Coulomb collision operator

R. D. Hazeltine

August 2, 2006

Abstract

Properties of the Coulomb collision operator that have been found useful in plasma kinetic theory and plasma transport theory are gathered and reviewed, with derivations. The use of velocity-space coordinates appropriate to a magnetized (drift-kinetic) plasma is emphasized. Little of the material is new, although the derivations may be more detailed than is customary.

Contents

1 Operator

1.1 Landau-Boltzmann form

The distribution function for plasma species a is denoted by $f_a(\boldsymbol{v})$. We use Boltzmann notation

$$f'_b \equiv f_b(\boldsymbol{v}')$$

Then the Coulomb collision operator acting on species a is

$$C_a = \sum_b C_{ab}$$

where[2]

$$C_{ab} = \frac{\gamma_{ab}}{2} \frac{\partial}{\partial v_\alpha} \int d^3v\,' U_{\alpha\beta} \left(f'_b \frac{\partial f_a}{\partial v_\beta} - \frac{m_a}{m_b} f_a \frac{\partial f'_b}{\partial v'_\beta} \right) \tag{1}$$

Here we have introduced the *scattering tensor*

$$U_{\alpha\beta} = \frac{u^2 \delta_{\alpha\beta} - u_\alpha u_\beta}{u^3}, \quad u_\alpha \equiv v_\alpha - v'_\alpha \tag{2}$$

and the constant

$$\gamma_{ab} \equiv \frac{4\pi e_a^2 e_b^2 \ln \Lambda}{m_a^2}$$

The charge and mass of species a are e_a and m_a; $\ln \Lambda$ is the Coulomb logarithm. In this and the following Section the components v_α are Cartesian; curvilinear coordinates are introduced in Section 3.

1.2 Coefficient

In terms of Braginskii's[2] electron-ion collision time τ_{ei}, we have

$$\gamma_{ei} = \frac{3\sqrt{\pi} v_{te}^3}{4 n_i \tau_{ei}} \tag{3}$$

where n_i is the ion density and

$$v_{ta} \equiv \sqrt{2T_a/m_a} \tag{4}$$

is the thermal speed. Similarly, in terms of the ion-ion collision time τ_i,

$$\gamma_{ii} = Z^2 (m_e/m_i)^2 \gamma_{ei} = \frac{3\sqrt{2\pi} v_{ti}^3}{4 n_i \tau_i}$$

where Z is the ionic charge.

The above formulae are in Gaussian units. In SI units,

$$\gamma_{ab} = \frac{e_a^2 e_b^2 \ln \Lambda}{4\pi \epsilon_0^2 m_a^2}$$

Some texts[3, 4] define γ with different factors of 2 and m.

1.3 Scattering and diffusion tensors

The definition (2) implies that

$$u_\alpha U_{\alpha\beta} = 0 \tag{5}$$

$$U_{\alpha\alpha} = 2/u \tag{6}$$

The scattering tensor also satisfies

$$U_{\alpha\beta} = \frac{\partial^2 u}{\partial u_\alpha \partial u_\beta} = \frac{\partial}{\partial u_\alpha}\left(\frac{u_\beta}{u}\right) \tag{7}$$

$$\frac{\partial U_{\alpha\beta}}{\partial u_\beta} = 2\frac{\partial}{\partial u_\alpha}\left(\frac{1}{u}\right) = \frac{\partial U_{\beta\beta}}{\partial u_\alpha} \tag{8}$$

$$\frac{\partial^2 U_{\alpha\beta}}{\partial u_\alpha \partial u_\beta} = -8\pi\delta(\boldsymbol{u}) \tag{9}$$

The averaged scattering tensor is the *diffusion tensor*

$$D_{b\,\alpha\beta} \equiv \int d^3v'\, f_b' U_{\alpha\beta} \tag{10}$$

We introduce the Maxwellian distribution

$$f_{Mb} \equiv \frac{n_b}{\pi^{3/2} v_{tb}^3} e^{-v^2/v_{tb}^2} \tag{11}$$

and consider the Maxwellian diffusion tensor

$$D_{Mb\,\alpha\beta} \equiv \int d^3v'\, f_{Mb}' U_{\alpha\beta}$$

In terms of the variable $s_\alpha \equiv v_\alpha / v_{tb}$ one finds that

$$D_{Mb\,\alpha\beta} = \frac{n_b}{2v_{tb}} \left[\delta_{\alpha\beta} M_1(s) - s_\alpha s_\beta M_2(s)\right] \tag{12}$$

with

$$M_1 = s^{-3}[s\ \mathrm{erf}'(s) + (2s^2 - 1)\,\mathrm{erf}(s)] \tag{13}$$

$$M_2 = 3s^{-5}[s\ \mathrm{erf}'(s) + (2s^2/3 - 1)\,\mathrm{erf}(s)] = -s^{-1} dM_1/ds \tag{14}$$

Here

$$\mathrm{erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s dt e^{-t^2}$$

denotes the error function and $\mathrm{erf}'(s) \equiv d\,\mathrm{erf}\,/ds$. The relation

$$s^3(M_1 - s^2 M_2) = 2(\mathrm{erf} - s\,\mathrm{erf}') \tag{15}$$

is often useful. The gradient of the Maxwellian diffusion tensor is surprisingly simple:

$$\frac{\partial D_{Mb\,\alpha\beta}}{\partial v_\beta} = 2\frac{n_b}{v_{tb}^2}\frac{s_\alpha}{s}\frac{\partial}{\partial s}\left(\frac{\mathrm{erf}(s)}{s}\right) \tag{16}$$

A derivation of this relation appears in Subsection 1.5.

From the well-known formulae[1],

$$\begin{aligned} \mathrm{erf}(s) &= 1 + \mathcal{O}(e^{-s^2}), \;\; \text{for } s \gg 1 \\ &= \frac{2}{\sqrt{\pi}}\left(s - \frac{s^3}{3} + \frac{s^5}{10} - \cdots\right), \;\; \text{for } s \ll 1 \end{aligned}$$

we obtain the asymptotic forms

$$M_1 \approx 2s^{-1}, \quad M_2 \approx 2s^{-3}, \text{ for } s \gg 1, \tag{17}$$

$$M_1 \approx 8/(3\sqrt{\pi}), \quad M_2 \approx 16/(15\sqrt{\pi}) \text{ for } s \ll 1, \tag{18}$$

1.4 Fokker-Planck (FP) form

We integrate by parts in (1) and use $\partial u/\partial \boldsymbol{v}' = -\partial u/\partial \boldsymbol{v}$ to obtain

$$C_{ab} = \frac{\gamma_{ab}}{2}\frac{\partial}{\partial v_\alpha}\left(D_{b\,\alpha\beta}\frac{\partial f_a}{\partial v_\beta} - \frac{m_a}{m_b} f_a \frac{\partial}{\partial v_\beta} D_{b\,\alpha\beta}\right) \tag{19}$$

This is standard Fokker-Planck form; the first parenthesized term represents diffusion and the second represents dynamical friction. An equivalent expression is

$$C_{ab} = \frac{\gamma_{ab}}{2}\frac{\partial}{\partial v_\alpha}\left[\frac{\partial}{\partial v_\beta}(f_a D_{b\,\alpha\beta}) - \left(1 + \frac{m_a}{m_b}\right) f_a \frac{\partial D_{b\,\alpha\beta}}{\partial v_\beta}\right] \tag{20}$$

1.5 Rosenbluth-MacDonald-Judd (RMJ) form

We introduce the functions

$$G_b \equiv \int d^3v'\, f_b'\, u$$

$$H_b \equiv \int d^3v'\, f_b'\, u^{-1}$$

and observe from (7) and (8) that

$$D_{b\,\alpha\beta} = \frac{\partial^2 G_b}{\partial v_\alpha \partial v_\beta} \tag{21}$$

$$\frac{\partial D_{b\,\alpha\beta}}{\partial v_\beta} = 2\frac{\partial H_b}{\partial v_\alpha} \tag{22}$$

Therefore (20) can be written as

$$C_{ab} = \frac{\gamma_{ab}}{2} \frac{\partial}{\partial v_\alpha} \left[\frac{\partial}{\partial v_\beta} \left(f_a \frac{\partial^2 G_b}{\partial v_\alpha \partial v_\beta} \right) - 2 \left(1 + \frac{m_a}{m_b} \right) f_a \frac{\partial H_b}{\partial v_\alpha} \right] \tag{23}$$

The original RMJ paper[5] expressed the *full* collision operator C_a for a two-species plasma, in terms of species-sums of G_b and H_b. We find the individual species version more convenient, and identify G_b and H_b with the so-called Rosenbluth potentials. The name stems from the identities

$$\nabla_{\boldsymbol{v}}^2 G_b \equiv \frac{\partial^2 G_b}{\partial v_\alpha \partial v_\alpha} = 2H_b \tag{24}$$

$$\nabla_{\boldsymbol{v}}^2 H_b = -4\pi f_b \tag{25}$$

which follow from (6) and (9), and which resemble Poisson's equation for the electrostatic potential.

