Relating parallel and perpendicular flows of particles and heat in a magnetized toroidal plasma

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The purpose of this Brief Communication is to emphasize the existence of a general relation between the parallel flows of heat and particles within flux surfaces and the transport of heat and particles across those flux surfaces predicted by neoclassical theory. The essential ingredients are a perspective that promotes the heat flow to the status of a fully independent dynamical variable and a unified treatment that makes no restriction regarding collisionality. Applied to well-known expressions from the literature, this approach provides a simple and explicit relation between parallel and radial flows that applies in all collisionality regimes. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2338821]

The present work stems from the observation[1] that in a neoclassical plasma the *radial* heat and particle fluxes can be expressed as linear combinations of the *parallel* heat and particle fluxes. This relation is significant both because it shows that perpendicular and parallel transport are inextricably linked and because it provides a means for calculating one in terms of the other. Additionally, the relation is universal in that it remains valid in all collisionality regimes, demonstrating that even a weakly collisional plasma will have both radial and parallel flows determined linearly from each other.

The discussion is based on well-known expressions for a neoclassical plasma; indeed, both parallel and radial transport have been treated quite extensively[1–3] and the existence of a universal relation between radial and parallel flows has previously been noticed and commented upon.[1] That radial transport is inevitably accompanied by significant transport within flux surfaces has perhaps, however, been underappreciated in the literature. Furthermore, there is an accumulating body of evidence indicating that careful treatment of parallel transport, in particular heat flow, is important to studies of magnetic island evolution and neoclassical tearing modes.[4–8] To address these concerns, we study the relation between parallel and radial transport with a focus on the role of the heat flow as an intrinsic dynamical variable. While this viewpoint is not altogether new, the transport relations derived are remarkably simple in both exposition and result, providing an explicit form for the universal relation noted previously in the literature.

The primary physical quantities in our discussion of transport are the particle flow

$$\mathbf{\Gamma} \equiv \int \mathbf{v} f d^3 v$$

and heat flow

$$\mathbf{q} \equiv \int \frac{m w^2}{2} \mathbf{w} f d^3 v, \tag{1}$$

where f is the phase-space distribution function, and the relative velocity, in terms of the particle flow $\mathbf{\Gamma}$ and density n, is $\mathbf{w} \equiv \mathbf{v} - \mathbf{\Gamma}/n$. The context of toroidal plasma confinement and nested magnetic flux surfaces naturally highlights the parallel and radial components

$$A_\parallel \equiv \mathbf{b} \cdot \mathbf{A},$$

$$A^r \equiv \langle \mathbf{A} \cdot \nabla r \rangle,$$

where $\mathbf{A}$ is any vector, $\mathbf{b}$ is the unit direction vector for the magnetic field $\mathbf{B}$, and we recall that in neoclassical theory only the flux-surface average of the radial components, indicated by the brackets, is calculated.

For simplicity, we consider a pure hydrogenic plasma and work in the rest frame of the ions. In this case, the plasma current coincides with the electron flow $\mathbf{\Gamma}$

$$\mathbf{J} = -e \mathbf{\Gamma}, \tag{2}$$

where $e > 0$ is the magnitude of the electron charge and, beginning here, all species-specific quantities without a subscript refer to the electrons.

Perhaps the simplest demonstration of a relation between radial and parallel transport is obtained for a strongly collisional plasma. Canonical references for the collision-dominated regime show that all components of the neoclassical fluxes can be written in terms of the parallel driving forces. Eliminating the forces in favor of the flows, $J_\parallel$, $q_\parallel$, Γ^r, and q^r, gives[3,9]

$$\begin{aligned} \Gamma^r_{\text{nc}} &= \frac{1}{\tau_e}\left[c^{\text{coll}}_{\Gamma\Gamma} \langle \Gamma_\parallel h \rangle + c^{\text{coll}}_{\Gamma q} \left\langle \frac{q_\parallel}{T} h \right\rangle \right] - \frac{c}{\chi'} \langle n E_\zeta \rangle, \\ \frac{q^r_{\text{nc}}}{T} &= \frac{1}{\tau_e}\left[c^{\text{coll}}_{q\Gamma} \langle \Gamma_\parallel h \rangle + c^{\text{coll}}_{qq} \left\langle \frac{q_\parallel}{T} h \right\rangle \right], \end{aligned} \tag{3}$$

where the "nc" subscript refers to the neoclassical flux, τ_e is the electron collision time, E_ζ the covariant toroidal component of the electric field, and the numerical coefficients are

