Isothermal tokamak

Peter J. Catto
Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

R. D. Hazeltine
Institute for Fusion Studies and Department of Physics, The University of Texas at Austin, Austin, Texas 78712

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An isothermal tokamak, Isomak, is investigated to demonstrate that nearly exact, rigidly toroidally rotating Maxwellian solutions exist for both the ions and the electrons. For the ions this Maxwellian solution is valid in the limit in which unlike collisions of the ions with the electrons are weak and the induced electric field unimportant, while for Maxwellian electrons unlike collisions can be retained as long as the friction with the ions is small (electron-ion collision frequency smaller than the electron gyrofrequency). In such cases magnetically confined, exponentially decaying density profiles are allowed, minimizing contact with the wall or limiter. Indeed, the near Maxwellian behavior assures that radial particle and heat fluxes are small. In fact, for specially tailored ion and electron current drives it is possible to maintain the Maxwellians as exact steady state solutions of the full ion and electron kinetic equations. Three reasons to consider an Isomak are its usefulness as an ideal tokamak reference, its possible relevance to lithium-walled tokamaks, and its value in checking codes in the isothermal limit. © *2006 American Institute of Physics*.
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I. INTRODUCTION

A. Isothermal confinement

By an isothermal tokamak we mean an axisymmetric, toroidal confinement system in which the temperature profile of each plasma species s is flat:

$$\nabla T_s = 0. \tag{1}$$

Confinement is then controlled by the density profile, with the density n becoming very small at the plasma boundary. We refer to such a device as an *Isomak*.

Although an isothermal tokamak has not been experimentally realized, there are several reasons to study such a device. First, in certain present tokamak experiments the temperature gradients are indeed relatively small over a portion of the plasma cross section.[1,2] In some proposed devices, such as those employing a liquid-metal boundary,[3] the near-isothermal region may include most of the plasma volume. It is therefore reasonable to examine the limiting case of Eq. (1).

Second, we will find that the analysis of an Isomak requires relatively few simplifying assumptions, and leads to a number of exact results. Such results can provide valuable checks of numerical simulations.

Finally, the Isomak is, in more than one sense, the ideal magnetic confinement device. In particular it lacks the free energy source that drives some of the most damaging instabilities in conventional tokamaks: it is much closer to thermal equilibrium. Similarly it lacks the dominant transport process of a conventional tokamak, ion heat conduction, displaying transport only on the much slower level associated with electron motion. Moreover it is especially amenable to steady-state operation, with self-driven toroidal current. This ideal character, for which we will find more evidence below, makes the isothermal tokamak an instructive reference point.

B. Geometry and notation

We use toroidal coordinates (r,θ,ζ), where the "radial" coordinate r is constant on toroidal magnetic surfaces and vanishes on the magnetic axis; the poloidal and toroidal angles are respectively θ and ζ; the former is assumed to vanish on the outside of the toroidal midplane. The confining field $\boldsymbol{B}=\boldsymbol{b}B$ has the form

$$\boldsymbol{B} = \nabla\zeta \times \nabla\psi(r) + I(r)\,\nabla\,\zeta,$$

where ψ, the poloidal flux, can also be identified with the toroidal component of the magnetic vector potential $\boldsymbol{A}$:

$$\psi = -R^2\,\nabla\,\zeta\cdot\boldsymbol{A}. \tag{2}$$

We will sometimes express $\boldsymbol{B}=\boldsymbol{B}_P+\boldsymbol{B}_T$ in terms of its poloidal ($\boldsymbol{B}_P=\nabla\zeta\times\nabla\psi$) and toroidal ($\boldsymbol{B}_T=I\nabla\zeta$) components. The scalar equilibrium is assumed to be axisymmetric: all equilibrium quantities can be expressed in terms of r and θ.

The total particle energy is denoted by

$$U = \tfrac{1}{2}mv^2 + e\Phi(r,\theta), \tag{3}$$

where m is the mass, $v=|\boldsymbol{v}|$ is the magnitude of the Cartesian velocity coordinate $\boldsymbol{v}$, and Φ is the electrostatic potential. The energy is not strictly constant, but it varies sufficiently slowly in tokamak equilibrium to be treated as a dynamical invariant. Similarly, axisymmetry makes the ζ component of the canonical angular momentum

$$p_\zeta = mR^2 \nabla \zeta \cdot \boldsymbol{v} - \left(\frac{e}{c}\right)\psi$$

a dynamical invariant. Here $R(r,\theta)$ is the major radius—the distance from the symmetry axis. We label this invariant using a conventional notation:

$$\psi_* = \psi - \left(\frac{mc}{e}\right)R^2 \nabla \zeta \cdot \mathbf{v}. \tag{4}$$

The fluid velocity for species s—the average of $\boldsymbol{v}$ over the s-species distribution function—is denoted by $\boldsymbol{V}_s$ and the thermal velocity by $v_{ts}=\sqrt{2T_s/m_s}$. Species subscripts s are suppressed whenever possible.

II. ION EQUILIBRIUM

A. Distribution function

The ion distribution $f_i(r,\theta,\boldsymbol{v})$ satisfies the steady-state kinetic equation

$$\boldsymbol{v} \cdot \nabla f_i + \frac{e}{m_i}(-\nabla\Phi + c^{-1}\boldsymbol{v}\times\boldsymbol{B}) \cdot \frac{\partial f_i}{\partial \boldsymbol{v}} = C_{ii}(f_i) \tag{5}$$

in an axisymmetric, isothermal system. Here C_{ii} is the collision operator for scattering of ions by ions.

Notice that Eq. (5) uses certain approximations. In particular, it neglects ion scattering by electrons, because the corresponding operator C_{ie} is smaller by a factor of $\sqrt{m_e/m_i}$ than C_{ii}.[4] It also neglects any nonelectrostatic electric field that might be present, as well as possible radiative sources of momentum ("current drive") or heat, because such effects are consistently ordered as comparable to C_{ie}. They are considered in Sec. IV.

The steady state assumption requires further comment. The absence of temperature gradients means that the drive terms for ion and electron temperature gradient modes are absent. However, other instabilities, such as the trapped electron mode (TEM) may be present. Any time variation associated with a zonal flow generated by secondary instabilities associated with the TEM is ignored.

