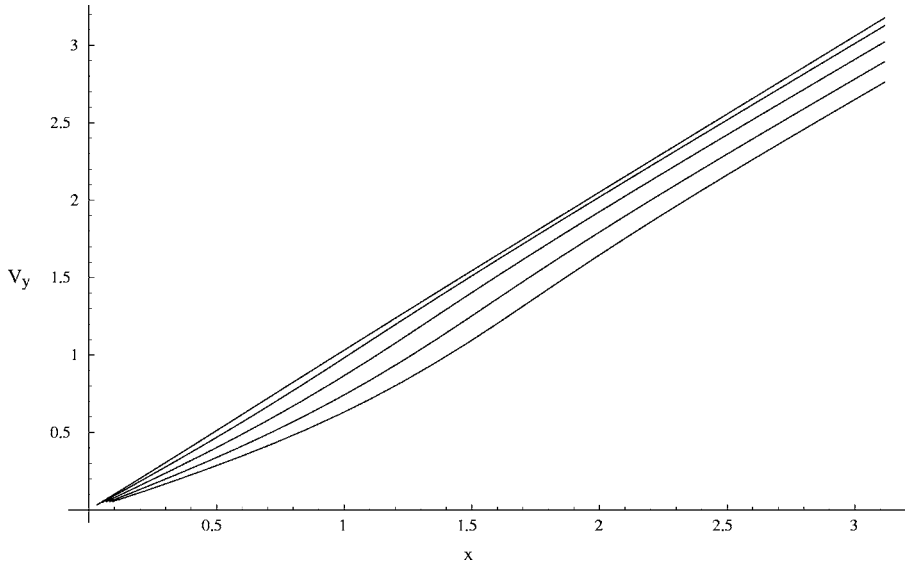

FIG. 3. Profiles of the velocity on a chord crossing the magnetic axis of the island (O-point). From top to bottom the profiles correspond to $\epsilon=0.1$, 0.3, 0.5, 0.7, and 0.9.

$$M(\psi) = A_0 + C\,\mathrm{sign}(x)\Theta(\psi_{\mathrm{sep}} - \psi)\int_{\psi_{\mathrm{sep}}}^{\psi} \frac{d\psi}{\langle B_y^4\rangle}, \tag{31}$$

where Θ is the unit step function. Note that in the above solution M is continuous and the jumps in M' at the separatrix are balanced; that is, they are equal and opposite on the two branches of the separatrix. The second term in Eq. (31) represents the sheared flow contribution created by the dragging of the island through the plasma caused by an external force. In the absence of external force the solution reduces to

$$M(\psi) = A_0,$$

a constant Mach number. This corresponds to a flow in which the velocity field is everywhere proportional to the transverse magnetic field, so that there is a circular convection pattern inside the separatrix and a sheared velocity away from the island. It is easy to see that the polarization current vanishes for such a flow pattern. In the following section, we will investigate the case of islands where the constant-psi approximation breaks down in order to see if the modification of the current distribution by the island may produce shear in the Mach number profile.

It is interesting to consider the corrections to the small Mach-number solution for small M'. We neglect the term proportional to I' (constant-ψ approximation) and seek a solution of the form $M=M_0+M_1(\psi)+\cdots$ where M_0 is a constant. We find

$$M_1 = \mathrm{sign}(x)\Theta(\psi_{\mathrm{sep}} - \psi)\int_{\psi_{\mathrm{sep}}}^{\psi} d\psi \frac{1 - M_0^2}{\langle B_y^4\rangle - M_0^2\langle B_y^2\rangle^2/\langle 1\rangle}.$$

We conclude that there is no first-order correction to the odd part of the velocity profile (the even part of M), and that the even part of the velocity profile remains qualitatively similar to the sub-Alfvénic solution.

VI. APPLICATION TO THE KELVIN-STUART ISLAND

In this section we consider the effect on the velocity profile of the inductive current, represented in Eq. (26) by the term proportional to $I'(\psi)$. In order to do this we use the Kelvin-Stuart island,[25] a well-known solution of the equilibrium equation for the following current profile:

$$J = (1 - \epsilon^2)e^{2\psi}.$$

The solution of the equilibrium equation,

$$\nabla^2\psi = -J,$$

is then

$$\psi = -\log(\cosh x - \epsilon\cos y).$$

Note that for this solution $B_y \sim 1$ and $\psi \sim x$ at large x whereas the velocity profile will vary linearly with x away from the island. Thus, for large x the Mach number varies like $M \sim x \sim \psi$, as can be verified by inspection of Eq. (26). The coefficients of the transport equation for this solution are given in terms of elliptic functions in the Appendix.

We have integrated the momentum transport equation numerically for the Kelvin-Stuart island. The resulting velocity profiles on a chord crossing the O-point of the island are shown in Fig. 3. In the limit of $\epsilon \to 0$, we recover the solution[18,19] $M=M_0$ constant. For $\epsilon \to 1$, however, there is a moderate depression of the Mach number inside the island corresponding to a flattening of the velocity profile in the vicinity of the island. We conclude that the violation of the constant-ψ approximation only has a moderate effect on the transport of momentum through the island. A corollary is that velocity shear will only have a modest effect on the size of the nonconstant-ψ island. Note that since most of the current is contained inside the island for large ϵ, the usual technique of determining the effect of the polarization current by asymptotic matching to external solutions is inapplicable.

