Revisiting linear gyrokinetics to recover ideal magnetohydrodynamics and missing finite Larmor radius effects

L. J. Zheng,[a)] M. T. Kotschenreuther, and J. W. Van Dam
Institute for Fusion Studies University of Texas at Austin, Austin, Texas 78712, USA

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The linear gyrokinetics theory in the axisymmetric configuration is revisited. It is found that the conventional gyrokinetic theory needs to be repaired in order to recover the linear magnetohydrodynamics from the gyrokinetics and to obtain the finite Larmor radius effect on the magnetohydrodynamic modes in an ordering-consistent manner. Two key inclusions are: (1) the solution of the equilibrium gyrokinetic distribution function is carried out to a sufficiently high order; (2) the gyrophase-dependent part of the perturbed distribution function is kept. The new gyrokinetic theory developed in this paper can be used to extend directly the magnetohydrodynamic stability analysis to the gyrokinetic one without invoking the hybrid kinetic-fluid hypothesis.
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I. INTRODUCTION

The retention of the kinetic effects on the magnetohydrodynamic (MHD) modes, especially the resistive wall modes, has become an important area in toroidal confinement studies. The gyrokinetic formalism provides the most efficient way of obtaining the appropriate reduced kinetic equations while still retaining the finite Larmor radius (FLR) effect. The Vlasov equation is a six-dimensional differential equation. By introducing particle motion (adiabatic) invariants and applying high gyrofrequency ordering, the gyrokinetic formalism effectively reduces the seven-dimensional substantial time derivative in the linearized Vlasov equation into a four-dimensional one (with velocity space derivatives eliminated) and therefore greatly simplifies the calculation of the kinetic effects.

The classic electrostatic gyrokinetic formalism was developed in 1960s.[1,2] Later, electrostatic gyrokinetics was extended to the electromagnetic one.[3,4] Most of the gyrokinetic treatments employ the eikonal ansatz for studying high-n modes (n is the toroidal mode number). Recently, a gyrokinetic formalism for long wavelength modes was developed in Refs. 5–8, in which a great effort has been made to derive the ideal MHD equations from gyrokinetic formalism. However, so far only partial MHD terms are shown to be recovered. To derive MHD equation from the gyrokinetic equation is not just of academic interest, but of practical importance. The recovery of the MHD can justify the validity and sufficiency of the perturbation expansion procedure in the gyrokinetic formalism. There is abundance evidence showing the coincidence between the experimental observations and the predictions of MHD theory. This is because in the presence of the strong magnetic field the ions are localized spatially to the magnetic field lines, so the ideal MHD formalism keeps the key mode features in the perpendicular direction. We also intend to use the gyrokinetic formalism derived in this paper to extend the linear MHD code AEGIS[9] into fully gyrokinetic one. It is therefore interesting to derive the MHD equations from the gyrokinetic analyses.

In this paper we construct a gyrokinetic formalism applicable to both long and short wavelength MHD modes in the axisymmetric configuration. Compared with the existing gyrokinetic theory, this newly derived gyrokinetic formalism can recover the linear MHD equations. To achieve it, it is shown that the solutions of both the equilibrium and perturbed distribution functions need to be improved. Two key modifications are made. First, the solution of the equilibrium gyrokinetic distribution function is carried out to a sufficiently high order; second, the gyrophase-dependent part of the perturbed distribution function is kept. In the conventional gyrokinetic theory, only the lowest-order equilibrium distribution function [i.e., $F_{g0}(\mathbf{X}_\perp, \mu, \epsilon)$, see Eq. (15) for definition] is used. Reference 10 shows that, if only the lowest-order equilibrium distribution function is used, even the MHD equilibrium equation cannot be recovered. This shows the necessity to include the higher-order equilibrium distribution function. The solution of the gyrophase-dependent part of the distribution function for electromagnetic gyrokinetic equation has been carried out previously for studying the cyclotron waves.[11,12] The necessity of including the gyrophase-dependent part of the distribution function to the FLR viscous tensor is also addressed recently in Ref. 13. In a recent effort to derive the MHD equations from the gyrokinetics, the gyrophase-dependent part of the perturbed distribution function is also calculated (see Sec. V for detailed discussion).[5–8] However, it should be pointed out that, only when both of the first-order equilibrium and the gyrophase-dependent part of distribution functions are taken into account simultaneously, the two perpendicular MHD equations of motion can be fully recovered.

The paper is arranged as follows. In the next section, the structure of the linear ideal MHD eigenmode equations is described; In Sec. III, the gyrokinetic equations are derived; in Sec. IV, the set of the eigenmode equations in the gyroki-

[a)]Electronic mail: lzheng@mail.utexas.edu

netic formalism is laid out and the recovery of the MHD equations is shown. The conclusions are given in the last section.

II. THE STRUCTURE OF THE LINEAR IDEAL MHD EIGENMODE EQUATIONS

In order to compare with the gyrokinetic formalism, we first describe the structure of the linear ideal MHD equations. For simplicity, we consider the axisymmetric equilibria, in which the equilibrium magnetic field can be represented as

$$\mathbf{B} = \nabla\phi \times \nabla\psi + g\,\nabla\,\phi, \tag{1}$$

where ψ is the poloidal magnetic flux, g is a function of ψ, and ϕ is the toroidal axisymmetric angle. We use the Princeton equilibrium and stability (PEST) coordinates with θ as the poloidal angle.[14] The Jacobian of the PEST coordinates is expressed as $\mathcal{J}=1/\nabla\psi\times\nabla\theta\cdot\nabla\phi=qR^2/g$, where q is the safety factor and R is the major radius.

The MHD equilibrium is governed by the force balance equation in the perpendicular direction and the free divergence of the current density for determining the parallel current (see, for instance, Ref. 10):

$$\mathbf{J}_\perp = \frac{\mathbf{B}\times\nabla P}{B^2}, \tag{2}$$

$$J_\parallel = -\frac{gP'}{B} - g'B, \tag{3}$$

where $\mathbf{J}=\nabla\times\mathbf{B}$ is the equilibrium current density, P represents the equilibrium plasma pressure, the subscripts "$\perp$" and "$\parallel$" denote the perpendicular and parallel to the equilibrium magnetic field, respectively, the prime symbol denotes the derivative with respect to ψ, and boldface is introduced to indicate the vector.

The linearized MHD equation for perturbation reads

$$-\rho_m\omega^2\boldsymbol{\xi} = \delta\mathbf{J}\times\mathbf{B} + \mathbf{J}\times\delta\mathbf{B} - \nabla\delta P, \tag{4}$$

where $\boldsymbol{\xi}$ represents the plasma displacement, $\delta\mathbf{B}=\nabla\times\boldsymbol{\xi}\times\mathbf{B}$ denotes the perturbed magnetic field, $\delta\mathbf{J}=\nabla\times\delta\mathbf{B}$ is the perturbed current density vector, $\delta P=-\boldsymbol{\xi}\cdot\nabla P-\gamma P\nabla\cdot\boldsymbol{\xi}$ represents the perturbed plasma pressure, ω is the mode frequency, ρ_m is the mass density, and γ represents the ratio of the specific heats. The MHD equation (4) has three projections. We introduce three unit vectors, i.e., $\mathbf{e}_b=\mathbf{B}/B$, $\mathbf{e}_1=\nabla\psi/|\nabla\psi|$, and $\mathbf{e}_2=\mathbf{e}_b\times\mathbf{e}_1$, to perform the projections. The $\mathbf{e}_2$ projection of the MHD equation (4) gives

$$\begin{aligned}\mathbf{e}_1\cdot\nabla\times\delta\mathbf{B} = &-\frac{gP'}{B^2}\mathbf{e}_1\cdot\delta\mathbf{B} - g'\mathbf{e}_1\cdot\delta\mathbf{B}\\ &+\frac{1}{B}\mathbf{e}_2\cdot\nabla(P'|\nabla\psi|\mathbf{e}_1\cdot\boldsymbol{\xi})\\ &+\gamma P\frac{1}{B}\mathbf{e}_2\cdot\nabla(\nabla\cdot\boldsymbol{\xi}) + \frac{\rho_m\omega^2}{B}\mathbf{e}_2\cdot\boldsymbol{\xi}.\end{aligned} \tag{5}$$

Similarly, the $\mathbf{e}_1$ projection of the MHD equation (4) yields

$$\begin{aligned}\mathbf{e}_2\cdot\nabla\times\delta\mathbf{B} = &-\frac{gP'}{B^2}\mathbf{e}_2\cdot\delta\mathbf{B} - g'\mathbf{e}_2\cdot\delta\mathbf{B} - \frac{P'|\nabla\psi|}{B^2}\mathbf{e}_b\cdot\delta\mathbf{B}\\ &-\frac{1}{B}\mathbf{e}_1\cdot\nabla(P'|\nabla\psi|\mathbf{e}_1\cdot\boldsymbol{\xi})\\ &-\gamma P\frac{1}{B}\mathbf{e}_1\cdot\nabla(\nabla\cdot\boldsymbol{\xi}) - \frac{\rho_m\omega^2}{B}\mathbf{e}_1\cdot\boldsymbol{\xi}.\end{aligned} \tag{6}$$

The $\mathbf{e}_b$ projection of the MHD equation (4) can be reduced, using $\nabla\cdot\boldsymbol{\xi}$ as an independent unknown, to

$$\gamma P\mathbf{B}\cdot\nabla\left(\frac{1}{B^2}\mathbf{B}\cdot\nabla\nabla\cdot\boldsymbol{\xi}\right) + \rho_m\omega^2\,\nabla\,\cdot\boldsymbol{\xi} = \rho_m\omega^2\,\nabla\,\cdot\boldsymbol{\xi}_\perp. \tag{7}$$

Noting that $\delta\mathbf{J}$ and $\delta\mathbf{B}$ are determined completely by $\boldsymbol{\xi}_\perp$, one can see that the set of equations (5)–(7) is complete for determining two components of $\boldsymbol{\xi}_\perp$ and one scalar $\nabla\cdot\boldsymbol{\xi}$.

Alternatively, one of the two perpendicular equations of motion, i.e., Eq. (5) or (6), can be replaced by the so-called vorticity equation, which is obtained by applying the operator $\nabla\cdot(\mathbf{B}/B^2)\times(\cdots)$ on Eq. (4),

$$\begin{aligned}-\nabla\cdot\frac{\mathbf{B}}{B^2}\times\rho_m\omega^2\boldsymbol{\xi} = &-\mathbf{B}\cdot\nabla\frac{\mathbf{B}\cdot\delta\mathbf{J}}{B^2} - \delta\mathbf{B}\cdot\nabla\sigma\\ &+\mathbf{J}\cdot\nabla\frac{\mathbf{B}\cdot\delta\mathbf{B}}{B^2} - 2\frac{\mathbf{B}\times\boldsymbol{\kappa}}{B^2}\cdot\nabla\delta P\\ &+2\frac{\mathbf{B}\times\nabla P}{B^4}\cdot\nabla\delta P - \frac{1}{B^2}\mathbf{J}\cdot\nabla\delta P,\end{aligned} \tag{8}$$

where $\sigma=\mathbf{J}\cdot\mathbf{B}/B^2$ and $\boldsymbol{\kappa}=\mathbf{e}_b\cdot\nabla\mathbf{e}_b$ is the magnetic field line curvature.

Equations (8), (5), and (7) describe three fundamental MHD waves: the shear Alfvén, the compressional Alfvén, and the parallel acoustic waves, respectively. Due to the particle localization on the magnetic field lines, one can expect the two perpendicular equations (5) and (6) can be recovered from the gyrokinetic equation, expect the plasma compressibility effect. Since the compressibility effect is related to the parallel motion and particles are not localized along the magnetic field, one can expect the necessity of the kinetic description for the compressibility effect. Nevertheless, one can expect that the coupling pattern of the perpendicular motion [the right-hand side (r.h.s.) of Eq. (7)] to the parallel one [the left-hand side (l.h.s.) of Eq. (7)] in the MHD description should be seen in the kinetic description. Noting also that the FLR effect specifies the difference between the fluid and magnetic field line displacements, one can expect the FLR modification of the inertia effect as well.

III. DERIVATION OF THE GYROKINETIC EQUATION

A. General formalism

In order to linearize the Vlasov equation, the distribution function F is decomposed to the equilibrium (F) and perturbed (δF) parts. The subscripts "i" and "e" are introduced to represent the corresponding quantities for ion and electron species, respectively. The linearized Vlasov equations for equilibrium and perturbation become

$$\bar{\mathcal{L}}_v F = 0, \tag{9}$$

$$\mathcal{L}_v \delta F = -\frac{e}{m_\rho}\delta \mathbf{a}\cdot \nabla_v F, \tag{10}$$

where $\bar{\mathcal{L}}_v=\mathbf{v}\cdot\nabla_x+(e/m_\rho)\mathbf{v}\times\mathbf{B}\cdot\nabla_v$, $\mathcal{L}_v=-i\omega+\mathbf{v}\cdot\nabla_x+(e/m_\rho)\mathbf{v}\times\mathbf{B}\cdot\nabla_v$, $\delta\mathbf{a}=\delta\mathbf{E}+\mathbf{v}\times\delta\mathbf{B}$, and $\delta\mathbf{E}$ is the perturbed electric field, e is the charge, m_ρ represents the mass, and ∇_x and ∇_v denote, respectively, the Laplace operators in the particle configuration $\mathbf{x}$ and velocity $\mathbf{v}$ spaces. Alternatively, the scalar $\delta\varphi$ and vector $\delta\mathbf{A}$ can be used to represent the perturbed electromagnetic fields: $\delta\mathbf{E}=-\nabla\delta\varphi+i\omega\delta\mathbf{A}$ and $\delta\mathbf{B}=\nabla\times\delta\mathbf{A}$.