On the other hand, we also note that Eq. (5) is valid for arbitrary gyroradius. While we assume the plasma to be strongly magnetized, and use a small gyroradius approximation to simplify certain fluid considerations, below, it does not play any role here. Similarly we compute the ion distribution without any ordering regarding the ion-ion collision frequency.

The solution to Eq. (5) is accomplished by constructing a Maxwellian distribution, with toroidal flow, exclusively from constants of the motion.[5–7] Because it depends only on dynamical constants, it is annihilated by the left-hand side of Eq. (5); because it is a (moving) Maxwellian it is also annihilated by the right-hand side of Eq. (5). One form of this distribution is

$$f_i = N\pi^{-3/2}v_t^{-3}e^{-U/T}e^{-e\omega\psi_*/cT}, \tag{6}$$

where ω is a constant rotation frequency and N is a normalization constant. This function is obviously an invariant, as desired, but it is not obviously a moving Maxwellian. To verify this latter property we next use ion force balance to find an appropriate choice for the constant N.

B. Ion force balance

Ion momentum in an axisymmetric, isothermal system with toroidal flow velocity $\boldsymbol{V}_i=\omega R^2\nabla\zeta$ is subject to four forces:

1. the magnetic force $en\boldsymbol{V}_i\times\boldsymbol{B}=en\omega\nabla\psi/c$;
2. the electrostatic force $en\nabla\Phi$;
3. the pressure gradient $\nabla p=T\nabla n$;
4. the centripetal force $mn\boldsymbol{V}_i\cdot\nabla\boldsymbol{V}_i=-mn\omega^2R\nabla R$.

Collisional friction with the electrons is not included in this list because, as we have noted, C_{ie} is consistently neglected in the order of interest. Therefore ion equilibrium, for arbitrary gyroradius, is characterized by the relation

$$-mn\omega^2R\nabla R + en\left(\nabla\Phi + \frac{\omega\nabla\psi}{c}\right) + T\nabla n = 0. \tag{7}$$

It follows that the choice

$$N = n\exp\left[\frac{e}{T}\left(\Phi + \frac{\omega\psi}{c}\right) - \frac{m}{2T}\omega^2R^2\right] \tag{8}$$

gives a constant normalization, as required: $\nabla N=0$. Thus our distribution function becomes

$$f_i = \pi^{-3/2}nv_t^{-3}e^{-m(\boldsymbol{v}-\omega R^2\nabla\zeta)^2/2T},$$

a Maxwellian with density n and rotation velocity

$$\boldsymbol{V}_i = \omega_i R^2 \nabla\zeta, \tag{9}$$

where ω can be computed from Eq. (7).

Since N is constant, we can equate the right-hand side of Eq. (8) to its value on the magnetic axis ($\psi=0$) and then solve for the ion density. Using a 0-subscript to distinguish parameter values on the axis, we find that f_i corresponds to the density profile

$$n = n_0\exp\left[\frac{e(\Phi_0-\Phi)}{T} - \frac{e\omega\psi}{cT} + \frac{m\omega^2(R^2-R_0^2)}{2T}\right]. \tag{10}$$

C. Toroidal flow

We recall the conventional definition of a magnetized plasma:

$$\delta_s \equiv \frac{\rho_s}{L} \ll 1,$$

where $\rho=v_{ts}/\Omega_s$ is the gyroradius (Ω_s is gyrofrequency of species s) and L is a scale length for variation of n or Φ. Applying this ordering to Eq. (7), we suppose that the density-gradient term is comparable to the magnetic field term and quickly find that

$$V_i \sim \delta_i v_{ti}. \tag{11}$$

It follows that the first term of Eq. (7) is consistently neglected: ion equilibrium is described by

$$en\left(\frac{\nabla\Phi+\omega_i\nabla\psi}{c}\right)+T_i\nabla n=0. \tag{12}$$

We assume that all terms in this relation are comparable, as is consistent with the conventional ordering,

$$\frac{e\Phi}{T}\sim 1. \tag{13}$$

Then the rigid-body rotation frequency is given by

$$\omega_i=-c\frac{d\Phi}{d\psi}-\frac{cT_i}{en}\frac{dn}{d\psi}, \tag{14}$$

the familiar combination of $E\times B$ and (isothermal) diamagnetic drifts.

While Eq. (7) also allows solutions with rapid toroidal flow,

$$V_i\sim v_{ti},$$

we do not consider the rapid-flow solutions here.[8,9]

D. Thermodynamic interpretation

We have derived Eq. (12) from force balance; however, it also can be interpreted thermodynamically.

Because magnetic fields do no work, true thermodynamic equilibrium, on time scales longer than the confinement time τ_c, cannot depend on the magnetic field. Magnetic confinement obviously pertains to time scales short compared to τ_c; thermodynamical methods are nonetheless applicable on such scales if they remain long compared to the Coulomb collision time. From Eq. (12) we see that on such an intermediate time scale the confining effect of the magnetic field enters force balance through the effective potential (interaction Lagrangian):[10]

$$\Phi_*=\Phi-c^{-1}\mathbf{V}_i\cdot\boldsymbol{A}.$$

Inserting Eq. (9) and recalling Eq. (2), we find $\Phi_*=\Phi+(\omega_i/c)\psi$, so that Eq. (12) can be expressed as

$$en\nabla\Phi_*+T_i\nabla n=0. \tag{15}$$

In the Appendix we show that this relation expresses the constancy of the chemical potential in a Maxwellian, isothermal plasma. Thus Eq. (12) can be interpreted as an expression of thermodynamic equilibrium.

III. ELECTRON EQUILIBRIUM

A. Lowest order form

We can construct an electron distribution that depends only on constants of the motion in precisely the same way: choosing f_e to have the form analogous to Eq. (6) (with the obvious changes in mass and charge) satisfies the left-hand side of the electron version of Eq. (5), as before. Thus the electron fluid has the lowest-order velocity

$$\boldsymbol{V}_{0e}=\omega_eR^2\nabla\zeta \tag{16}$$

with

$$\omega_e=-c\frac{d\Phi}{d\psi}+\frac{cT_e}{en}\frac{dn}{d\psi}. \tag{17}$$

Similarly electron force balance is described in lowest order by

$$-en\left(\frac{\nabla\Phi+\omega_e\nabla\psi}{c}\right)+T_e\nabla n=0. \tag{18}$$

It follows in particular that

$$n=n_0\exp\left[\frac{-e(\Phi_0-\Phi)}{T}+\frac{e\omega_e\psi}{cT}\right]. \tag{19}$$

We show presently that this result is consistent with quasineutrality and Eq. (10).