We use an M-subscript to distinguish the potentials evaluated on a Maxwellian distribution: $G_{Mb} = G_b(f_{Mb})$, etc. One finds

$$G_{Mb} = (n_b v_{tb}/2)[\mathrm{erf}'(s) + s^{-1}(1 + 2s^2)\,\mathrm{erf}(s)] \tag{26}$$

where $s = v/v_{tb}$ as usual. Similarly

$$H_{Mb} = (n_b/v_{tb}) s^{-1}\,\mathrm{erf}(s) \tag{27}$$

Note that substitution of (27) into (22) yields (16).

2 Symmetry of the collision operator

2.1 Scalar property

Functions and operators have of course different forms when they are expressed in terms of different coordinates. Thus if a point in velocity-space has the coordinates $\boldsymbol{v}$ in one system and $\bar{\boldsymbol{v}}$ in another, we expect the expressions of some function, such as a distribution function f, or an operator, such as the collision operator C, to have correspondingly distinct expressions, related by a *transformation law*,

$$f \to \bar{f}, C \to \bar{C}$$

Two types of transformation have special special importance: rotations and Galilean transformations. We will generally refer to a Galilean transformation as a "boost".

In relativistic theory, rotations and boosts are simply two types of homogeneous Lorentz transformations, and both can be treated simultaneously. However in non-relativistic theory they are distinct; in particular, rotations are linear and homogeneous, while boosts are inhomogeneous (affine).

The simplest transformation law is that of a *scalar* function $s(\boldsymbol{v})$:

$$\bar{s}(\bar{\boldsymbol{v}}) = s(\boldsymbol{v}) \tag{28}$$

We expect any physically meaningful quantity without coordinate indices to be scalar, under both rotations and boosts. Thus a distribution function is a scalar. Recall also that the full contraction of any even-rank tensor is necessarily a scalar under rotation; for example, if $T_{\alpha\beta}$ is a tensor under rotation, then the quantity

$$s = v_\alpha T_{\alpha\beta} v_\beta$$

is manifestly a scalar.

The collision operator is a *scalar operator*. To make this nomenclature explicit, we write the operator as

$$C_{ab} = C_{ab}[f_a, f_b](\boldsymbol{v})$$

It is thus a function*al* of the two distributions, but an ordinary function of velocity $\boldsymbol{v}$. The transformed operator of interest is

$$\bar{C}_{ab}[\bar{f}_a, \bar{f}_b](\bar{\boldsymbol{v}})$$

The scalar nature of C_{ab} is therefore expressed by:

$$\bar{C}_{ab}[\bar{f}_a, \bar{f}_b](\bar{\boldsymbol{v}}) = C_{ab}[f_a, f_b](\boldsymbol{v}) \tag{29}$$

This relation is to hold for both rotations and boosts in velocity space.

2.2 Invariance

A quantity $q(\boldsymbol{v})$ is *invariant* under some transformation $\boldsymbol{v} \to \bar{\boldsymbol{v}}$ if it satisfies

$$q(\boldsymbol{v}) = q(\bar{\boldsymbol{v}}) \tag{30}$$

For example, any function that depends on velocity only through is magnitude $v = |\boldsymbol{v}|$ is invariant under rotation. On the other hand the scalar mentioned above, $s(\boldsymbol{v}) = \boldsymbol{v} \cdot \boldsymbol{T} \cdot \boldsymbol{v}$, where $\boldsymbol{T}$ is a second-rank tensor, is not invariant under rotation:

$$s(\bar{\boldsymbol{v}}) = \bar{v}_\alpha T_{\alpha\beta} \bar{v}_\beta$$

When $\boldsymbol{T}$ is symmetric, this quantity becomes

$$s(\bar{\boldsymbol{v}}) = \boldsymbol{v} \cdot \bar{\boldsymbol{T}} \cdot \boldsymbol{v}$$

but it is invariant—equal to $s(\boldsymbol{v})$—only when $T_{\alpha\beta} = \delta_{\alpha\beta}$ is the identity matrix.

Being a scalar and being invariant are entirely independent properties. In particular (30) makes no reference to $\bar{q}$, so its transformation law is irrelevant. Thus one can determine invariance from the form of the function (or operator, or whatever) alone.

A quantity that is invariant under rotation will be called rotationally symmetric. A function that is rotationally symmetric has the same value at any two points that can be connected by a rotation. Distribution functions that are rotationally symmetric are often called *isotropic*; although the Maxwellian distribution is isotropic, most distribution functions are not.

The collision operator is both rotationally symmetric and invariant under Galilean transformations. That is, for either type of transformation,

$$C_{ab}[f_a, f_b](\boldsymbol{v}) = C_{ab}[\bar{f}_a, \bar{f}_b](\bar{\boldsymbol{v}}) \tag{31}$$

Verifying (31) is the main point of this Section.

To avoid confusion we remark that symmetry of the operator, in the sense of (31), does not imply symmetry of the velocity-function $C_{ab}(\boldsymbol{v})$, because the distributions are not in general symmetric.

2.3 Rotations

We denote the rotation matrix by $\boldsymbol{R}$, so that the basic transformation law is

$$\bar{\boldsymbol{v}} = \boldsymbol{R} \cdot \boldsymbol{v}$$

One must keep in mind that both sides of this equation refer to the same point in velocity space: the *coordinates* are transformed, while the velocity is fixed ("passive transformation").

Recall that the matrix inverse to $\boldsymbol{R}$ is the transposed matrix $\boldsymbol{R}^T$:

$$\boldsymbol{R}^T \cdot \boldsymbol{R} = \boldsymbol{I} \tag{32}$$

where $\boldsymbol{I}$ is the identity matrix. Thus

$$\boldsymbol{v} = \boldsymbol{R}^T \cdot \bar{\boldsymbol{v}} \tag{33}$$

and

$$\frac{\partial}{\partial v_\alpha} = \frac{\partial \bar{v}_\lambda}{\partial v_\alpha}\frac{\partial}{\partial \bar{v}_\lambda} = R_{\lambda\alpha}\frac{\partial}{\partial \bar{v}_\lambda}$$

That is

$$\frac{\partial}{\partial \boldsymbol{v}} = \boldsymbol{R}^T \cdot \frac{\partial}{\partial \bar{\boldsymbol{v}}} \tag{34}$$

That $\boldsymbol{v}$ and $\partial/\partial\boldsymbol{v}$ transform identically follows from the fact that the metric is the identity matrix: we are using Cartesian coordinates.

The scalar transformation law noted in (28) can now be written more explicitly as

$$f(\boldsymbol{R}^T \cdot \bar{\boldsymbol{v}}) = \bar{f}(\bar{\boldsymbol{v}}) \tag{35}$$

Similarly the scattering tensor $\boldsymbol{U}$ transforms according to

$$\bar{\boldsymbol{U}}(\bar{\boldsymbol{u}}) = \boldsymbol{R} \cdot \boldsymbol{U}(\boldsymbol{u}) \cdot \boldsymbol{R}^T$$

or

$$\boldsymbol{U}(\boldsymbol{R}^T \cdot \bar{\boldsymbol{u}}) = \boldsymbol{R}^T \cdot \bar{\boldsymbol{U}}(\bar{\boldsymbol{u}}) \cdot \boldsymbol{R} \tag{36}$$

Any quantity constructed by full contraction of higher-rank tensors will automatically transform under rotation as a scalar. Thus (1) shows immediately that the collision operator is a scalar under rotation: the scalar nature of f makes $\partial f/\partial \boldsymbol{v}$ a vector, the scattering tensor is indeed a tensor, and so on.

Another property of C_{ab} is also clear from (1): the operator is rotationally symmetric, in the sense of (31). The point is that the operator contains no information that distinguishes a particular direction in velocity space. Therefore observers of collisional effects with different orientations would see identical processes.

Despite the transparency of rotational symmetry, we next verify this property in detail, by expressing the right-hand side of (1) in terms of barred quantities. Notice that no transformation property of C_{ab} is involved, only its expression in terms of different coordinates. To express the collision operator in terms of $\bar{\boldsymbol{v}}$, we proceed as follows:

1. Express $\boldsymbol{v}$ in terms of $\bar{\boldsymbol{v}}$, introducing the factors of $\boldsymbol{R}^T$ as required by (33) and (34).
2. Similarly transform the dummy variable $\boldsymbol{v}'$, noting that $d^3v' = d^3\bar{v}'$. Thus all velocities $\boldsymbol{v}$ and $\boldsymbol{v}'$ are eliminated from C_{ab}, in favor of $\bar{\boldsymbol{v}}$ and $\bar{\boldsymbol{v}}'$.
3. Use (35) and (36) to express the distribution functions and scattering tensor in terms of the corresponding barred quantities.

