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Exact nonlinear solution for the Hall–Alfvén wave in partially ionized plasmas

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Abstract. It is shown that the exact nonlinear solution for the Hall–Alfvén waves can be obtained in a uniformly rotating weakly ionized plasma such as those which exist in various types of accretion disks. In addition this piece of work demonstrates a method of eliminating the inaccuracies embedded in the literature on this subject.

1. Introduction

The Alfvén waves have been invoked in various astrophysical situations for their diverse roles. They can accelerate particles such as in cosmic rays, heat plasmas such as in the Solar corona, and provide models for magnetohydrodynamic (MHD) turbulence such as in the Solar wind [1, 2]. The Alfvén waves of large amplitude are required for all of these purposes. The nonlinear couplings of the waves transfer energy to shorter spatial scales so that they can resonantly interact with the plasma particles resulting in their heating [3]. The heating processes becomes particularly favorable at short wavelengths. One way of obtaining short wavelength modes is to go beyond the ideal MHD and include non-ideal effects such as the Hall effect [4]. It is well known that the Alfvén waves are the exact solutions of the ideal MHD system. We have recently shown that the Alfvén waves of arbitrary amplitude form the normal modes even in the presence of the Hall effect [5]. In this paper we explore such a possibility for partially ionized rotating plasmas. The motivation behind this investigation is to study MHD processes in accretion disks such as the protoplanetary and protostellar disks. These disks have a low degree of ionization and therefore are weakly coupled with the magnetic field. The magnetic coupling, on the other hand, is absolutely essential for the outward transport of the angular momentum through processes such as the the magneto-rotational instability [6]. The Alfvén wave heating of protoplanetary disks has been an active area of study [7, 8] for additional heating and ionization in the disks. We determine the linear dispersion relation of the Hall–Alfvén waves in a uniformly rotating weakly ionized plasma including the electrical resistivity in Sec. 1. It is demonstrated in Sec. 2 that the linear dispersion relation is also valid in the nonlinear regime. We thus obtain the linear damping of these nonlinear waves in contrast to the earlier work [7, 8] on nonlinear damping of the linear waves and end the paper with some concluding remarks.

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2. Hall–Alfvén waves

The partially ionized plasma in the protoplanetary disks consists of electrons, heavy ions such as those of potassium and neutral hydrogen molecules along with a small percentage of helium molecules. The ions and the neutrals are coupled through collisions. The dynamics of such a system is essentially controlled by the neutrals, with the ions and electrons being treated as inertia-less [9, 10]. We write the equations in a dimensionless form by normalizing (as used in the astrophysical literature [9, 10] emphasizing the dynamics of the neutrals) the time and the space variables, respectively, with the gyro-period $\omega_{\rm cn}^{-1} = m_{\rm n}c/eB_0$, and the inertial length $\lambda_{\rm n} = c/\omega_{\rm pn}$, where $\omega_{\rm pn} = (4\pi e^2 n/m_{\rm n})^{1/2}$ is the plasma frequency and n and $m_{\rm n}$ are respectively the density and the mass. The index n refers to the neutral particles. The magnetic and the velocity fields are respectively normalized to the uniform ambient field B_0 and the Alfvén speed $V_{\rm A} = B_0/\sqrt{4\pi\rho}$, where ρ is the uniform mass density. Typically the Alfvén Mach number is larger than one for assumed subthermal magnetic fields. The resistivity η is normalized by $(\lambda_{\rm n}^2\omega_{\rm cn})$ In these units the following dimensionless equations

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [(\mathbf{V} - \epsilon\nabla \times \mathbf{B}) \times \mathbf{B}] + \eta\nabla^2\mathbf{B} \tag{2.1}$$

$$\frac{\partial(\nabla \times \mathbf{V})}{\partial t} = \nabla \times [\mathbf{V} \times (\nabla \times \mathbf{V}) - \mathbf{B} \times (\nabla \times \mathbf{B})] \tag{2.2}$$

constitute the dissipative Hall-MHD in the incompressible limit. Here $\epsilon = n/n_{\rm e}$ (In the approximate range 10^5–10^{10} in protoplanetary disks) and $n_{\rm e}$ is the electron density. Equation (2.2) has been obtained by taking the curl of the equation of motion of the neutral component. Thus, the neutral fluid behaves like a charged fluid due to the strong coupling with the ions. The electrical resistivity η is provided by the electron–neutral and the electron–ion collisions, the former being the dominant in weakly ionized plasmas. The equilibrium of the system is described by $\mathbf{B}_0 = \widehat{e}_z, \mathbf{V}_0 = r\Omega\widehat{e}_\theta$ with constant Ω and constant densities. We split the fields into their ambient and the fluctuating parts:

