Propagation of radially localized helicon waves in longitudinally nonuniform plasmas

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A gradient in the plasma density across the guiding magnetic field can support a low-frequency radially localized helicon (RLH) wave in a plasma column. If the radial density gradient changes along the magnetic field, this wave can undergo reflection and also excite conventional whistlers. This paper presents calculations of the corresponding reflection coefficient, including the effect of whistler radiation. It is shown that a sharp longitudinal density drop causes a nearly complete reflection of the RLH wave. The longitudinal wavelength of the excited whistlers is much greater than that of the RLH wave, and, as a result, only a small fraction of the RLH wave energy is transferred to the whistlers. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2212367]

I. INTRODUCTION

This work is motivated by the development of helicon plasma sources for plasma-based space thrusters.[1,2] Operation of such sources usually involves excitation of rf eigenmodes in the plasma.[3] Although helicon sources have been extensively studied in laboratory experiments,[4] the closed-end laboratory devices do not quite reproduce the conditions of a plasma thruster, because one end of the source must be open in a thruster. It is then important to understand whether having an open-end system would still allow the rf waves to be confined inside the source as they are in laboratory experiments.

The operational frequency of helicon sources is below the electron gyrofrequency and above the ion gyrofrequency. In uniform plasmas, whistler waves belong to this frequency range.[5] However, in actual helicon plasma sources, the plasma density is usually nonuniform across the equilibrium magnetic field. This nonuniformity can modify dramatically both the whistler wave structure and its dispersion relation.[6] A radial density gradient in a plasma cylinder can create a potential well for whistlers, allowing radially localized solutions.[7] We refer to such solutions as radially localized helicon (RLH) waves.

It is noteworthy that RLH waves must have a nonzero azimuthal mode number, m.[7] An essential element of RLH waves is the presence of a Hall current in the direction of the plasma density gradient, which vanishes for axisymmetric ($m=0$) modes. The radial Hall current generates an electron current along the equilibrium magnetic field to keep the divergence of the total plasma current equal to zero and prevent charge separation.

The RLH mode has been identified experimentally by its resonant response to the rf antenna.[8] The power balance analysis presented in Ref. 8 shows that this mode provides the dominant power deposition mechanism into the helicon discharge. The radial gradient of the plasma prohibits conventional (uniform plasma) whistlers in that particular experiment. The mode damping rate measured in Ref. 8 is relatively low, which allows the RLH wave to travel many times along the plasma without significant dissipation. This leaves no doubt that the wave is trapped axially along the magnetic-field lines. What is less obvious is whether the end walls or the plasma itself are responsible for such trapping. This distinction is crucial for space propulsion applications, where the plasma source is open downstream.

RLH waves may still be confined axially in an open-end system due to intrinsic nonuniformity of the plasma in the axial direction. The underlying reasons for such nonuniformity can be ion acceleration by the ambipolar electric field and radial plasma losses.[9] It is therefore important to examine the behavior of RLH waves in longitudinally nonuniform plasmas.

In this paper, we consider the RLH wave propagation within a slab model of the plasma cylinder, so that the x and y coordinates correspond to the radial and azimuthal coordinates, respectively. The z axis is directed along the unperturbed magnetic field. The equilibrium plasma density is assumed to be a function of x and z only, i.e., it is uniform along the y axis. The azimuthal wave number of the RLH wave reported in Ref. 8 is $m=+1$. The characteristic radial and azimuthal scales for this mode are comparable to each other and they are much shorter than the axial scale of the wave, since the plasma is elongated. In order to incorporate these essential elements into the slab model, we assume that the wave has a finite y component of the wave vector (k_y). This ensures that there is a nonzero component of the electron $(\mathbf{E}\times\mathbf{B})$-drift velocity along the density gradient (the x axis). We also assume that the characteristic longitudinal scale of the wave field is much longer than k_y^{-1}, whereas the characteristic transverse scales are comparable to k_y^{-1}.

We use the slab model to consider propagation and reflection of an RLH wave that is localized in x and travels in the z direction. We assume that the incident wave comes from $z\to-\infty$, where the unperturbed plasma density is uniform in z but nonuniform in x. The density is also assumed to be independent of z at $z\to+\infty$. The reflection takes place within a finite interval around $z=0$, where the plasma density is both x- and z-dependent. We show that a sharp drop in plasma density along the magnetic field causes a nearly com-

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plete reflection of the incident RLH wave. The reflection is generally accompanied by radiation of whistler waves whose wavelength is much greater than that of the RLH wave. We find that only a small fraction of the RLH wave energy is transferred to these large-scale whistlers.

The paper is organized as follows. In Sec. II, we derive a reduced equation that describes whistlers and RLH waves. In Sec. III, we consider reflection of RLH waves in a longitudinally nonuniform plasma. Section IV addresses the linear transformation of an RLH wave into large-scale whistlers. In Sec. V we summarize the results.