After this step the collision operator has the form

$$\begin{aligned} C_{ab} =& \frac{\gamma_{ab}}{2} \boldsymbol{R}^T \cdot \frac{\partial}{\partial \bar{\boldsymbol{v}}} \cdot \int d^3\bar{v}' \, \boldsymbol{R}^T \cdot \bar{\boldsymbol{U}} \cdot \boldsymbol{R} \\ & \cdot \left[\bar{f}_b(\bar{\boldsymbol{v}}') \boldsymbol{R}^T \cdot \frac{\partial \bar{f}_a(\bar{\boldsymbol{v}})}{\partial \bar{\boldsymbol{v}}} - \frac{m_a}{m_b} \bar{f}_a(\bar{\boldsymbol{v}}) \boldsymbol{R}^T \cdot \frac{\partial \bar{f}_b(\bar{\boldsymbol{v}}')}{\partial \bar{\boldsymbol{v}}'} \right] \end{aligned}$$

or, in view of (32),

$$\begin{aligned} C_{ab} =& \frac{\gamma_{ab}}{2} \boldsymbol{R}^T \cdot \frac{\partial}{\partial \bar{\boldsymbol{v}}} \cdot \int d^3\bar{v}' \, \boldsymbol{R}^T \cdot \bar{\boldsymbol{U}} \\ & \cdot \left[\bar{f}_b(\bar{\boldsymbol{v}}') \frac{\partial \bar{f}_a(\bar{\boldsymbol{v}})}{\partial \bar{\boldsymbol{v}}} - \frac{m_a}{m_b} \bar{f}_a(\bar{\boldsymbol{v}}) \frac{\partial \bar{f}_b(\bar{\boldsymbol{v}}')}{\partial \bar{\boldsymbol{v}}'} \right] \end{aligned} \tag{37}$$

To confirm that the remaining factors of $\boldsymbol{R}^T$ cancel, we consider the vector

$$\begin{aligned}
\boldsymbol{R}^T \cdot \frac{\partial}{\partial \bar{\boldsymbol{v}}} \cdot \int d^3\bar{v}'\, \boldsymbol{R}^T \cdot \bar{\boldsymbol{U}} &= R_{\lambda\kappa} \frac{\partial}{\partial \bar{v}_\lambda} \int d^3\bar{v}' R_{\alpha\kappa} \bar{U}_{\alpha\beta} \\
&= R_{\lambda\kappa} R_{\alpha\kappa} \frac{\partial}{\partial \bar{v}_\lambda} \int d^3\bar{v}'\, \bar{U}_{\alpha\beta} \\
&= \delta_{\lambda\alpha} \frac{\partial}{\partial \bar{v}_\lambda} \int d^3\, \bar{v}' \bar{U}_{\alpha\beta} \\
&= \frac{\partial}{\partial \bar{\boldsymbol{v}}} \cdot \int d^3\bar{v}'\, \bar{\boldsymbol{U}}
\end{aligned}$$

Therefore

$$C_{ab} = \frac{\gamma_{ab}}{2} \frac{\partial}{\partial \bar{\boldsymbol{v}}} \cdot \int d^3\bar{v}'\, \bar{\boldsymbol{U}} \cdot \left[\bar{f}_b(\bar{\boldsymbol{v}}') \frac{\partial \bar{f}_a(\bar{\boldsymbol{v}})}{\partial \bar{\boldsymbol{v}}} - \frac{m_a}{m_b} \bar{f}_a(\bar{\boldsymbol{v}}) \cdot \frac{\partial \bar{f}_b(\bar{\boldsymbol{v}}')}{\partial \bar{\boldsymbol{v}}'} \right] \quad (38)$$

Observing that the right-hand side of (38) coincides with $C_{ab}[\bar{f}_a, \bar{f}_b](\bar{\boldsymbol{v}})$, we conclude that (31) holds: the collision operator is invariant under rotation.

It is clear that the nature of the scattering tensor $\boldsymbol{U}$, depending on $\boldsymbol{v}$ and $\boldsymbol{v}'$ alone, is crucial to this result. For example the operator obtained from C_{ab} by the replacement

$$\boldsymbol{U} \rightarrow \boldsymbol{k}\boldsymbol{k}$$

where $\boldsymbol{k}$ is some fixed vector, such as the direction of an external field, would be a scalar like C_{ab}, but not invariant.

2.4 Boosts

Here we consider the coordinate transformation

$$\bar{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{V}$$

where $\boldsymbol{V}$ is some fixed velocity. Evidently the components $\boldsymbol{v}$ and $\bar{\boldsymbol{v}}$ are those see in two frames with relative velocity $\boldsymbol{V}$; again we emphasize that both sets of coordinates refer to the same point in velocity space. As before we can define $\bar{C}_{ab}$ to insure that the operator is a scalar under Galilean boosts. The more interesting point is its invariance under such boosts: the collisional process is described by the same operator in any inertial frame. This fact is not surprising but nonetheless we verify it in detail.

The scalar nature of the distribution is expressed by $\bar{f}(\bar{\boldsymbol{v}}) = f(\boldsymbol{v})$, or

$$\bar{f}_a(\bar{\boldsymbol{v}}) = f_a(\bar{\boldsymbol{v}} + \boldsymbol{V}) \quad (39)$$

Derivatives of the distributions satisfy $\partial/\partial\bar{\boldsymbol{v}} = \partial/\partial\boldsymbol{v}$ and the scattering tensor satisfies

$$\boldsymbol{U}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{v}}') \equiv \bar{\boldsymbol{U}} = \boldsymbol{U}(\boldsymbol{v}, \boldsymbol{v}')$$

With these facts, we can proceed as in the rotational case, expressing the operator C_{ab} in terms of the transformed velocity $\bar{\boldsymbol{v}}$ (without transforming the operator). After changing the integration variable as usual, $\boldsymbol{v}' \to \bar{\boldsymbol{v}}'$, one finds

$$\begin{aligned} C_{ab}[f_a, f_b](\boldsymbol{v}) &= \frac{\gamma_{ab}}{2}\frac{\partial}{\partial\bar{\boldsymbol{v}}} \cdot \int d^3\bar{v}'\,\bar{\boldsymbol{U}} \\ \cdot &\left[f_b(\bar{\boldsymbol{v}}' + \boldsymbol{V})\frac{\partial f_a(\bar{\boldsymbol{v}} + \boldsymbol{V})}{\partial\bar{\boldsymbol{v}}} - \frac{m_a}{m_b} f_a(\bar{\boldsymbol{v}} + \boldsymbol{V})\frac{\partial f_b(\bar{\boldsymbol{v}}' + \boldsymbol{V})}{\partial\bar{\boldsymbol{v}}'}\right] \end{aligned} \tag{40}$$

If we now use (39) to replace each distribution f by its transformed version $\bar{f}$, we see that the right-hand side of (40) becomes $C_{ab}[\bar{f}_a, \bar{f}_b](\bar{\boldsymbol{v}})$. Hence (31) pertains: the collision operator is invariant under Galilean transformations. The key underlying fact here is the dependence of the scattering tensor $\boldsymbol{U}$ on the velocity difference $\boldsymbol{v} - \boldsymbol{v}'$ alone.

3 Covariant expression

3.1 Notation

Next we express our key results in covariant form, valid for any velocity coordinates, (ξ^1, ξ^2, ξ^3). The metric tensor is denoted by $g_{\alpha\beta}$:

$$d\boldsymbol{v} \cdot d\boldsymbol{v} = g_{\alpha\beta} d\xi^\alpha d\xi^\beta$$

and its determinant is denoted by g, so that the Jacobian is $\sqrt{g}$:

$$d^3v = \sqrt{g}\, d^3\xi$$

The inverse matrix is $g^{\alpha\beta}$.

Covariant derivatives are denoted by commas. In particular, if S is a velocity-space scalar, then

$$S_{,\alpha,\beta} = \left(\frac{\partial S}{\partial \xi^\alpha}\right)_{,\beta} = \frac{\partial^2 S}{\partial\xi^\alpha \partial\xi^\beta} - \Gamma^\kappa_{\alpha\beta}\frac{\partial S}{\partial \xi^\kappa} \tag{41}$$

Here Γ is the Christoffel symbol, given by

$$\Gamma^\kappa_{\alpha\beta} = \frac{1}{2} g^{\kappa\lambda}\left(\frac{\partial g_{\lambda\beta}}{\partial\xi^\alpha} + \frac{\partial g_{\lambda\alpha}}{\partial\xi^\beta} - \frac{\partial g_{\alpha\beta}}{\partial\xi^\lambda}\right)$$

A scalar that is often useful is the kinetic energy,

$$w \equiv \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}$$

Notice that
$$v_\alpha = w_{,\alpha} \tag{42}$$
is a covariant vector, for any coordinate system. It's normalized version is of course $s_\alpha = w_{,\alpha}/v_t$ with contravariant components
$$s^\alpha \equiv v_t^{-1} g^{\alpha\beta} w_{,\beta} \tag{43}$$

A special case of (41) is
$$\begin{aligned} \nabla^2 S &= g^{\alpha\beta} S_{\alpha,\beta} \\ &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^\alpha} \sqrt{g}\, g^{\alpha\beta} \frac{\partial S}{\partial \xi^\beta} \end{aligned} \tag{44}$$

3.2 Covariant scattering and diffusion

We begin with the velocity-space scalar $u \equiv |\boldsymbol{v} - \boldsymbol{v}'|$, the distance between $\boldsymbol{v}$ and a fixed, chosen point $\boldsymbol{v}'$. Then we can define the covariant vector u_α by
$$u_\alpha = u u_{,\alpha}$$
where the derivative is taken with respect to $\boldsymbol{v}$ as usual. Similarly, the covariant definition of $U_{\alpha\beta}$ is
$$U_{\alpha\beta} \equiv u_{,\alpha,\beta} \tag{45}$$
Notice that the indices of $U_{\alpha\beta}$ are raised by the metric elements evaluated at $\boldsymbol{v}$, never at $\boldsymbol{v}'$.

