Effects of pressure gradient on existence of Alfvén cascade modes in reversed shear tokamak plasmas

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It is shown analytically that pressure gradient effects are favorable to the existence of Alfvén cascade (AC) modes in a tokamak plasma with reversed shear. What is crucial for obtaining the improved existence criterion of ACs is the averaged normal curvature. This term depends on the Shafranov shift, which contains a pressure gradient term that at sufficiently low frequency causes a cancellation in the mode existence criterion of all terms quadratic in the pressure gradient. The favorable criterion is then found to be proportional to the product of the pressure gradient and the inverse aspect ratio. Near the rational surface, there is one-to-one correspondence between Mercier stability and the AC mode existence. When the averaged curvature is favorable to Mercier modes, it is also favorable to the existence of AC. However, at higher frequencies the α^2 term can be unfavorable to mode existence. We show that when $\alpha > 3\epsilon$, that as $q_{\min}$ decreases from m/n, the cascade mode can easily satisfy its existence criterion at lower frequencies, but the existence criterion will fail before the frequency reaches the toroidal Alfvén eigenmode (TAE) gap, which occurs when $q_{\min}$ approaches $(m-1/2)/n$. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2196246]

I. INTRODUCTION

Alfvén cascade modes (or reversed shear Alfvén eigenmodes) have received a great deal of attention recently[1–11] since their interpretation[1] as Alfvén eigenmodes localized at the minimum q surface in tokamak plasmas with reversed shear profiles. These modes have long been observed in the experiments[2,9–11] and the frequencies of these modes typically increase as $q_{\min}$ drops in time according to $\omega^2 \sim [(n - m/q_{\min})V_A/R]^2 + \omega_{\rm geod}^2$, where $\omega_{\rm geod}$ is the geodesic curvature frequency. These cascade modes form near the region of zero magnetic shear and typically they are spontaneously excited by the presence of energetic particles. Their observation allows a precise determination of the value of the minimum safety factor in a tokamak discharge.

Theoretically it is found that the mode structure has a single dominant poloidal mode number, m, and that the frequencies of the modes are always slightly shifted above the maximum of the shear Alfvén continuum at the $q_{\min}$ surface. Thus, these modes have similar properties to the global Alfvén eigenmode (GAE).[12] However, fine-tuned corrections to the shear Alfvén wave terms of the MHD equations need to be taken into account to determine whether a global eigenmode can indeed be established. Such corrections arise from energetic particles,[1,4] toroidicity,[5] or plasma density gradient.[6] Recent numerical results from the NOVA (Ref. 13) code showed that the plasma pressure gradient has a favorable effect on the existence of Alfvén cascade modes,[7] a tendency that has been confirmed by other numerical codes such as LIGKA.[14] However, it has not been understood analytically why including the plasma thermal pressure gradient effect is favorable for the existence of these modes. Indeed, the work in Ref. 8 indicates that the pressure gradient effect is unfavorable to the establishment of the mode. Here, we reanalyze the MHD equations governing the Alfvén cascades and we find that the work in Ref. 8 needs to be modified somewhat. The principal modification is the inclusion of the averaged normal curvature from the interchange term. With the new terms we now find that including the pressure gradient terms allows an enhancement of the mode existence criterion.

We will use the results of the present theory to interpret published experimental observations in JET data as to why, as $q_{\min}$ decreases at higher q values, the cascade modes do not transform into toroidal Alfvén eigenmodes (TAEs) and why they do make this transformation at lower q values.

II. REDUCED MHD EQUATIONS

We start from the linearized momentum equation,

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\nabla \delta P + \delta \mathbf{J} \times \mathbf{B} + \mathbf{J} \times \delta \mathbf{B}, \tag{1}$$

where ρ is plasma mass density, $\boldsymbol{\xi}$ is the plasma displacement vector, $\mathbf{B}$ and $\mathbf{J}$ is the equilibrium magnetic field and plasma current, respectively, δP, $\delta\mathbf{B}$, and $\delta\mathbf{J}$ are the perturbed pressure, magnetic field, and plasma current, respectively. Using reduced MHD model for shear Alfvén waves in a low beta plasma, the plasma displacement can be written as

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$$\boldsymbol{\xi}=\frac{1}{B}\nabla U\times\mathbf{b}, \tag{2}$$

where U represents stream function of plasma displacement. Using the operation of $\nabla\cdot 1/B\mathbf{b}\times$ on both sides of Eq. (1), we get

$$-\nabla\cdot\left(\frac{\omega^2}{v_A^2}\nabla_\perp U\right)=\nabla\cdot\delta\mathbf{J}_\perp+\nabla\cdot\left[\frac{\mathbf{b}}{B}\times(\mathbf{J}\times\delta\mathbf{B})\right]+\nabla\cdot\frac{\nabla\delta P\times\mathbf{b}}{B}. \tag{3}$$