As in Ref. 4, we introduce the guiding center coordinates $(\mathbf{X},\mathbf{V})=(\mathbf{X},\varepsilon,\mu,\alpha)$, where $\mathbf{X}=\mathbf{x}+(1/\Omega)\mathbf{v}\times\mathbf{e}_b$, $\varepsilon=v^2/2$, $\mu=\mu_0+\tilde{\mu}_1+\bar{\mu}_1$, $\mu_0=v_\perp^2/2B$, $\tilde{\mu}_1=-\mathbf{v}_d\cdot\mathbf{v}_\perp/B-(v_\parallel/4\Omega B)(\mathbf{v}_\perp\times\mathbf{e}_b\cdot\nabla\mathbf{e}_b\cdot\mathbf{v}_\perp+\mathbf{v}_\perp\cdot\nabla\mathbf{e}_b\cdot\mathbf{v}_\perp\times\mathbf{e}_b)$, $\bar{\mu}_1=-(v_\parallel\mu_0/\Omega)\mathbf{e}_b\cdot\nabla\times\mathbf{e}_b$, $\mathbf{v}_d=\mathbf{e}_b\times[(v_\perp^2/2\Omega)\nabla\ln B+(v_\parallel^2/\Omega)\boldsymbol{\kappa}]$, α is the gyrophase, and $\Omega=eB/m_\rho$. The particle velocity can be decomposed as $\mathbf{v}_\perp=\mathbf{e}_1 v_\perp\cos\alpha+\mathbf{e}_2 v_\perp\sin\alpha+\mathbf{e}_b v_\parallel$. In the guiding center coordinates, the equilibrium and perturbed Vlasov equations (9) and (10) become[4]

$$\bar{\mathcal{L}}_g F_g = 0, \tag{11}$$

$$\mathcal{L}_g \delta F_g = -\frac{e}{m_\rho}\left(\delta\mathbf{a}\cdot\nabla_V+\frac{1}{\Omega}\delta\mathbf{a}\times\mathbf{e}_b\cdot\nabla_X\right)F_g, \tag{12}$$

where

$$\bar{\mathcal{L}}_g=\dot{\mathbf{X}}\cdot\nabla_X+\dot{\alpha}\frac{\partial}{\partial\alpha},$$
$$\mathcal{L}_g=-i\omega+\dot{\mathbf{X}}\cdot\nabla_X+\dot{\alpha}\frac{\partial}{\partial\alpha}. \tag{13}$$

Here, the dot represents the derivative along the unperturbed particle orbit, $\dot{\mathbf{X}}=v_\parallel\mathbf{e}_b+\mathbf{v}_D$, $\mathbf{v}_D=\mathbf{v}\cdot\mathbf{v}\times\nabla_x(\mathbf{e}_b/\Omega)$, $\dot{\alpha}=-\Omega(\mathbf{X})+\dot{\alpha}_1$, $\dot{\alpha}_1=\mathbf{v}\cdot\nabla_x\alpha+(1/\Omega)\mathbf{v}\times\mathbf{e}_b\cdot\nabla_x\Omega$, and $\nabla_x\alpha=(\nabla_x\mathbf{e}_2)\cdot\mathbf{e}_1+(v_\parallel/v_\perp^2)\nabla_x\mathbf{e}_b\cdot(\mathbf{v}_\perp\times\mathbf{e}_b)$. In $\dot{\alpha}_1$, the last term results from the guiding center transform of the gyrofrequency $\Omega(\mathbf{x})$. This correction is required, since $\mathbf{v}\cdot\nabla_x\alpha\sim(1/\Omega)\mathbf{v}\times\mathbf{b}\cdot\nabla_x\Omega$. We have also noted that the term $\dot{\mu}(\partial/\partial\mu)$ is one order smaller in the small Larmor radius ordering as compared to $\dot{\mathbf{X}}\cdot\nabla_X$ and therefore is neglected in $\bar{\mathcal{L}}_g$ and $\mathcal{L}_g$.

B. Equilibrium

We first solve the equilibrium gyrokinetic equation (11). Introducing L_p to represent the scale length of the equilibrium pressure and ρ to represent the Larmor radius, we adopt the following ordering assumption: $\rho_i/L_p\ll 1$. We also introduce L_B to represent the scale length of the magnetic field line curvature. We do not impose an ordering between L_p and L_B. However, L_p and L_B are kept explicitly in the ordering analyses, in order to make the specific physical ordering explicit, should the large aspect ratio configuration or the transport barrier physics is dealt with.

Corrected to order $\mathcal{O}(\rho/L_B)$, the equilibrium Vlasov equation (11) becomes

$$\left(v_\parallel\mathbf{e}_b\cdot\nabla_X-\Omega\frac{\partial}{\partial\alpha}\right)F_{g0}=0. \tag{14}$$

This gives the lowest-order equilibrium distribution function

$$F_{g0}=F_{g0}(\mathbf{X}_\perp,\mu,\varepsilon). \tag{15}$$

For simplicity, we use the isotropic Maxwellian distribution function as the lowest-order solution:

$$F_{g0}(\Psi,\varepsilon)=n_0(\Psi)\left(\frac{m_\rho}{2\pi T(\Psi)}\right)^{3/2}\exp\left\{-\frac{m_\rho\varepsilon}{T(\Psi)}\right\}, \tag{16}$$

where Ψ is the guiding center correspondent of ψ, n_0 is the plasma density, and T denotes the temperature.

Most of existing linear gyrokinetic theories use only the equilibrium solution of this order. As will be shown later, this treatment results in the loss of part of MHD effects and is also inconsistent for retaining the FLR effects. To derive the perturbed gyrokinetic equation ordering consistently, the next-order equilibrium distribution function F_{g1} is required. The solution of the equilibrium gyrokinetic equation of the next order has been described in the neoclassical transport theory.[10] The next-order gyrokinetic equation after gyrophase average reads

$$v_\parallel\mathbf{e}_b\cdot\nabla_X F_{g1}+\mathbf{v}_d\cdot\nabla_X F_{g0}=0.$$

Noting that $\mathbf{v}_d\cdot\nabla\Psi=v_\parallel\mathbf{e_b}\cdot\nabla_X(v_\parallel m_\rho g/eB)$, one obtains

$$F_{g1}=-v_\parallel\frac{g}{\Omega}\frac{\partial F_{g0}}{\partial\Psi}+\mathrm{sign}(v_\parallel)\bar{F}_{g1}(\Psi,\mu,\varepsilon), \tag{17}$$

where $\bar{F}_{g1}$ is the integration constant.[10] One can easily obtain the ordering estimate for F_{g1},

$$\frac{F_{g1}}{F_{g0}}\sim\frac{\rho L_B}{L_p^2}. \tag{18}$$

In passing, let us discuss briefly the necessity of the retention of the next-order equilibrium solution. We note first that Ref. 10 shows that, with and only with the first-order equilibrium distribution function in Eq. (17) included, the MHD equilibrium equations (2) and (3) can be recovered. In order to be self-contained, here we review the derivation of (2) and (3) in Ref. 10. Noting that F_{g1} does not contribute to the perpendicular current density due to being odd in $v_\parallel$, one has

$$\mathbf{J}_\perp=\sum_{i,e}e\int d^3v\,\mathbf{v}_\perp F_{g0}(\Psi,\varepsilon)$$
$$=\sum_{i,e}e\int d^3v\,\mathbf{v}_\perp\frac{1}{\Omega}\mathbf{v}\times\mathbf{e}_b\cdot\nabla F_{g0}(\Psi,\varepsilon)$$
$$=\frac{\mathbf{B}\times\nabla P}{B^2}. \tag{19}$$

This recovers MHD equilibrium Eq. (2). Noting also that F_{g0}

does not contribute to the parallel current density due to being even in $v_\parallel$, one has

$$J_\parallel = \sum_{i,e} e \int d^3v v_\parallel F_{g1}(\Psi,\varepsilon) = \sum_{i,e} e \int d^3v v_\parallel \left(- v_\parallel \frac{g}{\Omega}\frac{\partial F_{g0}}{\partial \Psi} + \mathrm{sign}(v_\parallel)\bar{F}_{g1}(\Psi,\mu,\varepsilon)\right) = -\frac{gP'}{B} - B\sum_{i,e} e \int d\mu d\varepsilon \bar{F}_{g1}(\Psi,\mu,\varepsilon). \quad (20)$$

Complete determination of the parallel current needs to derive $\bar{F}_{g1}$, which has been done in the neoclassical transport theory.[10] However, as justified in Ref. 10, one can generally impose that the surface function $\Sigma_{i,e} e \int d\mu d\varepsilon \bar{F}_{g1}(\Psi,\mu,\varepsilon) = g'$. By comparing Eq. (20) with Eq. (3), one can see that this is simply the requirement $\nabla \cdot \mathbf{J}=0$. Equation (20) recovers the other MHD equilibrium equation, Eq. (3). Equation (3) actually describes the so-called Pfirsch-Schlüter current. From the derivations of Eqs. (19) and (20) one can see that the lowest-order equilibrium distribution function in Eq. (15) does not produce the parallel current, although it gives rise to the diamagnetic current. The parallel equilibrium current can only be retained by including the first-order solution in Eq. (17). It is interesting to note that in the particle coordinates the total equilibrium distribution function $(F_{g0}+F_{g1})$ can be expressed in a form of the quasi-shifted Maxwellian $F_{g0}[\mathbf{x}_\perp,(\mathbf{v}-\mathbf{V}^k)^2/2]$. Here, $\mathbf{V}^k$ is a function of the space and velocity, instead of the fluid velocity as in the Braginskii two-fluid theory.[15] Note that the Braginskii two-fluid equations, as well as the gyroviscous tensor, are obtained by assuming that the equilibrium distribution function is a shifted Maxwellian. Without retaining the first-order equilibrium distribution function F_{g1}, the parallel component of $\mathbf{V}^k$ would vanish. These arguments show that, to recover the MHD, one has to keep the first-order equilibrium distribution function. Note also that in the another effort to derive MHD equation from gyrokinetics in Ref. 5, the equilibrium distribution function has been assumed to be a shifted Maxwellian by the parallel fluid velocity, in order to reproduce the parallel fluid velocity. Noting that $\mathbf{V}_\parallel^k$ is not a fluid velocity, the equilibrium distribution function in Ref. 5 is not the gyrokinetic equilibrium solution to the next order. The other effort to rederive the MHD equilibrium in Ref. 7 has assumed that there is no macroscopic parallel velocity. It therefore does not apply to tokamak physics.

C. Perturbed gyrokinetic equation

Now, we derive the perturbed linear gyrokinetic equation. Using the equilibrium distribution function in Eqs. (16) and (17) one can reduce the linear gyrokinetic equation Eq. (12). After extracting the adiabatic part of the perturbed distribution function,

$$\delta F = \frac{e}{m_\rho}\frac{\partial F_{g0}}{\partial \varepsilon}\delta\varphi(\mathbf{x}) + \delta H(\mathbf{X},\varepsilon,\mu,\alpha), \quad (21)$$

the gyrokinetic equation Eq. (12) is reduced to

$$\mathcal{L}_g \delta H(\mathbf{X}) = \mathcal{R}, \quad (22)$$

where

$$\begin{aligned}\mathcal{R} = & - i\omega \frac{e}{m_\rho}\frac{\partial F_{g0}}{\partial \varepsilon}\mathbf{v}_\perp \cdot \delta\mathbf{A}(\mathbf{x}) - i\omega \frac{1}{B}\mathbf{e}_b \times \nabla_X F_{g0} \cdot \delta \mathbf{A}(\mathbf{x}) \\ & + i(\omega - \omega_*^T)\frac{e}{m_\rho}\frac{\partial F_{g0}}{\partial \varepsilon}\delta\varphi(\mathbf{x}) + \frac{e}{m_\rho}\mathbf{v}\cdot\nabla_x\left(\frac{\partial F_{g0}}{\partial \varepsilon}\right)\delta\varphi(\mathbf{x}) \\ & + \left(\frac{g}{B}\frac{\partial F_{g0}}{\partial \Psi} + |v_\parallel|\frac{e}{m_\rho B}\frac{\partial \bar{F}_{g1}}{\partial \mu}\right)\mathbf{e}_b \times \mathbf{v}_\perp \cdot \delta\mathbf{B}(\mathbf{x}) \\ & + \frac{1}{B}\mathbf{v}_\perp \cdot \nabla_X F_{g0}\mathbf{e}_b \cdot \delta\mathbf{B}(\mathbf{x}) - \frac{g}{B}\frac{\partial F_{g0}}{\partial \Psi}\mathbf{e}_b \cdot [\nabla_x \delta\varphi(\mathbf{x}) \\ & - i\omega\delta\mathbf{A}(\mathbf{x})] + \mathrm{sign}(v_\parallel)\frac{e}{m_\rho B}\frac{\partial \bar{F}_{g1}}{\partial \mu}\mathbf{v}_\perp \cdot \nabla_x[\delta\varphi(\mathbf{x}) \\ & - i\omega\delta\mathbf{A}(\mathbf{x})] - i\omega\frac{e}{m_\rho}\frac{\partial F_{g0}}{\partial \varepsilon}\mathbf{v}_\parallel \cdot \delta\mathbf{A}(\mathbf{x}) \\ & - v_\parallel \frac{1}{B}\frac{\partial F_{g0}}{\partial \Psi}\nabla\Psi \cdot \delta\mathbf{B}(\mathbf{x}),\end{aligned} \quad (23)$$

with $\omega_*^T = [i/(\Omega \partial F_{g0}/\partial\varepsilon)]\mathbf{e}_b \times \nabla_X F_{g0} \cdot \nabla_X$. Here, the inclusion of F_{g1} on the r.h.s. of Eq. (12) has produced several new terms. These new terms result from the term $(e/m_\rho)\mathbf{v} \times \delta\mathbf{B}\cdot\nabla_V F_{g1}$ on the r.h.s. of Eq. (12), while there is no contribution from F_{g0} in this term due to its isotropic feature. Noting that the order of the first to the second terms on the r.h.s. of Eq. (12) is formally L_p/ρ_i and the order of F_{g1} to F_{g0} is given by Eq. (18), one can conclude that the F_{g1} contribution cannot be ignored for ordering consistency.

Next, we discuss the solution of the linearized gyrokinetic equation. Equation (22) can be formally written as

$$\mathcal{L}^1 \delta H - \Omega(\mathbf{X})\frac{\partial \delta H}{\partial \alpha} = \mathcal{R}, \quad (24)$$

where $\mathcal{L}^1 = -i\omega + \dot{\mathbf{X}}\cdot\nabla_X + \dot{\alpha}_1(\partial/\partial\alpha)$. To proceed further, the following ordering assumption is adopted:

$$\mathcal{L}^1/\Omega \sim \epsilon, \quad (25)$$

as in the conventional gyrokinetic theory. Although the last term in Eq. (24) $-\Omega(\partial/\partial\alpha)$ is the dominant term in the gyrokinetic ordering in Eq. (25), this does not imply $\partial\delta H/\partial\alpha = 0$ to the lowest order, since one cannot presume the ordering between the terms on l.h.s. and the source term on the r.h.s. of Eq. (24). As will become apparent in the analyses described in the next section, the gyrophase-dependent part of the distribution function should be solved, as well as the gyrophase-independent part, for ordering consistency. As the cyclotron frequency gyrokinetic formalism in Refs. 11 and 12, we use the Fourier decomposition method to solve Eq. (24),

$$\{\delta H, \mathcal{R}\} = \sum_k \{\delta H_k, \mathcal{R}_k\}\exp\{ik\alpha\}.$$

With ordering assumption in Eq. (25), Eq. (24) can be solved order by order. Expanding the perturbed distribution function as

$$\delta H_k = \delta H_k^{(0)} + \epsilon \delta H_k^{(1)} + \cdots ,$$

one has

$$\delta H_k^{(0)} = i\frac{1}{k\Omega}\mathcal{R}_k, \quad \text{for } k \neq 0,$$

$$\delta H_0^{(0)} = \mathcal{L}_{00}^{1\;-1}\left(\mathcal{R}_0 - \sum_l{}' \mathcal{L}_{0l}^1 \delta H_l^{(0)}\right),$$

$$\delta H_k^{(1)} = -i\frac{1}{k\Omega}\sum_l \mathcal{L}_{kl}^1 \delta H_l^{(0)}, \quad \text{for } k \neq 0, \tag{26}$$

$$\delta H_0^{(1)} = -\mathcal{L}_{00}^{1\;-1}\sum_l{}' \mathcal{L}_{0l}^1 \delta H_l^{(1)},$$

$$\cdots,$$

where Σ_l' represents a summation excluding $l=0$ and the matrix elements

$$\mathcal{L}_{lk}^1 = \frac{1}{2\pi}\int_0^{2\pi} d\alpha \exp\{-il\alpha\}\mathcal{L}^1 \exp\{ik\alpha\}.$$

D. Complete set of the eigenmode equations

To proceed further, the gauge for representing the electromagnetic potentials needs to be chosen. The Coulomb gauge has been used widely in the previous gyrokinetic theories. We avoid use of the Coulomb gauge, since it induces an additional differential equation to determine the relationship of three $\delta\mathbf{A}$ components and makes especially the numerical implementation complicated. For $n=0$ modes, the perturbed magnetic field can be represented by two scalars as in the equilibrium case in Eq. (1). We show that for $n\neq 0$ modes, one can also generally represents the magnetic field with two scalars. Note that the gauge invariance allows an arbitrary gradient to be added to the vector potential: $\delta\mathbf{A}+\nabla\delta\hat{\varphi} = (\delta A_\psi + \partial\delta\hat{\varphi}/\partial\psi)\nabla\psi + (\delta A_\theta + \partial\delta\hat{\varphi}/\partial\theta)\nabla\theta + (\delta A_\phi - in\delta\hat{\varphi})\nabla\phi$. Since $n\neq 0$, one can always find $\delta\hat{\varphi}$ to make $\delta A_\phi - in\delta\hat{\varphi}=0$. Therefore, for $n\neq 0$ modes, one can generally adopt a gauge $\delta A_\phi=0$; i.e., one can represent

$$\delta\mathbf{A} = \boldsymbol{\zeta}\times\mathbf{B}_\phi, \tag{27}$$

where $\mathbf{B}_\phi = g\nabla\phi$.