Some useful identities follow from (5) – (9):
$$u_\kappa g^{\kappa\alpha} U_{\alpha\beta} = 0 \tag{46}$$
$$U^{\alpha\beta}_{\ ,\beta} = g^{\alpha\beta}(2/u)_{,\beta} \tag{47}$$
$$U^{\alpha\beta}_{\ ,\alpha,\beta} = -8\pi\delta(\boldsymbol{u})/\sqrt{g} \tag{48}$$

These imply, as in (21) – (22),
$$D_{b\,\alpha\beta} = G_{b\,,\alpha,\beta} \tag{49}$$
$$D^{\alpha\beta}_{b,\alpha,\beta} = -8\pi f_b \tag{50}$$
$$D^{\ \ \beta}_{b\,\alpha\,,\beta} = 2H_{b\,,\alpha} \tag{51}$$
Note also that the covariant version of (6) implies
$$H_b = \frac{1}{2} g_{\alpha\beta} D^{\alpha\beta}_b \tag{52}$$

The vector given by (51) can be identified with dynamical friction; in fact a convenient notation is
$$\mathcal{F}^\alpha_{ab} = \left(1 + \frac{m_a}{m_b}\right) g^{\alpha\beta} H_{b\,,\beta} = \frac{1}{2}\left(1 + \frac{m_a}{m_b}\right) D^{\alpha\beta}_{b\,,\beta} \tag{53}$$

As usual we use an M-subscript to indicate evaluation on a Maxwellian distribution. The relation (12) has the covariant form

$$D_{Mb\,\alpha\beta} = \frac{n_b}{2v_{tb}}\left(g_{\alpha\beta}M_1 - v_{tb}^{-2}w_{,\alpha}w_{,\beta}M_2\right) \tag{54}$$

where the M's are given by (13) and (14) as before. This is indeed a covariant second-rank tensor, because w, M_1 and M_2 are all velocity-space scalars.

The Maxwellian diffusion tensor satisfies

$$D^{\alpha\beta}_{Mb\,,\alpha} = -\frac{m_b}{T_b}w_{,\alpha}D^{\alpha\beta}_{Mb} \tag{55}$$

Thus the dynamical friction force in the Maxwellian case is

$$\mathcal{F}^{\alpha}_{Mab} = -\frac{m_b}{2T_b}\left(1+\frac{m_a}{m_b}\right)w_{,\alpha}D^{\alpha\beta}_{Mb} \tag{56}$$

3.3 Covariant collision operator

The covariant FP operator—the covariant form of (20)—is

$$C_{ab} = \frac{\gamma_{ab}}{2}\left[\left(f_a D^{\alpha\beta}_b\right)_{,\beta} - 2f_a F^{\alpha}_{ab}\right]_{,\alpha} \tag{57}$$

The more explicit version,

$$C_{ab} = \frac{\gamma_{ab}}{2}\left[8\pi\frac{m_a}{m_b}f_a f_b + f_{a\,,\alpha,\beta}D^{\alpha\beta}_b + \left(1-\frac{m_a}{m_b}\right)f_{a\,,\beta}D^{\alpha\beta}_{b\,,\alpha}\right] \tag{58}$$

is sometimes useful. Here we used (50).

In terms of the Rosenbluth potentials we have

$$C_{ab} = \frac{\gamma_{ab}}{2}\left[\left(f_a g^{\alpha\kappa}g^{\beta\lambda}G_{b\,,\kappa,\lambda}\right)_{,\beta} - 2\left(1+\frac{m_a}{m_b}\right)f_a g^{\alpha\kappa}H_{b\,,\kappa}\right]_{,\alpha} \tag{59}$$

or

$$\begin{aligned} C_{ab} =& \frac{\gamma_{ab}}{2}\left[8\pi\frac{m_a}{m_b}f_a f_b + f_{a\,,\alpha,\beta}g^{\alpha\kappa}g^{\beta\lambda}G_{b\,,\kappa,\lambda}\right. \\ & \left. +2\left(1-\frac{m_a}{m_b}\right)f_{a\,,\kappa}g^{\kappa\lambda}H_{b\,,\lambda}\right] \end{aligned} \tag{60}$$

4 Coordinates for magnetized plasma

4.1 Spherical polar coordinates

These are the coordinates used by RMJ. We begin with

$$\boldsymbol{v} = \boldsymbol{b}v_\parallel + v_\perp(\boldsymbol{e}_1\cos\gamma + \boldsymbol{e}_2\sin\gamma) \tag{61}$$

Here the unit vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{b})$ for a local positive triplet, with $\boldsymbol{b}$ denoting the direction of the magnetic field; γ is the gyrophase angle; and the remaining notation is self-explanatory. We express the parallel and perpendicular speeds in terms of the Cosine of the pitch angle, $\xi \equiv v_\parallel / v$:

$$v_\parallel = v\xi, \;\; v_\perp = v\sqrt{1-\xi^2}$$

By spherical velocity coordinates, we mean the set

$$(\xi_1, \xi_2, \xi_3) = (v, \xi, \gamma)$$

The corresponding metric tensor is diagonal (i.e., the coordinates are orthogonal), with

$$g_{11} = 1, \;\; g_{22} = v^2/(1-\xi^2), \;\; g_{33} = v^2(1-\xi^2)$$

and therefore $\sqrt{g} = v^2$. Because $g_{\alpha\beta}$ is diagonal, its inverse $g^{\alpha\beta}$ is also diagonal, with

$$g^{\alpha\alpha} = \frac{1}{g_{\alpha\alpha}} \quad \text{(no sum)}$$

The only non-vanishing Christoffel symbols are:

$$\begin{aligned}
\Gamma^1_{22} &= -v/(1-\xi^2), \;\; \Gamma^1_{33} = -v(1-\xi^2) \\
\Gamma^2_{22} &= \xi/(1-\xi^2), \;\; \Gamma^2_{33} = \xi(1-\xi^2) \\
\Gamma^2_{12} &= \Gamma^2_{21} = 1/v, \;\; \Gamma^3_{23} = \Gamma^3_{32} = -\xi/(1-\xi^2), \;\; \Gamma^3_{13} = \Gamma^3_{31} = 1/v
\end{aligned}$$

We apply these formulae to the tensor $S_{,\alpha,\beta}$ of (41) and find

$$\begin{aligned}
S_{,1,1} &= \frac{\partial^2 S}{\partial v^2}, \\
S_{,2,2} &= \frac{\partial^2 S}{\partial \xi^2} + \frac{v}{1-\xi^2}\frac{\partial S}{\partial v} - \frac{\xi}{1-\xi^2}\frac{\partial S}{\partial \xi}, \\
S_{,3,3} &= \frac{\partial^2 S}{\partial \gamma^2} + v(1-\xi^2)\frac{\partial S}{\partial v} - \xi(1-\xi^2)\frac{\partial S}{\partial \xi} \\
S_{,1,2} &= S_{,2,1} = \frac{\partial^{2S}}{\partial v \partial \xi} - \frac{1}{v}\frac{\partial S}{\partial \xi} \\
S_{,1,3} &= S_{,3,1} = \frac{\partial^2 S}{\partial v \partial \gamma} - \frac{1}{v}\frac{\partial S}{\partial \gamma} \\
S_{,2,3} &= S_{,3,2} = \frac{\partial^2 S}{\partial \xi \partial \gamma} + \frac{\xi}{1-\xi^2}\frac{\partial S}{\partial \gamma}
\end{aligned}$$

We can combine these results with (44) to compute

$$\nabla^2 S = \frac{1}{v^2}\frac{\partial}{\partial v} v^2 \frac{\partial S}{\partial v} + \frac{1}{v^2}\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial S}{\partial \xi} + \frac{1}{v^2(1-\xi^2)}\frac{\partial^2 S}{\partial \gamma^2} \tag{62}$$