These results depend upon the neglect of unlike-species friction, included in the operator C_{ei}. We therefore use Eq. (11) to estimate

$$C_{ei}\sim f_e|\boldsymbol{V}_i-\boldsymbol{V}_e|/(v_{te}\tau_{ei})\sim f_e\delta_e/\tau_{ei},$$

where τ_{ei} is the electron-ion collision time of Braginskii.[4] We compare this quantity to dominant terms in the kinetic equation,

$$\boldsymbol{v}\cdot\nabla f_e\sim f_e\omega_{te},$$

where $\omega_{te}\equiv v_{te}/L$ is the transit frequency, to conclude that the ordering $\delta_e\ll\omega_{te}\tau_{ei}$ or, more simply,

$$\Omega_e\tau_{ei}\gg 1 \tag{20}$$

allows consistent neglect of C_{ei}. Here $\Omega_e=eB/m_ec$ is the electron gyrofrequency.

The same ordering applies to the neglected friction force in the electron force-balance equation. Denoting the friction force by

$$\boldsymbol{F}_e=\int d^3v m_e\boldsymbol{v}C_{ei}(f), \tag{21}$$

we find that

$$\frac{F_e}{n}\sim m_e|\boldsymbol{V}_i-\boldsymbol{V}_e|\tau_{ei}\sim\frac{\delta_em_ev_{te}}{\tau_{ei}}.$$

Nonetheless the electron version of Eq. (6) is only an approximate solution to the electron equation—a less generally accurate solution than its ion counterpart. One reason for its inaccuracy is that electron scattering by ions, while small compared to convection, is fully comparable to the like-species operator C_{ee}. It follows that the electron counterpart to Eq. (6) will not be annihilated by the full electron collision operator. A second reason pertains to external momentum or energy sources, such as nonelectrostatic electric fields or current drive terms. While such driving terms in the ion kinetic equation are consistently ordered with C_{ie} and neglected in lowest order, they are typically comparable to collisional friction in the electron equation.

Corrections to the distribution functions for both species are considered in Section IV. We will find that the corrections are indeed small: the equilibrium description derived here

remains valid in lowest order. However we will also find that the electron correction terms have important effects in some contexts.

In this regard we emphasize the 0-subscript on $\boldsymbol{V}_{0e}$; we find in Sec. IV that the electron flow includes a parallel correction term $\boldsymbol{V}_{1e}=\boldsymbol{b}V_{1e}$, where $\boldsymbol{b}=\boldsymbol{B}/B$, of significant size. A 0-subscript is *not* required for the ion flow, which is given by Eq. (9) to sufficient accuracy.

B. Plasma current

The distributions given by Eq. (6) and its electron counterpart include the diamagnetic flow, as well as the return flow, parallel to the magnetic field, needed to keep $\nabla\cdot\boldsymbol{J}=0$. Thus the lowest-order flows, from Eqs. (12) and (18), satisfy the isothermal force balance equation,

$$-en(\omega_i-\omega_e)\nabla\psi=c(T_i+T_e)\nabla n, \tag{22}$$

whose left-hand side is the $\boldsymbol{J}\times\boldsymbol{B}$ force and whose right-hand side is the isothermal pressure gradient.

However, as we have remarked, toroidal rotation does not fully represent electron flow. We will find in Sec. IV that the correction to the electron parallel flow, V_{1e}, can be comparable to $R\omega_e$. (There is also a correction to the electron perpendicular flow, corresponding to higher-order contributions to the pressure tensor, but this is smaller by a factor of δ_e.) Thus the Isomak equilibrium is characterized by the current density

$$\boldsymbol{J}=en(\omega_i-\omega_e)R^2\nabla\zeta-\boldsymbol{b}enV_{1e}. \tag{23}$$

Notice that, because of the rigid-body character of the toroidal rotation, this current does not vanish on the magnetic axis. The parallel component is

$$J_\parallel=en(\omega_i-\omega_e)\frac{I}{B}-enV_{1e}. \tag{24}$$

Ampère's law implies that the radial derivative of the toroidal field, measured by I', is proportional to the poloidal current:

$$\boldsymbol{J}\cdot\nabla\theta=-\frac{c}{4\pi}\frac{II'}{q\psi'R^2},$$

where $q(r)\equiv\boldsymbol{B}\cdot\nabla\zeta/\boldsymbol{B}\cdot\nabla\theta$ is the safety factor. Therefore Eq. (23) implies

$$I'(r)=\frac{4\pi enV_{1e}}{c\psi'B}.$$

It follows that nV_{1e}/B is constant on flux surfaces: the first-order flow is divergence-free. (The lowest-order flow, $\boldsymbol{V}_{0e}$, is similarly divergence-free.)

Ampère's law also implies that the toroidal current is given by

$$\boldsymbol{J}\cdot\nabla\zeta=\left(\frac{c}{4\pi}\right)R^{-2}\Delta^*\psi,$$

where $\Delta^*\equiv R^2\nabla\cdot(R^{-2}\nabla)$. Thus, in view of Eq. (23), the Grad-Shafranov equation[11,12] in an isothermal tokamak has the form

$$\Delta^*\psi=\left(\frac{4\pi en}{c}\right)\left[R^2(\omega_i-\omega_e)-\frac{IV_{1e}}{B}\right]. \tag{25}$$

Ohmic current enters this equation through the last term on the right-hand side. However, even without such current, Eq. (25) has solutions characterized by

$$\beta_P\equiv\frac{8\pi nT}{B_P^2}\sim 1,$$

as follows from Eq. (22). While the same conclusion pertains to a conventional tokamak, the fact that the toroidal current does not vanish on the magnetic axis should make such self-driven equilibria more accessible than in the conventional case.