$$\mathbf{B} = \widehat{e}_z + \mathbf{b}; \quad \mathbf{V} = \mathbf{V}_0 + \mathbf{v} \tag{2.3}$$

and substitute in (2.1) and (2.2) to get

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times [(\mathbf{v} - \epsilon\nabla \times \mathbf{b}) \times \widehat{e}_z + \mathbf{V}_0 \times \mathbf{b} + (\mathbf{v} - \epsilon\nabla \times \mathbf{b}) \times \mathbf{b}] + \eta\nabla^2\mathbf{b} \tag{2.4}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) = \nabla \times [&\mathbf{v} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{V}_0) \\ &+ \mathbf{V}_0 \times (\nabla \times \mathbf{v}) + (\nabla \times \mathbf{b}) \times \widehat{e}_z + (\nabla \times \mathbf{b}) \times \mathbf{b}].\end{aligned} \tag{2.5}$$

The linearized equations in cylindrical geometry turn out to be

$$\frac{\partial}{\partial t}\mathbf{b} = (\widehat{e}_z \cdot \nabla)[\mathbf{v} - \epsilon\nabla \times \mathbf{b}] - \Omega\mathbf{Z} + \eta\nabla^2\mathbf{b}, \tag{2.6}$$

$$\frac{\partial}{\partial t}\nabla \times \mathbf{v} = (\widehat{e}_z \cdot \nabla)[2\Omega\mathbf{v} + \nabla \times \mathbf{b}] - \Omega\mathbf{Y} \tag{2.7}$$

where

$$\mathbf{Z} = \widehat{e}_r\frac{\partial}{\partial\theta}\, b_r + \widehat{e}_\theta\frac{\partial}{\partial\theta}\, b_\theta + \widehat{e}_z\frac{\partial}{\partial\theta}\, b_z \tag{2.8}$$

and $\mathbf{Y}$ has the same form except that the components of $\mathbf{b}$ are replaced by those of $(\nabla \times \mathbf{v})$. In order to solve these equations we assume that $\mathbf{b} = \mathbf{b}(r)\exp(ikz + im\theta - i\omega t), \mathbf{v} = \mathbf{v}(r)\exp(ikz + im\theta - i\omega t)$ and substitute in (2.6) and (2.7) to arrive at

$$\mathbf{v} - \epsilon\nabla \times \mathbf{b} = -\frac{\omega_{\mathrm{m}}\mathbf{b}}{k} + i\frac{\eta}{k}\nabla^2\mathbf{b}, \tag{2.9}$$

$$2\Omega\mathbf{v} + \nabla \times \mathbf{b} = -\frac{\omega_{\mathrm{m}}}{k}\nabla \times \mathbf{v} \tag{2.10}$$

where $\omega_{\mathrm{m}} = (\omega - m\Omega)$ and ∇ is a full operator which has been retained for conciseness. Combining (2.9) and (2.10), we find

$$\nabla \times \nabla \times \mathbf{b} - (\omega_{\mathrm{m}}^2 - \beta - i\omega_{\mathrm{m}}\eta\nabla^2)(\epsilon\omega_{\mathrm{m}}k)^{-1}\nabla \times \mathbf{b} - 2\Omega\epsilon^{-1}[1 - i\eta\omega_{\mathrm{m}}^{-1}\nabla^2]\mathbf{b} = 0 \tag{2.11}$$

where $\beta = (2\Omega\epsilon + 1)k^2$. The solution of (2.11) is

$$\nabla \times \mathbf{b} = \alpha\mathbf{b} \tag{2.12}$$

and the corresponding linear dispersion relation is found to be

$$\omega_{\mathrm{m}}^2 + \omega_{\mathrm{m}}\left[\frac{2\Omega k}{\alpha} - \alpha\epsilon k + i\eta\alpha^2\right] - \beta + 2i\eta\alpha\Omega k = 0. \tag{2.13}$$

The eigenfunctions are the Chandrasekhar–Kendall functions described as

$$b_{\mathrm{r}} = \mu^{-2}\left(\frac{im\alpha b_z}{r} + ik\frac{\partial}{\partial r}b_z\right); \quad b_\theta = \mu^{-2}\left(-\alpha\frac{\partial}{\partial r}b_z - \frac{mkb_z}{r}\right); \quad b_z = AJ_{\mathrm{m}}(\mu r) \tag{2.14}$$

where $\mu^2 = \alpha^2 - k^2$ is to be interpreted as the radial wave number.