For shear Alfvén waves in low beta plasmas, the perturbed parallel magnetic field satisfies the following condition approximately:

$$\delta\mathbf{B}\cdot\mathbf{B}+\delta P=0. \tag{4}$$

Using Eq. (2), the perpendicular component of the perturbed magnetic field is given by

$$\delta\mathbf{B}_\perp=Q\mathbf{J}_\perp+\nabla Q\times\mathbf{B}. \tag{5}$$

Using Eq. (4) and (5) together with the quasineutrality condition $\nabla\cdot\delta\mathbf{J}=0$, and plasma equilibrium equation $\mathbf{J}\times\mathbf{B}=\nabla P$, Eq. (3) becomes

$$\begin{aligned}-\nabla\cdot\left(\frac{\omega^2}{v_A^2}\nabla_\perp U\right)=&-\mathbf{B}\cdot\nabla\frac{\delta\mathbf{J}\cdot\mathbf{B}}{B^2}+\frac{\delta P}{B}\mathbf{b}\cdot\nabla\left(\frac{J_\parallel}{B}\right)\\&-(Q\mathbf{J}_\perp+\nabla Q\times\mathbf{B})\cdot\nabla\left(\frac{J_\parallel}{B}\right)\\&-\delta P\mathbf{J}\cdot\nabla\left(\frac{1}{B^2}\right)-2\frac{\boldsymbol{\kappa}\cdot\nabla\delta P\times\mathbf{B}}{B^2}\\&-2\frac{\mathbf{J}_\parallel\cdot\nabla\delta P}{B^2},\end{aligned} \tag{6}$$

where $J_\parallel$ is the parallel equilibrium current, $\kappa=\mathbf{b}\cdot\nabla\mathbf{b}$ is the magnetic field curvature, and

$$Q=\frac{1}{B}\mathbf{b}\cdot\nabla U, \tag{7}$$

$$\delta P=\frac{1}{B}\mathbf{b}\times\nabla U\cdot\nabla P, \tag{8}$$

$$\begin{aligned}\delta\mathbf{J}\cdot\mathbf{B}&=\mathbf{J}\cdot\delta\mathbf{B}+\nabla\cdot[\delta\mathbf{B}\times\mathbf{B}]=J_\parallel\delta B_\parallel+\mathbf{J}\cdot\delta\mathbf{B}_\perp\\&+\nabla\cdot[Q\nabla P-B^2\nabla_\perp Q]=J_\parallel\delta B_\parallel+J_\perp^2Q+Q\nabla^2P\\&-\nabla\cdot(B^2\nabla_\perp Q).\end{aligned} \tag{9}$$

For the purpose of focusing on effects of pressure gradient, we have dropped the plasma compression term in the pressure perturbation. This eliminates the fluid compressibility and the sound wave. The effects of fluid compressibility will be considered in Sec. IV.

III. PRESSURE GRADIENT EFFECTS ON MODE EXISTENCE

Following the procedure of Breizman *et al.*,[5] we expand Eq. (6) to second order of inverse aspect ratio including the pressure gradient terms. The procedure is outlined briefly as follows. Consider Alfvén cascade modes with mode frequency close to the tip of Alfvén continuum at the $q_{\min}$ surface, $\omega\approx(n-m/q_{\min})v_A/R$. For simplicity, we assume high mode number $m\gg 1$. The plasma beta is assumed to be second order, $\beta\sim O(\epsilon^2)$, where $\epsilon=r/R$ is inverse aspect ratio. Keeping terms up to second order, Eq. (6) can be greatly simplified and written as

$$\nabla\cdot\left(\frac{\omega^2}{v_A^2}\nabla_\perp U\right)+\mathbf{B}\cdot\nabla\left(\frac{1}{B^2}\nabla\cdot B^2\nabla_\perp Q\right)-\nabla\left(\frac{J_\parallel}{B}\right)\cdot(\nabla Q\times\mathbf{B})+2\frac{\boldsymbol{\kappa}\cdot(\mathbf{B}\times\nabla\delta P)}{B^2}=0. \tag{10}$$

The neglected terms are either of higher order in ϵ or in $1/m$.