4.2 Quasi-invariant coordinates

When the electrostatic potential changes slowly along the guiding center orbit, the kinetic energy $w = v^2/2$ is nearly constant. Similarly for small gyroradius the magnetic moment $v_\perp^2/2B$ is approximately constant, where B is the magnetic field magnitude. Therefore the coordinates

$$(\xi^1, \xi^2, \xi^3) = (w, \lambda, \gamma) \tag{63}$$

are often convenient. Here $\lambda \equiv v_\perp^2/(2wB)$ is called the pitch-angle variable. Then

$$v_\| = \sigma\sqrt{2w(1-\lambda B)}, \quad v_\perp = \sqrt{2w\lambda B}$$

and the non-vanishing components of the metric tensor are

$$g_{11} = \frac{1}{2w}, \quad g_{22} = \frac{wB}{2\lambda(1-\lambda B)}, \quad g_{33} = 2w\lambda B$$

whence the Jacobian

$$\sqrt{g} = B\sqrt{\frac{w}{2(1-\lambda B)}} = \frac{Bw}{|v_\||}$$

Again the inverse tensor is trivial:

$$g^{\alpha\alpha} = \frac{1}{g_{\alpha\alpha}} \quad \text{(no sum)}$$

The non-vanishing Christoffel symbols are

$$\begin{aligned}
\Gamma^1_{11} &= -\frac{1}{2w}, \ \Gamma^1_{22} = -\frac{wB}{2\lambda(1-\lambda B)}, \ \Gamma^1_{33} = -2w\lambda B \\
\Gamma^2_{22} &= -\frac{1-\lambda B}{2\lambda(1-\lambda B)}, \ \Gamma^2_{33} = -2\lambda(1-\lambda B), \\
\Gamma^3_{13} &= \Gamma^3_{31} = \frac{1}{2w}, \ \Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2\lambda}, \ \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2w},
\end{aligned}$$

and the tensor $S_{,\alpha,\beta}$ has the following components:

$$\begin{aligned}
S_{,1,1} &= \frac{\partial^2 S}{\partial w^2} + \frac{1}{2w}\frac{\partial S}{\partial w} \\
S_{,2,2} &= \frac{\partial^2 S}{\partial \lambda^2} + \frac{1}{2\lambda(1-\lambda B)}\left[(1-2\lambda B)\frac{\partial S}{\partial \lambda} + wB\frac{\partial S}{\partial w}\right] \\
S_{,3,3} &= \frac{\partial^2 S}{\partial \gamma^2} + 2w\lambda B\frac{\partial S}{\partial w} + 2\lambda(1-\lambda B)\frac{\partial S}{\partial \lambda} \\
S_{,1,2} &= S_{,2,1} = \frac{\partial^2 S}{\partial \lambda \partial w} - \frac{1}{2w}\frac{\partial S}{\partial \lambda}, \\
S_{,1,3} &= S_{,3,1} = \frac{\partial^2 S}{\partial w \partial \gamma} - \frac{1}{2w}\frac{\partial S}{\partial \gamma} \\
S_{,2,3} &= S_{,3,2} = \frac{\partial^2 S}{\partial \lambda \partial \gamma} - \frac{1}{2\lambda}\frac{\partial S}{\partial \gamma}
\end{aligned}$$

We can substitute these expressions into (44) to compute

$$\nabla^2 S = \frac{2\sqrt{1-\lambda B}}{wB}\frac{\partial}{\partial\lambda}\lambda\sqrt{1-\lambda B}\frac{\partial S}{\partial\lambda} + \frac{2}{w^{1/2}}\frac{\partial}{\partial w}w^{3/2}\frac{\partial S}{\partial w} + \frac{1}{2w\lambda B}\frac{\partial^2 S}{\partial\gamma^2} \tag{64}$$

These formulae, or those of Subsection 4.1, can be directly substituted into (60), with $S = f_a$ and $S = G_b$. Thus we obtain an explicit expression for the general nonlinear collision operator in the two magnetized-plasma coordinate systems.

As a simple application of quasi-invariant coordinates, we verify (55). According to (51) it's left-hand side is (suppressing the species-subscript)

$$D^{\alpha\beta}_{M\ ,\beta} = g^{\alpha\kappa}D^{\beta}_{M\ \kappa,\beta} = 2g^{\alpha\kappa}H_{M\ ,\kappa}$$

In quasi-invariant coordinates, the covariant deriviative of H_M has only a single component, $H_{M\ ,\kappa} = \delta_{\kappa\,1}dH_M/dw$, and

$$\frac{d}{dw} = \frac{1}{v_t^2 s}\frac{d}{ds}$$

so (27) gives

$$\begin{aligned} D^{\alpha\beta}_{M\ ,\beta} &= \frac{2n}{v_t}g^{\alpha\kappa}\delta_{\kappa\,1}\frac{1}{v_t^2 s}\frac{d}{ds}\left(\frac{\mathrm{erf}}{s}\right) \\ &= \frac{2n}{v_t^3}g^{\alpha\,1}\frac{1}{s}\frac{d}{ds}\left(\frac{\mathrm{erf}}{s}\right) \end{aligned} \tag{65}$$

On the right-hand side we use (54) to write

$$-\frac{m}{T}v_t s_\beta D^{\alpha\beta}_{M} = -\frac{n}{v_t^2}s^\alpha(M_1 - s^2 M_2)$$

where s^α is the normalized contravariant velocity of (43). In quasi-invariant coordinates,

$$s^\alpha = v_t^{-1}g^{\alpha 1}$$

so (15) gives

$$-\frac{m}{T}v_t s_\beta D^{\alpha\beta}_{M} = -\frac{2n}{v_t^3}g^{\alpha 1}s^{-3}(\mathrm{erf} - s\,\mathrm{erf}') = \frac{2n}{v_t^3}g^{\alpha 1}\frac{1}{s}\frac{d}{ds}\left(\frac{\mathrm{erf}}{s}\right)$$

in agreement with (65).

5 Perturbed Maxwellian case

5.1 Maxwellian limit

When both distributions are Maxwellian, only energy-derivatives survive and (60) reduces to

$$\frac{C_{Mab}}{\gamma_{ab}} = 4\pi \frac{m_a}{m_b} f_{Ma} f_{Mb} + \left(1 - \frac{m_a}{m_b}\right) 2w \frac{dH_{Mb}}{dw} \frac{df_{Ma}}{dw} + \frac{dG_{Mb}}{dw} \frac{df_{Ma}}{dw} + 2w^2 \left(\frac{d^2 G_{Mb}}{dw^2} + \frac{1}{2w} \frac{dG_{Mb}}{dw}\right)\left(\frac{d^2 f_{Ma}}{dw^2} + \frac{1}{2w} \frac{df_{Ma}}{dw}\right) \tag{66}$$

Here

$$\frac{df_{Ma}}{dw} = -\frac{m_a}{T_a} f_{Ma}$$

while from (26) and the definitions of the M's we find that

$$\frac{dG_{Mb}}{dw} = \frac{n_b}{2v_{tb}} M_1, \tag{67}$$

$$\frac{d^2 G_{Mb}}{dw^2} = -\frac{n_b}{2v_{tb}^3} M_2 \tag{68}$$

Similarly (27) yields

$$2w \frac{dH_{Mb}}{dw} = \frac{n_b}{v_{tb}} s \frac{d}{ds} \left(\frac{\mathrm{erf}}{s}\right) \tag{69}$$

Substituting these expressions into (66) one finds, after some manipulation,

$$C_{Mab} = \bar{\nu}_{ab} \left(1 - \frac{T_b}{T_a}\right) f_{Ma} \left[\mathrm{erf}'(s) \left(1 + \frac{m_b T_a}{m_a T_b}\right) - \frac{\mathrm{erf}(s)}{s}\right] \tag{70}$$

where

$$\bar{\nu}_{ab} \equiv \frac{2\gamma_{ab} m_a n_b}{m_b v_{tb} v_{ta}^2} \tag{71}$$

5.2 Linear perturbation

We write

$$f_a = f_{Ma}(1 + \hat{f}_a)$$

and neglect terms of order $\hat{f}^2$ to obtain the linearized operator

$$C_{ab}^{\ell}(\hat{f}_a, \hat{f}_b) = C_{ab}(f_a, f_b) - C_{Mab} + \mathcal{O}(\hat{f}^2) \tag{72}$$

We begin with the Landau-Boltzmann form, (1), which reduces to

$$C_{ab}^{\ell} = \frac{\gamma_{ab}}{2} \frac{\partial}{\partial v_\alpha} \int d^3v\,' f_{Ma} f'_{Mb} U_{\alpha\beta} \mathcal{C}_{ab\,\beta} \tag{73}$$

with

$$\mathcal{C}_{ab\,\beta} \equiv \frac{\partial \hat{f}_a}{\partial v_\beta} - \frac{m_a}{m_b} \frac{\partial \hat{f}'_b}{\partial v'_\beta} - \frac{m_a v_\beta}{T_a} \left(1 - \frac{T_a}{T_b}\right) (\hat{f}_a + \hat{f}'_b) \tag{74}$$

This form evidently simplifies in the like-particle case. For unlike particles, one usually resorts to mass-ratio approximation, as discussed in Section 6. Therefore at this point we specialize to the like-particle operator, given by

$$C^{\ell} = \frac{\gamma}{2} \frac{\partial}{\partial v_\alpha} \int d^3v\,' f_M f'_M U_{\alpha\beta} \left(\frac{\partial \hat{f}}{\partial v_\beta} - \frac{\partial \hat{f}'}{\partial v'_\beta} \right) \tag{75}$$

with species labels suppressed.