$$c_{\Gamma\Gamma}^{\rm coll} = -0.656,$$

$$c_{\Gamma q}^{\rm coll} = +0.217,$$

$$c_{q\Gamma}^{\rm coll} = +0.543, \tag{4}$$

$$c_{qq}^{\rm coll} = -0.779.$$

Notice that the parallel components are flux-surface averaged with an additional factor of the poloidal gyrofrequency

$$h \equiv \frac{mc}{eB}\left(\frac{I}{\chi'}\right) \sim \frac{1}{\Omega_P},$$

which varies over the flux surface due to variation in $B \equiv \sqrt{\mathbf{B}\cdot\mathbf{B}}$. Here, $\chi(r)$ is the poloidal flux, $I(r)$ is the covariant toroidal component of $\mathbf{B}$, and the prime indicates the derivative with respect to the radial coordinate r.[10]

The significance of (3) is that the averaged radial fluxes can be determined from the parallel fluxes. We will indicate this schematically by

$$(\Gamma^r, q^r) \leftrightarrow (\Gamma_\parallel, q_\parallel). \tag{5}$$

Clearly, the conclusion is that if there is radial transport then there must be parallel transport, and vice versa. Furthermore, there is a simple rule to calculate one from the other. Thus, (5) both combines parallel and radial transport by demanding that they always occur together and separates them in the sense that either alone is sufficient to complete the description.

An obvious limitation to (3), however, is that it is derived for the collision-dominated regime. Since previous discussions of (5) point out its universal nature, we are motivated to construct such a generalization to (3). With this objective in mind, we recall a few features of basic neoclassical transport common to all collisionality regimes. The essential result for our purposes is that, for transport in a magnetized toroidal plasma, the parallel friction force and energy weighted friction, respectively,

$$F_\parallel \equiv \int m v_\parallel \mathcal{C}(f) d^3v,$$

$$G_\parallel \equiv \int \frac{1}{2} m v^2 v_\parallel \mathcal{C}(f) d^3v \tag{6}$$

determine the radial fluxes. This follows from the exact energy and momentum evolution equations, which are greatly simplified for neoclassical studies: the magnetized plasma and transport orderings are applied, axisymmetry invoked, and the flux surface average of the toroidal components is taken to isolate the radial fluxes. Separating out the "classical" contributions leaves the "neoclassical" fluxes[3,10]

$$\Gamma_{\rm nc}^r = \frac{1}{m}\langle F_\parallel h\rangle - \frac{c}{\chi'}\langle nE_\zeta\rangle,$$

$$\frac{q_{\rm nc}^r}{T} = \frac{1}{m}\langle H_\parallel h\rangle, \tag{7}$$

where we define the "heat friction" as

$$H_\parallel \equiv \frac{m}{T}G_\parallel - \frac{5}{2}F_\parallel$$

and have used the following vector component identities

$$A_{\parallel\zeta} = \frac{B_\zeta}{B}A_\parallel = \frac{I}{B}A_\parallel.$$

These relations for the radial fluxes of axisymmetric transport are well known and indicate explicitly which aspects of the parallel dynamics are crucial for determining the radial dynamics. Notice that with an expression for the collisional moments $(F_\parallel, G_\parallel)$ in terms of the fluid moments $(\Gamma_\parallel, q_\parallel)$, then (5) results immediately. Methods for constructing such an expression, a "friction-flow" relationship, exist in the literature, and we present a simple one below. Traditional analysis of (7), however, first decomposes the radial fluxes into "Pfirsch-Schlüter"and "banana-plateau" contributions which dominate in different collisionality regimes.[1,3] For example, the total radial heat flux

$$q_{\rm nc} \sim \left\langle \frac{H_\parallel}{B} \right\rangle$$

is decomposed into the banana-plateau

$$q_{\rm bp} \sim \langle H_\parallel B\rangle$$

and Pfirsch-Schlüter components

$$q_{\rm ps} \sim \left\langle H_\parallel \left(\frac{\langle B^2\rangle}{B} - B\right)\right\rangle,$$

so that

$$q_{\rm nc} \sim q_{\rm bp} + q_{\rm ps}.$$

The two contributions are then treated separately, usually by means of introducing coefficients to relate the right-hand sides to the parallel fluid moments: viscosity coefficients for the banana-plateau fluxes and friction coefficients for the Pfirsch-Schlüter fluxes. This decomposition of the collisionality regimes is motivated by physical considerations and the essential program of transport analysis, the relation of fluxes to driving forces. However, expressions for viscosity coefficients,[11] while valid for all collisionality regimes and arbitrary aspect ratio and used in neoclassical codes,[12] are evidently quite complex (see also Ref. 3). For our purpose of demonstrating a simple relation between flows, we instead return to (7), expressing the parallel frictional forces in terms of the parallel flows for all collisionalities. In other words, in terms of total flows with a weight of $1/B$ in the average, only the friction coefficients are necessary.