We now consider a large aspect ratio, low beta tokamak equilibrium with shifted circular flux surfaces. The shifted circle flux coordinates (r,θ,ζ) are used (see the Appendix). After expanding Eq. (10) to second order $O(\epsilon^2)$, we multiply Eq. (10) with $\mathcal{J}R_0\exp(im\theta)$ with $\mathcal{J}$ the Jacobian [assuming $U=\Sigma U_m(r)\exp(in\zeta-im\theta)$] and integrate over θ to arrive at the following mode equations:

$$(L_m^0+L_m^2)U_m+L_{m,m+1}^1U_{m+1}+L_{m,m-1}^1U_{m-1}=0, \tag{11}$$

$$(L_{m+1}^0+L_{m+1}^2)U_{m+1}+L_{m+1,m}^1U_m=0, \tag{12}$$

$$(L_{m-1}^0+L_{m-1}^2)U_{m-1}+L_{m-1,m}^1U_m=0, \tag{13}$$

where L^0, L^1, and L^2 are operator of zeroth, first, and second order, respectively, and are given by

$$L_m^0=\frac{\partial}{\partial r}(\bar\omega^2-k_m^2)r\frac{\partial}{\partial r}-\frac{m^2}{r}(\bar\omega^2-k_m^2), \tag{14}$$

$$\begin{aligned}L_{m,m\pm1}^1=&\bar\omega^2\left\{\frac{\partial}{\partial r}r(2\epsilon+\Delta')\frac{\partial}{\partial r}-\frac{m(m\pm1)}{r}(\epsilon-\Delta')\mp[\epsilon+(r\Delta')']m\frac{\partial}{\partial r}\right\}\\&-k_m\left\{\frac{\partial}{\partial r}r\Delta'\frac{\partial}{\partial r}k_{m\pm1}-\frac{m^2}{r}(\epsilon+\Delta')k_{m\pm1}\mp m[\epsilon+(r\Delta')']\frac{\partial}{\partial r}k_{m\pm1}\right\}\\&-\frac{m\alpha}{2q^2}\left(\frac{m}{r}\mp\frac{\partial}{\partial r}\right),\end{aligned} \tag{15}$$

$$L_m^2=\frac{\partial}{\partial r}\bar\omega^2 4\epsilon\Delta' r\frac{\partial}{\partial r}+\frac{m^2}{r}[4\epsilon(\epsilon+\Delta')\bar\omega^2+\bar\kappa_r\alpha/q^2], \tag{16}$$

where $\bar\omega=\omega/(V_A/R_0)$, $k_m=(n-m/q)$, $\alpha=-R_0q^2d\beta/dr$ is the normalized pressure gradient parameter and Δ' is the radial derivative of the Shafranov shift given by $\Delta'=\epsilon(\beta_p+l_i/2)$, with $\beta_p=2(\langle p\rangle-p)/B_p^2$ a measure of poloidal beta and $l_i=\langle B_p^2\rangle/B_p^2$ the internal inductance. Here, the bracket $\langle\ldots\rangle$ indicates a volume average within radius r. Finally, $\bar\kappa_r$ is the averaged normal curvature (normalized by $1/R_0$) and is given by Ref. 15 (see the Appendix),

$$\bar{\kappa}_r = \epsilon\left(1 - \frac{1}{q^2}\right) + \frac{1}{2}\alpha. \tag{17}$$

Equations (10)–(12) can be combined into a single equation for U_m,

$$(L_m^0 + L_m^2)U_m = \left(L_{m,m+1}^1 \frac{1}{L_{m+1,m+1}^0} L_{m+1,m}^1 + L_{m,m-1}^1 \frac{1}{L_{m-1,m-1}^0} L_{m-1,m}^1\right) U_m. \tag{18}$$

For high m, the mode is localized around $q=q_{\min}$ surface; thus, the radial derivative acts only on the perturbation to a good approximation. Then, we can simplify the operators as

$$L_{m\pm1}^0 = r(\bar{\omega}^2 - k_{m\pm1}^2)\left(D^2 - \frac{m^2}{r^2}\right), \tag{19}$$

$$L_{m,m\pm1}^1 = G_{0\pm} \pm G_{1\pm}D + G_{2\pm}D^2,. \tag{20}$$

$$L_{m\pm1,m}^1 = G_{0\pm} \mp G_{1\pm}D + G_{2\pm}D^2, \tag{21}$$

where the operator D denotes radial derivative and coefficients G_i are given by

$$G_{0\pm} = -\frac{m^2}{r}\left((\epsilon - \Delta')\bar{\omega}^2 + (\epsilon + \Delta')k_m k_{m\pm1} + \frac{\alpha}{2q^2}\right), \tag{22}$$

$$G_{1\pm} = -m\left((\epsilon + (r\Delta')')\bar{\omega}^2 - (\epsilon + (r\Delta')')k_m k_{m\pm1} - \frac{\alpha}{2q^2}\right), \tag{23}$$

$$G_{2\pm} = r(2\epsilon + \Delta')\bar{\omega}^2 - r\Delta' k_m k_{m\pm1}. \tag{24}$$

