Adiabatic bistable evolution of dynamical systems governed by a Hamiltonian with separatrix crossing

Oleg Polomarov and Gennady Shvets
Department of Physics and Institute of Fusion Studies, The University of Texas at Austin, Austin, Texas 78712

(Received 22 December 2005; accepted 29 March 2006; published online 19 May 2006)

Adiabatic evolution of a nonlinear resonantly driven dynamical system generic to a variety of plasma physics problems is studied. The corresponding Hamiltonian, depending on the strength and frequency of the slowly varying driver, has a variable number of fixed points and the dynamical system can be bistable due to repeated separatrix crossing in the phase space. It is analytically shown that the oscillation periods along the "sister" trajectories corresponding to the same value of the Hamiltonian are equal and the sum of the corresponding areas under them does not depend on the driver amplitude. As a consequence of that, the Hamiltonian of a bistable system always follows the same trajectory for an adiabatically varying driver, regardless of whether the system is excited or left quiescent. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2201927]

Resonant excitation of nonlinear dynamical systems is one of the most common uniting threads throughout plasma science. Best known examples include electron cyclotron resonance heating (ECRH) by radio frequency waves[1–5] and beat-wave excitation of electron plasma waves[6–9] in a plasma beatwave accelerator (PBWA). Numerous examples also exist outside of plasma physics. For example, the celestial dynamics of a system of planetary satellites[10,11] can be reduced to the generic problem of a nonlinear harmonically driven pendulum. All these vastly different nonlinear systems can be described by a generic Hamiltonian (sometimes[11] referred to as the second fundamental model for resonance):

$$H(I,\theta,\tau) = b(\tau)I - c(\tau)I^2 - a(\tau)\sqrt{2I}\sin\theta, \tag{1}$$

where I and θ are canonical conjugate variables, and b, c, and a are slowly varying parameters of the system and driver. The physical meaning of these variables can be clarified using near-resonant excitation of relativistic plasma waves by a laser beatwave pulse[7] as an example. In the context of a PBWA, θ is the relative phase between the excited plasma wave and the beatwave driver, I is proportional to the energy of the plasma wave, $a(\tau)$ is proportional to the slowly varying envelope of the laser beatwave, $b=1-\Delta\omega/\omega$ is the normalized to the electron plasma frequency ω_p detuning from the plasma resonance, and $c=3/16$ is the relativistic nonlinearity.

Earlier investigations of the Hamiltonian H in Refs. 3, 11, and 12 have revealed an important property of the phase space associated with it: depending on the values of a, b, and c, there are either one (elliptic) fixed point, or three (two elliptic, one hyperbolic) fixed points. The disappearance or reappearance of fixed points changes the topology of the phase space. This becomes important for dynamical systems described by Eq. (1) with slowly varying coefficients. For example, the nonlinear oscillator can be excited by a driver with the slowly varying amplitude and frequency. Two approaches to adiabatic excitation of nonlinear oscillations have been theoretically studied. One is the so-called autoresonant excitation: by slowly varying (or "chirping") the *driver frequency*, the relative frequency detuning b from the natural frequency of the small-amplitude oscillations can be swept across the exact resonance $b=0$. Such an approach was shown highly effective for the autoresonant excitation of the plasma waves[13,14] by a chirped laser beatwave and for the ECRH of plasma[4,5] in a nonuniform magnetic field. Autoresonant excitation does not rely on the topological change of the phase portrait. The second (bistability) approach, realized when the *driver amplitude* adiabatically slow evolves from zero to a certain above-threshold value a_{cr}, and then back to zero, does rely on the topological change of the phase space at $a=a_{\mathrm{cr}}$. When the phase space topology changes from the one with a single elliptic point to that with two elliptic points, the system must cross the separatrix and "choose" (depending on its position in the phase space at the time of the crossing) to be in one of the two accessible regions of the phase space. The bistability approach has been used for ECRH in a uniform magnetic field[1,2] and for the beatwave excitation of plasma waves based on relativistic bistability.[7,12]