Besides the ion and electron distribution functions $\delta f_{i,e}$, there are three independent field unknowns: two components of ζ and a scalar $\delta\varphi$. Two components of the Ampere's law and the quasineutrality condition, together with the ion and electron gyrokinetic equations, are used to construct the complete set of equations. Using the quasineutrality condition is equivalent to using the generalized Ohm's law. Ampere's law reads

$$\delta\mathbf{J}\cdot\mathbf{e}_{1,2} = \sum_{i,e} e\int d^3v\,\mathbf{v}_\perp\cdot\mathbf{e}_{1,2}\delta f(\mathbf{x}). \tag{28}$$

The quasineutrality condition is

$$\sum_{i,e} e\int d^3v\,\delta f(\mathbf{x}) = 0. \tag{29}$$

In Eqs. (28) and (29), the velocity space integration is performed in the particle coordinates, instead of in the guiding center coordinates.

Here, we make some remark about various approaches to construct the basic set of equations. In the conventional approach (see, for example, Ref. 16), one perpendicular component of Ampere's law and the vorticity equation are used to construct the basic set of equations. Instead, we use directly the two perpendicular components of Ampere's law for two reasons. First, from the MHD vorticity equation (8), one can see that both pressure and velocity moments need to be calculated to derive the kinetic vorticity equation. Note that calculating the velocity moment alone is equivalent to calculating the current density in Ampere's law. To avoid calculating two moments, a direct construction of the kinetic vorticity equation from the gyrokinetic equations has been used previously (see, for example, Refs. 5 and 16), by applying the operator $\Sigma_{i,e}e\int d^3v$ on the gyrophase-averaged gyrokinetic equation and using the quasineutrality condition for simplification. However, noting that the nabla operator in the term $\mathbf{v}_\parallel\cdot\nabla_{\mathbf{X}}$ in the gyrokinetic equation is in the guiding center coordinates, one cannot simply move $\mathbf{e}_b\cdot\nabla_{\mathbf{X}}$ out of the velocity space integration in a given particle coordinate without subtle elaboration for FLR modification. Note further that the term $\mathbf{v}_\parallel\cdot\nabla_{\mathbf{X}}$ produces the field line bending term of the shear Alfvén mode, which is generally much larger than the inertia term, tied to the FLR effects. These show that, to get a vorticity equation in the particle coordinates, a backward transform for the term $\mathbf{v}_\parallel\cdot\nabla_{\mathbf{X}}$ is needed in order to retain FLR effects consistently. This makes the vorticity equation approach nontrivial. This point has not been addressed in the previous derivation of the gyrokinetic vorticity equation in Refs. 16 and 5. Second, we note that the vorticity equation alone is not sufficient for the completeness of the basic set of equations. At least one perpendicular component of Ampere's law is needed. Since calculation of the two perpendicular components of Ampere's law are similar, our approach of using directly two perpendicular components of Ampere's law is therefore a straightforward approach for constructing the basic set of equations.

IV. EIGENMODE EQUATIONS AND RECOVERY OF LINEAR IDEAL MHD

In this section, we will investigate the solution of gyrokinetic equations and reduce Ampere's law Eq. (28) and the quasineutrality condition Eq. (29) to recover the linear ideal MHD equations. Although the theory outlined in the previous section is applicable to arbitrary FLR ordering, we restrict ourselves here only to keep the effects that are larger than or of the same order as the perpendicular inertia effect with diamagnetic frequency shift {i.e., ω^2 replaced by $\omega[\omega - \omega_{*1}(1+\eta_i)]$}, with $\omega_{*i} = -i(T_i/e_iB)(\partial\ln n_0/\partial\psi)\mathbf{e}_b\times\nabla\psi\cdot\nabla$ and $\eta_i = \partial\ln T_i/\partial\ln n_0$. In the ordering analyses, we assume that $\omega \gtrsim \omega_{*i}$. We also retain explicitly two different perpendicular wave lengths, using $\lambda_\perp$ and $\lambda_\wedge$ to denote, respectively, the perpendicular wavelengths normal and tangential to the magnetic surface. Noting that $\omega_{*i}\sim\rho_i/\lambda_\wedge$ as compared to the shear Alfvén frequency, the assumption $\omega\sim\omega_{*i}$ implies that effects of order $(\rho_i/\lambda_\wedge)^2$ are kept in the MHD equations. Specifically, we assume that $\rho_i/\lambda_\perp\ll 1$, $\lambda_\wedge\gtrsim\lambda_\perp$, $\lambda_\wedge\ll L_p$, and $L_B\gtrsim L_p$.

To recover MHD, we adopt the MHD gauge in this section,

$$\delta\mathbf{A}=\boldsymbol{\xi}\times\mathbf{B}, \tag{30}$$

in order to take advantage of displaying explicitly the term-by-term correspondences between the MHD and kinetic descriptions. As discussed in Ref. 17, the generality of the MHD gauge relies on the invertibility of $\mathbf{B}\cdot\nabla$, and consequently those modes involving the magnetic field reconnection are excluded from consideration. Nevertheless, noting the similarity of the MHD gauge in Eq. (30) and the general gauge in Eq. (27), the generalization from the MHD gauge to the general gauge for resistive MHD application is straightforward and will be discussed elsewhere. Note that the representation in Eq. (30) is valid for arbitrary toroidal mode number n.

A. Solution of the gyrokinetic equation

To calculate Ampere's law and the quasineutrality condition, both the first and zero (gyrophase averaged) harmonics of the gyrokinetic distribution functions are required. To solve the gyrokinetic equation (22), we extract the convective part of the distribution function from δH, by letting

$$\delta H(\mathbf{X})=-\boldsymbol{\xi}(\mathbf{X})\cdot\nabla F_{g0}+\delta G(\mathbf{X}). \tag{31}$$

Consequently, the gyrokinetic equation (22) is transformed to

$$\begin{aligned}\mathcal{L}_g\delta G(\mathbf{X}) = {} & -i\omega\frac{e}{m_\rho}\frac{\partial F_{g0}}{\partial\varepsilon}\mathbf{v}_\perp\cdot\delta\mathbf{A}(\mathbf{x})+\mathbf{v}_D\cdot\nabla_X[\boldsymbol{\xi}(\mathbf{X})\cdot\nabla F_{g0}]+i\omega[\boldsymbol{\xi}(\mathbf{x})-\boldsymbol{\xi}(\mathbf{X})]\cdot\nabla F_{g0}+i(\omega-\omega_*^T)\frac{e}{m_\rho}\frac{\partial F_{g0}}{\partial\varepsilon}\delta\varphi(\mathbf{x})\\ & +\frac{e}{m_\rho}\mathbf{v}\cdot\nabla_x\left(\frac{\partial F_{g0}}{\partial\varepsilon}\right)\delta\varphi(\mathbf{x})+\left(\frac{g}{B}\frac{\partial F_{g0}}{\partial\Psi}+|v_\parallel|\frac{e}{m_\rho B}\frac{\partial\bar{F}_{g1}}{\partial\mu}\right)\mathbf{e}_b\times\mathbf{v}_\perp\cdot\delta\mathbf{B}(\mathbf{x})+\frac{1}{B}\mathbf{v}_\perp\cdot\nabla_X F_{g0}\mathbf{e}_b\cdot\delta\mathbf{B}(\mathbf{x})\\ & -\frac{g}{B}\frac{\partial F_{g0}}{\partial\Psi}\mathbf{e}_b\cdot\nabla_x\delta\varphi(\mathbf{x})+\mathrm{sign}(v_\parallel)\frac{e}{m_\rho B}\frac{\partial\bar{F}_{g1}}{\partial\mu}\mathbf{v}_\perp\cdot[\nabla_x\delta\varphi(\mathbf{x})-i\omega\delta\mathbf{A}(\mathbf{x})]+v_\parallel\mathbf{e}_b\cdot[\nabla_x\boldsymbol{\xi}(\mathbf{x})\cdot\nabla F_{g0}-\nabla_X\boldsymbol{\xi}(\mathbf{X})\cdot\nabla F_{g0}].\end{aligned} \tag{32}$$

First, let us solve the first-harmonic solution of $\delta\widetilde{G}$, using the perturbation method outlined in the last section. The details are given in Appendix A. The solution is obtained from collecting the contributions from $\delta\widetilde{G}_{1a}$, $\delta\widetilde{G}_{1c}$, $\delta\widetilde{G}_{1e}$, $\delta\widetilde{G}_{1f}$, $\delta\widetilde{G}_{1g}$, and $\delta\widetilde{G}_{1\mathrm{conv}}$,

$$\begin{aligned}\delta\widetilde{G}_1(\mathbf{x}) = {} & -i\omega\frac{1}{B}\frac{\partial F_{g0}}{\partial\varepsilon}\mathbf{e}_b\times\mathbf{v}_\perp\cdot\delta\mathbf{A}+\frac{\omega^2}{\Omega}\frac{m_\rho}{T}F_{g0}v_\perp(-\sin\alpha\mathbf{e}_1+\cos\alpha\mathbf{e}_2)\cdot\boldsymbol{\xi}(\mathbf{x})-i\omega\frac{m_\rho}{T}F_{g0}v_\perp\cos\alpha\mathbf{e}_1\cdot\boldsymbol{\xi}\\ & -i\omega\frac{m_\rho}{T}F_{g0}\frac{3v_\perp^3}{8\Omega^2}B^{-3/2}\cos\alpha\{\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2\}:\nabla\nabla(B^{3/2}\mathbf{e}_1\cdot\boldsymbol{\xi})-i\omega\frac{m_\rho}{T}F_{g0}\cos\alpha\Bigg(\mathbf{e}_1\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}\Bigg\{-\frac{5v_\perp}{8\Omega^2}\mathbf{e}_1\cdot\left(\frac{v_\perp^2}{2}\nabla\ln B\right.\\ & \left.+v_\parallel^2\boldsymbol{\kappa}\right)+\frac{v_\perp^3}{8\Omega^2}\frac{\partial\ln n_0}{\partial\psi}\left[1+\eta\left(\frac{m_\rho\varepsilon}{T}-\frac{5}{2}\right)\right]-\frac{v_\perp^3}{16\Omega^2}\mathbf{e}_1\cdot(\nabla\ln B)\Bigg\}+\mathbf{e}_2\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}\Bigg[-\frac{5v_\perp}{8\Omega^2}\mathbf{e}_2\cdot\left(\frac{v_\perp^2}{2}\nabla\ln B\right.\\ & \left.+v_\parallel^2\boldsymbol{\kappa}\right)-\frac{9v_\perp^3}{16\Omega^2}\mathbf{e}_2\cdot(\nabla\ln B)\Bigg]+\frac{7v_\perp^3}{8\Omega^2}[(\mathbf{e}_1\cdot\nabla\mathbf{e}_1)\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}+(\mathbf{e}_2\cdot\nabla\mathbf{e}_2)\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}]\\ & +\mathbf{e}_1\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}\left[\frac{v_\perp}{8\Omega^2}\mathbf{e}_2\cdot\left(\frac{v_\perp^2}{2}\nabla\ln B+v_\parallel^2\boldsymbol{\kappa}\right)+\frac{7v_\perp^3}{16\Omega^2}\mathbf{e}_2\cdot(\nabla\ln B)\right]+\mathbf{e}_2\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}\Bigg[-\frac{v_\perp}{8\Omega^2}\mathbf{e}_1\cdot\left(\frac{v_\perp^2}{2}\nabla\ln B\right.\\ & \left.+v_\parallel^2\boldsymbol{\kappa}\right)-\frac{v_\perp^3}{8\Omega^2}\frac{\partial\ln n_0}{\partial\psi}\left[1+\eta\left(\frac{m_\rho\varepsilon}{T}-\frac{5}{2}\right)\right]+\frac{v_\perp^3}{16\Omega^2}\mathbf{e}_1\cdot(\nabla\ln B)\Bigg]+\frac{v_\perp^3}{8\Omega^2}[-(\mathbf{e}_2\cdot\nabla\mathbf{e}_1)\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}\\ & +(\mathbf{e}_1\cdot\nabla\mathbf{e}_2)\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}]\Bigg)-i\omega\frac{m_\rho}{T}F_{g0}v_\perp\sin\alpha\mathbf{e}_2\cdot\boldsymbol{\xi}-i\omega\frac{m_\rho}{T}F_{g0}\frac{3v_\perp^3}{8\Omega^2}B^{-3/2}\sin\alpha\{\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2\}:\nabla\nabla(B^{3/2}\mathbf{e}_2\cdot\boldsymbol{\xi})\\ & -i\omega\frac{m_\rho}{T}F_{g0}\sin\alpha\Bigg(\mathbf{e}_1\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}\left[-\frac{v_\perp}{8\Omega^2}\mathbf{e}_2\cdot\left(\frac{v_\perp^2}{2}\nabla\ln B+v_\parallel^2\boldsymbol{\kappa}\right)+\frac{v_\perp^3}{16\Omega^2}\mathbf{e}_2\cdot(\nabla\ln B)\right]\\ & +\mathbf{e}_2\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}\left\{\frac{v_\perp}{8\Omega^2}\mathbf{e}_1\cdot\left(\frac{v_\perp^2}{2}\nabla\ln B+v_\parallel^2\boldsymbol{\kappa}\right)-\frac{3v_\perp^3}{8\Omega^2}\frac{\partial\ln n_0}{\partial\psi}\left[1+\eta\left(\frac{m_\rho\varepsilon}{T}-\frac{5}{2}\right)\right]+\frac{7v_\perp^3}{16\Omega^2}\mathbf{e}_1\cdot(\nabla\ln B)\right\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{v_\perp^3}{8\Omega^2}[(\mathbf{e}_2\cdot\nabla\mathbf{e}_1)\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}-(\mathbf{e}_1\cdot\nabla\mathbf{e}_2)\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}]+\mathbf{e}_1\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}\left\{-\frac{5v_\perp}{8\Omega^2}\mathbf{e}_1\cdot\left(\frac{v_\perp^2}{2}\nabla\ \ln B+v_\parallel^2\boldsymbol{\kappa}\right)\right.\\
&\left.+\frac{5v_\perp^3}{8\Omega^2}\frac{\partial\ln n_0}{\partial\psi}\left[1+\eta\left(\frac{m_\rho\varepsilon}{T}-\frac{5}{2}\right)\right]-\frac{9v_\perp^3}{16\Omega^2}\mathbf{e}_1\cdot(\nabla\ln B)\right\}+\mathbf{e}_2\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}\left[-\frac{5v_\perp}{8\Omega^2}\mathbf{e}_2\cdot\left(\frac{v_\perp^2}{2}\nabla\ \ln B+v_\parallel^2\boldsymbol{\kappa}\right)\right.\\
&\left.-\frac{v_\perp^3}{16\Omega^2}\mathbf{e}_2\cdot(\nabla\ln B)\right]+\frac{7v_\perp^3}{8\Omega^2}[(\mathbf{e}_1\cdot\nabla\mathbf{e}_1)\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}+(\mathbf{e}_2\cdot\nabla\mathbf{e}_2)\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}]\Bigg)-i\omega\frac{|\nabla\psi|}{\Omega^2}\frac{\partial F_{g0}}{\partial\psi}\mathbf{v}_\perp\cdot\nabla(\mathbf{e}_1\cdot\boldsymbol{\xi})\\
&-i\frac{1}{B\Omega}(\omega-\omega_*^T)\frac{\partial F_{g0}}{\partial\varepsilon}\mathbf{v}_\perp\cdot\nabla\delta\varphi(\mathbf{x})-\frac{1}{B}\mathbf{v}\times\mathbf{e}_b\cdot\nabla\left(\frac{\partial F_{g0}}{\partial\varepsilon}\right)\delta\varphi(\mathbf{x})-\frac{3}{8B\Omega^2}v_\perp^3\mathbf{e}_1\cdot\nabla\left(\frac{\partial F_{g0}}{\partial\varepsilon}\right)\sin\alpha(\mathbf{e}_1\mathbf{e}_1\\
&+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla\delta\varphi(\mathbf{x})-\left(\frac{g}{\Omega B}\frac{\partial F_{g0}}{\partial\psi}+\frac{1}{B^2}\frac{\partial\bar{F}_{g1}}{\partial\mu}|v_\parallel|\right)\mathbf{v}_\perp\cdot\delta\mathbf{B}(\mathbf{x})-\frac{3}{8\Omega^3}\left(\frac{g}{B}\frac{\partial F_{g0}}{\partial\psi}+\frac{e}{m_\rho}|v_\parallel|\frac{\partial\bar{F}_{g1}}{\partial\mu}\right)v_\perp^2(\mathbf{e}_1\mathbf{e}_1\\
&+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\mathbf{e}_b\cdot\delta\mathbf{B}(\mathbf{x}))-\frac{|\nabla\psi|}{B\Omega}\frac{\partial F_{g0}}{\partial\psi}v_\perp\sin\alpha\mathbf{e}_b\cdot\delta\mathbf{B}(\mathbf{x})-\frac{3|\nabla\psi|}{8\Omega^3B}\frac{\partial F_{g0}}{\partial\psi}v_\perp^3\sin\alpha(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\mathbf{e}_b\cdot\delta\mathbf{B}(\mathbf{x}))\\
&-\frac{v_\perp}{\Omega}(\mathbf{e}_1\sin\alpha-\mathbf{e}_2\cos\alpha)\cdot\nabla\boldsymbol{\xi}\cdot\nabla F_{g0}-\frac{v_\perp^3}{8\Omega^3}|\nabla\psi|\frac{\partial F_{g0}}{\partial\psi}(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1\sin\alpha+\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\sin\alpha-\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\cos\alpha\\
&-\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2\cos\alpha)\,\vdots\,\nabla\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1)-\frac{v_\perp^3}{8\Omega^3}\frac{\partial F_{g0}}{\partial\psi}\mathbf{e}_1\cdot\nabla(|\nabla\psi|)(3\mathbf{e}_1\mathbf{e}_1\sin\alpha+\mathbf{e}_2\mathbf{e}_2\sin\alpha-2\mathbf{e}_1\mathbf{e}_2\cos\alpha):\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1)\\
&-\frac{v_\perp^3}{8\Omega^3}\frac{\partial F_{g0}}{\partial\psi}\mathbf{e}_2\cdot\nabla(|\nabla\psi|)(2\mathbf{e}_1\mathbf{e}_2\sin\alpha-\mathbf{e}_1\mathbf{e}_1\cos\alpha-3\mathbf{e}_2\mathbf{e}_2\cos\alpha):\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1)-\frac{v_\perp^3}{8\Omega^3}|\nabla\psi|\mathbf{e}_1\cdot\nabla\left(\frac{\partial F_{g0}}{\partial\psi}\right)\\
&\times(3\mathbf{e}_1\mathbf{e}_1\sin\alpha+\mathbf{e}_2\mathbf{e}_2\sin\alpha-2\mathbf{e}_1\mathbf{e}_2\cos\alpha):\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1).
\end{aligned}
\tag{33}
$$