VII. SUMMARY

We have extended the theory of momentum transport inside a magnetic island in order to allow the analysis of cases where commonly adopted assumptions are violated. Specifically we have considered cases where the constant-psi approximation is violated, where the Alfvénic Mach number

is of order unity, and where the aspect ratio of the island is finite. We find that the resulting velocity profiles are very similar to those found in the cases treated previously, except for a modest flattening of the velocity profile in the case of large Kelvin-Stuart islands encompassing most of the plasma current. In the remaining cases, the velocity remains proportional to the transverse magnetic field over a wide range of parameters and conditions. It follows that in a fully relaxed equilibrium and in the absence of momentum sources, velocity shear generally has only a modest influence on the stability of magnetic islands. We note, however, that nonrelaxed localized velocity profiles such as that considered by Ofman *et al.*[8] and Chen *et al.*[7] may influence stability, as reported by those authors.

The presence of external electromagnetic forces acting on the island is known to give rise to discontinuities in the velocity shear at the separatrix.[18] We have shown that the steady-state transport equation requires that these discontinuities be symmetric (i.e., that they be equal on the two branches of the separatrix). States with asymmetrical or unbalanced jumps, which were shown to be stabilizing by Smolyakov,[19] lead to the rapid spin-up around the O-point of the plasma inside the separatrix in a manner illustrated in Fig. 2. While such states are consistent with force-balance in the inviscid limit, they cannot be sustained in the presence of viscosity and will relax in a time comparable to the island momentum transport time μ/W^2.

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APPENDIX: GEOMETRIC COEFFICIENTS FOR THE KELVIN-STUART ISLAND

In this appendix we calculate the geometric coefficients appearing in the transport equation for the case of the Kelvin-Stuart solution. We find it convenient to change the integration variable to x so that

$$\langle f(x,y)\rangle = \int_{x_{\min}}^{x_{\max}} \frac{dx}{B_x} f[x, y(x,\psi)],$$

where $x_{\min}$ and $x_{\max}$ are the bounds of integration and

$$B_x = -\epsilon e^{\psi} \sin y = -e^{\psi}\sqrt{\epsilon^2 - (\cosh x - e^{-\psi})^2}.$$

Changing variables to $u=e^{-\psi}$ and $z=\cosh x$, we find

$$\langle f\rangle = u\int_{\mathrm{Max}[1,u-\epsilon]}^{u+\epsilon} dz \frac{f(z,u)}{\{(z^2-1)[\epsilon^2-(z-u)^2]\}^{1/2}}.$$

In order to express this integral in terms of elliptic integrals, we write the argument of the square root,

$$\alpha = (z^2-1)[\epsilon^2-(z-u)^2],$$

as a product two quadratic polynomials, $\alpha = Q_1 Q_2$. The polynomials must be chosen so that their roots are non-interlocking. For the exterior of the separatrix, this is achieved by the choice:

$$Q_1 = \epsilon^2 - (z-u)^2;$$

$$Q_2 = z^2 - 1.$$

We next seek linear combinations of these two quadratics that are perfect squares:

$$Q_1 - \lambda Q_2 = \epsilon^2 - u^2 + \lambda + 2uz - (1+\lambda)z^2.$$

This is a perfect square provided that

$$\lambda = \lambda_\pm = \tfrac{1}{2}[-(1+\epsilon^2-u^2) \pm \sqrt{(1+\epsilon^2-u^2)^2 - 4\epsilon^2}.$$

Hence,

$$Q_1 - \lambda_\pm Q_2 = -(1+\lambda_\pm)(z - u/(1+\lambda_\pm)^2.$$

This equation can be solved to express the Q's as a sum of perfect squares:

$$Q_1 = -\frac{\lambda_+(1+\lambda_-)}{\lambda_+-\lambda_-}\left(z - \frac{u}{1+\lambda_-}\right)^2 + \frac{\lambda_-(1+\lambda_+)}{\lambda_+-\lambda_-} \times \left(z - \frac{u}{1+\lambda_+}\right)^2;$$

$$Q_2 = \frac{(1+\lambda_+)}{\lambda_+-\lambda_-}\left(z - \frac{u}{1+\lambda_+}\right)^2 - \frac{(1+\lambda_-)}{\lambda_+-\lambda_-}\left(z - \frac{u}{1+\lambda_-}\right)^2.$$

We next change integration variables to t,

$$t = \frac{z - u/(1+\lambda_-)}{z - u/(1+\lambda_+)}.$$

In terms of this new integration variable the integral is easily transformed into canonical form. We find

$$\langle 1\rangle = \frac{2}{\sqrt{\lambda_+}} K\left(\frac{\lambda_-}{\lambda_+}\right),$$

where K is the complete elliptic integral of the first kind.

In the interior region, in order to keep the roots of Q_1 and Q_2 separated we choose

$$Q_1 = (1-z)(\epsilon - u + z);$$

$$Q_2 = (z-1)(\epsilon + u - z).$$

The linear combination $Q_1 + \lambda Q_2$ will be a perfect square whenever

$$\lambda_\pm = \frac{1 - u^2 + 6\epsilon + \epsilon^2 \pm 4\sqrt{\epsilon[(1+\epsilon)^2 - u^2]}}{(1-u-\epsilon)^2}.$$

Performing the same change of variable as in the exterior region, but with the above $\lambda_\pm$, we find

$$\langle 1\rangle = \frac{2}{(u+\epsilon-1)\sqrt{\lambda_+}} K\left(\frac{\lambda_-}{\lambda_+}\right).$$

The remaining integrals may be evaluated in a similar way.

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