Despite the generic nature of the Hamiltonian H and its ubiquity, some of its most important properties have not yet been uncovered. This work fills this gap by demonstrating two interesting analytic properties of the Hamiltonian. First, it is shown that in the adiabatic limit, when the bistability excitation mechanism is used, the value of H at a given amplitude a does not depend on the outcome of the separatrix crossing. The adiabatic limit here implies a vanishingly small rate of change of the driver, when the nonadiabatic effects due to separatrix crossing can be neglected.[16,17] Specifically, for driver amplitudes $a<a_{cr}$, there are two "sister" trajectories corresponding to the same value $H(I,\theta)\equiv h(a)$ and separated by the separatrix. Repeated separatrix crossings, due to varying driver $a(\tau)$, can result in switching from one "sister" trajectory to another, or in staying on the same "sister" trajectory. Nevertheless, in either case, the value of the Hamil-

tonian is the same and determined only by the driver strength for given initial condition. This important property was noted without proof[7] in the context of the excitation of a large amplitude relativistic plasma wave by a laser-driven beat-wave. Except being fundamentally interesting, this property can be of great help for a practical calculation. For example, the magnitude of the plasma wake behind the laser pulse ($\tau_w \to +\infty$, $a(\tau_w)=0$) can be easily calculated in an adiabatic limit by solving a simple algebraic equation $H[I,\theta,a(\tau_w)]=h_{\text{ini}ni}$, where $h_{\text{ini}ni}$ the initial value of Hamiltonian for $\tau_{\text{ini}}=-\infty$ (for initially quiescent plasma $h_{\text{ini}}=0$). This property can be similarly applied to the case of adiabatic ECRH of the plasma. Second, it is shown that the periods of motion along the two "sister" trajectories are identically equal.

We start by reviewing the basic properties of the phase portrait of the dynamical system governed by the Hamiltonian H with the constant coefficients a,b, and $c>0$. A particle representing the dynamical system moves according to the Hamilton's equations:

$$dI/d\tau = -\partial H/\partial\theta, \quad d\theta/d\tau = \partial H/\partial I. \tag{2}$$

The phase space trajectories in the I,θ plane are the constant energy contours $H(I,\theta)=h$. For $b>0$ and $0<a<a_{\text{cr}}$ there are three fixed points $(dI/d\tau=0, d\theta/d\tau=0)$: two stable elliptic points (classified according to the value of I as upper and lower) and one unstable hyperbolic point. With a replacement $I=u^2/2$, the hyperbolic u_h and the lower elliptic u_l points are given by $\theta=\pi/2$ and, correspondingly, the largest and smallest non-negative roots of the cubic equation $bu-cu^3-a\sin\pi/2=0$. For the upper elliptic point $\theta=3\pi/2$ and u_{up} is the non-negative root of $bu-cu^3-a\sin 3\pi/2=0$. The critical driver amplitude a_{cr} corresponds to merger and annihilation of the hyperbolic and lower elliptic points, and is given by $a_{\text{cr}}=2/3\sqrt{b^3/3c}$. A typical phase portrait for $c=3/16$, $b=0.05$, and $a=0.0085<a_{\text{cr}}\approx 0.01$ is shown in Fig. 1. Phase space trajectories corresponding to $H(I,\theta)=h$ are labeled according to the value of h. Because $b>0$ and $a<a_{\text{cr}}$, there are two elliptic and one hyperbolic fixed point in Fig. 1. Trajectory "4" (lower) belongs to the basin of the lower elliptic point while trajectories "1–3" are in the basin of the upper elliptic point. Trajectories "4" (upper) and "5" are the open trajectories encompassing both elliptic points. For certain values of h in the $H(u_l,\pi/2,a)<h<H(u_h,\pi/2,a)$ range there are two "sister" trajectories satisfying $H(I,\theta,a)=h$ exemplified by the trajectories "4." One (upper) trajectory is open and the other (lower) trajectory oscillates around the lower elliptic point u_l. The separatrix passing through the hyperbolic point and satisfying $H(u,\theta)=H(u_h,\pi/2)$ consists of two branches. The lower branch separates the basins of the lower and upper elliptic points. The upper branch separates the basin of the upper elliptic point from the region of open trajectories, or the region where trajectories rotate around all fixed points. Separatrix disappears for $a\geqslant a_{\text{cr}}$.