C. Density profile

We can solve Eq. (22) for the density

$$n=n_0\exp\left[-\frac{e\psi}{c}\left(\frac{\omega_i-\omega_e}{T_e+T_i}\right)\right]. \tag{26}$$

Thus, in the isothermal tokamak the plasma pressure is an exponential function of the poloidal flux. This profile can be seen to be consistent with Eq. (10), Eq. (19), and quasineutrality. By comparing Eq. (26) with either Eq. (10) or Eq. (19), one finds that the potential is given by

$$\frac{e(\Phi_0-\Phi)}{T_i}=\frac{T_e}{T_i+T_e}\left(\frac{\omega_i}{T_i}+\frac{\omega_e}{T_e}\right)\frac{e\psi}{c}. \tag{27}$$

Because the operator Δ^* is approximated by an ordinary Laplacian, it follows that the toroidal current $\boldsymbol{J}\cdot\nabla\zeta\propto\Delta^*\psi\propto\Delta^*\Phi$ is roughly proportional to the local charge density ("space charge").

IV. HIGHER-ORDER DISTRIBUTIONS

A. Exact kinetic equation

The exact kinetic equation, for either plasma species,

$$\frac{\partial f_s}{\partial t}+\boldsymbol{v}\cdot\nabla f_s+\frac{e_s}{m_s}(\boldsymbol{E}+c^{-1}\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f_s}{\partial\boldsymbol{v}}=C_s(f)$$

can be expressed as

$$\mathcal{L}_sf_s-C_{ss}f_s+\frac{e_s}{m_s}\boldsymbol{\mathcal{E}}\cdot\frac{\partial f_s}{\partial\boldsymbol{v}}=C_{s\bar{s}}(f_s,f_{\bar{s}}). \tag{28}$$

Here we have introduced the operator

$$\mathcal{L}_s\equiv\boldsymbol{v}\cdot\nabla+\frac{e_s}{m_s}(-\nabla\Phi+c^{-1}\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial}{\partial\boldsymbol{v}}, \tag{29}$$

as well as the notation

$$\mathcal{E}=\boldsymbol{E}+\nabla\Phi$$

for the nonelectrostatic part of the electric field. The like-particle (unlike-particle) collision operator is denoted by C_{ss} ($C_{s\bar{s}}$), so that

$$C_s = C_{ss} + C_{s\bar{s}}.$$

Finally the time derivative has been omitted since we are interested in a quasi-static state.

The moving-Maxwellian's described previously are solutions, for each species, to the equations

$$\mathcal{L}_s f_{0s}(U,\psi*) = 0 = C_{ss}(f_{0s}(U,\psi*)),$$

where

$$f_0(\boldsymbol{v}-\boldsymbol{V}) = N\pi^{-3/2}v_t^{-3}e^{-U/T}e^{-e\omega\psi_*/cT}.$$

The first-order distribution f_1 is defined by

$$f = f_0 + f_1.$$

To insure that the induced electric field is small compared to the Dreicer field, we assume

$$\left(\frac{e}{mv_{te}\nu_{ei}}\right)\mathcal{E} \sim \frac{f_1}{f_0}.$$

Then the linearized equation for f_1 has the form

$$\mathcal{L}_s f_{1s} - C_s f_{1s} = -\frac{e_s}{m_s}\mathcal{E}\cdot\frac{\partial f_{0s}}{\partial \boldsymbol{v}} + C_{s\bar{s}}(f_0). \tag{30}$$

The right-hand side of Eq. (30) shows the physical processes—external forces and unlike-species collisions—that drive the corrections f_1. Additional momentum or energy inputs, such as current drive or wave heating, could be lumped with $\mathcal{E}$, which would then become velocity dependent. Indeed one might choose the form of such inputs to annihilate the driving terms; this possibility is explored in Sec. V.

B. Electron driving term

The induced electric field in a tokamak is toroidal:

$$\boldsymbol{\mathcal{E}} = \mathcal{E}_T R\nabla\zeta = \frac{1}{c}\frac{\partial\psi}{\partial t}\nabla\zeta$$

with a parallel component $\mathcal{E}_\parallel \equiv \boldsymbol{b}\cdot\boldsymbol{\mathcal{E}} = (I/BR)\mathcal{E}_T$. It follows that

$$\frac{e}{m_e}\boldsymbol{\mathcal{E}}\cdot\frac{\partial f_{0e}}{\partial \boldsymbol{v}} = -\frac{e}{T_e}R\mathcal{E}_T f_{0e}\boldsymbol{v}\cdot\nabla\zeta. \tag{31}$$

In $\partial f_{0e}/\partial\boldsymbol{v}$ we have neglected a term proportional to ω_e, assuming that $V_e \ll v_{te}$.

To understand the form of the election-ion collision operator, it is convenient to expand the electron moving Maxwellian f_{0e} using $V_e \ll v_{te}$:

$$f_{0e} = f_{Me} + f_{de},$$

where the leading term is a Maxwellian at rest,

$$f_{Me} \equiv f_{0e}(\boldsymbol{V}_e = 0)$$

and

$$f_{de} \equiv f_{Me}\frac{m_e\boldsymbol{V}_{0e}\cdot\boldsymbol{v}}{T_e},$$

where $\boldsymbol{V}_{e0}$ is defined by Eq. (16). Now the linearized electron-ion collision operator can be written as

$$C_{ei}(f_e,f_i) = C_{ei}(f_{Me},f_{Mi}) + C_{ei}(f_{1e},f_{Mi}) + C_{ei}(f_{de},f_{Mi}) + C_{ei}(f_{Me},f_{di}+f_{1i}), \tag{32}$$

where f_{Mi} represents an ion Maxwellian at rest and

$$f_{di} \equiv f_{Mi}\frac{m_i\boldsymbol{V}_{0i}\cdot\boldsymbol{v}}{T_i}.$$

The first term in Eq. (32) represents energy exchange between the two species; measured by $m_e/(m_i\tau_{ei})$, this term is henceforth neglected. The second term, involving f_{1e}, is conveniently moved to the left-hand side of Eq. (30), where it combines with C_{ee} in a familiar way: we define the linearized electron collision operator

$$C_e^\ell = C_{ee}(f_{Me},f_{1e}) + C_{ee}(f_{1e},f_{Me}) + C_{ei}(f_{1e},f_{Mi}).$$