One direct calculation of friction coefficients expands the velocity dependence of the distribution function in a Laguerre/Sonine polynomial series (whose coefficients are related to the parallel flows) and integrates (6) inserting the series into the linearized Fokker-Planck collision operator[1,13]—a substantial task. We next present a simpler method that is also applicable for all collisionalities.

While the above discussion of radial transport was valid for any collisionality, it is possible to treat the parallel dynamics with even greater generality; it will not be necessary to use the transport ordering, invoke axisymmetry, average over flux surfaces, or even to assume the plasma is magnetized as before. The price paid for such generality will come in the form of an approximation: the representation of particular function by a two-term polynomial. Our results will therefore be approximate rather than exact.

Consider, for example, the expression for the parallel heat flux

$$\frac{q_\parallel}{T} = v_t^4 \int z_\parallel \left(z^2 - \frac{5}{2} \right) f d^3 z,$$

which follows from (1) under the restriction that the lowest-order pressure tensor is isotropic and that the plasma flows are small in the sense that $\Gamma \ll n v_t$, where we normalize the velocity

$$z \equiv \frac{v}{v_t} = v \sqrt{\frac{m}{2T}}.$$

Both of these conditions can hold for less restrictive orderings, but in particular, they are valid for neoclassical transport. Defining the generalized Spitzer function f_{s2} as the solution to

$$\mathcal{C}(f_{s2}) = z_\parallel \left(z^2 - \frac{5}{2} \right) f_{\mathrm{M}},$$

where $\mathcal{C}$ is the linearized collision operator and the Maxwellian distribution is

$$f_{\mathrm{M}} \equiv n \left(\frac{m}{2\pi T} \right)^{3/2} \exp(-z^2),$$

the heat flux can be rewritten

$$\frac{q_\parallel}{T} = v_t^4 \int \mathcal{C}(f_{s2}) \frac{f}{f_M} d^3 z.$$

Self-adjointness of the collision operator[10] implies that the argument of $\mathcal{C}$ can be switched, giving

$$\frac{q_\parallel}{T} = v_t^4 \int \frac{f_{s2}}{f_{\mathrm{M}}} \mathcal{C}(f) d^3 z. \tag{8}$$

Although the Spitzer function f_{s2} essentially corresponds to the actual distribution function in the context of *collision-dominated* parallel transport, here it is interpreted simply as the solution to a particular mathematical equation. Thus, expanding a smooth f_{s2}/f_{M} (but not the *actual* distribution function f) in velocity powers with coefficients a_n,

$$f_{s2} = z_\parallel f_{\mathrm{M}} \sum_n a_n z^{2n}$$

shows that $q_\parallel$ can be written as a sum

$$\frac{q_\parallel}{T} = v_t^4 \sum_n a_n \int z^{2n} z_\parallel \mathcal{C}(f) d^3 z, \tag{9}$$

where the integrals are, by definition, the parallel collisional moments. Similar arguments indicate that with another Spitzer function defined by

$$\mathcal{C}(f_{s1}) = z_\parallel f_{\mathrm{M}},$$

the parallel particle flux can also be written as a different sum of parallel collisional moments. Higher fluid moments may be treated in the same way using Spitzer functions defined with appropriate inhomogeneous source terms.

These results depend only on the self-adjointness of the collision operator and smoothness of the Spitzer functions, and imply, fairly generally, that each parallel moment can be thought of as a certain linear combination of all the parallel collisional moments. Notice that to reach this conclusion, *it has not been necessary to say anything about the actual distribution function or the collisionality*; we have only converted moments of the unspecified quantity f into moments of the unspecified quantity $\mathcal{C}(f)$ using purely mathematical objects (generalized Spitzer functions).

However, while (8) is exact, without an expression for f_{s2} it is merely formal manipulation. A useful approximation is provided by the significance of the particle and heat flows, which implies that the collisional moments $F_\parallel$ and $G_\parallel$ are determined by $\Gamma_\parallel$ and $q_\parallel$. Thus, it follows from (9) that to the extent that the Spitzer functions f_{s1} and f_{s2} can be accurately represented using only the lowest two terms

$$f_{\mathrm{si}} \sim (a_{i0} + a_{i1} z^2) z_\parallel f_{\mathrm{M}}, \tag{10}$$

there is a qualitatively correct expression relating $\Gamma_\parallel$ and $q_\parallel$ linearly to $F_\parallel$ and $G_\parallel$. Since the Spitzer functions are simply particular mathematical functions whose forms do not change, this result clearly holds in all collisionality regimes.

