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# Equilibrium through self-organization and a stability condition derived from generalized self-organization theory

KONDOH YOSHIOMI[1], TAKAHASHI TOSHIKI[1] and JAMES W. VAN DAM[2]

[1]Deptartment of Electronic Engineering, Gunma University, Kiryu, Gunma 376-8515, Japan
(*kondohy@el.gunma-u.ac.jp*)

[2]Institute for Fusion Studies, The University of TX at Austin, TX 78712, USA

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**Abstract.** The equilibria and stability of the least-dissipated relaxed state are derived from a generalized theory of self-organization for open or closed general nonlinear and dissipative dynamical systems. An application of the general theory is presented for the two-fluid model of plasmas. The concept of selective decay, together with helicity invariance, is analytically shown to have no connection with relaxed states derived from helicity invariance.

## 1. Introduction

A recent generalized theory of self-organization for finding the decaying self-similar states [1–3] has been shown to incorporate a previous theory [4] for obtaining the minimum dissipative state of magnetic energy, by means of which the 'Taylor state' [5] $\nabla \times \mathbf{B} = \lambda \mathbf{B}$ can be derived. The generalized theory is also applicable to other relaxed states, such as non-Taylor states with non-uniform resistivity [6], and to dissipative dynamical structures, such as the soliton solutions of the viscid Korteweg–de Vries equation [7] and the vortex solutions of the two-dimensional incompressible viscous fluid equations [8, 9]. In this paper, we describe equilibrium states and their stability condition as derived from this generalized theory; we also present an application of the theory to the two-fluid model of plasmas. We also critique the traditional theories in [5, 10–12] to derive 'relaxed states' by showing that the concept of selective decay together with that of helicity invariance is unrelated to relaxed states (cf. [13]). Note that their theories are neither based on a variational principle, used in the classical mechanics, nor on an energy principle, used in an ideal Magnetohydrodynamic (MHD) plasma [14], either of which leads to dynamical equations for the time evolution of the system of interest.

## 2. Equilibria and stability of relaxed states

First we briefly present the generalized theory of self-organization [1–3]. The generalized theory considers a set of $N$ dynamical variables $\mathbf{q} \equiv \mathbf{q}[\xi^k] \equiv (q_1[\xi^k], \ldots, q_N[xi^k])$, with $M$-dimensional independent variables $[\xi^k]$ $(k = 1, 2, \ldots, M)$. It deals with a general nonlinear set of $N$ simultaneous equations, for an open or closed dynamical system, written as follows:

$$\partial q_i[\xi^k]/\partial \xi^j = D_i^j[\mathbf{q}], \tag{2.1}$$

where $\xi^j$ is a fixed independent variable among $[\xi^k]$, such as time $t$, and $D_i^j[\mathbf{q}]$ $(i = 1, 2, \ldots, N)$ represents dynamical operators that include both nonlinear and dissipative terms for the change of $q_i$ along $\xi^j$. Since the self-organized states must be the configurations that change the least along $\xi^j$ during the evolution of the dynamical system, the best algorithm to judge and identify the self-organized states is that of finding those states for which the rate of change of global auto-correlations for multiple dynamical field quantities is minimized:

$$\min\left|\frac{\int q_i[\xi^j]q_i[\xi^j + (\triangle\xi^j/\tau_{ci})]|J_{k\neq j}|\prod_{k\neq j} d\xi^k}{\int (q_i[\xi^k])^2|J_{k\neq j}|\prod_{k\neq j} d\xi^k} - 1\right|. \tag{2.2}$$

From (2.1) and (2.2), we obtain two equivalent equations for identifying the self-organized states [1–3, 9]. Using variational calculus for the first and the second variations of the two equivalent equations, we obtain the self-organized states, denoted by the superscript #, for the minimum rate of change and also their stability condition, as follows:

$$q_i^{j\#}[\xi^k] = U_{i1}[\xi^k] = \exp(-\tau_{ci}\Lambda_{i1}\xi^j)U_{i1}[\xi^k_{k\neq j}]. \tag{2.3}$$