Then, the right-hand side (RHS) of Eq. (18) can be written as

$$\mathrm{RHS} = \frac{1}{r(\bar{\omega}^2 - k_{m+1}^2)}\left[G_{2+}^2 D^2 + \frac{(m+1)^2}{r^2}G_{2+}^2 + 2G_{0+}G_{2+} - G_{1+}^2\right] + \frac{1}{r(\bar{\omega}^2 - k_{m-1}^2)}\left[G_{2-}^2 D^2 + \frac{(m-1)^2}{r^2}G_{2-}^2 + 2G_{0-}G_{2-} - G_{1-}^2\right]. \tag{25}$$

As pointed out previously,[5] it is remarkable that the terms proportional to the inverse of operator $D^2 - m^2/r^2$ do not appear in the above equation. It can be shown that these terms cancel if it is assumed that Δ' is linear in r [i.e., $(r\Delta')' \approx 2\Delta'$]. Thus, for simplicity, we assume Δ' is linear in r from now on, and it then follows that $\Delta' = (\epsilon + \alpha)/4$.

Taking advantage of $\bar{\omega}^2 \approx k_m^2$ and $\bar{\omega}^2 - k_{m\pm1}^2 \approx -1/q^2 \pm 2k_m/q$, Eq. (25) becomes

$$\mathrm{RHS} = 2r\bar{\omega}^2\left[\epsilon^2 + 2\Delta'\epsilon + \frac{(\epsilon + \Delta')^2}{4k_m^2 q^2 - 1}\right]D^2 + 2\bar{\omega}^2\frac{m^2}{r}\left[\epsilon^2 + 2\Delta'\epsilon - \frac{\Delta'^2}{4k_m^2 q^2 - 1}\right] + 2\frac{m^2}{r}\frac{2\Delta' k_m^2 \alpha - \frac{1}{4q^2}\alpha}{4k_m^2 q^2 - 1}. \tag{26}$$

Substituting Eq. (16) and Eq. (26) into Eq. (18), we arrive at the final expression for the mode equation with pressure gradient effects,

$$\frac{\partial}{\partial r}(\bar{\omega}^2 - k_m^2)r\frac{\partial}{\partial r}U_m - \frac{m^2}{r}(\bar{\omega}^2 - k_m^2)U_m = -\frac{m^2}{r}\left[\frac{2(\epsilon^2 + 2\Delta'\epsilon)\bar{\omega}^2}{1 - 4k_m^2 q^2} - \frac{\alpha^2/2q^2}{1 - 4k_m^2 q^2} + \frac{4\Delta'\bar{\omega}^2\alpha}{1 - 4k_m^2 q^2} + \frac{1}{q^2}\bar{\kappa}_r\alpha\right]U_m. \tag{27}$$

This completes our derivation for the eigenmode equation for the cascade modes with pressure gradient effects.

Equation (27) extended previous work to full pressure gradient effects. The first term on the right-hand side of Eq. (27) is the second-order toroidicity term originally derived in Ref. 5. The second term comes from the mode coupling term due to the curvature. This is the α^2 term derived by Breizman *et al.*[8] and it is always unfavorable to the existence of the cascade modes. The third and the fourth terms are the new pressure gradient terms obtained in this work. In particular, the new average curvature term dominates over the α^2 term. It is the main pressure gradient term responsible for the favorable effect on the existence of the cascade modes. To see this more clearly, we consider the low-frequency limit $\bar{\omega}^2 \sim k_m^2 \ll 1/4q^2$. Then, Eq. (27) becomes

$$\frac{\partial}{\partial r}(\bar{\omega}^2 - k_m^2)r\frac{\partial}{\partial r}U_m - \frac{m^2}{r}(\bar{\omega}^2 - k_m^2)U_m = -\frac{m^2}{r}\left[-\alpha^2/2q^2 + \frac{1}{q^2}\bar{\kappa}_r\alpha\right]U_m = -\frac{m^2}{r}\left[\left(1 - \frac{1}{q^2}\right)\epsilon\alpha\right]U_m. \tag{28}$$