II. BASIC EQUATIONS

The waves under consideration belong to the frequency range of

$$\omega_{ce} \gg \omega \gg \omega_{ci}, \tag{1}$$

where ω is the wave frequency, and $\omega_{ce} \equiv |e|B_0/m_e c$ and $\omega_{ci} \equiv |e|B_0/m_i c$ are the electron and ion gyrofrequencies. Note that the quantities ω_{ce} and ω_{ci} are positive in our notation. The waves are described by the following linearized equations:[10]

$$\nabla \times \mathrm{B} = \frac{4\pi}{c}\mathrm{j}, \tag{2}$$

$$\nabla \times \mathrm{E} = -\frac{1}{c}\frac{\partial \mathrm{B}}{\partial t}, \tag{3}$$

$$0 = \mathrm{E} - \frac{1}{|e|n_0 c}\mathrm{j} \times \mathrm{B}_0, \tag{4}$$

where n_0 and $\mathbf{B}_0 \equiv B_0 \mathbf{e}_z$ are the equilibrium density and magnetic field and $\mathbf{B}$, $\mathbf{E}$, and $\mathbf{j}$ are the perturbed magnetic field, electric field, and electron current.

Equations (2) and (3) are Maxwell's equations without the displacement current. The displacement current can be neglected when the plasma is sufficiently dense, so that $\omega_{pe} \gg \omega_{ce}$, where $\omega_{pe} \equiv \sqrt{4\pi n_0 e^2/m_e}$ is the electron plasma frequency. Equation (4) is the electron equation of motion in the zero-inertia limit, which is justified at $\omega \ll \omega_{ce}$. This equation implies that the plasma current across the magnetic field is the electron Hall current. It also shows that the perturbed electric field $\mathbf{E}$ is orthogonal to $\mathbf{B}_0$. The z component of $\mathbf{E}$ vanishes due to high electron conductivity along the magnetic-field lines, provided that the characteristic scale lengths of the wave fields and plasma density are greater than the plasma skin depth c/ω_{pe}. Note that this assumption automatically excludes electrostatic modes from the analysis.

Since the plasma is uniform in the y direction, we use a Fourier expansion of the perturbed fields to select a single harmonic that depends on y as $e^{ik_y y}$. The time dependence is assumed to be $e^{-i\omega t}$. Elimination of $\mathbf{B}$ and $\mathbf{j}$ reduces Eqs. (2)–(4) to a closed set of equations for E_x and E_y,

$$ik_y \frac{\partial E_y}{\partial x} + k_y^2 E_x = \frac{\partial^2 E_x}{\partial z^2} - \frac{\omega}{\omega_{ce}}\frac{\omega_{pe}^2}{c^2} iE_y, \tag{5}$$

$$ik_y \frac{\partial E_x}{\partial x} - \frac{\partial^2 E_y}{\partial x^2} = \frac{\partial^2 E_y}{\partial z^2} + \frac{\omega}{\omega_{ce}}\frac{\omega_{pe}^2}{c^2} iE_x. \tag{6}$$

As explained in the Introduction, we will treat the z derivatives as being smaller than k_y and the x derivatives as being comparable to k_y. In this limit, the eigenfrequency of a whistler wave satisfies the condition

$$\omega \ll \omega_{ce}\frac{k_y^2 c^2}{\omega_{pe}^2}, \tag{7}$$

where $k_y^2 c^2/\omega_{pe}^2 \ll 1$, and is given by[5]

$$\omega \approx \omega_{ce}\frac{\sqrt{k_x^2 + k_y^2}|k_z|c^2}{\omega_{pe}^2}, \tag{8}$$

with $k_z = -i\partial/\partial z$ and $k_x = -i\partial/\partial x$. We assume that condition (7) holds for all waves under consideration, which makes it convenient to introduce a dimensionless function

$$G \equiv \frac{\omega}{\omega_{ce}}\frac{\omega_{pe}^2}{k_y^2 c^2} \ll 1 \tag{9}$$

that will be treated as a small quantity.

Next, we use Eq. (5) to express E_x in Eq. (6) in terms of E_y, $\partial E_y/\partial x$, and $\partial^2 E_x/\partial z^2$. Collecting E_x terms on the left-hand side and E_y terms on the right-hand side, we find that Eqs. (5) and (6) transform to

$$ik_y^2 E_x - i\frac{\partial^2 E_x}{\partial z^2} = k_y \frac{\partial E_y}{\partial x} + k_y^2 G E_y, \tag{10}$$

$$ik_y \frac{\partial^2}{\partial z^2}\left(\frac{\partial E_x}{\partial x}\right) - ik_y^2 G \frac{\partial^2 E_x}{\partial z^2} = k_y^2 \frac{\partial^2 E_y}{\partial z^2} - k_y^3 \frac{\partial G}{\partial x} E_y + k_y^4 G^2 E_y. \tag{11}$$

We now make use of the smallness of the z derivatives and G. To lowest order in these small quantities, we put $\partial/\partial z = 0$ and $G = 0$ in Eq. (10) to find that

$$ik_y E_x = \frac{\partial E_y}{\partial x}. \tag{12}$$

We use this relation to eliminate E_x from Eq. (11) and we omit the second term on the left-hand side of Eq. (11), since for $G \ll 1$ it is much smaller than the first term. We thereby reduce Eq. (11) to