$$0 < \Lambda_{i1} \leqslant \lambda_{i1}. \tag{2.4}$$

Here, $\lambda_{i1}$ and $\Lambda_{i1}$ are the smallest positive eigenvalues for the normalized solutions of $\delta D_i^{j\#}[\mathbf{u}] + \tau_{ci}\lambda_{im}u_{im}[\xi^k_{k\neq j}] = 0$ with boundary conditions for $\delta q_i^{j\#}[\xi^k_{k\neq j}]$ and the eigenvalue equation $D_i^{j\#}[\mathbf{U}] + \tau_{ci}\Lambda_{im}U_{im}[\xi^k] = 0$ with boundary conditions for $q_i^{j\#}[\xi^k_{k\neq j}]$. All $N$ self-organized profiles of $q_i^{j\#}[\xi^k]$ given by (2.3) correspond to the most stable equilibrium states within the characteristic length $\tau_{ci}$, because they are guaranteed to change more slowly than any other states due to the $N$ stability conditions (2.4). Since each profile $U_{im}[\xi^k]$ can appear at different points along the $\xi^j$ axis, the dynamical system evolves to a cycle around each ${q_i}^{j\#}[\xi^k]$, just as with the phenomena of sawteeth oscillations in a fusion plasma.

We apply the generalized theory to the two-fluid model for a fully ionized, compressible, resistive, viscid MHD fusion plasma. The plasma is described by momentum and energy conservation laws for electrons and ions, and by Maxwell's equations with the displacement current neglected. Using the functional to calculate the self-organized states with the minimum rate of change of global auto-correlations for multiple physical quantities, and taking account of quasi-neutrality and negligibly small electron mass, we obtain the following simultaneous eigenvalue equations:

$$\begin{aligned}&\nabla(p_\parallel + p_\perp) + \Sigma_{\alpha=\mathrm{i,e}}[\nu_\alpha\nabla\times\nabla\times\mathbf{u}_\alpha - 4\nu_\alpha\nabla(\nabla\cdot\mathbf{u}_\alpha)/3 - \nabla\nu_\alpha(\nabla\cdot\mathbf{u}_\alpha)/3 + \mathbf{u}_\alpha\nabla^2\nu_\alpha\\ &\quad -\nabla(\mathbf{u}_\alpha\cdot\nabla\nu_\alpha) + \nabla\times(\nabla\nu_\alpha\times\mathbf{u}_\alpha)] + \mathbf{D}_\mathrm{V} - \mathbf{j}\times\mathbf{B} = \Sigma_{\alpha=\mathrm{i,e}}\Lambda_{\mathrm{m}\alpha}m_\alpha n_\alpha\mathbf{u}_\alpha,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\nabla\times\{\eta_\parallel\nabla\times\mathbf{B}_\parallel + \eta_\perp\nabla\times\mathbf{B}_\perp + \mu_0(1/en_e)[\nu_\mathrm{i}\nabla\times\nabla\times\mathbf{u}_\mathrm{i}\\ &\quad -4\nu_\mathrm{i}\nabla(\nabla\cdot\mathbf{u}_\mathrm{i})/3 - \nabla\nu_\mathrm{i}(\nabla\cdot\mathbf{u}_\mathrm{i})/3 + \nabla\times(\nabla\nu_\mathrm{i}\times\mathbf{u}_\mathrm{i}) - \nabla(\mathbf{u}_\mathrm{i}\cdot\nabla\nu_\mathrm{i})\\ &\quad +\mathbf{u}_\mathrm{i}\nabla^2\nu_\mathrm{i} + \mathbf{D}_\mathrm{Vi}] - \mathbf{u}_\mathrm{e}\times\mathbf{B}\} + \mu_0\nabla n_\mathrm{e}\times\nabla p_e/\mathrm{e}n_e^2 = \mu_0\Lambda_\mathrm{B}\mathbf{B},\end{aligned} \tag{2.6}$$

$$\begin{aligned}&\Sigma_{\alpha=\mathrm{i,e}}\{\nabla\cdot(p_\alpha\mathbf{u}_\alpha)/(\gamma-1) + p_\alpha(\nabla\cdot\mathbf{u}_\alpha) - \Sigma_{j,k}^3\,\Pi_{\alpha,j,k}\,\partial u_{\alpha,j}/\partial x_k\\ &\quad -\nabla\cdot(\kappa_{\alpha\parallel}\nabla_\parallel T_\alpha + \kappa_{\alpha\perp}\nabla_\perp T_\alpha)\} - (\eta_\parallel {j_\parallel}^2 + \eta_\perp {j_\perp}^2) = \Lambda_\mathrm{p}\Sigma_{\alpha=\mathrm{i,e}}\,p_\alpha/(\gamma-1).\end{aligned} \tag{2.7}$$