A similar dispersion relation was obtained in the slab geometry [11, 12]. If we drop the properties related to the cylindrical geometry in (2.13) by taking $\Omega = 0$ (the azimuthal mode number m automatically disappears since it appears only in the form $m\Omega$), we recover the dispersion relation of the Alfvén wave including constant resistivity in the slab geometry.

3. Exact nonlinear dispersion relation

Here we show that the solution (2.12) along with the linearized equations (2.9) and (2.10) for the fluctuations $\mathbf{v}$ and $\mathbf{b}$ are also the solution of the complete nonlinear equations (2.4) and (2.5). The nonlinear terms from (2.4) and (2.5) are $(\mathbf{v} - \epsilon\nabla\times\mathbf{b})\times\mathbf{b}$ and $(\mathbf{v} \times (\nabla \times \mathbf{v}) + (\nabla \times \mathbf{b}) \times \mathbf{b})$. One can easily check that the nonlinear terms vanish for the relationships of $\mathbf{b}$ and $\mathbf{v}$ given in (2.9), (2.10) and (2.12). Thus, we conclude that the dispersion relation (2.13) and the eigenfunctions (2.14) represent the exact solutions of the incompressible dissipative Hall-MHD of the weakly ionized uniformly rotating plasma for fluctuations of arbitrary amplitudes.

4. Hall–Alfvén waves in accretion disks

Accretion disks are generally differentially rotating systems at the Keplerian frequency $\Omega(r) = \sqrt{GMr^{-3}}$ where G is the gravitational constant, M is the mass of the central compact object such as a star and r is the radial position of the orbiting matter. The importance of the resistive and the Hall effects in protoplanetary weakly ionized disks has been recently emphasized [9, 10]. In [9, 10] as well as in more recent studies [13, 14], linear or nonlinear fluctuations with

only axial wave vector k_z along the ambient magnetic field B_0 are prescribed for wave propagation, neglecting completely the azimuthal and the radial variations in a cylindrical system. Even when the radial variation is considered, it has been done by Fourier analyzing in the radial coordinate. Now it is well known that an inhomogeneous system such as one rotating with speed $V_\theta = \Omega r$ even with constant Ω is not an autonomous system in r and should not be Fourier analyzed in the radial coordinate. We wish to point out that the complete neglect of the radial and the azimuthal variations violates the $\nabla \cdot \mathbf{b} = 0$ condition along with the z component of the induction equation. Both of these maladies can be cured as we have shown in the previous section. Our solution (2.12) automatically satisfies the divergence condition. Further we find that the radial variation can only be neglected if one retains the azimuthal variation and the azimuthal mode number m in this case must be equal to ± 1. This can be easily seen by writing the components of (2.12) for $\partial/\partial r = 0$. We see that the only consistent solution is $b_z = 0, \alpha = \pm k, m = \pm 1$; the corresponding dispersion relation being

$$\omega_{\mathrm{m}}^2 + 2\Omega\omega_{\mathrm{m}}\left[1 - \frac{\epsilon k^2}{2\Omega} + \frac{i\eta k^2}{2\Omega}\right] - [(1 + 2\Omega\epsilon)k^2 - 2i\Omega\eta k^2] = 0 \tag{4.1}$$

and the eigenfunctions are $\mathbf{b} = \text{constant}\ [\widehat{e}_r + i\widehat{e}_\theta]\exp(-i\omega t + i\theta + ikz)$ representing circularly polarized waves. The constancy of Ω is a good assumption over the wavelength scales of interest for the microscopic processes. We find that these nonlinear waves are strongly damped for the relevant parameters. Thus, if somehow these waves could be excited, they can damp and heat the plasma to further ionize it.

5. Conclusion

We have furnished the correct analysis of the Hall-MHD of rotating weakly ionized dissipative plasmas, a system of extreme importance for accretion disks of various types. The exact nonlinear solution of the wave will go way beyond the usual linear and the so-called local solutions in delineating the nonlinear phenomena in accretion disks and their role in heating and ionizing the cold weakly ionized disks.

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