After writing (75) in covariant form, we integrate by parts and then simplify using (55). The result is

$$\begin{aligned} C^{\ell} = \gamma f_M \Bigg[& 4\pi f_M \hat{f} + \frac{1}{2} \left(\hat{f}_{,\alpha,\beta} - \frac{2m}{T} w_{,\alpha} \hat{f}_{,\beta} \right) D_M^{\alpha\beta} \\ & - \frac{m}{T} H_1 + \frac{1}{2} \left(\frac{m}{T}\right)^2 w_{,\alpha} w_{,\beta} D_1^{\alpha\beta} \Bigg] \end{aligned} \tag{76}$$

Here

$$\begin{aligned} H_1 &\equiv \int d^3v\,' f_M \hat{f} u^{-1} && (77) \\ D_1^{\alpha\beta} &\equiv g^{\alpha\kappa} g^{\beta\lambda} \int d^3v\,' f_M \hat{f} u_{,\kappa,\lambda} && (78) \end{aligned}$$

The factor of 2 in the third term of (76) deserves comment. It comes from the sum of two contributions, one being the obvious derivative of the Maxwellian. But there is an equal contribution, because of (55), from the derivative of the Maxwellian diffusion tensor.

Finally we introduce

$$G_1 \equiv \int d^3v\,' f_M \hat{f} u \tag{79}$$

and recall (50) in order to express (76) in the form

$$\begin{aligned} C^{\ell} = & \gamma f_M \frac{n}{v_t} \left[\frac{2\,\mathrm{erf}'}{v_t^2} \hat{f} + \frac{1}{4} \left(g^{\alpha\beta} M_1 - s^\alpha s^\beta M_2 \right) \left(\hat{f}_{,\alpha,\beta} - \frac{4}{v_t} s_\alpha \hat{f}_{,\beta} \right) \right] \\ & + \gamma f_M \frac{m}{T} \left(s^\alpha s^\beta G_{1\,,\alpha,\beta} - H_1 \right) \end{aligned} \tag{80}$$

5.3 Linear operator in quasi-invariant coordinates

We use (76) to write the linear operator in quasi-invariant coordinates. The result is simplified using (15) and the identity

$$1 - \frac{3}{2}\lambda B = \sqrt{1-\lambda B}\frac{\partial}{\partial\lambda}\lambda\sqrt{1-\lambda B}$$

Thus we find

$$\begin{aligned} C^\ell = \gamma f_M \frac{n}{v_t}\Bigg[&\frac{M_1}{2wB}\left(\sqrt{1-\lambda B}\frac{\partial}{\partial\lambda}\lambda\sqrt{1-\lambda B}\frac{\partial\hat f}{\partial\lambda} + \frac{1}{4\lambda B}\frac{\partial^2\hat f}{\partial\gamma^2}\right)\\ &+\frac{\operatorname{erf}(s) - s\operatorname{erf}'(s)}{s}\frac{T}{m}\frac{\partial^2\hat f}{\partial w^2} + \frac{2s\operatorname{erf}'(s) - \operatorname{erf}(s)}{s}\frac{\partial\hat f}{\partial w}\\ &+\frac{m}{T}\operatorname{erf}'(s)\hat f\Bigg] + \gamma f_M\frac{m}{T}\left[\left(2w\frac{\partial^2 G_1}{\partial w^2} + \frac{\partial G_1}{\partial w}\right) - H_1\right] \end{aligned} \tag{81}$$

We could also use (24) and (64) to write

$$\begin{aligned} H_1 &= \frac{1}{2}\nabla^2 G_1\\ &= \frac{\sqrt{1-\lambda B}}{wB}\frac{\partial}{\partial\lambda}\lambda\sqrt{1-\lambda B}\frac{\partial G_1}{\partial\lambda}\\ &+ \frac{1}{w^{1/2}}\frac{\partial}{\partial w}w^{3/2}\frac{\partial G_1}{\partial w} + \frac{1}{4w\lambda B}\frac{\partial^2 G_1}{\partial\gamma^2} \end{aligned} \tag{82}$$

and thus express the linear operator in terms of G_1 alone; however the form (81) is usually most convenient.

5.4 Linear operator in spherical coordinates

Returning to (80), we insert the geometrical coefficients from Subsection 4.1 to obtain the linear operator

$$\begin{aligned} \frac{C^\ell}{\gamma f_M} = \frac{n}{4v_t}&\left[M_1\nabla^2\hat f - s^2 M_2\frac{\partial^2\hat f}{\partial v^2} - \frac{2m}{T}(M_1 - s^2 M_2)v\frac{\partial\hat f}{\partial v}\right]\\ &+ 4\pi f_M\hat f + \frac{m}{T}\left(s^2\frac{\partial^2 G_1}{\partial v^2} - H_1\right) \end{aligned} \tag{83}$$

The first term here could be made explicit by means of (62).

6 Small mass-ratio approximations

6.1 Scattering of electrons by ions

For simplicity and concreteness we consider here a plasma with electrons and a single ion species; the relevant small parameter is m_e/m_i.

All small mass-ratio approximations are based on the assumptions that both distributions are nearly Maxwellian, with comparable temperatures. Thus thermal speeds can be defined for both species, and $v_{te} \gg v_{ti}$.

Cartesian coordinates

To find the approximate form of C_{ei} we can begin with (1), using Cartesian coordinates. After dropping the explicit m_e/m_i we have

$$C_{ei} = \frac{\gamma_{ei}}{2} \frac{\partial}{\partial v_\alpha} \int d^3v\,'\, f_i' U_{\alpha\beta} \frac{\partial f_e}{\partial v_\beta} \tag{84}$$

The sole additional approximation is to write

$$\int d^3v\,' f_i' U_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{v}') \approx U_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{V}_i)$$

where $\boldsymbol{V}_i$ is the mean ion flow,

$$n_i \boldsymbol{V}_i \equiv \int d^3v\, f_i \boldsymbol{v},$$

and we implicitly neglect terms involving more than one power of V_i, assuming $V_i \ll v_{ti}$. One then finds that

$$U_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{V}_i) = V_{\alpha\beta} + V_{\alpha\beta}^{(1)}$$

where

$$V_{\alpha\beta} = U_{\alpha\beta}(\boldsymbol{v}' \to 0) = \frac{v^2\delta_{\alpha\beta} - v_\alpha v_\beta}{v^3}$$

and

$$V_{\alpha\beta}^{(1)} = \frac{v_\alpha V_{i\beta} + v_\beta V_{i\alpha} + \delta_{\alpha\beta} \boldsymbol{V}_i \cdot \boldsymbol{v}}{v^3} - 3\frac{v_\alpha v_\beta \boldsymbol{V}_i \cdot \boldsymbol{v}}{v^5}$$

Notice that (5) implies

$$v_\alpha V_{\alpha\beta}^{(1)} = V_{i\alpha} V_{\alpha\beta} \tag{85}$$

The nonlinear C_{ei} is now given by

$$C_{ei} = \frac{\gamma_{ei}}{2} \frac{\partial}{\partial v_\alpha} U_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{V}_i) \frac{\partial f_e}{\partial v_\beta} \tag{86}$$

The linear version has particular importance. Using

$$\frac{\partial f_e}{\partial v_\beta} = f_{Me} \left[\frac{\partial \hat{f}_e}{\partial v_\beta} - \frac{2v_\beta}{v_{te}^2} f_{Me} \left(1 + \hat{f}_e\right) \right]$$

an obvious consequence of (8),

$$\frac{\partial V_{\alpha\beta}}{\partial v_\alpha} = -\frac{2v_\beta}{v^3}$$

and (85), we find

$$C_{ei}^\ell = \frac{\gamma_{ei}}{2} f_{Me} \frac{\partial}{\partial v_\alpha} \left(V_{\alpha\beta} \frac{\partial \hat{f}_e}{\partial v_\beta} - \frac{4V_{i\,\alpha}}{v_{te}^2 v} \right) \tag{87}$$