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 E_y}{\partial x^2} - k_y^2 E_y\right) = -k_y^3 \frac{\partial G}{\partial x} E_y + k_y^4 G^2 E_y. \tag{13}$$

It is convenient to eliminate k_y from Eq. (13) by replacing x and z by x/k_y and z/k_y, with x and y being dimensionless variables from now on. Then Eq. (13) takes the form

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 E}{\partial x^2} - E\right) = -\frac{\partial G}{\partial x} E + G^2 E, \tag{14}$$

where the subscript y is omitted for brevity, so that $E \equiv E_y$. As shown below, Eq. (14) describes both the whistler waves and the slab version of the RLH waves.

In the case of a longitudinally uniform plasma, $\partial^2/\partial z^2$ can be replaced by $-\kappa_z^2$, with $E \propto e^{i\kappa_z z}$. Consequently, Eq. (14) takes the form of a one-dimensional Schrödinger's equation for a "particle" with an "energy" $\varepsilon_0=-1$,

$$\frac{\partial^2 E}{\partial x^2} + [\varepsilon_0 - U(x)]E = 0, \tag{15}$$

where

$$U(x) \equiv \frac{-1}{\kappa_z^2}\left(G^2 - \frac{\partial G}{\partial x}\right) \tag{16}$$

is the effective potential energy. If the plasma is also uniform in x, then $U(x)$ is a constant and Eq. (15) describes "free motion" with a constant "momentum," i.e. $E \propto e^{i\kappa_x x}$, with

$$\kappa_x = \pm\sqrt{\frac{G^2}{\kappa_z^2} - 1}. \tag{17}$$

Equation (17) is equivalent to the whistler dispersion relation in the limit of small longitudinal wave numbers.[5] "Free motion" implies that the potential energy is lower than ε_0, which requires $G > |\kappa_z|$.

In a nonuniform plasma, the dominant contribution to the potential energy tends to come from $\partial G/\partial x$, as the G^2 term is relatively small for $G \ll 1$. We observe that a monotonically decreasing density profile can create a potential well. For example, $G(x) \propto \text{const-arctan}(x)$ produces a potential well located at $x=0$. Depending on the depth of the well, there can be one or more bound states. The depth of the well scales as κ_z^{-2}, so that for any bound state there exists a value of κ_z such that the bound state energy is equal to ε_0.

Consider a step-like density profile that creates a δ well,[11]

$$U(x) \approx \frac{1}{\kappa_z^2}\frac{\partial G}{\partial x} \approx -\frac{|\Delta G|}{\kappa_z^2}\delta(x). \tag{18}$$

The energy of the only bound state in this well is $-|\Delta G|/2\kappa_z^2$. The requirement that this energy equals $\varepsilon_0=-1$ selects the following value of κ_z:

$$\kappa_z^2 = |\Delta G|/2. \tag{19}$$

This equation reproduces the dispersion relation for the RLH waves derived in Ref. 7. We thus conclude that, in the context of the quantum-mechanical analogy, RLH waves represent discrete bound states, whereas whistler waves belong to the continuous spectrum.

III. NONRADIATIVE PROPAGATION OF RLH WAVES

In this section, we consider propagation of RLH waves in a longitudinally nonuniform plasma in the regime where whistler waves can be neglected. A generic density profile of interest is shown in Fig. 1. It is nonuniform in the x direction, with $\partial n/\partial x < 0$, so that the plasma can support an RLH wave. The density is z-dependent only within a finite interval around $z=0$. We assume that there is an incident RLH wave with a given frequency ω at $z \to -\infty$.

Under the condition $G \ll 1$, RLH waves are typically much shorter than whistler waves. It follows from Eq. (17) that the longitudinal scale length of whistler waves is at least $1/G$. On the other hand, the characteristic longitudinal scale length of RLH waves is $1/\sqrt{|\Delta G|}$ [see Eq. (19)]. For a nonuniformity with $|\Delta G| \sim G$, RLH waves are indeed much shorter than whistler waves. Longitudinal scales of RLH and whistler waves can become comparable only in the case of very weak nonuniformity with $|\Delta G| \leqslant G^2$.

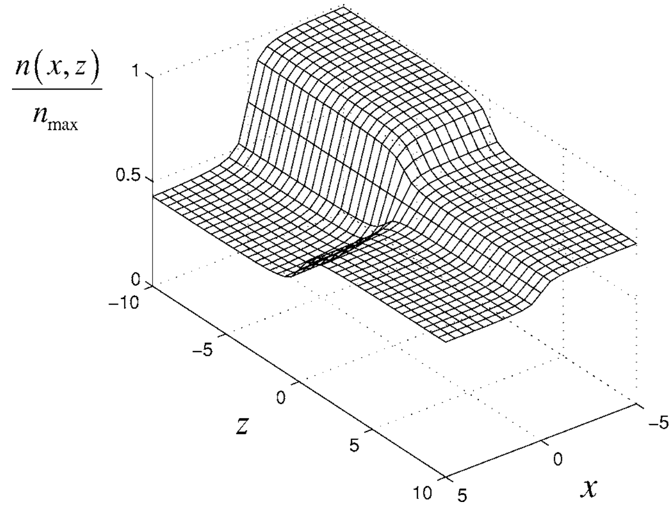

FIG. 1. Normalized plasma density profile with a gradient along the x axis. The density gradient changes in the vicinity of $z=0$. The density is normalized to $n_{\max} \equiv [n(x,z)]_{x\to-\infty;z\to-\infty}$ and coordinates x and z are normalized to $1/k_y$.