Next, we derive the gyrophase-averaged gyrokinetic equation. Summarizing the calculations in Appendix B, one can find

$$
\begin{aligned}
(\mathbf{v}_\parallel\cdot\nabla-i\omega-i\omega_d)\delta G_0(\mathbf{X})=&-i\omega\mu_0B\frac{\partial F_{g0}}{\partial\varepsilon}\nabla_\perp\cdot\boldsymbol{\xi}-i\omega\frac{\partial F_{g0}}{\partial\varepsilon}(\mu_0B-v_\parallel^2)\boldsymbol{\kappa}\cdot\boldsymbol{\xi}+i\omega_d\boldsymbol{\xi}\cdot\nabla F_{g0}+i(\omega-\omega_*^T)\frac{e}{m_\rho}\frac{\partial F_{g0}}{\partial\varepsilon}\delta\varphi\\
&-\mu_0\mathbf{e}_1\cdot\nabla\left(\frac{\partial F_{g0}}{\partial\varepsilon}\right)\mathbf{e}_2\cdot\nabla\delta\varphi-\frac{v_\perp^2}{\Omega}\mathbf{e}_1\cdot\nabla_XF_{g0}\mathbf{e}_2\cdot\nabla\left(\frac{1}{B}\mathbf{e}_b\cdot\delta\mathbf{B}\right)+\frac{v_\parallel v_\perp^2}{\Omega}(\mathbf{e}_1\mathbf{e}_1\\
&+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\mathbf{e}_b\cdot\nabla\boldsymbol{\xi}\cdot\nabla F_{g0})-\frac{v_\parallel v_\perp^2}{\Omega}[(\mathbf{e}_1\cdot\nabla\mathbf{e}_b)\cdot\nabla(\mathbf{e}_1\cdot\nabla\boldsymbol{\xi}\cdot\nabla F_{g0})\\
&+(\mathbf{e}_2\cdot\nabla\mathbf{e}_b)\cdot\nabla(\mathbf{e}_2\cdot\nabla\boldsymbol{\xi}\cdot\nabla F_{g0})],
\end{aligned}
\tag{34}
$$

where $\omega_d=i\mathbf{v}_d\cdot\nabla$.

B. Ampere's law

Using the first-harmonic solution $\delta\widetilde{G}_1(\mathbf{x})$ of the gyrokinetic equation in Eq. (33) and the gyrophase-averaged distribution function $G_0(\mathbf{X})$, governed by Eq. (34), one can calculate two perpendicular components of the current density in Ampere's law in Eq. (28). The $\delta\widetilde{G}_1(\mathbf{x})$ contribution has been given in Appendix A. The $\delta G_0(\mathbf{x})$ contribution can be expressed as follows:

$$
\sum_{i,e}e\int d^3v\mathbf{v}_\perp\cdot\mathbf{e}_{1,2}\delta G_0(\mathbf{X})\bigg|_{\mathbf{x}}=\sum_{i,e}e\int d^3v\mathbf{v}_\perp\cdot\mathbf{e}_{1,2}\frac{1}{\Omega}\mathbf{v}\times\mathbf{e}_b\cdot\nabla\delta G_0(\mathbf{x})=-\sum_{i,e}m_\rho\int d^3v\mu_0\mathbf{e}_{2,1}\cdot\nabla\delta G_0(\mathbf{x}).
\tag{35}
$$

Combining the contributions from individual terms given in Appendix A and Eq. (35), one obtains the two components of the Ampere's law in the kinetic description:

$$
\begin{aligned}
\mathbf{e}_1\cdot\nabla\times\delta\mathbf{B} ={}& -\frac{gP'}{B^2}\mathbf{e}_1\cdot\delta\mathbf{B} - g'\mathbf{e}_1\cdot\delta\mathbf{B} + \frac{1}{B}\mathbf{e}_2\cdot\nabla(P'|\nabla\psi|\mathbf{e}_1\cdot\boldsymbol{\xi}) - \sum_{i,e} m_\rho \int d^3v\,\mu_0\mathbf{e}_2\cdot\nabla\delta G_0(\mathbf{x}) + \frac{\omega^2}{B}\rho_m\mathbf{e}_2\cdot\boldsymbol{\xi} \\
& - i\omega\frac{|\nabla\psi|}{B\Omega_i}P_i'\mathbf{e}_1\cdot\nabla(\mathbf{e}_1\cdot\boldsymbol{\xi}) + in_0m_{\rho i}[\omega-\omega_{*i}(1+\eta_i)]\frac{1}{B^2}\mathbf{e}_1\cdot\nabla\delta\varphi - i\omega\frac{2n_0Te_i}{m_\rho}\frac{3}{4\Omega_i^2}B^{-3/2}\{\mathbf{e}_1\mathbf{e}_1 \\
& + \mathbf{e}_2\mathbf{e}_2\}:\nabla\nabla(B^{3/2}\mathbf{e}_1\cdot\boldsymbol{\xi}) - i\omega\frac{2n_0Te_i}{m_\rho}\Bigg\{\mathbf{e}_1\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}\Bigg[-\frac{1}{8\Omega_i^2}\mathbf{e}_1\cdot\Bigg(6\,\nabla\,\ln B+\frac{5}{2}\boldsymbol{\kappa}\Bigg)+\frac{1}{4\Omega_i^2}\frac{\partial\ln n_0}{\partial\psi}(1+\eta_i)\Bigg] \\
& + \mathbf{e}_2\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}\Bigg[-\frac{1}{8\Omega_i^2}\mathbf{e}_2\cdot\Bigg(14\,\nabla\,\ln B+\frac{5}{2}\boldsymbol{\kappa}\Bigg)\Bigg]+\frac{7}{4\Omega_i^2}[(\mathbf{e}_1\cdot\nabla\mathbf{e}_1)\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}+(\mathbf{e}_2\cdot\nabla\mathbf{e}_2)\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}] \\
& + \mathbf{e}_1\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}\frac{1}{8\Omega_i^2}\mathbf{e}_2\cdot\Bigg(8\,\nabla\,\ln B+\frac{1}{2}\boldsymbol{\kappa}\Bigg)+\mathbf{e}_2\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}\Bigg[-\frac{1}{16\Omega_i^2}\mathbf{e}_1\cdot\boldsymbol{\kappa}-\frac{1}{4\Omega_i^2}\frac{\partial\ln n_0}{\partial\psi}(1+\eta_i)\Bigg]+\frac{1}{4\Omega_i^2} \\
& [-(\mathbf{e}_2\cdot\nabla\mathbf{e}_1)\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}+(\mathbf{e}_1\cdot\nabla\mathbf{e}_2)\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}]\Bigg\} - \frac{3}{2\Omega_i}\Bigg[\frac{gn_0T_i^2}{B^2\Omega_im_{\rho i}}\frac{\partial\ln n_0}{\partial\psi}(1+2\eta_i) \\
& - 4\pi m_{\rho i}\int d\varepsilon d\mu_0\mu_0\bar{F}_{g1}\Bigg](\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\mathbf{e}_1\cdot\delta\mathbf{B}) + \frac{n_0e_i}{2\Omega_i^3}\Bigg(\frac{T}{m_{\rho i}}\Bigg)^2|\nabla\psi|\frac{\partial\ln n_0}{\partial\psi}(1+2\eta)(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 \\
& + \mathbf{e}_2\mathbf{e}_2\mathbf{e}_2)\,\vdots\,\nabla\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) + \frac{n_0e_i}{\Omega_i^3}\Bigg(\frac{T}{m_{\rho i}}\Bigg)^2\frac{\partial\ln n_0}{\partial\psi}(1+2\eta)\mathbf{e}_1\cdot\nabla(|\nabla\psi|)\mathbf{e}_1\mathbf{e}_2:\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) \\
& + \frac{n_0e_i}{2\Omega_i^3}\Bigg(\frac{T}{m_{\rho i}}\Bigg)^2\frac{\partial\ln n_0}{\partial\psi}(1+2\eta)\mathbf{e}_2\cdot\nabla(|\nabla\psi|)(\mathbf{e}_1\mathbf{e}_1+3\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) + \frac{n_0e_i}{\Omega_i^3}\Bigg(\frac{T}{m_{\rho i}}\Bigg)^2|\nabla\psi|\Bigg\{\Bigg(\frac{\partial\ln n_0}{\partial\psi}\Bigg)^2\Bigg(1 \\
& + \frac{15}{2}\eta^2+4\eta\Bigg)+\frac{\partial^2\ln n_0}{\partial\psi^2}(1+2\eta)+2\frac{\partial\ln n_0}{\partial\psi}\Bigg(2\frac{\partial\eta}{\partial\psi}-\eta\frac{7}{2}\frac{\partial\ln T}{\partial\psi}\Bigg)\Bigg\}\mathbf{e}_1\mathbf{e}_2:\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1),
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
\mathbf{e}_2\cdot\nabla\times\delta\mathbf{B} ={}& -\frac{gP'}{B^2}\mathbf{e}_2\cdot\delta\mathbf{B} - g'\mathbf{e}_2\cdot\delta\mathbf{B} - \frac{P'|\nabla\psi|}{B^2}\mathbf{e}_b\cdot\delta\mathbf{B} - \frac{1}{B}\mathbf{e}_1\cdot\nabla(P'|\nabla\psi|\mathbf{e}_1\cdot\boldsymbol{\xi}) + \sum_{i,e} m_\rho\int d^3v\,\mu_0\mathbf{e}_1\cdot\nabla\delta G_0(\mathbf{x}) \\
& - \frac{\omega^2}{B}\rho_m\mathbf{e}_1\cdot\boldsymbol{\xi} - i\omega\frac{|\nabla\psi|}{B\Omega_i}P_i'\mathbf{e}_2\cdot\nabla(\mathbf{e}_1\cdot\boldsymbol{\xi}) + in_0m_{\rho i}[\omega-\omega_{*i}(1+\eta_i)]\frac{1}{B^2}\mathbf{e}_2\cdot\nabla\delta\varphi - i\omega\frac{2n_0Te_i}{m_\rho}\frac{3}{4\Omega_i^2}B^{-3/2}\{\mathbf{e}_1\mathbf{e}_1 \\
& + \mathbf{e}_2\mathbf{e}_2\}:\nabla\nabla(B^{3/2}\mathbf{e}_2\cdot\boldsymbol{\xi}) - i\omega\frac{2n_0T_ie_i}{m_{\rho i}}\Bigg\{-\mathbf{e}_1\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}\frac{1}{16\Omega_i^2}\mathbf{e}_2\cdot\boldsymbol{\kappa}+\mathbf{e}_2\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}\Bigg[\frac{1}{8\Omega_i^2}\mathbf{e}_1\cdot\Bigg(8\,\nabla\,\ln B+\frac{1}{2}\boldsymbol{\kappa}\Bigg) \\
& - \frac{3}{4\Omega_i^2}\frac{\partial\ln n_0}{\partial\psi}(1+\eta_i)\Bigg]+\frac{1}{4\Omega_i^2}[(\mathbf{e}_2\cdot\nabla\mathbf{e}_1)\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}-(\mathbf{e}_1\cdot\nabla\mathbf{e}_2)\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}]+\mathbf{e}_1\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi} \\
& \Bigg[-\frac{1}{8\Omega_i^2}\mathbf{e}_1\cdot\Bigg(14\,\nabla\,\ln B+\frac{5}{2}\boldsymbol{\kappa}\Bigg)+\frac{5}{4\Omega_i^2}\frac{\partial\ln n_0}{\partial\psi}(1+\eta_i)\Bigg]-\mathbf{e}_2\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}\frac{1}{8\Omega_i^2}\mathbf{e}_2\cdot\Bigg(6\,\nabla\,\ln B+\frac{5}{2}\boldsymbol{\kappa}\Bigg) \\
& + \frac{7}{4\Omega_i^2}[(\mathbf{e}_1\cdot\nabla\mathbf{e}_1)\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}+(\mathbf{e}_2\cdot\nabla\mathbf{e}_2)\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}]\Bigg\}+\frac{3}{2}n_0m_{\rho i}\frac{|\nabla\psi|T_i}{B^3e_i}\frac{\partial\ln n_0}{\partial\psi}(1+\eta_i)(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla\delta\varphi \\
& - \frac{3}{2\Omega_i}\Bigg[\frac{gn_0T_i^2}{B^2\Omega_im_{\rho i}}\frac{\partial\ln n_0}{\partial\psi}(1+2\eta_i)-4\pi m_{\rho i}\int d\varepsilon d\mu_0\mu_0\bar{F}_{g1}\Bigg](\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\mathbf{e}_2\cdot\delta\mathbf{B}) \\
& - \frac{3|\nabla\psi|e_in_0}{2B\Omega_i^3}\Bigg(\frac{T_i}{m_{\rho i}}\Bigg)^2\frac{\partial\ln n_0}{\partial\psi}(1+2\eta_i)(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\mathbf{e}_b\cdot\delta\mathbf{B}) - \frac{n_0e_i}{2\Omega_i^3}\Bigg(\frac{T}{m_{\rho i}}\Bigg)^2\frac{\partial\ln n_0}{\partial\psi}(1+2\eta)(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1 \\
& + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2)\,\vdots\,\nabla\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) - \frac{n_0e_i}{2\Omega_i^3}\Bigg(\frac{T}{m_{\rho i}}\Bigg)^2\frac{\partial\ln n_0}{\partial\psi}(1+2\eta)\mathbf{e}_1\cdot\nabla(|\nabla\psi|)(3\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) \\
& - \frac{n_0e_i}{\Omega_i^3}\Bigg(\frac{T}{m_{\rho i}}\Bigg)^2\frac{\partial\ln n_0}{\partial\psi}(1+2\eta)\mathbf{e}_2\cdot\nabla(|\nabla\psi|)\mathbf{e}_1\mathbf{e}_2:\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) - \frac{n_0e_i}{2\Omega_i^3}\Bigg(\frac{T}{m_{\rho i}}\Bigg)^2|\nabla\psi|\Bigg\{\Bigg(\frac{\partial\ln n_0}{\partial\psi}\Bigg)^2\Bigg(1+\frac{15}{2}\eta^2 \\
& + 4\eta\Bigg)+\frac{\partial^2\ln n_0}{\partial\psi^2}(1+2\eta)+\frac{\partial\ln n_0}{\partial\psi}\Bigg(2\frac{\partial\eta}{\partial\psi}-\eta\frac{7}{2}\frac{\partial\ln T}{\partial\psi}\Bigg)\Bigg\}(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1).
\end{aligned}
\tag{37}
$$