The remaining two terms on the right-hand side of Eq. (32) can be evaluated from the well-known, small mass-ratio form of the operator,[4] with the result

$$C_{ei}(f_{de},f_{Mi}) + C_{ei}(f_{Me},f_{di}+f_{1i}) = \frac{\nu_{ei}v_{te}}{v^3}f_{Me}(\boldsymbol{v})\boldsymbol{v}\cdot(\boldsymbol{V}_i - \boldsymbol{V}_{0e}), \tag{33}$$

where we use a convenient abbreviation

$$\nu_{ei} \equiv \frac{3\sqrt{\pi}}{2\tau_{ei}}.$$

In combining these results it is convenient to express the velocity coordinate as

$$\boldsymbol{v} = \boldsymbol{b}u + \boldsymbol{v}_\perp$$

and to recall Eq. (24) for the parallel current density. Then we have

$$\frac{e}{m_e}\boldsymbol{\mathcal{E}}\cdot\frac{\partial f_{0e}}{\partial\boldsymbol{v}} + C_{ei}(f_{0e}) = f_{0e}(uQ_e + \boldsymbol{v}_{\perp\zeta}Q_{\perp e}), \tag{34}$$

where $\boldsymbol{v}_{\perp\zeta} = \boldsymbol{v}_\perp\cdot(R\nabla\zeta)$,

$$Q_e = -\frac{e}{T_e}\mathcal{E}_\parallel + \frac{\nu_{ei}v_{te}}{v^3}\frac{I}{B}(\omega_i - \omega_e), \tag{35}$$

and

$$Q_{\perp e} = -\frac{e}{T_e}\mathcal{E}_T + \frac{\nu_{ei}v_{te}}{v^3}R(\omega_i - \omega_e).$$

To avoid massive electron runaway, the electric force due to $\mathcal{E}$ cannot exceed the collisional friction force:

$$e\mathcal{E} \sim \nu_{ei}m_e v_{te}\delta_e.$$

This estimate, when inserted into Eq. (31), confirms the ordering that was anticipated above.

C. Ion driving term

We represent electron scattering of ions by the operator

$$C_{ie}(f) = \frac{\boldsymbol{F}_e}{m_i n} \cdot \frac{\partial f}{\partial \boldsymbol{v}},$$

where $\boldsymbol{F}_e$ is the friction force given by Eq. (21). This form omits terms responsible for energy exchange (temperature equilibration), but otherwise captures the key physics. A familiar calculation[4] shows that, when both distributions are moving Maxwellians,

$$\boldsymbol{F}_e = \frac{m_e n}{\tau_{ei}}(\boldsymbol{V}_i - \boldsymbol{V}_{0e}).$$

Significantly, this simple form for the friction force is accurate only because the temperatures are constant. Neglecting the V_i term in $\partial f_{0i}/\partial bv$ as a small correction we have, in lowest order,

$$C_{ie}(f_{0i}) = -\frac{m_e}{T_i \tau_{ei}} f_{0i} \boldsymbol{v} \cdot (\boldsymbol{V}_i - \boldsymbol{V}_{0e}). \tag{36}$$

Combining this expression with the ion version of Eq. (31), we obtain the ion driving term

$$-\frac{e}{m_i}\boldsymbol{\mathcal{E}} \cdot \frac{\partial f_{0i}}{\partial \boldsymbol{v}} + C_{ie}(f_{0e}) = f_{0i}(uQ_i + \boldsymbol{v}_{\perp\zeta} Q_{\perp i}), \tag{37}$$

where

$$Q_i(r,\theta) = \frac{e}{T_i}\mathcal{E}_\parallel - \frac{m_e}{T_i \tau_{ei}}\frac{I}{B}(\omega_i - \omega_e) \tag{38}$$

and

$$Q_{\perp i} = \frac{e}{T_i}\mathcal{E}_T - \frac{m_e}{T_i \tau_{ei}} R(\omega_i - \omega_e).$$

Unlike its electron counterpart, Q_i is independent of velocity. In other respects, however, the two driving terms are very similar.

V. TAILORED EQUILIBRIA

A. Ion tailoring

It is instructive to demonstrate that it is possible to maintain the rigidly toroidally rotating Maxwellian by introducing sources and sinks in the kinetic equations. The appropriately chosen momentum sources and sinks must modify the drive Q's on the right side of the gyroaveraged kinetic equations for the perturbed ion and electron distribution functions to make the new drive vanish. The gyrophase dependent contributions from the Q's are smaller in the gyroradius expansion so there is no need to modify them. They only result in small higher order corrections to the perpendicular flows of the order of the classical electron particle flux and Ware-Galeev pinch effect.[13,14]

The sources and sinks required to remove the gyrophase independent terms in the drive are extremely difficult to obtain in an actual experiment, but they provide an ideal to strive toward for isothermal operation. Moreover, for numerical work it is possible to make these choices to see if a code is behaving properly. In the absence of sources or sinks the kinetic equations for the perturbed distribution functions can be solved by standard techniques as outlined in Sec. VI.

We first consider the gyroaveraged perturbed ion kinetic equation. To keep the ions Maxwellian we need to add a momentum source S_i that modifies Q_i and that depends only on the spatial radial and poloidal variables; no velocity dependence is allowed. This source, to be inserted on the right-hand side of the perturbed ion kinetic equation, has the form

$$S_i = v_\parallel \frac{\mathcal{F}_i f_{Mi}}{n T_i},$$

where the force density $\mathcal{F}_i$ that must be applied to make f_{1i} vanish and thereby maintain a rigidly rotating Maxwellian is

$$\mathcal{F}_i = \frac{m_e n}{\tau_{ei} B} I(\omega_i - \omega_e) - en\mathcal{E}_\parallel.$$

In an experiment the ion momentum input would have to come from specially tailored neutral beams or radio frequency driven ions, and the need for a precise form would, of course, present a formidable practical challenge. However, with this choice the ions remain Maxwellian to an extremely high degree and are therefore only subject to collisional ambipolar particle transport. Thus approaching this ideal would appear to carry substantial benefit.

In addition we recall that constant ion temperature insures there will be no ion temperature gradient driven modes. Consequently, the ions are expected to drive much less anomalous transport—something that could be verified by numerical simulation.