This form can also be derived from the Galilean invariance of C_{ei}, as expressed by (31). Under the transformation $\boldsymbol{v} \to \bar{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{V}_i$ the ion distribution

$$f_i(\boldsymbol{v}) \approx f_{Mi}(\boldsymbol{v} - \boldsymbol{V}_i)$$

becomes

$$\bar{f}_i(\bar{\boldsymbol{v}}) = f_i(\boldsymbol{v}) = f_i(\bar{\boldsymbol{v}} + \boldsymbol{V}_i) = f_{Mi}(\bar{\boldsymbol{v}}),$$

a Maxwellian at rest. The electron distribution,

$$f_e(\boldsymbol{v}) = f_{Me}(\boldsymbol{v})[1 + \hat{f}_e(\boldsymbol{v})]$$

becomes

$$\begin{aligned}
\bar{f}_e(\bar{\boldsymbol{v}}) &= f_e(\boldsymbol{v}) = f_e(\bar{\boldsymbol{v}} + \boldsymbol{V}_i) \\
&= f_{Me}(\bar{\boldsymbol{v}} + \boldsymbol{V}_i)[1 + \hat{f}_e(\bar{\boldsymbol{v}} + \boldsymbol{V}_i)] \\
&\approx f_{Me}(\bar{\boldsymbol{v}}) \left(1 - \frac{m_e \bar{\boldsymbol{v}} \cdot \boldsymbol{V}_i}{T_e} \right) [1 + \hat{f}_e(\bar{\boldsymbol{v}})] \\
&= f_{Me}(\bar{\boldsymbol{v}}) \left[1 - \frac{m_e \bar{\boldsymbol{v}} \cdot \boldsymbol{V}_i}{T_e} + \hat{f}_e(\bar{\boldsymbol{v}}) \right] \qquad (88)
\end{aligned}$$

Therefore (31) implies, in an abbreviated notation,

$$C_{ei}^\ell(\hat{f}_e; V_i) = C_{ei}^\ell(\hat{f}_e - m_e \boldsymbol{V}_i \cdot \boldsymbol{v}/T_e; V_i = 0) \tag{89}$$

It is not hard to verify that (87) satisfies this relation.

We take note here of the approximate Rosenbluth potentials:

$$\begin{aligned}
G_i &\approx n_i v \left(1 - v^{-2} \boldsymbol{V}_i \cdot \boldsymbol{v} \right) \qquad (90) \\
H_i &\approx (n_i/v) \left(1 + v^{-2} \boldsymbol{V}_i \cdot \boldsymbol{v} \right) \qquad (91)
\end{aligned}$$

These relations imply

$$D_{i\alpha\beta} = \frac{\partial^2 G_i}{\partial v_\alpha \partial v_\beta} \approx n_i U_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{V}_i)$$

and

$$\frac{\partial H_i}{\partial v_\beta} \approx \left(\frac{v_\beta - V_{i\beta}}{v^3} + 3 \frac{\boldsymbol{V}_i \cdot \boldsymbol{v} v_\beta}{v^5} \right)$$

In view of (24), this quantity can be written as

$$\frac{\partial H_i}{\partial v_\beta} = \frac{n_i}{2}\frac{\partial}{\partial v_b}U_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{V}_i) \tag{92}$$

It is then straightforward to confirm that (90) – (92) yield (86).

Quasi-invariant coordinates

Here, in addition to using quasi-invariant coordinates (w, λ, γ), we assume that the kinetic equation ("drift-kinetic equation") involves only the gyrophase average of the distribution:

$$f(w, \lambda, \gamma) \approx \bar{f}(w, \lambda)$$

The overbar is henceforth suppressed. For such a distribution, the mean velocity is parallel to the magnetic field:

$$\boldsymbol{V} = \boldsymbol{b}V_\parallel$$

Drift-kinetic theory treats perpendicular flow, which can be comparable to $V_\parallel$, by means other than the kinetic equation. Using (90) and (91) we find

$$\begin{aligned} C_{ei} = \gamma_{ei}n_i \Bigg[&\frac{\xi}{wB}\left(\frac{1}{\sqrt{2w}} + \frac{V_{\parallel i}\sigma\xi}{2w}\right)\frac{\partial}{\partial\lambda}\lambda\xi\frac{\partial f_e}{\partial\lambda} \\ &- \frac{V_{\parallel i}\sigma\xi}{2w}\frac{\partial}{\partial w}\left(f_e + 2\lambda\frac{\partial f_e}{\partial\lambda}\right)\Bigg] \end{aligned} \tag{93}$$

where $\xi \equiv \sqrt{1 - \lambda B}$ as in Subsection 4.2.

The linear version of this operator is

$$C_{ei}^\ell = \gamma_{ei}n_i f_{Me}\frac{\xi}{2w^2}\left(\frac{\sqrt{2w}}{B}\frac{\partial}{\partial\lambda}\lambda\xi\frac{\partial\hat{f}_e}{\partial\lambda} + \frac{m_e V_{\parallel i}\sigma w}{T_e}\right) \tag{94}$$

Although C_{ei} is a rotationally symmetric operator, the right-hand side of (94) is not a symmetric function of velocity, as manifested by the factor σ that occurs with $V_{\parallel i}$. Ion parallel flow breaks the symmetry: $\bar{f}_i(\bar{\boldsymbol{v}}) = f_i(\boldsymbol{v}) \neq f_i(\bar{\boldsymbol{v}})$. This circumstance was anticipated at the end of subsection 2.2.

The coefficient γ_{ei} contains a factor $e_i^2 = Z^2e^2$ and therefore can exceed γ_{ee} for multiply charged ion species. In other respects however, we see that

$$C_{ei} \sim C_{ee}$$

That is, electrons are scattered by ions at a rate comparable to their self-scattering.

6.2 Scattering of ions by electrons

A convenient starting point for C_{ie} is (60); using Cartesian coordinates for simplicity we have

$$\begin{aligned} C_{ie} = &\frac{\gamma_{ie}}{2}\left[8\pi\frac{m_i}{m_e}f_if_e + \frac{\partial^2 f_i}{\partial v_\alpha \partial v_\beta}\frac{\partial^2 G_e}{\partial v_\alpha \partial v_\beta}\right. \\ &\left. + 2\left(1-\frac{m_i}{m_e}\right)\frac{\partial f_i}{\partial v_\beta}\frac{\partial H_e}{\partial v_\beta}\right] \end{aligned} \tag{95}$$

In the first two terms on the right-hand side we need retain only the contributions from the electron Maxwellian; the non-Maxwellian electron distribution $\hat{f}_e$ has no special qualitative significance and offers only a small correction. Thus

$$\frac{m_i}{m_e}f_ef_i \approx \frac{m_i}{m_e}\frac{n_e}{v_{te}^3}f_i$$

since $v/v_{te} \sim v_{ti}/v_{te}$ is neglected. For the second term, involving $G_e \approx G_{Me}$, we use (50), (54) and (18) to find

$$\frac{\partial^2 G_e}{\partial v_\alpha \partial v_\beta} \approx \delta_{\alpha\beta}\frac{4n_e}{3\sqrt{\pi}v_{te}}$$

The last term in (95), involving H_e, is slightly more complicated. The point is that a straightforward mass-ratio expansion of H_e gives

$$\frac{\partial H_e}{\partial v_\beta} = \int d^3v' f_e' |v'|^{-3}(v_\beta' + \cdots) \tag{96}$$

Here the leading term is of order $n_e v_{te}^{-2}\hat{f}_e$: it is relatively large with respect to the mass ratio, but can contribute only in the case of a non-isotropic perturbation to the electron distribution. Thus the perturbation $\hat{f}_e$ does have special qualitative significance in this case, and we must retain both H_{Me} and H_{1e}. For H_{Me} we use (27) to find

$$\frac{\partial H_{Me}}{\partial v_\beta} \approx -\frac{4n_e}{3\sqrt{\pi}v_{te}^3}v_\beta$$

The lowest-order contribution to the correction is, as noted above,

$$\frac{\partial H_{1e}}{\partial v_\beta} = \int d^3v' f_e' \frac{v_\beta'}{v'^3}$$

In Section 7 we show that this quantity is related to the collisional friction force $\boldsymbol{F}_{ei}$:

$$\frac{\partial H_{1e}}{\partial v_\beta} = \frac{n_e V_{i\beta}}{\gamma_{ei}\tau_e n_i} - \frac{F_{ei\,\beta}}{\gamma_{ei}n_i m_e} \tag{97}$$

After combining these results and recalling (3), we obtain the operator

$$\begin{aligned} C_{ie} &= \frac{m_e n_e}{m_i n_i \tau_{ei}} \left[3f_i + \frac{T_e}{m_i} \frac{\partial^2 f_i}{\partial v_\alpha \partial v_\beta} + (v_\beta - V_{i\beta}) \frac{\partial f_i}{\partial v_\beta} \right] + \frac{F_{ei\,\beta}}{m_i n_i} \frac{\partial f_i}{\partial v_\beta} \\ &= \frac{m_e n_e}{m_i n_i \tau_{ei}} \frac{\partial}{\partial v_\beta} \left[(v_\beta - V_{i\beta}) f_i + \frac{T_e}{m_i} \frac{\partial f_i}{\partial v_\beta} \right] + \frac{F_{ei\,\beta}}{m_i n_i} \frac{\partial f_i}{\partial v_\beta} \end{aligned} \tag{98}$$