where $\mathbf{D}_V \equiv \Sigma_{\alpha=i,e} m_\alpha n_\alpha (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha$ is the quadratic terms in $\mathbf{u}_\alpha$. The self-organized states with minimum dissipation depend on the spatial profiles of resistivity $\eta$ and viscosity $\nu$, as has been found in experimental measurements [15] and in simulation results [6]. From (2.6), we obtain the Taylor state $\nabla \times \mathbf{B} = \lambda \mathbf{B}$ in the limiting case of uniform $\eta$ and $\nu$, $\mathbf{u} = 0$, and for a quasi-steady zero-pressure plasma, as well as in [4]. We also find the non-Taylor state, e.g., written as $\nabla \times \mathbf{B} \cong \sqrt{\mu_0 \Lambda_B/\eta}\, \mathbf{B}$ with non-uniform $\eta$ [6]. On the other hand, if we consider a simplified axisymmetric case with moderate profiles of viscosity, negligibly small electron viscosity and momentum, and divergent-free flows, we obtain from (2.5) and (2.6) a set of simplified equilibrium and eigenvalue equations of shear flows and magnetic fields, written respectively, as: (A) $\nabla p = \mathbf{j}_t \times \mathbf{B}_p + \mathbf{j}_p \times \mathbf{B}_t + m_i n_i (\mathbf{u}_i \cdot \nabla)\mathbf{u}_i$; and (B) $\nu_i \nabla \times \nabla \times \mathbf{u}_{i\phi} = \Lambda_{mi\phi} m_i n_i \mathbf{u}_{i\phi}$, $\nabla \times [\eta \nabla \times \mathbf{B}_\phi + (\mu_0/en_e)\nu_i \nabla \times \nabla \times \mathbf{u}_i] = \mu_0 \Lambda_{B\phi} \mathbf{B}_\phi$. Here, the subscripts t and p respectively denote the toroidal and the poloidal components, and the subscript $\phi$ takes both of t and p. From the decomposed equations in (A) and (B), we obtain simultaneous equations of the minimally dissipated stable equilibrium with shear flow for plasma confinement systems of (a) the Tokamak, the reversed field pinch and (b) the field reversed configuration, respectively, as follows:

$$\nabla p = \mathbf{j} \times \mathbf{B} + m_i n_i (\mathbf{u}_i \cdot \nabla)\mathbf{u}_i, \quad \nu_i \nabla \times \nabla \times \mathbf{u}_{i\phi} = \Lambda_{mi\phi} m_i n_i \mathbf{u}_{i\phi}.$$

$$\nabla \times [\eta \nabla \times \mathbf{B}_\phi + (\mu_0/en_e)\nu_i \nabla \times \nabla \times \mathbf{u}_i] = \mu_0 \Lambda_{B\phi} \mathbf{B}_\phi. \tag{2.8}$$

$$\nabla p = \mathbf{j}_t \times \mathbf{B}_p + m_i n_i (\mathbf{u}_i \cdot \nabla)\mathbf{u}_i, \quad \nu_i \nabla \times \nabla \times \mathbf{u}_{it} = \Lambda_{mi\phi} m_i n_i \mathbf{u}_{it}.$$

$$\nabla \times [\eta \nabla \times \mathbf{B}_p + (\mu_0/en_e)\nu_i \nabla \times \nabla \times \mathbf{u}_{it}] = \mu_0 \Lambda_{B\phi} \mathbf{B}_p. \tag{2.9}$$

We obtained numerical solutions of (2.8) and (2.7), which are published in [16].