Having reviewed the relevant features of the phase portraits of the dynamical system described by the stationary $H(I,\theta)$, we prove a rather remarkable property of the "sister" trajectories: their oscillation periods are identically equal,

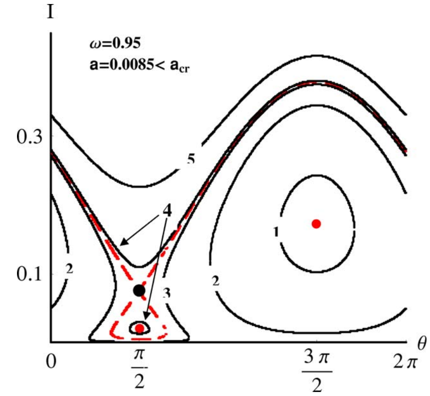

FIG. 1. (Color online) Constant energy contours (solid lines) $H(I,\theta)=h_i$ labeled by their index i: "1"—$h_1=7\times 10^{-3}$, "2"—$h_2=2\times 10^{-3}$, "3"—$h_3=-4\times 10^{-4}$, "4"—$h_4=-7.5\times 10^{-4}$, and "5"—$h_5=-5\times 10^{-3}$. Red dashed line: separatrix; large black dot: the hyperbolic point; smaller red dots: upper and lower elliptic points. Note that there are two "sister" trajectories corresponding to h_4. Parameters of the Hamiltonian: $a=0.0085$, $b=0.05$, and $c=3/16$.

$$T_1(h) = T_2(h), \tag{3}$$

where 1 and 2 label, respectively, the lower and the upper trajectories. Equation (3) hold for any values of a, b, and c for which "sister" trajectories exist. Equality (3) is proven by integration of the Eq. (4), obtained by expressing $a\cos\theta$ from $H(I,\theta,a)=h$ and substituting it into Eq. (2):

$$dI/dt = \pm\sqrt{2Ia^2 - (h - bI + cI^2)^2}. \tag{4}$$

To simplify Eq. (4) we introduce the turning points $I_1<I_2<I_3<I_4$ such that $I_1<I<I_2$ for the lower trajectory and $I_3<I<I_4$ for the upper trajectory. I_1, I_2, I_3, and I_4 are found as roots of the quatric equation

$$0 = 2Ia^2 - (h - bI + cI^2)^2 \equiv c^2(I_4 - I)(I_3 - I)(I_2 - I)(I - I_1), \tag{5}$$

and the oscillation periods are found as

$$T_1 = \frac{2}{c}\int_{I_1}^{I_2} \frac{dI}{\sqrt{(I_4 - I)(I_3 - I)(I_2 - I)(I - I_1)}}, \tag{6}$$

for the lower trajectory, and

$$T_2 = \frac{2}{c}\int_{I_3}^{I_4} \frac{dI}{\sqrt{(I_4 - I)(I_3 - I)(I_2 - I)(I - I_1)}}, \tag{7}$$

for the upper trajectory. Simple algebraic manipulations[18] result in

$$T_1 = T_2 \equiv \frac{4}{c\sqrt{(I_4 - I_2)(I_3 - I_1)}}\mathbf{K}\left(\frac{(I_4 - I_3)(I_2 - I_1)}{(I_4 - I_2)(I_3 - I_1)}\right), \tag{8}$$

where $\mathbf{K}[m]$ is the complete elliptic integral[18] of the first kind. To our knowledge, this simple property of the equal energy ("sister") trajectories has never been shown. Later we use Eq. (3) to prove an important property of the Hamiltonian H with slowly varying driver amplitude $a(\tau)$.

When the Hamiltonian evolves adiabatically in time, any given phase space trajectory is deformed so as to preserve its adiabatic invariant,[15]

$$J=\frac{2}{2\pi}\int_{I_{\min}}^{I_{\max}} I\frac{d\theta}{dI}dI=\frac{1}{2\pi}\int_{I_{\min}}^{I_{\max}}\frac{h+I(b-3cI)}{\sqrt{2a^2I-[h+I(-b+cI)]^2}}, \tag{9}$$

where $I_{\min}$ and $I_{\max}$ are the minimal and maximal values of I for the particle trajectory, and the $J=\frac{1}{2\pi}\oint I\,d\theta$ integral is taken in the direction of the phase flow. The two "sister" trajectories change so as to remain the solutions of the $H(u,\theta,\tau)=h_{1,2}(\tau)$ and also to conserve the corresponding adiabatic invariants $J_{1,2}=\frac{1}{2\pi}\oint I\,d\theta$. For fixed b and c actions, $J_{1,2}\equiv J_{1,2}(h,a)$ are functions of slowly evolving driver amplitude a. Remarkably, $h_1(\tau)=h_2(\tau)$, as shown later.