These two equations are the kinetic counterparts of two perpendicular components [Eqs. (5) and (6)] of the MHD velocity momentum equations.

C. The quasineutrality condition

To determine $\delta\varphi$, the quasineutrality condition Eq. (29) is needed. Inserting Eqs. (21) and (31) into the quasineutrality condition Eq. (29) yields

$$\delta\varphi(\mathbf{x}) = -\frac{1}{\sum_{i,e} n_0 e^2/T}\sum_{i,e} e \int d^3v\, \delta G_0. \tag{38}$$

Here, we have noted that for our ordering requirement, no FLR expansion is needed for δG_0, and we have considered also the fact that the contribution from the gyrophase-dependent part of the perturbed distribution function is negligible. The parallel-electric-field effect enters into both the gyrophase-averaged gyrokinetic equation (34) and the two perpendicular components of Ampere's law: Eqs. (36) and (37).

First, we discuss the $\delta\varphi$ effect in the gyrokinetic equation (34). Inserting Eq. (38) into the terms containing $\delta\varphi$ in Eq. (34), one can find that those terms become of the same order as the term $-i\omega\delta G_0$. This shows that the parallel-electric-field effect should be kept as soon as the wave-particle resonance effect is taken into account. This fact is especially relevant to the study of the kinetic stabilization of the resistive wall modes, in which the particle-wave resonance effect is critical. Comparing Eqs. (36) and (37) with the MHD counterparts [Eqs. (5) and (6)], one can see that δG_0 plays the role of $\nabla\cdot\boldsymbol{\xi}$. The sound-wave resonance in the ideal MHD case in Eq. (7) is replaced by the wave-particle resonance in the kinetic description in Eq. (34). Note that the sound wave resonance in the ideal MHD case is related to the sideband resonance.[18] Therefore, the parallel-electric-field effect comes mainly from the side band.

Next, we discuss the $\delta\varphi$ effect appearing explicitly in the two perpendicular components of Ampere's law [Eqs. (36) and (37)]. In this case, the leading parallel electric field effect on the main harmonic of the Ampere's law is directly through the main harmonic of $\delta\varphi$. Note that $\nabla\cdot\boldsymbol{\xi}$ on the right-hand side of Eq. (34) equals $-2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}$ to the leading order for the modes with frequencies much lower than the compressional Alfvén frequency. In the vicinity of the rational surfaces $\mathbf{v}_\parallel\cdot\nabla \lesssim \omega$, the main harmonic of the perturbed distribution function δG_0 is of order $F_{g0}(L_p/\omega L_B^2)\xi_\wedge$. Inserting this estimate into the quasineutrality condition Eq. (38), one can find that the main harmonic of $\delta\varphi$ is of order $(T/e)\times(L_p/L_B^2)\xi_\wedge$. With this ordering estimate, one can find that the term containing explicitly $\delta\varphi$ in Eq. (36) is of order $(\omega_{*i}/\omega)(L_p/L_B)^2(\lambda_\wedge/\lambda_\perp)$ as compared to the MHD perpendicular inertia term [fifth term on the r.h.s. of Eq. (36)]. This gives the condition for which the parallel-electric-field effect of the main harmonic should be kept. The main harmonic of kinetic-parallel-electric-field effect is relevant to study the modes of the resistive MHD type; for example, the field line reconnection phenomenon.

D. Discussion of the basic set of equations in the gyrokinetic description

We first discuss the structure similarity of the basic set of equations between the ideal MHD and the current gyrokinetic descriptions. In the ideal MHD description, there are three unknowns: two components of $\boldsymbol{\xi}_\perp$ and scalar $\nabla\cdot\boldsymbol{\xi}$. They are governed by the three projections of the MHD momentum equation: Eqs. (5)–(7). In the gyrokinetic description, two perpendicular components of $\boldsymbol{\xi}_\perp$ remain, but the scalar $\nabla\cdot\boldsymbol{\xi}$ is replaced by the ion and electron gyrophase-averaged perturbed distribution functions δG_0. Note that in the kinetic description, $\boldsymbol{\xi}$ represents the field line displacement—a field variable; instead, in the ideal MHD description $\boldsymbol{\xi}$ corresponds to the fluid velocity moment. The proportionality between $\nabla\cdot\boldsymbol{\xi}$ and $-\delta G_0$ can be envisaged by the fluid continuity equation, noting that the convective part of the distribution function has been extracted in Eq. (31). Correspondingly, the two perpendicular projections of the MHD momentum equation: Eqs. (5) and (6) are replaced by two perpendicular projections of Ampere's law [Eqs. (36) and (37)]; the parallel projection of the MHD momentum equation (7) is replaced by gyrophase-averaged gyrokinetic equations for ion and electron species [Eq. (34)]. In the kinetic description, there is one more unknown $\delta\varphi$, describing the parallel electric field effect. $\delta\varphi$ is governed by the quasineutrality condition in Eq. (38).

Next, let us discuss term by term correspondences between the ideal MHD and the current gyrokinetic descriptions. Comparing the two perpendicular components of the ideal MHD momentum equations (5) and (6) with the two projections of Ampere's law in kinetic description [Eqs. (36) and (37)], one can see that all MHD terms in Eqs. (5) and (6), except the plasma compressibility term (proportional to $\nabla\cdot\boldsymbol{\xi}$), are recovered in the kinetic equations (36) and (37). Note that the MHD fluid description is based on the particle spatial localization assumption. In the collisionless Vlasov equation description, however, the spatial localization can only be expected in the perpendicular direction due to the strong magnetic field, while particles can move freely in the parallel direction. Therefore, a fully kinetic description is needed in the parallel direction. Consequently, the terms due to the plasma compressibility effect in the ideal MHD description [the fourth term of Eq. (5) and the fifth term of Eq. (6)] are replaced by the kinetic moments for plasma compressibility [the fourth term of Eq. (36) and the fifth term of Eq. (37), respectively]. Interestingly, There is also structure similarity between the MHD equation of the parallel motion [Eq. (7)] and the ion gyrophase-averaged gyrokinetic equation (34). Note that the kinetic plasma compressibility terms in Eq. (36) and Eq. (37) depend only on the even part (with respect to the parallel velocity) of the gyrophase-averaged distribution function. The gyrophase-averaged gyrokinetic equation governing the even part can be derived from Eq. (34) as given in Eq. (B1) in Appendix B. To recover the MHD, the limit $\omega \gg \omega_d$, ω_* should be taken. In this limit, ω_d on the l.h.s. of Eq. (34) [(or Eq. (B1)] can be dropped and only the first two terms on the r.h.s. of Eq. (34) need to be kept. Noting $m_{\rho i} \gg m_{\rho e}$, only ion distribution function needs to be kept. With these simplifications, the term by term correspondences between the MHD equation of parallel motion [Eq. (7)] and the ion gyrokinetic equation (34) [with Eq. (B1) used to construct the even part of the distribution function] are obvious, with the particle velocity replaced by the proper thermal velocity, and the fact noted that the second term on the r.h.s. of Eq. (34) vanishes with the thermal velocity replacement. It should be pointed out that the recovery of the similarity between the MHD and kinetic parallel descriptions is realized until the order of the second term on the r.h.s. of Eq. (34). This is particularly relevant for low frequency MHD modes, for which one usually has $\nabla\cdot\boldsymbol{\xi}_\perp \sim -2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}$, and therefore the first and second terms on the r.h.s. of Eq. (34) become of the same order.

In the collisionless kinetic description, the ions and electrons move individually along the field lines, instead of collectively as a fluid element. The different responses of the ions and electrons to the electromagnetic perturbations causes the charge separation and thus the excitation of the parallel electric field. This has led the electrostatic scalar potential $\delta\varphi$ to appear in both the perpendicular [Eqs. (36) and (37)] and parallel [Eq. (34)] equations of motion.

Besides, in the MHD description, the perpendicular motion of a fluid element is regarded to be the same as the field line displacement. In the kinetic description, however, the perpendicular motion of a fluid element is considered to be different from the field line displacement due to the FLR effect. The FLR effect leads to the changes: $\omega^2 \to \omega[\omega - \omega_{*i}(1+\eta_i)]$ in the inertia terms [the fifth and sixth terms on the r.h.s. of Eq. (36) and the sixth and seventh terms on the r.h.s. of Eq. (37)] and contributes as well additional FLR effect terms in Eqs. (36), (37), and (34).

Comparing to the existing gyrokinetic theories, in which the only FLR effect in this order is $\omega^2 \to \omega[\omega - \omega_{*i}(1+\eta_i)]$ in the inertia terms.[16] In our case, however, one can see that several FLR effect-related terms appearing in Eqs. (36), (37), and (34). The first two MHD terms on the r.h.s. of both Eq. (36) and Eq. (37) are also missing in the previous gyrokinetic formulation. They are recovered here by taking into account the first-order correction of the equilibrium distribution function and the gyrophase-dependent part of the perturbed distribution function. These two MHD terms result from $\mathbf{J}_0 \times \delta\mathbf{B}$ term in the perpendicular force balance equations (5) and (6).

V. CONCLUSIONS

In this paper, we revisit the linear gyrokinetic theory, so that the linear ideal MHD is recovered from our newly derived gyrokinetic formalism and the FLR effects are retained fully. We find that the $\mathbf{J}_0 \times \delta\mathbf{B}$ effect on the perpendicular force balance (or the perpendicular Ampere's law) is not retained fully in the conventional gyrokinetic formalism. Because of this the MHD terms in the perpendicular momentum equation cannot be recovered completely in the conventional gyrokinetic formalism. We find also that in the conventional gyrokinetic formalism the FLR effects are not retained fully. In the conventional gyrokinetic formalism the lowest-order FLR correction is given only by the so-called diamagnetic frequency shift; i.e., replacing ω^2 with $\omega[\omega - \omega_{*i}(1+\eta_i)]$ in the inertia term. We show that in this same ordering there are actually several additional FLR terms. The conventional type of the FLR modification is valid only in the cylinder limit. In passing, we note that the Braginskii gyroviscous tensor exhibits also the complexity nature of the FLR effect.[15] In particular, it can also be proved that the conventional type of the FLR modification is valid only in the cylinder limit in the Braginskii two-fluid description.[19]

Two key modifications are made in our new gyrokinetic theory: First, the solution of the equilibrium gyrokinetic distribution function is carried out to sufficiently high order; Second, the gyrophase-dependent part (i.e., δH_k with $k \neq 0$) of the perturbed distribution function is retained. To recover the structure similarity between the MHD and kinetic parallel descriptions, the coupling of the gyrophase-dependent part of the distribution function to the unperturbed gyrophase variation in the first order (i.e., $\dot{\alpha}_1$) needs to be taken into account as well in deriving the gyrophase-averaged gyrokinetic equation [see Eq. (B3)].