As a check on (98), we consider the strictly Maxwellian limit, with $V_i = 0 = F_{ei}$:

$$\begin{aligned} C_{Mie} &= \frac{m_e n_e}{m_i n_i \tau_{ei}} \frac{\partial}{\partial v_\alpha} \left(v_\alpha f_{Mi} + \frac{T_e}{m_i} \frac{\partial f_{Mi}}{\partial v_\alpha} \right) \\ &= \frac{m_e n_e}{m_i n_i \tau_{ei}} \left(1 - \frac{T_e}{T_i} \right) f_{Mi} \left(3 - \frac{2v^2}{v_{ti}^2} \right) \end{aligned} \tag{99}$$

To compare this result with the corresponding expansion of (70), we write the latter as

$$C_{Mie} = \frac{2\gamma_{ie} m_i n_e}{m_e v_{te}^3} \left(1 - \frac{T_e}{T_i} \right) \mathcal{C}(s) \tag{100}$$

where

$$\mathcal{C}(s) = \left(1 + \frac{m_i T_e}{m_e T_i} \right) \mathrm{erf}'(s) - \frac{m_i T_e}{m_e T_i} \frac{\mathrm{erf}(s)}{s}$$

and expand $\mathcal{C}$ for $s \ll 1$:

$$\begin{aligned} \mathcal{C} &= \frac{2}{\sqrt{\pi}} \left[1 - \frac{m_i T_e}{m_e T_i} \left(\frac{2}{3} s^2 + \mathcal{O}(s^4) \right) \right] \\ &= \frac{2}{3\sqrt{\pi}} \left[3 - 2 \frac{v^2}{v_{ti}^2} \left(1 + \mathcal{O}(s^2) \right) \right] \end{aligned}$$

In view of (3), substitution of this result into (100) reproduces (99).

7 Moments of the operator

The zeroth moment of the Coulomb collision operator is easily seen to vanish:

$$\int d^3v\, C_{ab} = 0,$$

for any physically reasonable distributions. The other moments of general interest are the collisional momentum exchange, or "friction force,"

$$\boldsymbol{F}_{ab} \equiv \int d^3v\, m_a \boldsymbol{v} C_{ab} \tag{101}$$

and the collisional energy exchange

$$W_{ab} \equiv \int d^3v\, \frac{1}{2} m_a v^2 C_{ab} \tag{102}$$

We consider these moments in order.

7.1 Friction force

It is convenient to write the collision in RMJ form, using (23). After substitution into (101) and partial integration, we find

$$\boldsymbol{F}_{ab} \; = -m_a \gamma_{ab} \left(1 + \frac{m_a}{m_b}\right) \int d^3v\, f_a \frac{\partial H_b}{\partial \boldsymbol{v}} \tag{103}$$

To proceed further, we resort to mass-ratio approximation. It is awkward to apply the mass-ratio expansion directly to (103), because of singularities that appear in the expansion of $|\boldsymbol{v} - \boldsymbol{v}'|^{-3}$, so we use (87). (The following argument has appeared previously[3]; see also[2].) After integration by parts we have

$$F_{ei\,\gamma} = -\frac{\gamma_{ei} n_i}{2} \int d^3\, v \left(V_{\alpha\beta} \frac{\partial \hat{f}_e}{\partial v_\alpha} - 4 \frac{V_{i\,\beta}}{v_{te}^2 v} \right) \frac{\partial}{\partial v_\beta} v_{\gamma f_{Me}}$$

But

$$\frac{\partial}{\partial v_\beta} v_{\gamma f_{Me}} = f_{Me} \left(\delta_{\beta\gamma} - 2 \frac{v_\beta v_\gamma}{v_{te}^2} \right)$$

and

$$\int d^3v\, f_{Me} \frac{1}{v} \left(\delta_{\beta\gamma} - 2 \frac{v_\beta v_\gamma}{v_{te}^2} \right) = \frac{2\sqrt{\pi}}{3} \frac{n_e}{v_{te}} \delta_{\beta\gamma}$$

Therefore, in view of (3) and (5),

$$F_{ei} = -\frac{\gamma_{ei} n_i}{2} \int d^3v\, f_{Me} V_{\alpha\gamma} \frac{\partial \hat{f}_e}{\partial v_\alpha} + \frac{m_e n_e}{\tau_{ei}} V_{i\,\gamma} \tag{104}$$

Another integration by parts, with (8), gives an equally useful form:

$$\boldsymbol{F}_{ei} = -\gamma_{ei} n_i \int d^3v\, f_e \frac{\boldsymbol{v}}{v^3} + \frac{m_e n_e}{\tau_{ei}} V_{i\,\gamma} \tag{105}$$

Notice that (105) confirms (97). The appearance of τ_{ei} with unit coefficient reflects the historical definition of the collision time ("momentum exchange time").

7.2 Energy exchange

We compute the energy exchange assuming all species have Maxwellian distributions in a common rest frame. The simplest calculation uses (99), but it is somewhat instructive to eschew mass-ratio approximation and use (70). Thus we have

$$\begin{aligned}
W_{ab} &= 4\pi \frac{m_a}{2} v_{tb}^5 \int ds\, s^4 C_{Mab} \\
&= -2\frac{\gamma_{ab} n_a n_b m_a^2}{v_{ta} m_b} \left(\frac{v_{tb}}{v_{ta}}\right)^4 \left(1 - \frac{T_b}{T_a}\right) \\
&\times \int_0^\infty ds\, s^4 \operatorname{erf}'(\beta s) \left[\frac{\operatorname{erf}(s)}{s} - \left(1 + \frac{m_b T_a}{m_a T_b}\right) \operatorname{erf}'(s)\right] \qquad (106)
\end{aligned}$$

Here

$$\beta \equiv \frac{v_{tb}}{v_{ta}}$$

so that, in particular,

$$f_{Ma} = \frac{n_a}{2\pi v_{ta}^3} \operatorname{erf}'(\beta s)$$

and furthermore,

$$1 + \frac{m_b T_a}{m_a T_b} = \frac{1+\beta^2}{\beta^2}$$

It is not hard to show, by partial integration, that

$$\int_0^\infty ds\, s^3 \operatorname{erf}'(\beta s) \operatorname{erf}(s) = \frac{2}{\sqrt{\pi}} \frac{2 + 3\beta^2}{4\beta^4 (1+\beta^2)^{3/2}}$$

and that

$$\int_0^\infty ds\, s^4 \operatorname{erf}'(\beta s) \operatorname{erf}'(s) = \frac{2}{\sqrt{\pi}} \frac{3}{(1+\beta^2)^{5/2}}$$

After substitution in (106) we find

$$W_{ab} = -4\gamma_{ab} n_a n_b \frac{m_a \beta^4}{m_b v_{ta}^3} (T_a - T_b) \mathcal{W}$$

where $\mathcal{W}$ represents the square-bracketted quantity in (106), coming from the various integrals:

$$\begin{aligned}
\mathcal{W} &= \frac{2}{\sqrt{\pi}} \left[\frac{2 + 3\beta^2}{4\beta^4 (1+\beta^2)^{3/2}} - \frac{1+\beta^2}{\beta^2} \frac{3}{4(1+\beta^2)^{5/2}}\right] \\
&= \frac{2}{\sqrt{\pi}} \frac{(2 + 3\beta^2 - 3\beta^2)}{4\beta^4 (1+\beta^2)^{3/2}} \qquad (107)
\end{aligned}$$

The sole noteworthy feature of the calculation is the cancellation, in (107), of the two β^2-terms in the numerator; since $\beta^2 \sim m_a/m_b$, the result is to insert a perhaps unanticipated factor m_b/m_a into W_{ab}. We have

$$W_{ab} = -4\gamma_{ab} n_a n_b \frac{m_a}{m_b} \frac{T_a - T_b}{(v_{ta}^2 + v_{tb}^2)^{3/2}} \tag{108}$$

which pertains for arbitrary mass ratio. The small mass-ratio limit of W_{ei} coincides with the appropriate moment of (99). Note that the right-hand side of (108) is antisymmetric under species interchange,

$$W_{ab} = -W_{ba}$$

as required by energy conservation.

References

[1] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions*. National Bureau of Standards, 1964.

[2] S. I. Braginskii. Transport processes in a plasma. In M. A. Leontovich, editor, *Reviews of Plasma Physics*, volume 1. Consultants Bureau, New York, 1965.

[3] R. D. Hazeltine and J. D. Meiss. *Plasma Confinement*. Frontiers in Physics. Addison-Wesley, Redwood City, 1992.

[4] R. D. Hazeltine and F. L. Waelbroeck. *The Framework of Plasma Physics*. Perseus Books, Reading, Massachussetts, 1998.

[5] M. N. Rosenbluth, William M. MacDonald, and David L. Judd. Fokker-Planck equation for an inverse-square force. *Physical Review*, 107:1–6, 1957.