To evaluate the particle transport associated with the gyrophase dependent drive on the right side, we solve the lowest-order kinetic equation to find the leading gyrophase-dependent part of the perturbed ion distribution function, namely,

$$\tilde{f}_{1i} = -\frac{Q_{\perp i}}{\Omega_i} f_{0i} R \nabla \zeta \cdot \boldsymbol{v} \times \boldsymbol{b},$$

where $\Omega_i = eB/m_i c$ is the ion gyrofrequency (with e the magnitude of the charge on an electron). The associated perpendicular particle flux is

$$\Gamma_\perp \equiv \int d^3 v \boldsymbol{v} f_{1i} = -\frac{p_i Q_{\perp i}}{m_i \Omega_i R B} \nabla \psi = -\frac{(T_i + T_e)}{m_e \Omega_e^2 \tau_{ei}} \nabla n - \frac{cn\mathcal{E}_T}{RB^2} \nabla \psi, \tag{39}$$

representing classical perpendicular transport in combination with the classical pinch.

B. Electron tailoring

Next we consider the gyrophase independent portion of the perturbed electron kinetic equation, where inserting a velocity and spatially dependent momentum *sink* is required to maintain Maxwellian electrons. The required electron momentum sink to be added to Q_e has the form

$$S_e = v_\parallel \frac{\mathcal{F}_e f_{Me}}{nT_e},$$

where the velocity-dependent force density $\mathcal{F}_e$, needed to make f_{1e} vanish and maintain a rigidly rotating electron Maxwellian, is given by

$$\mathcal{F}_e = \frac{3\sqrt{\pi} m_e v_{te}^3}{4e^2 n \tau_{ei} v^3 B} I(\omega_i - \omega_e) - en\mathcal{E}_\parallel.$$

In an actual experiment electron momentum would have to be removed from the very slow electrons and added to the very fast ones by carefully controlled radio frequency interactions with the electrons—a task that is unlikely to be possible in practice. If it were possible, the electrons would not only be free from electron temperature-gradient-driven anomalous transport, but they would remain Maxwellian and presumably only subject to classical collisional particle transport and the Ware-Galeev pinch. To see this we next consider the lowest-order gyrophase dependent solution to the perturbed electron kinetic equation

$$\tilde{f}_{1e} = -\frac{Q_{\perp e}}{\Omega_e} f_{0e} R \nabla \zeta \cdot \boldsymbol{v} \times \boldsymbol{b},$$

where $\Omega_e = eB/m_e c$. Forming the perpendicular particle flux gives the expected result:

$$\Gamma_\perp = \frac{v^2 Q_{\perp e}}{3\Omega_e RB} \nabla \psi = -\frac{(T_i + T_e)}{m_e \Omega_e^2 \tau_{ei}} \nabla n - \frac{cn\mathcal{E}_T}{RB^2} \nabla \psi.$$

Finally we notice that if the induced electric field vanishes, the ion momentum source and electron momentum sink forces needed for steady state operation approach

$$\mathcal{F}_i \to \frac{m_e n}{\tau_{ei} B} I(\omega_i - \omega_e), \tag{40}$$

$$\mathcal{F}_e \to \frac{3\sqrt{\pi} n m_e v_{te}^3}{4\tau_{ei} B v^3} I(\omega_i - \omega_e). \tag{41}$$

These choices are required to maintain steady state operation of an isothermal tokamak with toroidally rotating Maxwellian ions and electrons. A particle source is also required, but this source is too small to enter the kinetic equation to the order considered.

C. Energy transport

To verify that there is no like-particle collisional energy transport associated with the preceding choices for the sources and sinks we can use a standard moment approach to write the collisional part of the perpendicular ion radial heat flux as

$$\left\langle \int d^3v \left(\frac{m_i v^2}{2}\right) f_i \boldsymbol{v} \cdot \nabla \psi \right\rangle = -\left(\frac{m_i^2 c}{2e}\right) \left\langle \int d^3v R^2 \nabla \zeta \cdot \boldsymbol{v} v^2 C_i^\ell (f_{1i}) \right\rangle,$$

where the angular brackets denote a flux surface average. We have tailored the ion momentum source to make the gyroaveraged part of f_{1i} vanish, and $C_i^\ell(\tilde{f}_{1i}) = 0$ by momentum conservation in ion-ion collisions (recall that $Q_{\perp i}$ does not depend on velocity). Therefore, as expected, no collisional neoclassical or classical ion heat flux occurs:

$$\boldsymbol{q}_{\perp i} = \int d^3v f_i \boldsymbol{v}_\perp \frac{(m_i v^2 - 5T_i)}{2} = 0.$$

Turning to the electrons, a similar procedure yields, for the collisional portion of the perpendicular electron heat flux,

$$\left\langle \int d^3v (m_e v^2/2) f_e \boldsymbol{v} \cdot \nabla \psi \right\rangle - \left(\frac{m_e^2 c}{2e}\right) \times \left\langle \int d^3v R^2 \nabla \zeta \cdot \boldsymbol{v} v^2 C_e^\ell (f_{1e}) \right\rangle.$$

The electron momentum sink makes the gyroaveraged part of f_{1e} vanish so we need only consider the gyrophase dependent terms $\tilde{f}_{1e}$ in the collision operators. But the rotational invariance of the linearized collision operators means that $C_e^\ell(\tilde{f}_{1e}) = C_{ee}^\ell(\tilde{f}_{1e}) + C_{ei}^\ell(\tilde{f}_{1e}) \propto \boldsymbol{v} \times \boldsymbol{b} \cdot \nabla \zeta$. Therefore the gyrophase average annihilates the $\tilde{f}_{1e}$ terms as well. We conclude, as anticipated, that there is no classical or neoclassical like-particle contribution to the perpendicular heat flux in the Isomak.