The difference in scales implies that RLH waves are only weakly coupled to whistler waves. We can then limit our consideration just to RLH waves and neglect in Eq. (14) the G^2 term that is only important for whistler waves. For the sake of simplicity, we assume that the density changes in the x direction on a scale smaller than the transverse scale of RLH waves. In this case, the transverse density gradient can be treated as a δ function, so that

$$\frac{\partial G}{\partial x} \approx -|\Delta G(z)|\delta(x), \tag{20}$$

where it is explicitly assumed that $\partial n(x,z)/\partial x < 0$. This simplification allows us to construct an x-localized solution in the form

$$E(x,z) = E_0(z)e^{-|x|}, \tag{21}$$

which automatically satisfies Eq. (14), without the G^2 term, for $x>0$ and $x<0$. To find an equation for the amplitude $E_0(z)$, we integrate Eq. (14) over a small vicinity of $x=0$, which yields

$$\frac{\partial^2 E_0}{\partial z^2} = -\frac{|\Delta G(z)|}{2}E_0. \tag{22}$$

If the plasma density is a smooth function of z with a scale length much greater than $1/\sqrt{|\Delta G|}$, then the WKB approximation applies to the incident RLH wave, indicating that the wave propagates along z without much reflection. For shorter plasma density scale lengths, reflection becomes increasingly important.

The reflection is most pronounced when the density has a discontinuity at $z=0$, such that

$$|\Delta G(z)| = \theta(-z)|\Delta G_1| + \theta(z)|\Delta G_2|, \tag{23}$$

where $\theta(z)$ is the step function and $|\Delta G_1|$ and $|\Delta G_2|$ are the values of $|\Delta G|$ at $z<0$ and $z>0$, respectively. The corresponding solution of Eq. (22) is

$$E_0(z) = \begin{cases} E_* e^{iz\sqrt{|\Delta G_1|/2 - |x|}} + E_1 e^{-iz\sqrt{|\Delta G_1|/2-|x|}}, & z<0; \\ E_2 e^{iz\sqrt{|\Delta G_2|/2-|x|}}, & z>0; \end{cases} \tag{24}$$

where E_* is a given amplitude of the incident wave and E_1 and E_2 are the amplitudes of the reflected and transmitted RLH waves that satisfy matching conditions for E_0 and $\partial E_0/\partial z$ at $z=0$. The ensuing reflection coefficient R and transmission coefficient T are

$$R = \frac{|E_1|^2}{|E_*|^2} = \left(\frac{1-\eta}{1+\eta}\right)^2, \tag{25}$$

$$T = 1 - \frac{|E_1|^2}{|E_*|^2} = \frac{4\eta}{(1+\eta)^2}, \tag{26}$$

where $\eta \equiv \sqrt{|\Delta G_2/\Delta G_1|}$. As seen from these expressions, the reflection approaches 100% and the transmitted power goes to zero for $\eta \to 0$.

To be more precise, Eqs. (25) and (26) need to be corrected if the transverse density jump at $z>0$ is so small that $|\Delta G_2| \leqslant G^2$. In this case, the transmitted RLH wave is no longer much shorter than the whistler waves radiated from the vicinity of the $z=0$ point. Although the whistler waves need to be taken into consideration to find the correct reflection and transmission coefficients, the reflection is still going to be very close to 100%, because the longitudinal scales of the *incident* RLH wave and the whistler waves are significantly different.

In order to address the situation with small $|\Delta G_2|$ in more detail, we consider a special case in which $\Delta G_2=0$. Then the plasma at $z>0$ can support only whistler waves and all the transmitted power is attributed to radiation of whistlers. This case involves a technical challenge, because one has to take into account the large-scale component of the wave field both at $z>0$ and $z<0$. The next section presents an analytical treatment of this problem. Our calculations show that the radiated power is indeed smaller than the reflected power and the ratio of the two is of the order of $G^2/|\Delta G_1|$.

IV. RADIATION OF WHISTLERS

In order to examine the transformation of RLH waves into large-scale whistler waves, we retain the G^2 term in Eq. (14) and put $\Delta G_2=0$ in the expression for $\partial G/\partial x$ [see Eqs. (20) and (23)], so that

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 E}{\partial x^2} - E\right) - G^2 E = \theta(-z)\,\delta(x) E |\Delta G_1|. \tag{27}$$

Similarly to the previous section, we are interested in the limiting case of $|\Delta G_1| \gg G^2$. However, we make an additional simplifying assumption that $|\Delta G_1| \ll |G|$, which allows us to treat G^2 as a constant and makes Eq. (27) parity-invariant with respect to x.