It is also interesting to compare the current results with the existing works in deriving MHD from gyrokinetics. References 5–8 have used the so-called gyrocenter method to recover ideal MHD. Instead, we use the guiding center transform for this purpose. References 5–8 recover only part of MHD; instead, we recover MHD fully. We also recover the missing FLR effects. For the compressional Alfvén mode, for example, comparing the resulting Eq. (93) in Ref. 8 with the corresponding equation (36) in our paper, one can see that Ref. 8 derives only two MHD terms: the term on the left and the fifth term on the right of our Eq. (36). Reference 5 tried to recover the MHD vorticity equation (8). However, the derivation is based upon the assumption that $\delta\mathbf{A}_\perp = 0$. Even if the $\delta\mathbf{A}_\perp \neq 0$ case were picked up in their formalism, as was done in the earlier paper,[16] the MHD vorticity equation could not be recovered properly. This is related to the recovery of the term $\mathbf{J}\cdot\nabla(\mathbf{B}\cdot\delta\mathbf{B})/B^2$ in the MHD vorticity equation (8). This is because the equilibrium distribution function given in Ref. 5 is not a proper solution of the gyrokinetic equation to the next order, as discussed in Sec. III B. Besides, as discussed in Sec. III D, the commutation of $\mathbf{b}\cdot\nabla_X$ with the velocity space integration at the particle coordinates has caused the corresponding FLR effects lost in the Refs. 5 and 16.

The basic set of kinetic equations derived in this paper includes kinetic effects: such as the particle-wave resonance, the trapped particle effect, the parallel electric field, the FLR effect, etc. Our concrete derivation is carried out only to the order of the perpendicular inertia effect with the diamagnetic frequency shift modification. Higher-order correction can be obtained based on the general formalism laid out in Sec. III. The gyrokinetics is of general importance for studying the stability of the magnetically confined plasmas. Here, we point out especially that, due to the recovery of the linear ideal MHD in our kinetic description, our theory can be used to extend MHD codes to kinetic ones with limited modifications. The kinetic formalism laid out in this paper is "hybrid" in appearance, but fully kinetic in essence. This feature is particularly useful for developing global code to study for example the resistive wall mode. The kinetic equations derived in this paper are being used to extend the adaptive MHD code AEGIS[9] to the kinetic one.

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APPENDIX A: THE SOLUTION OF THE FIRST HARMONIC GYROKINETIC EQUATION AND AMPERE'S LAW

In this appendix, we detail the solutions of the first harmonic of the gyrokinetic equations and compute their contributions to Ampere's law. The formal solution of the gyrokinetic equation has been given in Eq. (26). Here, we describe the explicit solutions. We denote the contributions of individual terms on the r.h.s. of (32) to the first harmonic of the perturbed distribution function as $\delta\tilde{G}_{1a},\delta\tilde{G}_{1b},\ldots$, with subscripts $a,b,\ldots$ representing the sequence order of the terms. The corresponding perturbed current moments are denoted by $\delta j_{\cdots;e_1,e_2}=\Sigma_{i,e}e\int d^3v(\mathbf{v}_\perp\cdot\mathbf{e}_{1,2})\delta\tilde{G}_{1\ldots}$.

The first-harmonic contribution of the first term on the r.h.s. of Eq. (32) is determined by

$$\mathcal{L}_g\delta\tilde{G}_{1a}=-i\omega\frac{e}{m_\rho}\frac{\partial F_{g0}}{\partial\varepsilon}\mathbf{v}_\perp\cdot\delta\mathbf{A}(\mathbf{x}).$$

As will be seen, the MHD inertia effect [the second term in Eq. (A1)] is of order $(\omega/\omega_{*i})(\rho_i^2/\lambda_\wedge L_p)$. Therefore, the solution of $\delta\tilde{G}_{1a}$ has to be carried out until this order,

$$\begin{aligned}
\delta\tilde{G}_{1a}(\mathbf{x}) ={}& -i\omega\frac{1}{B}\frac{\partial F_{g0}}{\partial\varepsilon}\mathbf{e}_b\times\mathbf{v}_\perp\cdot\delta\mathbf{A}+\frac{\omega^2}{\Omega}\frac{m_\rho}{T}F_{g0}v_\perp(-\sin\alpha\mathbf{e}_1+\cos\alpha\mathbf{e}_2)\cdot\boldsymbol{\xi}(\mathbf{x})-i\omega\frac{m_\rho}{T}F_{g0}v_\perp\cos\alpha\mathbf{e}_1\cdot\boldsymbol{\xi}\\
&-i\omega\frac{m_\rho}{T}F_{g0}\frac{3v_\perp^3}{8\Omega^2}B^{-3/2}\cos\alpha\{\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2\}:\nabla\nabla(B^{3/2}\mathbf{e}_1\cdot\boldsymbol{\xi})-i\omega\frac{m_\rho}{T}F_{g0}\cos\alpha\Bigg(\mathbf{e}_1\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}\Bigg\{-\frac{5v_\perp}{8\Omega^2}\mathbf{e}_1\cdot\Bigg(\frac{v_\perp^2}{2}\nabla\ln B\\
&+v_\parallel^2\boldsymbol{\kappa}\Bigg)+\frac{v_\perp^3}{8\Omega^2}\frac{\partial\ln n_0}{\partial\psi}\left[1+\eta\left(\frac{m_\rho\varepsilon}{T}-\frac{5}{2}\right)\right]-\frac{v_\perp^3}{16\Omega^2}\mathbf{e}_1\cdot(\nabla\ln B)\Bigg\}+\mathbf{e}_2\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}\Bigg[-\frac{5v_\perp}{8\Omega^2}\mathbf{e}_2\cdot\Bigg(\frac{v_\perp^2}{2}\nabla\ln B\\
&+v_\parallel^2\boldsymbol{\kappa}\Bigg)-\frac{9v_\perp^3}{16\Omega^2}\mathbf{e}_2\cdot(\nabla\ln B)\Bigg]+\frac{7v_\perp^3}{8\Omega^2}[(\mathbf{e}_1\cdot\nabla\mathbf{e}_1)\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}+(\mathbf{e}_2\cdot\nabla\mathbf{e}_2)\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}]\\
&+\mathbf{e}_1\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}\left[\frac{v_\perp}{8\Omega^2}\mathbf{e}_2\cdot\left(\frac{v_\perp^2}{2}\nabla\ln B+v_\parallel^2\boldsymbol{\kappa}\right)+\frac{7v_\perp^3}{16\Omega^2}\mathbf{e}_2\cdot(\nabla\ln B)\right]+\mathbf{e}_2\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}\Bigg\{-\frac{v_\perp}{8\Omega^2}\mathbf{e}_1\cdot\Bigg(\frac{v_\perp^2}{2}\nabla\ln B\\
&+v_\parallel^2\boldsymbol{\kappa}\Bigg)-\frac{v_\perp^3}{8\Omega^2}\frac{\partial\ln n_0}{\partial\psi}\left[1+\eta\left(\frac{m_\rho\varepsilon}{T}-\frac{5}{2}\right)\right]+\frac{v_\perp^3}{16\Omega^2}\mathbf{e}_1\cdot(\nabla\ln B)\Bigg\}+\frac{v_\perp^3}{8\Omega^2}[-(\mathbf{e}_2\cdot\nabla\mathbf{e}_1)\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}\\
&+(\mathbf{e}_1\cdot\nabla\mathbf{e}_2)\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}]\Bigg)-i\omega\frac{m_\rho}{T}F_{g0}v_\perp\sin\alpha\mathbf{e}_2\cdot\boldsymbol{\xi}-i\omega\frac{m_\rho}{T}F_{g0}\frac{3v_\perp^3}{8\Omega^2}B^{-3/2}\sin\alpha\{\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2\}:\nabla\nabla(B^{3/2}\mathbf{e}_2\cdot\boldsymbol{\xi})\\
&-i\omega\frac{m_\rho}{T}F_{g0}\sin\alpha\Bigg(\mathbf{e}_1\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}\left[-\frac{v_\perp}{8\Omega^2}\mathbf{e}_2\cdot\left(\frac{v_\perp^2}{2}\nabla\ln B+v_\parallel^2\boldsymbol{\kappa}\right)+\frac{v_\perp^3}{16\Omega^2}\mathbf{e}_2\cdot(\nabla\ln B)\right]\\
&+\mathbf{e}_2\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}\left\{\frac{v_\perp}{8\Omega^2}\mathbf{e}_1\cdot\left(\frac{v_\perp^2}{2}\nabla\ln B+v_\parallel^2\boldsymbol{\kappa}\right)-\frac{3v_\perp^3}{8\Omega^2}\frac{\partial\ln n_0}{\partial\psi}\left[1+\eta\left(\frac{m_\rho\varepsilon}{T}-\frac{5}{2}\right)\right]+\frac{7v_\perp^3}{16\Omega^2}\mathbf{e}_1\cdot(\nabla\ln B)\right\}\\
&+\frac{v_\perp^3}{8\Omega^2}[(\mathbf{e}_2\cdot\nabla\mathbf{e}_1)\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}-(\mathbf{e}_1\cdot\nabla\mathbf{e}_2)\cdot\nabla\mathbf{e}_1\cdot\boldsymbol{\xi}]+\mathbf{e}_1\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}\Bigg\{-\frac{5v_\perp}{8\Omega^2}\mathbf{e}_1\cdot\left(\frac{v_\perp^2}{2}\nabla\ln B+v_\parallel^2\boldsymbol{\kappa}\right)\\
&+\frac{5v_\perp^3}{8\Omega^2}\frac{\partial\ln n_0}{\partial\psi}\left[1+\eta\left(\frac{m_\rho\varepsilon}{T}-\frac{5}{2}\right)\right]-\frac{9v_\perp^3}{16\Omega^2}\mathbf{e}_1\cdot(\nabla\ln B)\Bigg\}+\mathbf{e}_2\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}\Bigg[-\frac{5v_\perp}{8\Omega^2}\mathbf{e}_2\cdot\left(\frac{v_\perp^2}{2}\nabla\ln B+v_\parallel^2\boldsymbol{\kappa}\right)\\
&-\frac{v_\perp^3}{16\Omega^2}\mathbf{e}_2\cdot(\nabla\ln B)\Bigg]+\frac{7v_\perp^3}{8\Omega^2}[(\mathbf{e}_1\cdot\nabla\mathbf{e}_1)\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}+(\mathbf{e}_2\cdot\nabla\mathbf{e}_2)\cdot\nabla\mathbf{e}_2\cdot\boldsymbol{\xi}]\Bigg).
\end{aligned}\tag{A1}$$

Here, the second term on the r.h.s. derives from the first-order correction $-i\omega\delta\tilde{G}_{1a}$, which, as will be seen, gives rise to the MHD inertia term. Here, $\delta\tilde{G}_{1a}$ represents the first term on the r.h.s. of Eq. (A1). In obtaining Eq. (A1), the contribution from the term $\dot{\alpha}_1(\partial\delta\tilde{G}_{1a}/\partial\alpha)$ is also retained, but the first-harmonic contribution of the term $\dot{\mathbf{X}}\cdot\nabla_X\delta\tilde{G}_{1a}$ is dropped, due to it being odd in $v_\parallel$. Using quasineutrality condition, one can prove that the leading order contribution of $\delta\tilde{G}_{1a}$ (i.e., the first term on the r.h.s.) to the current moment vanishes. The current moments from $\delta\tilde{G}_{1a}$ are given as follows:

$$\delta j_{1a;\mathbf{e}_1} = \frac{\omega^2 n_0 m_\rho}{B}\mathbf{e}_2\cdot\boldsymbol{\xi}(\mathbf{x}) - i\omega\frac{2n_0 Te_i}{m_\rho}\frac{3}{4\Omega_i^2}B^{-3/2}\{\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2\}:\boldsymbol{\nabla\nabla}(B^{3/2}\mathbf{e}_1\cdot\boldsymbol{\xi}) - i\omega\frac{2n_0 Te_i}{m_\rho}\Bigg\{\mathbf{e}_1\cdot\boldsymbol{\nabla}\mathbf{e}_1\cdot\boldsymbol{\xi}\Bigg[-\frac{5}{8\Omega_i^2}\mathbf{e}_1\cdot\Bigg(\boldsymbol{\nabla}\ln B + \frac{1}{2}\boldsymbol{\kappa}\Bigg) + \frac{1}{4\Omega_i^2}\frac{\partial\ln n_0}{\partial\psi}(1+\eta_i) - \frac{1}{8\Omega_i^2}\mathbf{e}_1\cdot(\boldsymbol{\nabla}\ln B)\Bigg] + \mathbf{e}_2\cdot\boldsymbol{\nabla}\mathbf{e}_1\cdot\boldsymbol{\xi}\Bigg[-\frac{5}{8\Omega_i^2}\mathbf{e}_2\cdot\Bigg(\boldsymbol{\nabla}\ln B+\frac{1}{2}\boldsymbol{\kappa}\Bigg) - \frac{9}{8\Omega_i^2}\mathbf{e}_2\cdot(\boldsymbol{\nabla}\ln B)\Bigg] + \frac{7}{4\Omega_i^2}[(\mathbf{e}_1\cdot\boldsymbol{\nabla}\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{e}_1\cdot\boldsymbol{\xi} + (\mathbf{e}_2\cdot\boldsymbol{\nabla}\mathbf{e}_2)\cdot\boldsymbol{\nabla}\mathbf{e}_1\cdot\boldsymbol{\xi}] + \mathbf{e}_1\cdot\boldsymbol{\nabla}\mathbf{e}_2\cdot\boldsymbol{\xi}\Bigg[\frac{1}{8\Omega_i^2}\mathbf{e}_2\cdot\Bigg(\boldsymbol{\nabla}\ln B+\frac{1}{2}\boldsymbol{\kappa}\Bigg)+\frac{7}{8\Omega_i^2}\mathbf{e}_2\cdot(\boldsymbol{\nabla}\ln B)\Bigg] + \mathbf{e}_2\cdot\boldsymbol{\nabla}\mathbf{e}_2\cdot\boldsymbol{\xi}\Bigg[-\frac{1}{8\Omega_i^2}\mathbf{e}_1\cdot\Bigg(\boldsymbol{\nabla}\ln B+\frac{1}{2}\boldsymbol{\kappa}\Bigg) - \frac{1}{4\Omega_i^2}\frac{\partial\ln n_0}{\partial\psi}(1+\eta_i) + \frac{1}{8\Omega_i^2}\mathbf{e}_1\cdot(\boldsymbol{\nabla}\ln B)\Bigg] + \frac{1}{4\Omega_i^2}[-(\mathbf{e}_2\cdot\boldsymbol{\nabla}\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{e}_2\cdot\boldsymbol{\xi} + (\mathbf{e}_1\cdot\boldsymbol{\nabla}\mathbf{e}_2)\cdot\boldsymbol{\nabla}\mathbf{e}_2\cdot\boldsymbol{\xi}]\Bigg\}, \tag{A2}$$