If the system undergoes separatrix crossing at $\tau=\tau_s$, then at that moment the lower and upper "sister" trajectories coincide with the lower and upper branches of the separatrix and become a constant energy contour, $H[I_s(\theta),\theta,\tau_s]=h_s$. Actions $J_{1,2}(\tau_s)$ can be easily computed, noting that the separatrix passes through the hyperbolic point $(I_s,\theta_s=\pi/2)$ and expressing the Hamiltonian h_s and driver a_s through I_s as $a_s=\sqrt{2I_s}(b-2cI_s)$ and $h_s=bI_s-cI_s^2-a_s\sqrt{2I_s}$:

$$J_1=-\frac{1}{2\pi}\int_{I_1}^{I_s}\frac{b/c-3(I+I_s)}{\sqrt{(I_2-I)(I-I_1)}}dI \tag{10}$$

for the lower separatrix branch and

$$J_2=\frac{1}{2\pi}\int_{I_s}^{I_2}\frac{b/c-3(I+I_s)}{\sqrt{(I_2-I)(I-I_1)}}dI \tag{11}$$

for the upper branch of the separatrix. Here $I_2=b/c-I_s+2\sqrt{I_s(b-2cI_s)/c}$ and $I_1=b/c-I_s-2\sqrt{I_s(b-2cI_s)/c}$ are the highest and lowest points on the separatrix, correspondingly. Equations (10) and (11) can be simplified[18] to

$$J_1^{(s)}=\frac{b}{2c}-\frac{3}{2c\pi}\sqrt{(b-2cI_s)(6cI_s-b)}-\frac{b}{2c\pi}\sin^{-1}\left(\frac{1}{2cI_s}\sqrt{(b-2cI_s)(6cI_s-b)}\right), \tag{12}$$

and

$$J_2^{(s)}=-\frac{b}{2c}-\frac{3}{2c\pi}\sqrt{(b-2cI_s)(6cI_s-b)}-\frac{b}{2c\pi}\sin^{-1}\left(\frac{1}{2cI_s}\sqrt{(b-2cI_s)(6cI_s-b)}\right). \tag{13}$$

From Eqs. (13) and (14), it follows that

$$\Delta J(h_s(a),a)\equiv J_1[h_s(a),a]-J_2[h_s(a),a]=b/c. \tag{14}$$

Because $J_1^{(s)}>0$ and $J_2^{(s)}<0$, the geometric interpretation of Eq. (14) is that the sum of the areas enclosed by the lower and upper branches of the separatrix is equal to $2\pi b/c$ (3 and 10). Note, that for fixed b and c, separatrix actions $J_{1,2}^{(s)}\equiv J_{1,2}^{(s)}(a)$ and $h_s(a)$ are functions of the slowly evolving driver amplitude a through $I_s(a)$.

Using Eq. (3) of the equality of oscillation periods along the "sister" trajectories and the relationship between the adiabatic invariant and the oscillation period, $T_{1,2}(h,a)=2\pi\,\partial J_{1,2}/\partial h$ (Ref. 15), we obtain $\Delta J(h,a)\equiv J_1(h,a)-J_2(h,a)=C(a)$ for all values of h, where $C(a)$ is a function of a. Using Eq. (14) as a "boundary condition," and the fact that $h_s(a)$ is a properly defined function for all $a<a_{\rm cr}$, we obtain $C(a)\equiv b/c$. The above proof and notations implied that a is a variable and b,c are constants. Clearly, the proof can be generalized to express an important property of the adiabatic invariants of the "sister" trajectories valid for any energy h and any set of parameters a,b,c for which those trajectories exist:

$$J_1(h,a,b,c)-J_2(h,a,b,c)=b/c. \tag{15}$$

Equation (15) is an important generalization of Eq. (14) and is employed later to demonstrate that when the driver amplitude $a(\tau)$ varies adiabatically with time, and the dynamical system crosses the separatrix, the system always ends up on one of the two "sister" trajectories. The implication of this is that the energy of the system is determined only by the instantaneous value of a and its initial adiabatic invariant J_0, but not by the particular path taken by the system. The specific path determines on which of the two "sister" trajectories the system ends up and, therefore, the final value of its adiabatic invariant. It does not affect the value of the Hamiltonian, $h\equiv h(a,J_0)$. To prove this statement, note that the adiabatic change of a implies that the values of the adiabatic invariants of the two "sister" trajectories $H(I,\theta,\tau)=h$ do not change. From $J(h,a)_{1,2}=\text{Const}_{1,2}$, it follows that

$$\frac{dh_{1,2}}{da}=-\frac{\partial J_{1,2}}{\partial a}\Bigg/\frac{\partial J_{1,2}}{\partial h}. \tag{16}$$