Similarly to Eq. (21), the incident RLH solution of Eq. (27) is an even function of x. It is therefore appropriate to limit our consideration to solutions that are symmetric in x. The corresponding Fourier representation of E is

$$E_{\kappa_x}(z) = \int_{-\infty}^{+\infty} E(x,z)\cos(\kappa_x x)\,dx, \tag{28}$$

with E_{κ_x} satisfying the Fourier-transformed Eq. (27),

$$(\kappa_x^2+1)\frac{\partial^2 E_{\kappa_x}}{\partial \zeta^2} + E_{\kappa_x} = -\theta(-\zeta)\lambda^2 \int_{-\infty}^{+\infty} E_{\kappa_x'} d\kappa_x', \tag{29}$$

where

$$\zeta \equiv zG, \tag{30}$$

$$\lambda \equiv \sqrt{\frac{|\Delta G_1|}{2\pi G^2}} \gg 1. \tag{31}$$

In this notation, the field of the incident RLH wave is $E_{\kappa_x}(\zeta) = \psi_{\kappa_x} e^{i\kappa_z \zeta/G}$, where

$$\psi_{\kappa_x} = \frac{1}{\kappa_x^2 + 1 - G^2\kappa_z^2} \approx \frac{1}{\kappa_x^2+1} \tag{32}$$

and the value of κ_z is given by straightforwardly modified Eq. (19),

$$|\Delta G_1| = 2|\kappa_z|\sqrt{\kappa_z^2 - G^2} \approx 2\kappa_z^2. \tag{33}$$

Note that the amplitude of the incident wave is not normalized. The field of the reflected wave can then be written as $E_{\kappa_x}(\zeta) = \alpha\psi_{\kappa_x} e^{-i\kappa_z\zeta/G}$, where α is a complex amplitude that remains to be found. It is important to point out that α is independent of κ_x.

Taking into account the known structure of the RLH waves, we represent E_{κ_x} as a sum of three terms,

$$E_{\kappa_x}(\zeta) = \psi_{\kappa_x}(e^{i\kappa_z\zeta/G} + \alpha e^{-i\kappa_z\zeta/G})\,\theta(-\zeta) + F_{\kappa_x}(\zeta). \tag{34}$$

The last term describes the radiated whistler waves that accompany reflection of the incident RLH wave from the density discontinuity at $\zeta=0$. The problem now reduces to finding $F_{\kappa_x}(\zeta)$ and α from Eq. (29).

The field of the radiated whistler wave at $\zeta>0$, where the right-hand side of Eq. (29) vanishes, can easily be expressed in terms of the total field at $\zeta=0$,

$$F_{\kappa_x}(\zeta) = E_{\kappa_x}(0)e^{is\zeta}, \tag{35}$$

where

$$s \equiv 1/\sqrt{\kappa_x^2+1}, \tag{36}$$

and s is positive, because there are no whistler waves coming from $\zeta=+\infty$.

For $\zeta<0$, Eqs. (29) and (34) yield

$$\frac{\partial^2 F_{\kappa_x}}{\partial \zeta^2} + s^2 F_{\kappa_x} = \frac{Q(\zeta)}{\kappa_x^2+1}, \tag{37}$$

where $Q(\zeta)$ is a functional of $F_{\kappa_x}(\zeta)$,

$$Q(\zeta) \equiv -\lambda^2 \int_{-\infty}^{+\infty} F_{\kappa_x'}(\zeta) d\kappa_x'. \tag{38}$$

Equation (37) describes radiation of whistler waves by a self-consistently determined source $Q(\zeta)$. This equation gives the following expression for $F_{\kappa_x}(\zeta)$ in terms of $Q(\zeta)$:

$$F_{\kappa_x}(\zeta) = A_{\kappa_x} e^{is\zeta} + B_{\kappa_x} e^{-is\zeta} - \frac{i}{2s}\left[\int_{-\infty}^{z} \frac{Q(\xi)}{\kappa_x^2+1} e^{is(\zeta-\xi)} d\xi + \int_{z}^{0} \frac{Q(\xi)}{\kappa_x^2+1} e^{is(\xi-\zeta)} d\xi\right], \tag{39}$$

where A_{κ_x} and B_{κ_x} are functions of κ_x only. The continuity conditions for E_{κ_x} and $\partial E_{\kappa_x}/\partial\zeta$ at $\zeta=0$ yield two equations for A_{κ_x} and B_{κ_x},

$$\psi_{\kappa_x}(1+\alpha) + A_{\kappa_x} + B_{\kappa_x} - \frac{i}{2s}\int_{-\infty}^{0} \frac{Q(\xi)}{\kappa_x^2+1} e^{-is\xi} d\xi = E_{\kappa_x}(0), \tag{40}$$

$$\frac{\kappa_z}{G}\psi_{\kappa_x}(1-\alpha) + s(A_{\kappa_x} - B_{\kappa_x}) - \frac{i}{2}\int_{-\infty}^{0} \frac{Q(\xi)}{\kappa_x^2+1} e^{-is\xi} d\xi = sE_{\kappa_x}(0). \tag{41}$$

