Electrostatic potential fluctuations in a Maxwellian plasma

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The spatial correlation function of a Maxwellian plasma with perturbations arising in the electrostatic potential due to random ion density fluctuations is examined. The entropy is found from the one-particle distribution function using the Shannon formula and then, using the Einstein method, the probability distribution for the electrostatic potential fluctuation is determined. This straightforward procedure is demonstrated to be a powerful tool in studying plasma correlation functions when the system entropy can be computed. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2167585]

The Einstein method provides a straightforward way[1,2] of computing fluctuation spectra based on the inversion of Boltzmann's entropy formula, assuming that the entropy of the system is known.[3] Here, the procedure is used to study the electrostatic potential fluctuations arising in an ideal plasma due to random motions. The applicability of the Einstein method depends on the knowledge of the entropy; we use the Shannon formula[4] to compute the entropy from the one-particle distribution function.[5] Our result is not surprising: the correlation function decays over a Debye length. Our main goal is to illustrate the power of the Einstein method in calculating the spatial correlations of an electrostatic field.

Let a system be parametrized by a set of variables $\boldsymbol{\gamma}=\{\gamma_i\}$ such that $\boldsymbol{\gamma}=0$ when the system is in thermal equilibrium. For sufficiently small γ_i the entropy of the system can be defined for nonequilibrium states, such that it satisfies the Boltzmann formula

$$S(\boldsymbol{\gamma}) = \log P(\boldsymbol{\gamma}),$$

where P is the probability of observing the state having the fluctuation $\boldsymbol{\gamma}$ from the equilibrium state. Inverting the Boltzmann formula produces the Einstein distribution,

$$P(\boldsymbol{\gamma}) = e^{S(\boldsymbol{\gamma})}, \tag{1}$$

from which all the statistical properties of the γ_i can be computed if the system entropy is known. The spatial correlation function, in terms of the probability distribution, Eq. (1), is then

$$[\gamma(\boldsymbol{x})\gamma(\boldsymbol{x}')] \equiv c(\boldsymbol{x},\boldsymbol{x}') = \int D[\boldsymbol{\gamma}]\gamma(\boldsymbol{x})\gamma(\boldsymbol{x}')e^{S(\boldsymbol{\gamma})}, \tag{2}$$

where $\int D[\boldsymbol{\gamma}]$ denotes a functional integral.

Since the entropy of the system is a maximum in the equilibrium state, it can contain no terms linear in the fluctuating parameters, γ_i. We have already assumed that the γ_i are small, so we can conclude that the probability distribution is well approximated by the quadratic terms of the entropy. If S_0 denotes the equilibrium entropy, then the nonequilibrium state has an entropy

$$S(\boldsymbol{\gamma}) = S_0 + \Delta S,$$

where

$$\Delta S = -\frac{1}{2}\int d^3x\, d^3x'\ \gamma(\boldsymbol{x})\sigma(\boldsymbol{x},\boldsymbol{x}')\gamma(\boldsymbol{x}'), \tag{3}$$

thus defining the entropy kernel, σ. Our restriction to quadratic terms implies Gaussian statistics; the spatial correlation function then satisfies the integral equation

$$\int d^3x\ \sigma(\boldsymbol{x},\boldsymbol{x}'')c(\boldsymbol{x}'',\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{4}$$

Note that the Einstein method is not capable of describing the relaxation process, only the final equilibrium state. Indeed, this indifference to the relaxation mechanism is a virtue of the method.

The usefulness of the Einstein method is dependent upon knowledge of the system entropy, a generally formidable task. We use the Shannon formula[4] to find the nonequilibrium entropy,

$$S = -\int d^3x\, d^3v\, f \log\left[\left(\frac{h}{m}\right)^3 f\right], \tag{5}$$

where $f(\boldsymbol{x},\boldsymbol{v})$ is the one-particle distribution function, m is the particle mass, and h is Planck's constant. We will consider a distribution function of the form

$$f(\boldsymbol{x},\boldsymbol{v}) = f_{M0}(\boldsymbol{v})[1+\alpha+\hat{f}(\boldsymbol{x},\boldsymbol{v})], \tag{6}$$

where $\alpha=\Delta n(\boldsymbol{x})/n_0$ is the normalized fluctuation in the density, $\hat{f}\ll 1$ describes the plasma response to that fluctuation, and f_{M0} is the spatially homogeneous Maxwellian,

$$f_{M0} \equiv n_0\left(\frac{m}{2\pi T}\right)^{3/2} e^{-mv^2/2T}.$$

Here, n_0 is the lowest-order constant density, and T is the (constant) temperature in units of energy.