From Eq. (15), it immediately follows that $\partial J_1/\partial h=\partial J_2/\partial h$ and $\partial J_1/\partial a=\partial J_2/\partial a$ because the rhs of Eq. (15) is independent of h and a. Finally, combining these equalities with Eq. (16) implies that the rates of change of the Hamiltonian with varying driver amplitude a are the same for both "sister" trajectories:

$$dh_1/da=dh_2/da. \tag{17}$$

Therefore, if at some time τ two particles occupy "sister" trajectories, they will remain on the equal-energy ("sister") trajectory for as long as $a(\tau)$ is changing adiabatically.

To be specific, assume that the system initially resides on the lower "sister" trajectory. Consider a driver amplitude increasing past the certain value a_s for which the system crosses the separatrix *into* the basin of the upper elliptic point. When the driver amplitude reaches the maximum and decreases again to $a=a_s$, the system crosses the separatrix again *from* the basin of the upper elliptic point. It can now end up either on its initial (lower) trajectory, or on its "sister" upper trajectory. From that point on, the system remains on the chosen trajectory for $a<a_s$. Neglecting small nonconservation of the adiabatic invariant during the separatrix crossing,[16,17,19] we conclude from Eq. (17) that the Hamiltonian of the system remains the same regardless of whether it returns back to its initial trajectory, or is excited into the "sister" trajectory.

To confirm these analytic predictions, we have numerically integrated Eq. (2) with the variable driver amplitude $a(\tau)=a_m\exp(-\tau^2/\tau_L^2)$, where $a_m=0.03>a_s$ and $\tau_L=5\times10^5$.

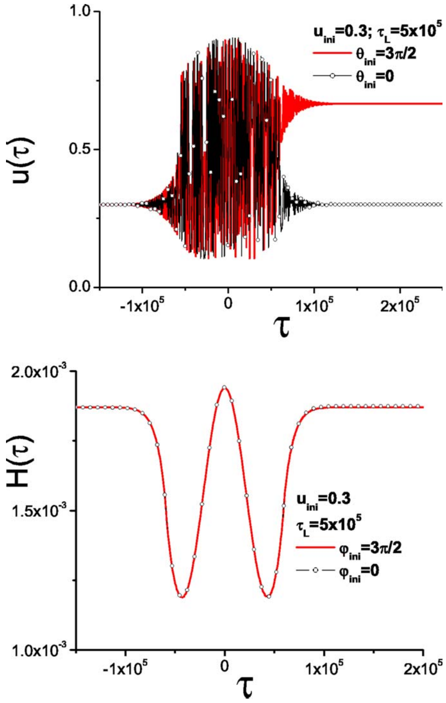

FIG. 2. (Color online) The slow amplitude $u(\tau)$ and the Hamiltonian $H(\tau)$ along the trajectories $u(\tau),\theta(\tau)$ with different initial conditions for adiabatically varying diver $a(\tau)$.

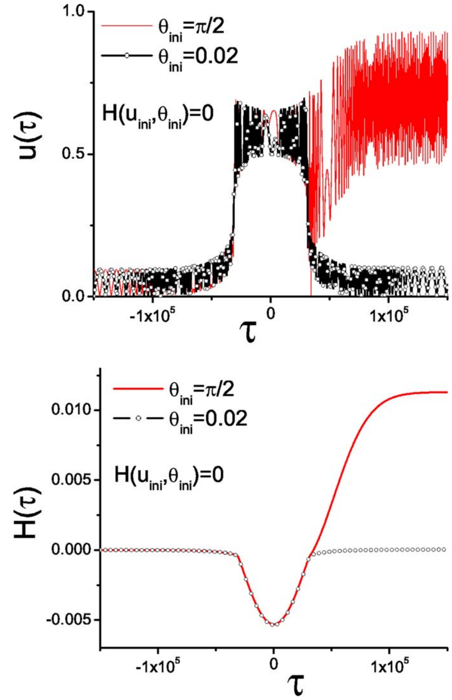

FIG. 3. (Color online) The slow amplitude $u(\tau)$ and the Hamiltonian $H(\tau)$ along the system trajectory $u(\tau),\theta(\tau)$ for adiabatically varying detuning $b(\tau)$.