We now eliminate A_{κ_x} and B_{κ_x} from Eq. (39), which gives

$$F_{\kappa_x}(\zeta) = E_{\kappa_x}(0) e^{is\zeta} - \psi_{\kappa_x}\left[(1+\alpha)\cos(s\zeta) + \frac{i\kappa_z}{sG}(1-\alpha)\sin(s\zeta)\right] + \int_{z}^{0} Q(\xi)\frac{\sin(s[\xi-\zeta])}{s(\kappa_x^2+1)} d\xi. \tag{42}$$

There are no whistler waves coming from $\zeta=-\infty$ and, therefore, all terms proportional to $e^{is\zeta}$ on the right-hand side of Eq. (42) must vanish as $\zeta\to-\infty$, which determines $E_{\kappa_x}(0)$,

$$E_{\kappa_x}(0) = \frac{\psi_{\kappa_x}}{2}\left[(1+\alpha) + \frac{\kappa_z}{sG}(1-\alpha)\right] - \frac{i}{2}\int_{-\infty}^{0} \frac{Q(\xi)e^{-is\xi}}{s(\kappa_x^2+1)} d\xi. \tag{43}$$

In order to find α, we note that Eq. (43) must give a finite value of $\int_{-\infty}^{+\infty} E_{\kappa_x}(0) d\kappa_x$. On the other hand, it follows from Eq. (43) that $E_{\kappa_x}(0)$ is proportional to $1/|\kappa_x|$ for large values of $|\kappa_x|$,

$$E_{\kappa_x}(0) \approx \frac{\kappa_z}{2|\kappa_x|G}\left[(1-\alpha) - \frac{iG}{\kappa_z}\int_{-\infty}^{0} Q(\xi) d\xi\right]. \tag{44}$$

For the integral $\int_{-\infty}^{+\infty} E_{\kappa_x}(0) d\kappa_x$ to converge, the square-bracket term in Eq. (44) must vanish. This requirement yields

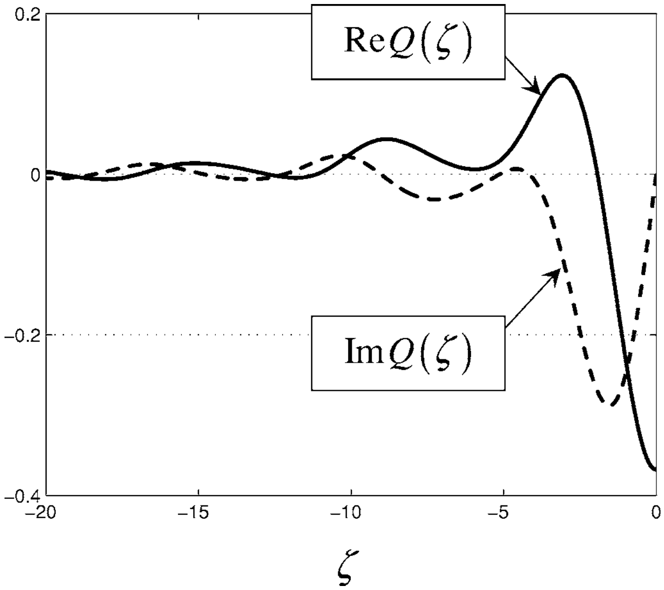

FIG. 2. Real and imaginary parts of the solution of Eq. (47) in the limiting case of $\lambda \gg 1$.

$$\alpha = 1 - \frac{iG}{\kappa_z}\int_{-\infty}^{0} Q(\xi) d\xi \approx 1 - \frac{1}{\lambda}\left(\frac{i}{\sqrt{\pi}}\int_{-\infty}^{0} Q(\xi) d\xi\right). \tag{45}$$

Next, we use Eqs. (43) and (45) to eliminate $E_{\kappa_x}(0)$ and α from Eq. (42),

$$F_{\kappa_x}(\zeta) = -\psi_{\kappa_x} e^{-is\zeta} + \frac{i}{2}\left(\frac{G}{\kappa_z} + \frac{1}{s}\right)\psi_{\kappa_x} e^{-is\zeta} \times \int_{-\infty}^{0} Q(\xi) d\xi - \frac{i}{2}\int_{\zeta}^{0} \frac{Q(\xi)e^{is(\xi-\zeta)}}{s(\kappa_x^2+1)} d\xi - \frac{i}{2}\int_{-\infty}^{\zeta} \frac{Q(\xi)e^{is(\zeta-\xi)}}{s(\kappa_x^2+1)} d\xi. \tag{46}$$