Note that the α^2 term has been canceled by the pressure-gradient-dependent part of the average curvature using Eq. (17). What remains is a purely linear term in pressure gradient which is favorable for the mode existence when $q > 1$. This condition corresponds exactly to the stability condition of Mercier modes at low shear, i.e., $q > 1$ for stability. We note for comparison that Eq. (28) can be used at low shear to obtain the MHD stability criterion of Mercier modes. In particular, the Mercier condition yields stability for a tokamak when $q > 1$, due to the favorable normal curvature, while the α^2 terms coming from the sideband coupling is canceled by the beta dependence of the equilibrium Shafranov shift. This cancellation has been found by others[15,16] and it appears to be a general property of Mercier modes.

Following Berk *et al.*,[1] Eq. (27) can be expanded about the zero shear point and written in the following dimensionless form:

$$\frac{\partial}{\partial x}(S+x^2)\frac{\partial}{\partial x}U_m+(Q-S-x^2)U_m=0, \tag{29}$$

where $x=m(r-r_0)/r_0$ with r_0 being the radius of minimum q,

$$S=\frac{mq_0^2}{(-k_{m0})r_0^2q_0''}(\bar{\omega}^2-k_{m0}^2), \tag{30}$$

$$Q=Q_{\text{tor}}+Q_{\text{pressure}}, \tag{31}$$

$$Q_{\text{tor}}=2\frac{mq_0^2(-k_{m0})}{r_0^2q_0''}\frac{(\epsilon^2+2\Delta'\epsilon)}{1-4k_{m0}^2q_0^2}, \tag{32}$$

$$Q_{\text{pressure}}=\frac{mq_0^2}{(-k_{m0})r_0^2q_0''}\left[\frac{4\Delta'\bar{\omega}^2\alpha-\alpha^2/2q_0^2}{1-4k_{m0}^2q_0^2}+\frac{1}{q_0^2}\bar{\kappa}_r\alpha\right], \tag{33}$$

with $k_{m0}=n-m/q_0$ and q_0 the minimum q. Here, Q_{tor} is the toroidicity term derived previously[5] and Q_{pressure} is the new term due to the pressure gradient. Note that the singularity at $1-4k_{m0}^2q^2=0$ arises because the mode frequency is then in the TAE band, and one of the sideband harmonics needs to be treated differently. The singularity of $k_{m0}=0$ in Q_{pressure} arises because of the failure of the approximation $|k_{m0}| \gg (1/4m)r_0^2q_0''\sqrt{Q_{\text{pressure}}}$, which was assumed in obtaining Eq. (29).

It can be shown that existence of the mode localized at $r=r_0$ requires $Q>1/4$.[1] For $(m-1/2)/n<q_0<m/n$, where $-k_m$ and $1-4k_{m0}^2q^2$ are positive and the mode frequency is below the TAE mode frequency, we have $Q_{\text{tor}}>0$ which is favorable for existence of cascade modes as shown previously. The pressure gradient term Q_{pressure} is typically positive since usually $q_0>1$ for reversed shear q profiles. In particular, near a rational surface where k_m goes to zero and the frequency is small, we have $Q_{\text{tor}}/Q_{\text{pressure}}\sim 4k_{m0}^2q_0^2 \ll 1$ for $\alpha \sim \epsilon$. Thus, the pressure gradient term *is much larger than the toroidicity term* in this limit and is strongly favorable for the existence of these cascade modes.

For finite mode frequency, a general formula for Q can be written using Eq. (17),

$$Q=-2\left(\frac{mq_0^2k_{m0}}{r_0^2q_0''}\right)\times\frac{(3\epsilon-\alpha)(\alpha+\epsilon)+4(1-1/q_0^2)[1/(4k_{m0}^2q_0^2)-1]\epsilon\alpha}{1-4k_{m0}^2q_0^2}, \tag{34}$$

where we have taken into account the implicit pressure gradient dependence in Q_{tor} via Δ'. It is interesting to note that, at $q_0>1$, a small pressure gradient is always favorable to mode existence for any mode frequency below the TAE's.

Finally, it is interesting to examine the case of cascade modes above TAE frequency (i.e., $4k_{m0}^2q_0^2>1$, where $\bar{\omega}^2 >1/(4q_0^2)$). In this case Q_{tor} is now positive in the range $q_0 >m/n$, where $k_m>0$. Then, whether the pressure gradient term is favorable or not depends on the value of mode frequency. Using Eq. (34), a small pressure gradient is always favorable at $1<q_0<\sqrt{2}$. For $q_0<1$ or $q_0>\sqrt{2}$, a small pressure gradient is favorable at $k_{m0}^2<(1-1/q_0^2)/(2q_0^2-4)$ and unfavorable at $k_{m0}^2>(1-1/q_0^2)/(2q_0^2-4)$.