$$\delta j_{1a;\mathbf{e}_1} = -\frac{\omega^2 n_0 m_\rho}{B}\mathbf{e}_1\cdot\boldsymbol{\xi} - i\omega\frac{2n_0 Te_i}{m_\rho}\frac{3}{4\Omega_i^2}B^{-3/2}\{\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2\}:\boldsymbol{\nabla\nabla}(B^{3/2}\mathbf{e}_2\cdot\boldsymbol{\xi}) - i\omega\frac{2n_0 T_i e_i}{m_{\rho i}}\Bigg\{\mathbf{e}_1\cdot\boldsymbol{\nabla}\mathbf{e}_1\cdot\boldsymbol{\xi}\Bigg[-\frac{1}{8\Omega_i^2}\mathbf{e}_2\cdot\Bigg(\boldsymbol{\nabla}\ln B+\frac{1}{2}\boldsymbol{\kappa}\Bigg) + \frac{1}{8\Omega_i^2}\mathbf{e}_2\cdot(\boldsymbol{\nabla}\ln B)\Bigg] + \mathbf{e}_2\cdot\boldsymbol{\nabla}\mathbf{e}_1\cdot\boldsymbol{\xi}\Bigg[\frac{1}{8\Omega_i^2}\mathbf{e}_1\cdot\Bigg(\boldsymbol{\nabla}\ln B+\frac{1}{2}\boldsymbol{\kappa}\Bigg) - \frac{3}{4\Omega_i^2}\frac{\partial\ln n_0}{\partial\psi}(1+\eta_i) + \frac{7}{8\Omega_i^2}\mathbf{e}_1\cdot(\boldsymbol{\nabla}\ln B)\Bigg] + \frac{1}{4\Omega_i^2}[(\mathbf{e}_2\cdot\boldsymbol{\nabla}\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{e}_1\cdot\boldsymbol{\xi} - (\mathbf{e}_1\cdot\boldsymbol{\nabla}\mathbf{e}_2)\cdot\boldsymbol{\nabla}\mathbf{e}_1\cdot\boldsymbol{\xi}] + \mathbf{e}_1\cdot\boldsymbol{\nabla}\mathbf{e}_2\cdot\boldsymbol{\xi}\Bigg[-\frac{5}{8\Omega_i^2}\mathbf{e}_1\cdot\Bigg(\boldsymbol{\nabla}\ln B+\frac{1}{2}\boldsymbol{\kappa}\Bigg) + \frac{5}{4\Omega_i^2}\frac{\partial\ln n_0}{\partial\psi}(1+\eta_i) - \frac{9}{8\Omega_i^2}\mathbf{e}_1\cdot(\boldsymbol{\nabla}\ln B)\Bigg] + \mathbf{e}_2\cdot\boldsymbol{\nabla}\mathbf{e}_2\cdot\boldsymbol{\xi}\Bigg[-\frac{5}{8\Omega_i^2}\mathbf{e}_2\cdot\Bigg(\boldsymbol{\nabla}\ln B+\frac{1}{2}\boldsymbol{\kappa}\Bigg) - \frac{1}{8\Omega_i^2}\mathbf{e}_2\cdot(\boldsymbol{\nabla}\ln B)\Bigg] + \frac{7}{4\Omega_i^2}[(\mathbf{e}_1\cdot\boldsymbol{\nabla}\mathbf{e}_1)\cdot\boldsymbol{\nabla}\mathbf{e}_2\cdot\boldsymbol{\xi} + (\mathbf{e}_2\cdot\boldsymbol{\nabla}\mathbf{e}_2)\cdot\boldsymbol{\nabla}\mathbf{e}_2\cdot\boldsymbol{\xi}]\Bigg\}. \tag{A3}$$

Here, one can see that the first terms on the r.h.s. of Eqs. (A2) and (A3) correspond to the ideal MHD inertia term. The second term on the right is formally of order $(\omega_{*i}/\omega)\times(L_p/\lambda_\perp)$ larger than the ideal MHD inertia term. However, in the vorticity equation $\boldsymbol{\nabla}\cdot\delta\mathbf{j}=0$, they become of the same order for the case with mode frequency much lower than the compressional Alfvén mode frequency. In the case with mode frequency being of the same order as the compressional Alfvén mode frequency, one has $\omega \gg \omega_{*i}$ and the second term is negligible. The rest terms are of the same order as the inertia term.

The first-harmonic contribution of the second term on the r.h.s. of Eq. (32), $\delta\tilde{G}_{1b}$, is of order $(L_p/L_B)(\lambda_\wedge/\rho)\times(\omega_{*i}/\omega)^2$ as compared to the second term on the r.h.s. of Eq. (A1). However, its leading order is odd in $v_\parallel$ and therefore has no contribution to the perturbed current moment. The next nonvanishing correction of the first harmonic is reduced by order $\rho^2/\lambda_\perp^2$. Therefore, the first-harmonic contribution of the second term on the r.h.s. of Eq. (32) can be ignored as compared to the inertia term.

The first-harmonic contribution of the third term on the r.h.s. of Eq. (32) is determined by

$$-\Omega\frac{\partial\delta\tilde{G}_{1c}}{\partial\alpha} = -i\omega|\boldsymbol{\nabla}\psi|\frac{\partial F_{g0}}{\partial\psi}\frac{1}{\Omega}\mathbf{v}\times\mathbf{e}_b\cdot\boldsymbol{\nabla}(\mathbf{e}_1\cdot\boldsymbol{\xi}).$$

One can find that $\delta\tilde{G}_{1c}$ is of order ω_{*i}/ω as compared to the second term on the r.h.s. of Eq. (A1). Therefore, only the lowest order solution needs to be determined:

$$\delta\tilde{G}_{1c} = -i\omega\frac{|\boldsymbol{\nabla}\psi|}{\Omega^2}\frac{\partial F_{g0}}{\partial\psi}\mathbf{v}_\perp\cdot\boldsymbol{\nabla}(\mathbf{e}_1\cdot\boldsymbol{\xi}).$$

The corresponding current moment is given by

$$\delta j_{1c;e_1,e_2} = -i\omega\frac{|\boldsymbol{\nabla}\psi|}{B\Omega_i}P_i'\mathbf{e}_{1,2}\cdot\boldsymbol{\nabla}(\mathbf{e}_1\cdot\boldsymbol{\xi}).$$

This term gives the so-called $\omega_{*i}(1+\eta_i)$ modification to the ideal MHD inertia term, noting that $\mathbf{e}_1\cdot\boldsymbol{\nabla}(\mathbf{e}_1\cdot\boldsymbol{\xi})\approx-\mathbf{e}_2\cdot\boldsymbol{\nabla}(\mathbf{e}_2\cdot\boldsymbol{\xi})$ for the modes with frequencies much lower than the compressional Alfvén frequency.

The first-harmonic contribution of the fourth term on the r.h.s. of Eq. (32) is determined by

$$-\Omega\frac{\partial\delta\widetilde{G}_{1d}}{\partial\alpha}=-i(\omega-\omega_*^T)\frac{e}{m_\rho}\frac{\partial F_{g0}}{\partial\varepsilon}\frac{1}{\Omega}\mathbf{v}\times\mathbf{e}_b\cdot\nabla\delta\varphi(\mathbf{X}),$$

where the guiding center expansion of $\delta\varphi(\mathbf{x})$ has been made and only leading-order contribution is kept. The solution is

$$\delta\widetilde{G}_{1d}=-i\frac{1}{B\Omega}(\omega-\omega_*^T)\frac{\partial F_{g0}}{\partial\varepsilon}\mathbf{v}_\perp\cdot\nabla\delta\varphi(\mathbf{x}). \tag{A4}$$

The leading-order contribution of $\delta\widetilde{G}_{1d}$ to two perpendicular components of the perturbed current moment are as follows:

$$\delta j_{1d;e_1,e_2}=in_0m_i[\omega-\omega_{*i}(1+\eta_i)]\frac{1}{B^2}\mathbf{e}_{1,2}\cdot\nabla\delta\varphi.$$

This term represents the parallel-electric-field modification to the MHD inertia term. As is analyzed in Sec. IV C, this term can be of the same order as the inertia term near the singular layers.

The first-harmonic contribution of the fifth term on the r.h.s. of Eq. (32) is determined by

$$-\Omega\frac{\partial\delta\widetilde{G}_{1e}}{\partial\alpha}=\frac{e}{m_\rho}\mathbf{v}\cdot\nabla_x\left(\frac{\partial F_{g0}}{\partial\varepsilon}\right)\delta\varphi(\mathbf{x}).$$

One can find that $\delta\widetilde{G}_{1e}$ is of order $(\lambda_\perp\lambda_\wedge/\rho_i^2)(\omega_{*i}/\omega)$, as compared to $\delta\widetilde{G}_{1d}$ in Eq. (A4). Therefore, the solution has to be carried out until the second order, yielding

$$\begin{aligned}\delta\widetilde{G}_{1e}(\mathbf{x})=&-\frac{1}{B}\mathbf{v}\times\mathbf{e}_b\cdot\nabla\left(\frac{\partial F_{g0}}{\partial\varepsilon}\right)\delta\varphi(\mathbf{x})\\&-\frac{3}{8B\Omega^2}v_\perp^3\mathbf{e}_1\cdot\nabla\left(\frac{\partial F_{g0}}{\partial\varepsilon}\right)\\&\times\sin\alpha(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla\delta\varphi.\end{aligned}$$

Using the quasineutrality condition, one can prove that the leading-order contribution of $\delta\widetilde{G}_{1e}$ to the current moment vanishes. The remaining contribution is

$$\delta j_{1e;e_1}=0,$$

$$\begin{aligned}\delta j_{1e;e_2}=&\frac{3}{2}n_0m_\rho\frac{|\nabla\psi|T_i}{B^3e_i}\frac{\partial\ln n_0}{\partial\psi}(1+\eta_i)\\&\times(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla\delta\varphi.\end{aligned}$$

The first-harmonic contribution of the sixth term on the r.h.s. of Eq. (32) is determined by

$$-\Omega\frac{\partial\delta\widetilde{G}_{1f}}{\partial\alpha}=\left(\frac{g}{B}\frac{\partial F_{g0}}{\partial\Psi}+|v_\parallel|\frac{e}{m_\rho B}\frac{\partial\bar{F}_{g1}}{\partial\mu}\right)\mathbf{e}_b\times\mathbf{v}_\perp\cdot\delta\mathbf{B}(\mathbf{x}). \tag{A5}$$

First, we note that, according to Eq. (18), the two terms in the parenthesis are of the same order. One can find that $\delta\widetilde{G}_{1f}$ is of order $(\lambda_\wedge^2/\rho_i^2)(\omega_{*i}/\omega)^2$ as compared to the second term on the r.h.s. of Eq. (A1). Here, an order estimate, i.e., $\delta\mathbf{B}_\perp\sim(B/L_B)\xi_\wedge$, has been used. Therefore, the solution of Eq. (A5) has to be carried out until the second order, yielding

$$\begin{aligned}\delta\widetilde{G}_{1f}(\mathbf{x})=&-\frac{1}{\Omega}\left(\frac{g}{B}\frac{\partial F_{g0}}{\partial\psi}+|v_\parallel|\frac{e}{m_\rho B}\frac{\partial\bar{F}_{g1}}{\partial\mu}\right)\mathbf{v}_\perp\cdot\delta\mathbf{B}(\mathbf{x})\\&-\frac{3}{8\Omega^3}\left(\frac{g}{B}\frac{\partial F_{g0}}{\partial\psi}+|v_\parallel|\frac{e}{m_\rho B}\frac{\partial\bar{F}_{g1}}{\partial\mu}\right)\\&\times v_\perp^2(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla[\mathbf{v}_\perp\cdot\delta\mathbf{B}(\mathbf{x})].\end{aligned}$$

The current moment is given by

$$\begin{aligned}\delta j_{1f;e_1,e_2}=&-\frac{gP'}{B^2}\mathbf{e}_{1,2}\cdot\delta\mathbf{B}-g'\mathbf{e}_{1,2}\cdot\delta\mathbf{B}\\&-\frac{3}{2\Omega_i}\left(\frac{1}{B^2}\frac{gn_0T_i^2}{\Omega_im_{\rho i}}\frac{\partial\ln n_0}{\partial\psi}(1+2\eta_i)\right.\\&\left.-4\pi m_{\rho i}\int d\varepsilon d\mu_0\mu_0\bar{F}_{g1}\right)\\&\times(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\mathbf{e}_{1,2}\cdot\delta\mathbf{B}).\end{aligned}$$

The first-harmonic contribution of the seventh term on the r.h.s. of Eq. (32) is determined by

$$-\Omega\frac{\partial\delta\widetilde{G}_{1g}}{\partial\alpha}=\frac{1}{B}\mathbf{v}_\perp\cdot\nabla_XF_{g0}\mathbf{e}_b\cdot\delta\mathbf{B}(\mathbf{x}). \tag{A6}$$

One can find that $\delta\widetilde{G}_{1g}$ is of order $(P/B^2)(\lambda_\wedge\lambda_\perp/\rho_i^2)\times(\omega_{*i}/\omega)^2$ as compared to the second term on the r.h.s. of Eq. (A1). Here, an order estimate, i.e., $\delta\mathbf{B}_\parallel\sim B\nabla\cdot\boldsymbol{\xi}_\perp\sim(B/L_B)\xi_\wedge$, is employed, which corresponds to the case with mode frequency much lower than the compressional Alfvén mode frequency.[18] In the case with mode frequency being of the same order as the compressional Alfvén mode frequency, one has $\omega\gg\omega_{*i}$, the seventh term becomes negligible. Therefore, the solution of Eq. (A6) has to be carried out until the second order, yielding

$$\begin{aligned}\delta\widetilde{G}_{1g}(\mathbf{x})=&-\frac{|\nabla\psi|}{B\Omega}\frac{\partial F_{g0}}{\partial\psi}v_\perp\sin\alpha\mathbf{e}_b\cdot\delta\mathbf{B}(\mathbf{x})\\&-\frac{3|\nabla\psi|}{8\Omega^3B}\frac{\partial F_{g0}}{\partial\psi}v_\perp^3\\&\times\sin\alpha(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\mathbf{e}_b\cdot\delta\mathbf{B}).\end{aligned}$$

The current moment is given by

$$\delta j_{1g;e_1}=0,$$

$$\begin{aligned}\delta j_{1g;e_2}=&-\frac{P'|\nabla\psi|}{B^2}\delta\mathbf{B}\cdot\mathbf{e}_b-\frac{3|\nabla\psi|e_in_0}{2B\Omega_i^3}\left(\frac{T_i}{m_{\rho i}}\right)^2\\&\times\frac{\partial\ln n_0}{\partial\psi}(1+2\eta_i)(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\mathbf{e}_b\cdot\delta\mathbf{B}).\end{aligned}$$

The eighth term on the r.h.s. of Eq. (32) is of order $(\omega_{*i}/\omega)(\lambda_\wedge/a)$ (a represents the minor radius) as compared to the forth term on the r.h.s. of Eq. (32). Therefore, $\delta\widetilde{G}_{1h}$ can be dropped.

The ninth term on the r.h.s. of Eq. (32) is of the same order as the term giving rise to the second term on the r.h.s.

of Eq. (A1). However, its leading order is odd in $v_\parallel$ and therefore has no contribution to the perturbed current moment.

Based on the same reason as analyzed for fifth term, the first-harmonic contribution of the tenth term on the r.h.s. of Eq. (32) can be dropped.