In Fig. 2 the slow amplitudes $u(\tau)$ and the Hamiltonians $h(\tau)$ corresponding to the initial amplitude $u_{\text{ini}ni}=0.3$ and two initial phases $\theta=0$ and $\theta=3\pi/2$, are plotted. Clearly, the Hamiltonian evolves in the same way along the trajectories that correspond to the same initial value $h_{\text{ini}}=h(u_{\text{ini}})$, even if these trajectories diverge after a bifurcation: the trajectory with $\theta_{\text{ini}ni}=3\pi/2$ acquires the final amplitude much larger than initial $u(\tau=+\infty)>u_{\text{ini}}$ and the trajectory with $\theta_{\text{ini}}=0$ returns to its initial value $u(\tau=+\infty)=u_{\text{ini}}$. For nonvanishing $1/\tau_L$, the difference between the Hamiltonians corresponding to these two initial phases is due[16,17] to the finite rate of the driver evolution. The adiabatic approach is applicable for $T_{1,2}d|\ln a|/d\tau \ll 1$. An order of magnitude estimate results in $\tau_L \gg 2\pi/b$.

Note that the property of our Hamiltonian to remain the same after a bifurcation is a particular property of our system for the case of adiabatically varying driver $a(\tau)$. Consider a slow varying detuning $b(\tau)=0.1-0.1\exp(-\tau^2/\tau_L^2)$, with $\tau_L=5\times 10^5$ and constant driver amplitude $a=0.02$. It is seen from Fig. 3, where the slow amplitudes $u(\tau)$ and the Hamiltonians $h(\tau)$ for the same h_{ini} but two different θ_{ini} are plotted, that the values of the Hamiltonian after the bifurcation do not coincide. This is because the rhs of Eq. (15) depends on b and the equation similar to Eq. (16) yields $dh_1/db \neq dh_2/db$.

In conclusion, a specific nonlinear Hamiltonian, exhibiting separatrix crossing, is analyzed. The Hamiltonian has a variable number of fixed points and can have the "sister" trajectories (multitrajectory solutions of $H(I,\theta)=h$), depending on the values of its parameters. The bistable evolution of the resulting dynamical system is analyzed and it is analytically shown that periods of oscillations along the "sister" trajectories are equal, and the sum of corresponding areas under them does not depend on the driver amplitude. As a result, in the adiabatic limit the Hamiltonian changes in the same way regardless of the multiple separatrix crossing of the system.

[1] W. M. Nevins, T. D. Ronglien, and B. I. Cohen, Phys. Rev. Lett. **59**, 60 (1987).
[2] I. A. Kotel'nikov and G. V. Stupakov, Phys. Fluids B **2**, 871 (1990).
[3] I. A. Kotel'nikov and G. V. Stupakov, J. Plasma Phys. **45**, 19 (1991).
[4] E. V. Suvorov and M. D. Tokman, Fiz. Plazmy **14**, 250 (1988).
[5] A. I. Neishtadt and A. V. Timofeev, Sov. Phys. JETP **66**, 973 (1987).
[6] T. Tajima and J. M. Dawson, Phys. Rev. Lett. **43**, 267 (1979).
[7] G. Shvets, Phys. Rev. Lett. **93**, 195004 (2004).
[8] R. J. Noble, Phys. Rev. A **32**, 460 (1985).
[9] W. B. Mori, IEEE Trans. Plasma Sci. **PS-15**, 88 (1987).
[10] A. I. Neishtadt, Prikl. Mat. Mekh. **39**, 621 (1975).
[11] J. Henrard and A. Lemaitre, Celest. Mech. **30**, 197 (1983).
[12] S. Kalmykov, O. Polomarov, D. Korobkin, J. Otwinowski, J. Power, and G. Shvets, Philos. Trans. R. Soc. London, Ser. A **364**, 725 (2006).
[13] M. Deutsch, B. Meerson, and J. E. Golub, Phys. Fluids B **3**, 1773 (1991).
[14] R. R. Lindberg, A. E. Charman, J. S. Wurtele, and L. Friedland, Phys. Rev. Lett. **93**, 055001 (2004).
[15] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, New York, 1976), p. 156.
[16] A. V. Timofeev, Sov. Phys. JETP **48**, 656 (1978).
[17] J. L. Tennyson, J. R. Cary, and E. F. Escande, Phys. Rev. Lett. **56**, 2117 (1986).
[18] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1994), p. 290.
[19] J. R. Cary, D. F. Escande, and J. L. Tennyson, Phys. Rev. A **34**, 4256 (1986).