IV. COMPRESSIBILITY EFFECTS

Recall that we have so far neglected plasma compressibility in the pressure perturbation given in Eq. (8) for simplicity. When the compressibility term is retained it is found that the shear Alfvén wave continuum can no longer go to zero frequency at a rational surface but reaches a minimum frequency due to the excitation of a fast acoustic continuum mode.[17] The frequency of this mode is that of the geodesic acoustic mode[18] and the eigenmode frequencies we seek lie above this frequency.[8] Thus, at low frequency, we need to modify our analysis. Thus retaining the compressibility term, the perturbed pressure is now given by

$$\delta P=-\boldsymbol{\xi}\cdot\nabla P-\Gamma P\nabla\cdot\boldsymbol{\xi}=\frac{1}{B}\mathbf{b}\times\nabla U\cdot\nabla P -\Gamma P\left[\frac{\mathbf{J}\cdot\nabla U}{B^2}+\frac{2\boldsymbol{\kappa}\cdot(\mathbf{B}\times\nabla\delta U)}{B^2}-\frac{J_{\parallel}Q}{B}\right], \tag{35}$$

where Γ is the ratio of specific heat. Since the plasma pressure is of second order, only the curvature term in Eq. (35) remains after the expansion of Eq. (10) to second order. Thus, the compressibility results in additional terms only for the second-order operator L_m^2, while L_m^0 and L_m^1 remain the same. L_m^2 becomes

$$L_m^2=\frac{\partial}{\partial r}\bar{\omega}^2 4\epsilon\Delta' r\frac{\partial}{\partial r}+\frac{m^2}{r}[4\epsilon(\epsilon+\Delta')\bar{\omega}^2+\bar{\kappa}_r\alpha/q^2] -\frac{2\Gamma P}{B^2}\left[r\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial^2}{\partial\theta^2}\right]. \tag{36}$$

Correspondingly, Eq. (27) becomes

$$\frac{\partial}{\partial r}(\bar{\omega}^2-\bar{\omega}_{\text{geod}}^2-k_m^2)r\frac{\partial}{\partial r}U_m-\frac{m^2}{r}(\bar{\omega}^2-\bar{\omega}_{\text{geod}}^2-k_m^2)U_m =\frac{m^2}{r}\left[\frac{2(\epsilon^2+2\Delta'\epsilon)\bar{\omega}^2}{4k_{m0}^2q^2-1}+\frac{4\Delta'\bar{\omega}^2\alpha-\alpha^2/2q^2}{4k_{m0}^2q^2-1} -\frac{1}{q^2}\bar{\kappa}_r\alpha\right]U_m, \tag{37}$$

where $\bar{\omega}_{\text{geod}}=\omega_{\text{geod}}/(V_A/R_0)$ and $\omega_{\text{geod}}=\sqrt{2\Gamma P/\rho}/R_0$ is the geodesic acoustic frequency. Note that $\bar{\omega}_{\text{geod}}=\sqrt{\Gamma\beta}\ll 1$. When the mode frequency is close to the geodesic acoustic frequency, Eq. (38) reduces to

$$\frac{\partial}{\partial r}(\bar{\omega}^2-\bar{\omega}_{\text{geod}}^2-k_m^2)r\frac{\partial}{\partial r}U_m-\frac{m^2}{r}(\bar{\omega}^2-\bar{\omega}_{\text{geod}}^2-k_{m0}^2)U_m =-\frac{m^2\epsilon}{rq^2}\left(1-\frac{1}{q^2}\right)\alpha U_m. \tag{38}$$

It should be noted that the definition of $\bar{\omega}_{\text{geod}}^2$ differs from Ref. 8 by a term that is proportional to $1/2q^2$. To obtain this term, it is necessary to include an additional displacement along the magnetic field line in the MHD equations in the same way it was performed in Ref. 8.