In addition, we need to consider the first-harmonic contribution of the convective term $-\boldsymbol{\xi}(\mathbf{X})\cdot\nabla F_{g0}$; i.e., the first term on the r.h.s. of Eq. (31). Note that $\boldsymbol{\xi}(\mathbf{X})$ in the convective term is given in the guiding center coordinates. Converting back to the particle coordinates, the first-harmonic contribution of the convective term becomes

$$\delta\widetilde{G}_{1\text{conv}} = -\frac{v_\perp}{\Omega}(\mathbf{e}_1\sin\alpha - \mathbf{e}_2\cos\alpha)\cdot\nabla\boldsymbol{\xi}\cdot\nabla F_{g0} - \frac{v_\perp^3}{8\Omega^3}|\nabla\psi|\frac{\partial F_{g0}}{\partial\psi}(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1\sin\alpha + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\sin\alpha - \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\cos\alpha - \mathbf{e}_2\mathbf{e}_2\mathbf{e}_2\cos\alpha) \vdots \nabla\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) - \frac{v_\perp^3}{8\Omega^3}\frac{\partial F_{g0}}{\partial\psi}\mathbf{e}_1\cdot\nabla(|\nabla\psi|)(3\mathbf{e}_1\mathbf{e}_1\sin\alpha + \mathbf{e}_2\mathbf{e}_2\sin\alpha - 2\mathbf{e}_1\mathbf{e}_2\cos\alpha):\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) - \frac{v_\perp^3}{8\Omega^3}\frac{\partial F_{g0}}{\partial\psi}\mathbf{e}_2\cdot\nabla(|\nabla\psi|)(2\mathbf{e}_1\mathbf{e}_2\sin\alpha - \mathbf{e}_1\mathbf{e}_1\cos\alpha - 3\mathbf{e}_2\mathbf{e}_2\cos\alpha):\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) - \frac{v_\perp^3}{8\Omega^3}|\nabla\psi|\mathbf{e}_1\cdot\nabla\left(\frac{\partial F_{g0}}{\partial\psi}\right)(3\mathbf{e}_1\mathbf{e}_1\sin\alpha + \mathbf{e}_2\mathbf{e}_2\sin\alpha - 2\mathbf{e}_1\mathbf{e}_2\cos\alpha):\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1).$$

Here, we note that the first term of this solution is of order $(L_p\lambda_\wedge/\rho_i^2)(\omega_{*i}/\omega)^2$ as compared to the inertia term [second term on the r.h.s. of Eq. (A1)] and the second term is a correction of order $(\rho_i/\lambda_\perp)^2$ to the first term. The $\delta\widetilde{G}_{1\text{conv}}$-induced current moment components are given as follows:

$$\delta j_{1\text{conv};\mathbf{e}_1} = \frac{1}{B}\mathbf{e}_2\cdot\nabla\boldsymbol{\xi}\cdot\nabla P + \frac{n_0e_i}{2\Omega_i^3}\left(\frac{T}{m_{\rho i}}\right)^2|\nabla\psi|\frac{\partial\ln n_0}{\partial\psi}(1+2\eta)(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_2\mathbf{e}_2)\vdots\nabla\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) + \frac{n_0e_i}{\Omega_i^3}\left(\frac{T}{m_{\rho i}}\right)^2\frac{\partial\ln n_0}{\partial\psi}(1+2\eta)\mathbf{e}_1\cdot\nabla(|\nabla\psi|)\mathbf{e}_1\mathbf{e}_2:\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) + \frac{n_0e_i}{2\Omega_i^3}\left(\frac{T}{m_{\rho i}}\right)^2\frac{\partial\ln n_0}{\partial\psi}(1+2\eta)\mathbf{e}_2\cdot\nabla(|\nabla\psi|)(\mathbf{e}_1\mathbf{e}_1 + 3\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) + \frac{n_0e_i}{\Omega_i^3}\left(\frac{T}{m_{\rho i}}\right)^2|\nabla\psi|\left\{\left(\frac{\partial\ln n_0}{\partial\psi}\right)^2\left(1+\frac{15}{2}\eta^2+4\eta\right) + \frac{\partial^2\ln n_0}{\partial\psi^2}(1+2\eta) + 2\frac{\partial\ln n_0}{\partial\psi}\left(2\frac{\partial\eta}{\partial\psi} - \eta\frac{7}{2}\frac{\partial\ln T}{\partial\psi}\right)\right\}\mathbf{e}_1\mathbf{e}_2:\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1), \tag{A7}$$

$$\delta j_{1\text{conv};\mathbf{e}_2} = -\frac{1}{B}\mathbf{e}_1\cdot\nabla\boldsymbol{\xi}\cdot\nabla P - \frac{n_0e_i}{2\Omega_i^3}\left(\frac{T}{m_{\rho i}}\right)^2\frac{\partial\ln n_0}{\partial\psi}(1+2\eta)(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2)\vdots\nabla\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) - \frac{n_0e_i}{2\Omega_i^3}\left(\frac{T}{m_{\rho i}}\right)^2\frac{\partial\ln n_0}{\partial\psi}(1+2\eta)\mathbf{e}_1\cdot\nabla(|\nabla\psi|)(3\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) - \frac{n_0e_i}{\Omega_i^3}\left(\frac{T}{m_{\rho i}}\right)^2\frac{\partial\ln n_0}{\partial\psi}(1+2\eta)\mathbf{e}_2\cdot\nabla(|\nabla\psi|)\mathbf{e}_1\mathbf{e}_2:\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1) - \frac{n_0e_i}{2\Omega_i^3}\left(\frac{T}{m_{\rho i}}\right)^2|\nabla\psi|\left\{\left(\frac{\partial\ln n_0}{\partial\psi}\right)^2\left(1+\frac{15}{2}\eta^2+4\eta\right) + \frac{\partial^2\ln n_0}{\partial\psi^2}(1+2\eta) + \frac{\partial\ln n_0}{\partial\psi}\left(2\frac{\partial\eta}{\partial\psi} - \eta\frac{7}{2}\frac{\partial\ln T}{\partial\psi}\right)\right\}(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\boldsymbol{\xi}\cdot\mathbf{e}_1). \tag{A8}$$

Here, the second terms on the r.h.s. of Eqs. (A7) and (A8) are formally of order $(\omega_{*i}/\omega)^2(L_p/\lambda)$ larger than the ideal MHD inertia term. However, in the vorticity equation $\nabla\cdot\delta\mathbf{j}=0$, it becomes of the same order as the inertia term, noting that $(\mathbf{e}_1\cdot\nabla\mathbf{e}_2\cdot\nabla - \mathbf{e}_2\cdot\nabla\mathbf{e}_1\cdot\nabla)\mathbf{e}_1\cdot\boldsymbol{\xi}(\mathbf{x})$ cancel in the leading order.

APPENDIX B: REDUCTION OF THE GYROPHASE-AVERAGED GYROKINETIC EQUATION

In this appendix, we show the reduction of the gyrophase-averaged gyrokinetic equation for calculating the pressure moment in Ampere's law and the density moment in the quasineutrality condition.

First, we note that only the even part $[\delta G_0^e = (\delta G_0(v_\parallel) + \delta G_0(-v_\parallel))/2]$ of the distribution function with respect to the parallel velocity is needed for calculating the pressure and density moments. Inspecting the r.h.s. (denoted by $\mathcal{R}_g$) of Eq. (32), one can see that $\mathcal{R}_g$ contains both even ($\mathcal{R}_g^e$) and odd ($\mathcal{R}_g^o$) parts with respect to $v_\parallel$. The gyrophase-averaged gyrokinetic equation for the even part can be formally written as[20]

$$v_{\parallel}\mathbf{e}_b\cdot\nabla\frac{v_{\parallel}}{i(\omega-\omega_d)}\mathbf{e}_b\cdot\nabla\delta G_0^e-i(\omega-\omega_d)\delta G_0^e$$
$$=\mathcal{R}_g^e+v_{\parallel}\mathbf{e}_b\cdot\nabla\frac{1}{i(\omega-\omega_d)}\mathcal{R}_g^o. \tag{B1}$$

This expression shows that the contribution of the even part ($\mathcal{R}_g^o$) is of order $(\omega/\omega_t^2)\mathcal{R}_g^e$ and the odd part ($\mathcal{R}_g^o$) is enhanced by order $(\lambda_\wedge/\rho)(L_p/L_B)(\omega_*/\omega)$, as compared to the even part ($\mathcal{R}_g^e$). With these ordering estimates, we can analyze Eq. (32) term by term.

The gyrophase-average $\langle\cdots\rangle_\alpha$ of the first term on the r.h.s. of Eq. (32) yields

$$\left\langle -i\omega\frac{e}{m_\rho}\frac{\partial F_{g0}}{\partial\varepsilon}\mathbf{v}_\perp\cdot\delta\mathbf{A}(\mathbf{x})\right\rangle_\alpha$$
$$=-i\omega\mu_0B\frac{\partial F_{g0}}{\partial\varepsilon}\nabla_\perp\cdot\boldsymbol{\xi}-i\omega\mu_0\frac{\partial F_{g0}}{\partial\varepsilon}\boldsymbol{\xi}\cdot\nabla B$$
$$-i\omega\frac{\partial F_{g0}}{\partial\varepsilon}\left(\mu_0B-\frac{v_{\parallel}^2}{2}\right)\boldsymbol{\kappa}\cdot\boldsymbol{\xi}. \tag{B2}$$

Note that, in the case with compressional Alfvén mode suppressed, one has $\nabla_\perp\cdot\boldsymbol{\xi}\sim\xi_\wedge/L_B$. Therefore, the calculation has been carried out one order further in Eq. (B2). As discussed in Sec. IV D, the first term on the r.h.s. corresponds to the term on the r.h.s. of Eq. (7), which gives rise to the so-called apparent mass effect. Therefore, the first term (with $\nabla_\perp\cdot\boldsymbol{\xi}\sim\xi_\wedge/L_B$ assumed) can be used as a reference term.

There is one more contribution of the same type as Eq. (B2). Note that $\delta\widetilde{G}_{1a}$ in Eq. (A1) can couple to the last term in Eq. (13), yielding

$$-\left\langle\left(\mathbf{v}\cdot\nabla_x\alpha+\frac{1}{\Omega}\mathbf{v}\times\mathbf{e}_b\cdot\nabla_x\Omega\right)\frac{\partial\delta\widetilde{G}_{1a}}{\partial\alpha}\right\rangle_\alpha$$
$$=i\omega\mu_0\frac{\partial F_{g0}}{\partial\varepsilon}\boldsymbol{\xi}\cdot\nabla B+i\omega\frac{\partial F_{g0}}{\partial\varepsilon}\frac{v_{\parallel}^2}{2}\boldsymbol{\kappa}\cdot\boldsymbol{\xi}. \tag{B3}$$

The gyrophase average of the second term on the r.h.s. of Eq. (32) yields $i\omega_d\boldsymbol{\xi}\cdot\nabla F_{g0}$. One can prove that this term is of order ω_*/ω, as compared to the first term in Eq. (B2) and therefore should be kept.

The gyrophase average of the third term on the r.h.s. of Eq. (32) is of order $(\rho^2/\lambda_\perp\lambda_\wedge)(L_B/L_p)$ and therefore can be neglected.

Inserting the quasineutrality condition Eq. (38) into the fourth term on the r.h.s. of Eq. (32), one can see that this term becomes of the same order as $\omega\delta G_0$ and therefore should be kept.

The gyrophase average of the fifth term on the r.h.s. of Eq. (32) yields

$$\left\langle\frac{e}{m_\rho}\mathbf{v}\cdot\nabla_x\left(\frac{\partial F_{g0}}{\partial\varepsilon}\right)\delta\varphi(\mathbf{x})\right\rangle_\alpha$$
$$=-\mu\mathbf{e}_1\cdot\nabla\left(\frac{\partial F_{g0}}{\partial\varepsilon}\right)\mathbf{e}_2\cdot\nabla\delta\varphi(\mathbf{X}).$$

This term is of order ω_*/ω as compared to the fourth term on the r.h.s. of Eq. (32) and therefore should be kept.

The gyrophase average of the sixth term on the r.h.s. of Eq. (32) is of order $(\omega_*/\omega)(\lambda_\wedge/L_p)$, as compared with the first term on the r.h.s. of Eq. (B2) and therefore can be neglected.

The gyrophase average of the seventh term on r.h.s. of Eq. (32) yields

$$\left\langle\frac{1}{B}\mathbf{v}_\perp\cdot\nabla_XF_{g0}\mathbf{e}_b\cdot\delta\mathbf{B}(\mathbf{x})\right\rangle$$
$$=-\frac{v_\perp^2}{\Omega}\mathbf{e}_1\cdot\nabla_XF_{g0}\mathbf{e}_2\cdot\nabla\left(\frac{1}{B}\mathbf{e}_b\cdot\delta\mathbf{B}(\mathbf{X})\right).$$

This term is of order ω_*/ω, as compared with the first term in Eq. (B2) and therefore is kept. Here, an order estimate, i.e., $\delta\mathbf{B}_{\parallel}\sim B\nabla\cdot\boldsymbol{\xi}_\perp\sim(B/L_B)\xi_\wedge$, is employed. This applies to the modes with frequencies much lower than the compressional Alfvén mode frequency. If the mode frequencies are of the same order as the compressional Alfvén mode frequency, one has $\omega\gg\omega_{*i}$ and this term becomes negligible.

The eighth term on the r.h.s. of Eq. (32) is of order $(\omega_*/\omega)(\lambda_\wedge/a)$ as compared to the fourth term on the r.h.s. of Eq. (32) and therefore can be neglected.

The ninth term on the r.h.s. of Eq. (32) is odd in $v_{\parallel}$. Therefore, in the following ordering estimate, the $v_{\parallel}$-odd enhancement in the discussion of Eq. (B1) needs to be taken into account. The $i\omega\delta\mathbf{A}$ part of the ninth term is of order $\lambda_\wedge/a$ as compared to the first term on the r.h.s. of Eq. (B2). The $\nabla\delta_x\varphi$ part is of order $(\omega_*/\omega)^2(\lambda_\wedge^2/\lambda_\perp L_B)$, as compared to the fourth term on the r.h.s. of Eq. (32), taking into consideration that $[(\mathbf{e}_2\cdot\nabla)(\mathbf{e}_1\cdot\nabla)-(\mathbf{e}_1\cdot\nabla)(\mathbf{e}_2\cdot\nabla)]\delta\varphi\sim(1/\lambda_\perp L_B)\delta\varphi$. Therefore, the ninth term can be neglected.

Finally, the last term on the r.h.s. of Eq. (32) is of order $(\omega_*/\omega)(\lambda_\wedge/\lambda_\perp)(L_p/L_B)$ as compared to the first term on the r.h.s. of Eq. (32), with the $v_{\parallel}$-odd enhancement in the discussion of Eq. (B1) taken into account. Therefore, this term needs to be retained. The gyrophase average of the last term yields

$$\langle v_{\parallel}\mathbf{e}_b\cdot(\nabla_x\boldsymbol{\xi}(\mathbf{x})\cdot\nabla F_{g0}-\nabla_X\boldsymbol{\xi}(\mathbf{X})\cdot\nabla F_{g0})\rangle_\alpha$$
$$=\frac{v_{\parallel}v_\perp^2}{\Omega}(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2):\nabla\nabla(\mathbf{e}_b\cdot\nabla\boldsymbol{\xi}\cdot\nabla F_{g0})$$
$$-\frac{v_{\parallel}v_\perp^2}{\Omega}[(\mathbf{e}_1\cdot\nabla\mathbf{e}_b)\cdot\nabla(\mathbf{e}_1\cdot\nabla\boldsymbol{\xi}\cdot\nabla F_{g0})$$
$$+(\mathbf{e}_2\cdot\nabla\mathbf{e}_b)\cdot\nabla(\mathbf{e}_2\cdot\nabla\boldsymbol{\xi}\cdot\nabla F_{g0})].$$

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