When unlike-particle collisions are included, the form of $\tilde{f}_{1i}$ shows that the perpendicular ion heat flux remains zero. In view of Eq. (22), there remains an off-diagonal, inward classical electron heat flux[4]

$$\boldsymbol{q}_{\perp e} = -\frac{3nT_e(\omega_i - \omega_e)}{\Omega_e \tau_{ei} B} \nabla \psi = \frac{3nT_e(T_i + T_e)}{2m_e \Omega_e^2 \tau_{ei}} \nabla n.$$

D. Partial tailoring

The preceding discussion assumes that precise sources and sinks can be introduced to maintain the ion and electron distribution functions as drifting Maxwellians with small gyrophase corrections. However, less special choices are possible that allow small gyrophase independent corrections $\bar{f}_{1s}$ to the Maxwellians having prescribed moments that vanish. For example, if an ion momentum source can be introduced such that

$$\left\langle \int d^3v R^2 \nabla \zeta \cdot \boldsymbol{v} v^2 C_i^\ell (\bar{f}_{1i}) \right\rangle = 0, \tag{42}$$

then there will still be no neoclassical or classical ion heat flux. [The rotational invariance of $C_i^\ell(\tilde{f}_{1i})$ insures that there will be no collisional contribution from $\tilde{f}_{1i}$.] Once this integral condition is imposed, the non-Maxwellian contribution to f_i cannot yield radial ion heat flux. Hence one has considerable freedom in choosing the applied ion force density $\mathcal{F}_i$: for any function that satisfies the integral constraint Eq. (42), there is no radial ion heat flux, and f_{1i} need not vanish.

Similar freedom exists for the electrons: if the electron force density $\mathcal{F}_e$ satisfies the electron version of Eq. (42), then the only electron transport process is the frictional, off-diagonal heat flux. Alternatively, one could choose $\mathcal{F}_e$ to

tailor the current density. For example, by choosing $\mathcal{F}_e$ to cancel the induced electric term we can make the Isomak behave as if it were a bootstrap-driven tokamak.

VI. PERTURBATION THEORY

A. Small gyroradius limit

Here we consider the solution to Eq. (30) in the limit of vanishing gyroradius: $\delta \to 0$.

To begin we transform velocity coordinates in Eq. (30),

$$(\boldsymbol{x},\boldsymbol{v}) \to (U,\mu,\gamma,\boldsymbol{x}),$$

where U is the total energy as before, $\mu=(1/2B)mv_\perp^2$ is the magnetic moment, and γ is the gyrophase angle. The operator $\mathcal{L}$ is therefore replaced by

$$\mathcal{L}f_1 = \boldsymbol{v}(U,\mu,\gamma)\cdot\nabla f_1 + \dot{U}\frac{\partial f_1}{\partial U} + \dot{\mu}\frac{\partial f_1}{\partial \mu} + \dot{\gamma}\frac{\partial f_1}{\partial \gamma},$$

where

$$\boldsymbol{v}(U,\mu,\gamma) = \boldsymbol{b}u(U,\mu) + \boldsymbol{v}_\perp(\mu,\gamma).$$

The fact that the gyrophase changes on the fastest time scale,

$$\dot{\gamma} \sim \Omega,$$

implies that the gyrophase dependence of f_1 is weak. Hence the zero-gyroradius limit of our kinetic equation is obtained by performing a gyrophase average and neglecting contributions from the gyrophase dependence of f_1. Since the $\dot{U}$ and the gyrophase average of $\dot{\mu}$ are small, we obtain in this way the lowest-order guiding-center kinetic equation,

$$u\nabla_\parallel f_1 - C^\ell(f_1) = uf_0Q, \tag{43}$$

where the function Q is given by Eq. (35) for the electrons and Eq. (38) for the ions. Note that the functions $Q_\perp$ no longer appear: the coefficient $\boldsymbol{v}_{\perp\zeta}$ is annihilated by the gyrophase average.

It is now convenient to transform velocity variables again. The point is that the only motion entering Eq. (43) is parallel streaming, and the electrostatic potential is constant under such motion. Therefore we can replace U by the kinetic energy variable $w=U-e\Phi=mv^2/2$; it is also convenient to replace μ by the pitch-angle variable $\lambda=\mu/w$. After this transformation,

$$(\boldsymbol{x},U,\mu) \to (\boldsymbol{x},w,\lambda),$$

the parallel velocity becomes

$$u = \sigma\sqrt{(2/m)w}\,\xi, \tag{44}$$

where

$$\xi \equiv \sqrt{1-\lambda B} \tag{45}$$

and $\sigma=\pm 1$.

We recall here the critical value of λ,

$$\lambda_c = \min\left\{\frac{1}{B}\right\},$$

that delineates the trapped ($\lambda \geqslant \lambda_c$) and passing ($\lambda < \lambda_c$) regions of phase space. In the trapped region, one defines the bounce angle θ_b by

$$u(w,\lambda,\pm\theta_b) = 0.$$

The region $|\theta| > \theta_b$ is not accessible to trapped particles.

B. Orbital average

The orbital average, or trajectory average, of some function f on phase space is denoted by $\langle f\rangle_o$. Its key property is that it annihilates the parallel-streaming term in the kinetic equation: for any function f that satisfies the bounce condition,

$$\langle u\nabla_\parallel f\rangle_o = 0. \tag{46}$$

The bounce condition, stating that f evaluated at the bounce angle must be even in σ, is necessarily satisfied by any physical distribution function in the trapped region.

The orbital average is closely related to an ordinary flux-surface average, which we denote by $\langle\ldots\rangle_\theta$. However, the latter must be modified to take into account the inaccessibility of the full range of θ in the trapped region. Thus we write

$$\langle f\rangle_\theta = \int_{-\theta_b}^{\theta_b}\frac{d\theta}{2\theta_b}\frac{f}{\boldsymbol{B}\cdot\nabla\theta}.$$

Notice that this becomes the usual flux-surface average in the passing region: $\lambda<\lambda_c \Rightarrow \theta_b=\pi$.

The explicit expression of the orbital average has different forms in the trapped and passing regions of phase space.[15] In the passing region, the orbital average is given by

$$\langle f\rangle_o = \frac{\langle fB/u\rangle_\theta}{\langle B/u\rangle_\theta}, \quad \text{for } \lambda < \lambda_c.$$

In the trapped region, in order to satisfy Eq. (46), we must also include a sum over σ:

$$\langle f\rangle_o = \frac{1}{2}\sum_\sigma \frac{\langle fB/|u|\rangle_\theta}{\langle B/|u|\rangle_\theta}, \quad \text{for } \lambda > \lambda_c.$$

One property of the orbital average is especially useful: if the function g has the form

$$g(r,\theta,w,\lambda,\sigma) = uG(r,\theta,w),$$

then

$$\langle g\rangle_o = \Theta(\lambda_c-\lambda)\sigma\frac{\langle GB\rangle_\theta}{\langle B/|u|\rangle_\theta}, \tag{47}$$

where Θ is a step function.