Finally, we integrate Eq. (46) over κ_x to obtain an integral equation for $Q(\zeta)$,

$$\frac{Q(\zeta)}{\lambda^2} = \int_{-\infty}^{+\infty} \frac{e^{-is\zeta} d\kappa_x}{\kappa_x^2+1} - \frac{i}{2}\left[\int_{-\infty}^{+\infty} \frac{e^{-is\zeta} d\kappa_x}{\sqrt{\kappa_x^2+1}}\right] \times \int_{-\infty}^{0} Q(\xi) d\xi + \frac{i}{2}\int_{-\infty}^{+\infty} d\kappa_x \int_{\zeta}^{0} \frac{Q(\xi)e^{is(\xi-\zeta)}}{\sqrt{\kappa_x^2+1}} d\xi + \frac{i}{2}\int_{-\infty}^{+\infty} d\kappa_x \int_{-\infty}^{\zeta} \frac{Q(\xi)e^{is(\zeta-\xi)}}{\sqrt{\kappa_x^2+1}} d\xi. \tag{47}$$

In deriving Eq. (47), we have taken into account that $\kappa_z \gg sG$, so that $\psi_k \approx 1/(\kappa_x^2+1)$. After $Q(\zeta)$ is found from Eq. (47), one can readily calculate the amplitude of the reflected wave from Eq. (45).

Being interested in the limiting case of $\lambda \gg 1$, we neglect the left-hand side in Eq. (47). The resulting equation has a universal form, as it does not contain λ. It can be solved iteratively, following the procedure described in the Appendix. The corresponding solution of Eq. (47) is shown in Fig. 2. It yields the following value of α [see Eq. (45)]:

$$\alpha \approx 1 - \frac{0.377}{\lambda}. \tag{48}$$

The ensuing reflection and transmission coefficients are

$$R = |\alpha|^2 \approx 1 - \frac{0.754}{\lambda}, \tag{49}$$

$$T = 1 - R \approx \frac{0.754}{\lambda}. \tag{50}$$

V. SUMMARY

We have considered the propagation of waves with frequencies in the helicon range ($\omega_{ce} \gg \omega \gg \omega_{ci}$) through a magnetized nonuniform plasma, where the equilibrium magnetic field is directed along the z axis and the equilibrium plasma density is a function of x and z only. The waves under consideration have a finite y component of the wave vector (k_y) and their characteristic longitudinal scale is much longer than k_y^{-1}, whereas the characteristic transverse scales are comparable to k_y^{-1}.

There are two distinct types of waves in the considered frequency range: RLH and whistler waves. In the case of a longitudinally uniform plasma with a density gradient across the magnetic field, the wave equation takes the form of a one-dimensional Schrödinger's equation [Eq. (15)]. We have shown that RLH waves represent discrete bound states, whereas whistler waves belong to the continuous spectrum.

In Secs. III and IV we used a slab configuration to analyze the reflection of an RLH wave traveling in the z direction in a longitudinally nonuniform plasma. The plasma density is independent of z at $z \to +\infty$, so that reflection takes place within a finite interval around $z=0$, where the plasma density is both x- and z-dependent. In general, the reflection of RLH waves is accompanied by the radiation of whistlers.

The wave field structure is decided by the profile of a single dimensionless function $G \equiv (\omega/\omega_{ce})(\omega_{pe}^2/k_y^2 c^2)$ that is proportional to the plasma density [Eq. (9)]. This function is a small quantity for the waves under consideration ($G \ll 1$) and, as a result, the RLH and whistler waves have different longitudinal scale lengths and are only weakly coupled. In the case of $|\Delta G| \sim G$, where ΔG corresponds to the density drop across the magnetic field, RLH waves are much shorter than whistler waves.

Taking advantage of weak coupling between RLH waves and whistlers, we first address the problem of RLH reflection without radiation of whistlers. The reflection off the longitudinal nonuniformity is most pronounced when its scale length is much shorter than the longitudinal scale length of the incident wave. The corresponding reflection coefficient is

$$R = \left(\frac{1-\eta}{1+\eta} \right)^2, \tag{51}$$

where $\eta \equiv \sqrt{|\Delta G_2 / \Delta G_1|}$, and $|\Delta G_1|$ and $|\Delta G_2|$ correspond to the density drop across the magnetic field at $z<0$ and $z>0$, respectively. The reflection approaches 100% for $\eta \to 0$. Equation (51) is valid only if the transverse density jump at $z>0$ is not very small ($|\Delta G_2| > G^2$).

Radiated whistler waves must be taken into account in the case of a very weak nonuniformity ($|\Delta G_2| \leqslant G^2$), when the longitudinal scale length of the radiated whistlers becomes comparable to that of the transmitted RLH wave. In order to address the situation with small ΔG_2 in more detail, we have considered a special case in which $\Delta G_2 = 0$. In this case, the plasma at $z>0$ can support only whistler waves and all the transmitted power is attributed to the radiation of whistlers. The ensuing reflection coefficient [Eq. (49)] is close to unity, because the longitudinal scales of the *incident* RLH wave and the radiated whistlers are significantly different.