V. CONCLUSIONS

In conclusion, it is shown that the plasma pressure gradient is favorable to the existence of cascade modes, especially when $q_{\min}$ is slightly less than the fraction m/n that characterizes a rational surface. In this region a pressure gradient produces the dominant term in the existence criterion even when α/ϵ is small. This analytic result is compatible with previously reported numerical results. The favorable effect comes mainly from the averaged magnetic curvature in the interchange term when $q>1$. In the limit of low shear near a rational surface, there is one-to-one correspondence between the Mercier criterion and existence condition of cascade modes. The Mercier condition is stable at small shear due to the favorable averaged normal curvature in a tokamak when $q>1$, and this term is proportional to the criterion we are obtaining for the existence of the cascade mode. For finite alpha the Mercier criterion gives a direct destabilizing α^2 term (which is related to the α^2 term obtained in Ref. 8 that was unfavorable to mode existence) because a net lower MHD energy perturbation can be found for a mode that is not quite a flute. However, near the rational surface this direct α^2 term is canceled by the modification of the equilibrium that comes from the beta dependence of the Shafranov shift.[15,16] Near a rational surface our mode existence criterion produces a similar cancellation of the α^2 term for the very same reason. For $q<1$ the normal averaged curvature term produces an unstable Mercier criterion and an alpha term that is unfavorable to the cascade mode existence criterion. Thus, there is a complete parallel between MHD stability and cascade mode existence near a rational surface.

Away from the rational surface, an α^2 does persist because the frequency is finite (recall that the α^2 cancellation is only complete when ω^2 approaches zero). We then find that, as the decreasing $q_{\min}$ approaches $(m-1/2)/n$, where the cascade mode frequency approaches the TAE gap, the unfavorable nature of the α^2 term is the strongest with respect to the existence of the cascade mode. Indeed, if $\alpha>3\epsilon$, we see from Eq. (34) that the mode existence criterion will fail at frequencies below the TAE gap and the cascade mode cannot enter the TAE gap. Interestingly, it has been shown in Ref. 19 that the condition where TAE modes do not exist at low shear is when $\alpha>3\epsilon$. This previous calculation, together with Eq. (34), then suggests that in the low shear limit, when $\alpha>3\epsilon$, a cascade can exist when

$$k_m^2 < \frac{\frac{1}{q_0^2}\left(1-\frac{1}{q_0^2}\right)}{4\left(1-\frac{1}{q_0^2}\right)+(\alpha-3\epsilon)(\alpha+\epsilon)} \tag{39}$$

and oscillate at a frequency below the TAE gap, but its frequency cannot reach the TAE gap. Indeed, the criterion from Ref. 19 indicates that the TAE mode does not even exist for such values of α, which then implies that the frequency of the cascade is limited to

$$\frac{\omega^2}{\omega_{\mathrm{TAE}}^2} < \frac{4\left(1-\frac{1}{q_0^2}\right)}{4\left(1-\frac{1}{q_0^2}\right)+(\alpha-3\epsilon)(\alpha+\epsilon)}. \tag{40}$$

This last result suggests an explanation for cascade observations often found in JET experimental data and is illustrated in the data published in Ref. 1 and Ref. 5 for JET discharge #49382 and #53487, respectively. The data show the cascade modes emerging at low frequency and going to higher frequency as $q_{\min}$ falls with time. The starting frequency has been interpreted as being close to the geodesic curvature frequency[8] and the frequency rises towards the TAE gap. At the lower $q_{\min}$ values the cascade modes clearly transform into TAE modes, which then persist at a relatively constant frequency as $q_{\min}$ continues to decrease. However, at earlier times the discharge has higher $q_{\min}$ values, and at those times the cascade mode frequency, for the higher m-number excitations, abruptly terminates as the frequency approaches the TAE gap. Recall that α is proportional to $q_{\min}^2$ and in the experiment $q_{\min}^2$ is changing appreciably while other parameters, such as a pressure profile, likely vary more slowly. Thus, we suggest that the maximum cascade frequency predicted by Eq. (40), when $\alpha>3\epsilon$, which can occur at the higher $q_{\min}$ values, may be the basis for why the cascade frequencies cannot penetrate into the TAE gap. Certainly this conjecture requires further scrutiny, but if correct the understanding of this transition in shear reversed discharges may be a valuable diagnostic tool.

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APPENDIX: FLUX COORDINATES

For completeness, here we describe the flux-type large aspect ratio coordinates (r,θ,ζ) (Ref. 20) used in expanding the equation to second order in inverse aspect ratio, where r is the minor radius, θ is a poloidal angle, and ζ is the toroidal angle. These coordinates are defined via cylindrical coordinates (R,Z,ϕ) as

$$R = R_0 + r\cos\theta - \Delta(r) + r\eta(r)(\cos 2\theta - 1), \tag{A1}$$

$$Z = r\sin\theta + r\eta(r)\sin 2\theta, \tag{A2}$$

$$\phi = -\zeta, \tag{A3}$$

where $\eta=(\epsilon+\Delta')/2$, Δ' is the radial derivative of Shafranov shift. For low beta, large aspect ratio circular tokamak equilibria, $\Delta'=\epsilon(\beta_p+l_i/2)$, with $\beta_p=2(\langle p\rangle-p)/B_p^2$ a measure of poloidal beta and $l_i=\langle B_p^2\rangle/B_p^2$ the internal inductance. Here, the bracket $\langle\ldots\rangle$ indicates a volume average within radius r.