C. Relation to conventional theory

The dominant distributions f_0, which constitute the main new result of this work, are fully nonlinear solutions to the Boltzmann kinetic equation, for arbitrary gyroradius. These solutions exist only in the isothermal state and are very different from the distribution functions studied in neoclassical

kinetic theory: the latter are linearized about stationary Maxwellians, and are valid only asymptotically, in the limit of small gyroradius.

However, our analysis of the correction distribution functions f_1 uses a linear perturbation theory that is clearly similar to the neoclassical version. It is therefore not surprising that our correction function f_1 bears a simple relation to the corresponding first-order distribution, denoted here by f_{nc}, that is studied in conventional theory,[16,17] viz.,

$$f_1 + \frac{mI}{TB} u\omega = f_{nc}.$$

Indeed, after expressing Eq. (43) in terms of f_{nc}, it is straightforward to show that the equation coincides with the isothermal limit of the linearized kinetic equation solved in the neoclassical literature.

A well-known neoclassical result is that the ion perturbation has the form

$$f_{nci} = \frac{m_i I}{T_i B} u\omega_i + g_i,$$

where the function g_i is proportional to the ion temperature gradient. It follows that $g_i = f_{1i} = 0$ in the isothermal case. This is consistent with the present analysis, since Eq. (43), and the fact that $\tau_i \sim \tau_{ii} \sim (m_i/m_e)^{1/2} \tau_{ei}$ imply

$$f_{1i} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \frac{|\mathbf{V}_i - \mathbf{V}_e|}{v_{ti}} f_{0i}.$$

Thus standard neoclassical theory omits the correction f_{1i} as being small in the square root of the mass ratio.

The electron case is more interesting. We have to solve

$$u\nabla_\parallel f_{1e} - C_e^\ell(f_{1e}) = u f_{Me}\left[-\frac{e}{T_e}\mathcal{E}_\parallel + \frac{\nu_{ei} v_{te}}{v^3}\frac{I}{B}(\omega_i - \omega_e)\right].$$

It can be inferred[4] from Eq. (33), together with the definition of the Spitzer function f_S,[18]

$$C_e^\ell(f_S) = \frac{e}{T_e}\mathcal{E}_\parallel u f_{Me},$$

that this kinetic equation is equivalent to

$$u\nabla_\parallel f_{1e} - C_e^\ell\left[f_{1e} - \frac{m_e}{T_e}\frac{I}{B}u(\omega_i - \omega_e) f_{Me} + f_S\right] = 0, \tag{48}$$

whose orbital average,

$$\left\langle C_e^\ell\left[f_{1e} - \frac{m_e}{T_e}\frac{I}{B}u(\omega_i - \omega_e) f_{Me} + f_S\right]\right\rangle_o = 0, \tag{49}$$

is the isothermal version of a well-known constraint, sometimes called the "banana constraint," of neoclassical theory. Thus the function f_{1e} contains all the familiar effects of toroidicity on transport at low collision frequency: the (isothermal version of the) bootstrap current enters through the term involving $\omega_i - \omega_e$, the conductivity reduction effect enters through the Spitzer function, and so on.

In other words, the present formulation reproduces the isothermal version of all neoclassical processes.

VII. SUMMARY

The strongest classical or neoclassical transport process in a tokamak is ion thermal conduction. The ion radial heat flow, which is proportional to the ion temperature gradient, is faster than all other transport processes by a factor of $(m_i/m_e)^{1/2}$. It is therefore obvious that a tokamak without temperature gradients will display relatively weak collisional transport.

The present work has explored a circumstance that is less obvious and more significant: that the ion kinetic equation can be solved exactly in the isothermal, axisymmetric case. Our nonlinear solution Eq. (6) is valid for arbitrary gyroradius, arbitrary collisionality, and arbitrary aspect ratio. The corresponding solution for the electrons is not exact in this strong sense, because it neglects electron friction with ions, but is nonetheless a close approximation, especially at low collisionality.

The equilibrium implied by this exact solution is characterized by the following:

1. rigid-body ion toroidal rotation, as shown by Eq. (14);
2. diamagnetic current that remains finite on the magnetic axis, as shown by Eq. (23);
3. a density profile that depends exponentially on the poloidal flux, as shown by Eq. (26);
4. a specified radial electrostatic potential profile that is proportional to the poloidal flux, as shown by Eq. (27).

Note that all of these properties differ markedly from the conventional tokamak case.

Although it is likely to support certain microinstabilities, such as the TEM mode, one would expect the isothermal state to be relatively stable and less prone to plasma turbulence. The isothermal state has fewer sources of free energy; indeed, as noted in Eq. (15), it is close to thermodynamic equilibrium in an important sense.

In addition to displaying the nonlinear isothermal distribution function and examining the key features of the associated equilibrium, we have examined the corrections to the distribution function that are needed to account for unlike-species collisions. These corrections are small for both species, but negligible only for the ions. We show that they can be computed by straightforward extension of standard neoclassical procedures.

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APPENDIX: THE CHEMICAL POTENTIAL

Here we compute the chemical potential for a Maxwellian plasma, thus verifying the interpretation of Eq. (15). We begin with the well-known Maxwellian entropy density[4]

$$s = n \log\left(\frac{T^{3/2}}{n}\right) + nK,$$

where the constant K is determined by quantum mechanics but arbitrary here. Convenient thermodynamic variables are the density n and the internal energy density $u=(3/2)nT(n,u)+en\Phi(\boldsymbol{x})$. Thus

$$T(n,u) = \frac{2}{3}\left(\frac{u}{n} - e\Phi\right).$$

Note that the scalar field Φ is a function of position only and not a thermodynamic potential.

The chemical potential μ is defined by[19]

$$\frac{\mu}{T} = \frac{\partial s}{\partial n},$$

where the derivative is taken at constant u. Thus, after absorbing all constants multiplying n into K, we find

$$s = -n \log n + \frac{3}{2} n \log\left(\frac{u}{n} - e\Phi\right) + nK$$

and

$$\mu = KT + T \log\left(\frac{T^{3/2}}{n}\right) - e\Phi.$$

Therefore, when T is constant,

$$\nabla\mu = -T\nabla \log n - e\nabla\Phi. \tag{A1}$$

To reproduce Eq. (15) we simply replace Φ by the effective potential Φ_*.

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