Finally, it must be pointed out that the smallness of the dimensionless function G implies that the waves under consideration are elongated, such that the characteristic scale length of the wave field along the magnetic-field lines is much bigger than the characteristic scale length across the field lines. Indeed, it follows from the dispersion relation for the RLH waves [see Eq. (19)] that the condition $G \ll 1$ translates into $\kappa_z \ll 1$, where κ_z is the longitudinal component of the wave vector normalized to k_y. The extent of the RLH wave elongation determines the coupling between the RLH and the whistler waves. For example, the fraction of the incident RLH wave energy that goes into the radiated whistlers is roughly equal to κ_z for a nonuniformity with $|\Delta G| \sim G$ [see Eqs. (31) and (50)]. Therefore, coupling between RLH waves and whistlers should be stronger in compact plasma sources with an order of unity aspect ratio.

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APPENDIX: ITERATIVE PROCEDURE FOR SOLVING EQ. (47)

Being interested in the limiting case of $\lambda \gg 1$, we neglect the left-hand side in Eq. (47). To improve the convergence of integrals in Eq. (47), we differentiate Eq. (47) twice with respect to ζ. The resulting equation has a universal form, as it does not contain λ,

$$\begin{aligned} -\int_{-\infty}^{+\infty} \frac{e^{-is\zeta} d\kappa_x}{(\kappa_x^2+1)^2} &= \pi Q(\zeta) - \frac{i}{2}\left[\int_{-\infty}^{+\infty} \frac{e^{-is\zeta} d\kappa_x}{(\kappa_x^2+1)^{3/2}} \right] \\ &\quad \times \int_{-\infty}^{0} Q(\xi) d\xi + \frac{i}{2} \int_{-\infty}^{+\infty} d\kappa_x \int_{\zeta}^{0} \frac{Q(\xi) e^{is(\xi-\zeta)}}{(\kappa_x^2+1)^{3/2}} d\xi \\ &\quad + \frac{i}{2} \int_{-\infty}^{+\infty} d\kappa_x \int_{-\infty}^{\zeta} \frac{Q(\xi) e^{is(\zeta-\xi)}}{(\kappa_x^2+1)^{3/2}} d\xi. \end{aligned} \tag{A1}$$

It is convenient to eliminate oscillations in $Q(\zeta)$ by introducing a new function,

$$W(\zeta) \equiv Q(\zeta)\mathrm{e}^{i\zeta}. \tag{A2}$$

Equation (A1) transforms to

$$-\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-i(s-1)\zeta} d\kappa_x}{(\kappa_x^2+1)^2} = \pi W(\zeta) - R(\zeta)\int_{-\infty}^{0} W(\xi)\mathrm{e}^{-i\xi} d\xi + \int_{\zeta}^{0} R(\zeta-\xi)W(\xi)d\xi + \int_{-\infty}^{\zeta} A(\xi-\zeta)W(\xi)d\xi, \tag{A3}$$

where

$$R(\zeta) \equiv \frac{i}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-i(s-1)\zeta} d\kappa_x}{(\kappa_x^2+1)^{3/2}}, \tag{A4}$$

$$A(\zeta) \equiv \frac{i}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-i(s+1)\zeta} d\kappa_x}{(\kappa_x^2+1)^{3/2}}. \tag{A5}$$

We solve Eq. (A3) numerically, using consecutive iterations. Our first step is to find $W_0(\zeta)$ satisfying the equation

$$-\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-i(s-1)\zeta} d\kappa_x}{(\kappa_x^2+1)^2} = \pi W_0(\zeta) + \int_{\zeta}^{0} R(\zeta-\xi)W_0(\xi)d\xi. \tag{A6}$$

We then use $W_0(\zeta)$ to compute the terms in Eq. (A3) that did not enter Eq. (A6). Our next step, which is the first iteration, is to find $W_1(\zeta)$ satisfying the equation

$$-\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-i(s-1)\zeta} d\kappa_x}{(\kappa_x^2+1)^2} = \pi W_1(\zeta) - R(\zeta)\int_{-\infty}^{0} W_0(\xi)\mathrm{e}^{-i\xi} d\xi + \int_{\zeta}^{0} R(\zeta-\xi)W_1(\xi)d\xi + \int_{-\infty}^{\zeta} A(\xi-\zeta)W_0(\xi)d\xi. \tag{A7}$$

After that, we replace $W_0(\zeta)$ by $W_1(\zeta)$ and $W_1(\zeta)$ by $W_2(\zeta)$. This gives us an equation for our second iteration that we solve to find $W_2(\zeta)$. We continue this procedure until we achieve desirable precision.

Since $|W(\zeta)| \to 0$ as $\zeta \to -\infty$, we can truncate the interval on which the equation is solved,

$$\zeta \in [\zeta_*, 0]. \tag{A8}$$

We take $\zeta_* = -(N-1)\delta = -80$, where $\delta = 0.02$ is the step size and $N = 4001$ is the number of grid points on a uniform grid that we use. We compute α after each iteration and we determine the convergence by tracking the value of α. It takes 15 iterations for the relative change in α to become smaller than 1%. The solution of Eq. (47) after 15 iterations is shown in Fig. 2.

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