To determine the numerical coefficients that best represent the exact Spitzer functions f_{s1} and f_{s2}, we make use of a Laguerre decomposition. The associated Laguerre polynomials $L_n^{3/2}$ form a complete orthogonal set such that any "sufficiently" smooth function properly integrable over the infinite interval $(0,\infty)$ can be expressed as an infinite sum of the polynomials.[14] Since $L_0^{3/2}(x)=1$ and $L_1^{3/2}(x)=\frac{5}{2}-x$, the approximate Spitzer functions (10) are linear combinations of these lowest two polynomials with argument z^2. We choose the coefficients a_{ij}, so that they match the corresponding Laguerre coefficients of the numerically calculated solutions.[15] By orthogonality, they are each given by a single integral of the numerical data weighted by $L_0^{3/2}$ or $L_1^{3/2}$. Using these in (10), integration and inversion of (8) and its analog for $\Gamma_\parallel$ becomes

$$\begin{aligned} F_\parallel &= \frac{m}{\tau_e} \left(-0.665 \Gamma_\parallel + 0.230 \frac{q_\parallel}{T} \right), \\ G_\parallel &= \frac{T}{\tau_e} \left(-1.08 \Gamma_\parallel - 0.282 \frac{q_\parallel}{T} \right), \end{aligned} \tag{11}$$

and the parallel part of the problem is completed. Remember that we have employed a Laguerre decomposition to approximate the generalized Spitzer functions, but not the distribution function itself. As discussed, the Spitzer functions and their Laguerre coefficients are the same for all collisionalities. Since our model functions make use of those coefficients, this result also holds in all collisionality regimes, another explicit confirmation of the known universality of the friction coefficients.[1] In fact, because it was not necessary to resort to the transport ordering or flux-surface averages, (11) may also have relevance in studies of particle and heat flow more general than that of neoclassical transport.

Now the parallel and radial results can be combined. Substituting the friction-flow relationship (11) into the radial fluxes (7) gives

$$\begin{aligned} \Gamma^r_{\mathrm{nc}} &= \frac{1}{\tau_e}\left[c_{\Gamma\Gamma}\langle \Gamma_\| h\rangle + c_{\Gamma q}\left\langle \frac{q_\|}{T}h\right\rangle\right] - \frac{c}{\chi'}\langle nE_\zeta\rangle, \\ \frac{q^r_{\mathrm{nc}}}{T} &= \frac{1}{\tau_e}\left[c_{q\Gamma}\langle \Gamma_\| h\rangle + c_{qq}\left\langle \frac{q_\|}{T}h\right\rangle\right], \end{aligned} \tag{12}$$

where

$$\begin{aligned} c_{\Gamma\Gamma} &= -0.665, \\ c_{\Gamma q} &= +0.230, \\ c_{q\Gamma} &= +0.582, \\ c_{qq} &= -0.857. \end{aligned} \tag{13}$$

Notice the location of the collision frequency $\nu \sim 1/\tau_e$. Thus, in a weakly collisional plasma the parallel fluxes are the substantial factor $\mathcal{O}(\Omega_P/\nu)$ larger than the radial fluxes.

For comparison, recall the coefficients for large collisionality (4). Our expressions match the known results in the collision-dominated regime to within 10%, but are also approximately valid in all collisionality regimes. In addition, they are consistent with a "universal" Ohm's law—the linear transport relation between the parallel current, electric field, and radial fluxes. To reduce our expressions to an Ohm's law, the parallel heat term is algebraically eliminated between the two equations of (12) resulting in

$$-\langle \Gamma_\| h\rangle = 1.97\tau_e\left(\frac{c}{\chi'}\langle nE_\zeta\rangle + \Gamma^r_{\mathrm{nc}} + 0.268\frac{q^r_{\mathrm{nc}}}{T}\right), \tag{14}$$

which, since (2) holds in the ion rest frame, is in approximate agreement with previous derivations.[1,16,17] Naturally, when the known expressions for the radial fluxes Γ^r_{nc} and q^r_{nc} in the banana regime are inserted into (14), the resulting current is very nearly the bootstrap current, verifying that the simple relation (12) includes important neoclassical effects.

To summarize, we have presented an explicit relation, valid for all collisionalities, between parallel and radial transport of particles and heat. Our approach differs from previous discussions in that the preferential treatment of the flows as intrinsic variables allows for a simple derivation and result.

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