## 3. Critique of traditional derivations for relaxed states with the use of helicities

The concept of selective decay is the final theoretical basis in [5, 10–12], where the first basis of conserved helicities within ideal plasmas is used, but it is analyticaly proved to be incorrect in [3]. Defining a canonical momentum $\mathbf{P}_\alpha \equiv m_\alpha \mathbf{u}_\alpha + q_\alpha \mathbf{A}$, a generalized vorticity $\mathbf{\Omega}_\alpha \equiv \nabla \times \mathbf{P}_\alpha$, and global and dimensional quantities $K_\mathbf{B}$, $K_\mathbf{j}$, $K_\alpha$, $W_\mathbf{B}$, $W_\mathbf{j}$, $W_{\Omega\alpha}$, which are respectively volume integrals on the magnetic, the current and the selfhelicities, i.e. $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{B} \cdot \mathbf{j}$ and $\mathbf{P}_\alpha \cdot \mathbf{\Omega}_\alpha$, and auto-correlations of the magnetic field, the current and the generalized vorticity, i.e., $\mathbf{B} \cdot \mathbf{B}$, $\mathbf{j} \cdot \mathbf{j}$ and $\mathbf{\Omega}_\alpha \cdot \mathbf{\Omega}_\alpha$, and the kinetic and the magnetic energies $W_{km}$, and using the same mathematical process as in [5, 10–12], we can obtain the following selective decay relations among the time derivatives of these quantities with respect to perturbed field quantities, determined uniquely by the numbers of curl operator in their definition for the MHD and the two-fluid plasmas, respectively, as follows:

$$\partial K_\mathbf{B}/\partial t < \partial W_\mathbf{B}/\partial t < \partial K_\mathbf{j}/\partial t < \partial W_\mathbf{j}/\partial t, \tag{3.1}$$

$$\partial K_\alpha/\partial t < \partial W_{km}/\partial t < \partial W_{\Omega\alpha}/\partial t. \tag{3.2}$$

Using any one of the following three conjectures defined by $\{K_\mathbf{B} = \text{constant and min } W_\mathbf{B}\}$, $\{W_\mathbf{B} = \text{constant and min } K_\mathbf{j}\}$, and $\{K_\mathbf{j} = \text{constant and min } W_\mathbf{j}\}$ connecting with (3.1) and also any one of their reversal conjectures defined by $\{W_\mathbf{B} = \text{constant}$

and min $K_{\mathbf{B}}\}$, $\{K_{\mathbf{j}} =$ constant and min $W_{\mathbf{B}}\}$, and $\{W_{\mathbf{j}} =$ constant and min $K_{\mathbf{j}}\}$ having no relations with (3.1), we can directly derive the Taylor state $\nabla \times \mathbf{B} = \lambda \mathbf{B}$. In the same way, using any one of the following two conjectures defined by $\{K_\alpha =$ constant and min $W_{\rm km}\}$ and $\{W_{\rm mk} =$ constant and min $W_{\Omega\alpha}\}$ connecting with (3.2) and also any one of their reversal conjectures defined by $\{W_{\rm mk} =$ constant and min $K_\alpha\}$ and $\{W_{\Omega\alpha} =$ constant and min $W_{\rm km}\}$, having no relations with (3.2), we can directly derive 'relaxed states for two-fluid model of plasmas' [10–12]. Hence, the concept of selective decay has no connection with so-called relaxed states in [5, 10–12]. The fusion plasma is also known to be described for simulations by a set of charge, mass, momentum, and energy conservation laws, and Maxwell's equations to follow its dynamic evolution. However, replacement of any element in the set by the helicity conservation law [5] makes the evolution untraceable due to this non-physical law. Thus, the theoretical basis for relaxed states derived on the basis of conserved helicities and the concept of selective decay is questionable.

## 4. Conclusions

From a recent generalized theory of self-organization, we have derived eigenvalue equations to obtain the equilibrium for the least dissipated stable relaxed state. We applied this theory to the two-fluid plasma model (cf. (2.5), (2.6), and (2.7)). For a particular simplified axisymmetric case with moderate viscosity profiles, we derived equilibrium equations for the reversed field pinch and the field reversed configuration with shear flow due to viscosity (cf. (2.8) and (2.9)). The generalized theory of self-organization [1–3, 13] is a unifying theory for apparently different systems of minimum dissipation of magnetic energy in [4] and of minimal (enstrophy/energy) based on the selective theory in [16], as analytically proved in [9].

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