With these coordinates, the magnetic field, Jacobian, and metric coefficients can be written as

$$\mathbf{B} = \frac{rB_0}{q} \nabla r \times \nabla(q\theta - \zeta), \tag{A4}$$

$$\mathcal{J} = rR_0(1 + 2\epsilon \cos\theta), \tag{A5}$$

$$g^{rr} = 1 + 2\Delta' \cos\theta, \tag{A6}$$

$$g^{\theta\theta} = \frac{1}{r^2}[1 - 2(\epsilon + \Delta')\cos\theta], \tag{A7}$$

$$g^{r\theta} = g^{\theta r} = -\frac{1}{r}[\epsilon + (r\Delta')']\sin\theta, \tag{A8}$$

$$g^{\zeta\zeta} = \frac{1}{R_0^2}[1 - 2\epsilon\cos\theta]. \tag{A9}$$

Using these coordinates, we can write the following operators and the perturbed variables Q and δP:

$$\mathbf{B}\cdot\nabla = \frac{rB_0}{\mathcal{J}}\left(\frac{\partial}{\partial\zeta} + \frac{1}{q}\frac{\partial}{\partial\theta}\right), \tag{A10}$$

$$\mathcal{J}\nabla \cdot F\nabla_\perp = \frac{\partial}{\partial r}\mathcal{J}Fg^{rr}\frac{\partial}{\partial r} + \frac{\partial}{\partial r}\mathcal{J}Fg^{r\theta}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\theta}\mathcal{J}Fg^{\theta r}\frac{\partial}{\partial r} + \frac{\partial}{\partial\theta}\mathcal{J}Fg^{\theta\theta}\frac{\partial}{\partial\theta}, \tag{A11}$$

$$Q = \frac{rB_0}{\mathcal{J}B^2}\left(\frac{\partial}{\partial\zeta} + \frac{1}{q}\frac{\partial}{\partial\theta}\right)U, \tag{A12}$$

$$\delta P = -\frac{B_\phi R}{\mathcal{J}B^2}\frac{\partial U}{\partial\theta}P'. \tag{A13}$$

Note that in Eq. (A11), we have dropped some second-order terms since they cancel out in Eq. (10) due to $\bar{\omega} \approx k_m$. Furthermore, we have also neglected a third-order term in Eq. (A13). Finally, the curvature term in Eq. (10) can be written as

$$\frac{\boldsymbol{\kappa}\cdot(\mathbf{B}\times\nabla\delta P)}{B^2} = -\frac{\kappa_r B_\phi R}{\mathcal{J}B^2}\frac{\partial\delta P}{\partial\theta} + \frac{\kappa_\theta B_\phi R}{\mathcal{J}B^2}\frac{\partial\delta P}{\partial r}, \tag{A14}$$

where κ_r and κ_θ are the radial and poloidal components of the curvature, respectively. Here, we have also dropped a few higher order terms related to the toroidal component of the curvature. Using equilibrium force balance for a large aspect ratio circular tokamak, the radial and poloidal component of the curvature can be written as

$$\kappa_r \approx -\frac{1}{R}\frac{\partial R}{\partial r} - \frac{r}{q^2R^2} \tag{A15}$$

and

$$\kappa_\theta \approx -\frac{1}{R}\frac{\partial R}{\partial\theta}. \tag{A16}$$

Using the flux coordinates defined above, we obtain the following explicit expression for the averaged normal curvature:

$$\bar{\kappa}_r = R_0\frac{\int \mathcal{J}\kappa_r d\theta}{\int \mathcal{J}d\theta} = \frac{\epsilon}{2} + \frac{3}{2}\Delta' + \frac{1}{2}r\Delta'' - \frac{\epsilon}{q^2}.$$

$$= \epsilon\left(1 - \frac{1}{q^2}\right) + \frac{1}{2}\alpha, \tag{A17}$$

where we have used the Grad-Shafranov equation and have assumed low shear $s \ll 1$. It should be pointed out that the above equation is valid for general current and pressure profile without assuming Δ' is linear in r.

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