Putting the perturbed distribution function, Eq. (6), into the Shannon formula, Eq. (5), neglecting terms linear in the fluctuation α ($\hat{f}$ is also linear in α) since they do not contrib-

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ute to the spatial integral, and assuming that the quadratic terms dominate the statistics, the result is conveniently expressed as

$$S = S_0 + \Delta S,$$

where S_0 is the lowest-order, constant Maxwellian term corresponding to the entropy of an ideal gas. The fluctuations are thus described by the second-order correction to the entropy,

$$\Delta S = -\frac{1}{2}\int d^3x\, d^3v\, f_{M0}(\alpha + \hat{f})^2. \tag{7}$$

The application of the Einstein method is indifferent to the particular source of the fluctuation; all that matters is that a random fluctuation, Δn_i, of the ion density from the equilibrium density profile has occurred. We are then concerned with how the plasma will respond to this perturbation. The localized buildup of excess charge associated with the ion density perturbation will induce an electrostatic perturbation, $\mathbf{\Delta\Phi}$. We write the normalized fluctuation in the distribution function, indicated in Eq. (6), as

$$\frac{f - f_{M0}}{f_{M0}} = \alpha + \hat{f}, \tag{8}$$

where $\alpha = \Delta n / n_0$ is the normalized ion density perturbation imposed on the system and $\hat{f}$ is the normalized plasma response.

In general, the normalized response is found from kinetic theory, but here we suppose that the time scale of the fluctuations is slow enough to allow the ion species to respond adiabatically. In other words, the plasma response is determined through the Maxwell-Boltzmann law

$$n = n_0 e^{e\Phi/T} + \Delta n, \tag{9}$$

where n_0 varies slowly on the scale length of the fluctuations. The perturbation described by Eqs. (8) and (9) is justified by the presence of collisions, which force the distribution to remain a perturbed Maxwellian. Note that the $\hat{f}$ implicit in Eq. (9) is independent of velocity, and therefore unaffected by the collision operator. Thus Eqs. (8) and (9) describe the unique response that satisfies the collisional kinetic equation.

In the absence of collisions, Eqs. (8) and (9) provide one solution to the kinetic equation, but not the only one. Indeed, the equilibrium state corresponding to collisionless relaxation is a subject of considerable controversy.[6]

We next Taylor expand the exponential and compare the result with Eq. (8) to find that

$$\hat{f} = \frac{e\Phi}{T} \equiv \phi, \tag{10}$$

and Eq. (7) becomes

$$\Delta S = -\frac{1}{2}\int d^3x\, n_0(\alpha + \phi)^2.$$

This relation implies that the quantity $\gamma_0 = \alpha + \phi$ has vanishing correlation length—it is "delta correlated." The above expression for the fluctuation in entropy can be written as

$$\Delta S = -\frac{1}{2}\int d^3x\, d^3x'\; \sigma_0(\boldsymbol{x},\boldsymbol{x}')\gamma_0(\boldsymbol{x})\gamma_0(\boldsymbol{x}'), \tag{11}$$

where, allowing for slow variation of n_0,

$$\sigma_0(\boldsymbol{x},\boldsymbol{x}') = n_0(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{12}$$

Putting Eq. (12) into Eq. (4) shows that, indeed, $\gamma_0(\boldsymbol{x})$ is delta correlated.

Since the perturbation γ_0 is delta correlated, the correlation function of any perturbation γ that is linearly related to γ_0 can easily be found. Suppose that

$$\gamma(\boldsymbol{x}) = \int d^3x'\; h(\boldsymbol{x},\boldsymbol{x}')\gamma_0(\boldsymbol{x}'), \tag{13}$$

where the function h describes some integrodifferential operator that is constant on the ensemble. With

$$[\gamma_0(\boldsymbol{x})\gamma_0(\boldsymbol{x}')] = c_0(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{x}'),$$

using Eq. (13) in Eq. (2), shows that γ has the correlation function

$$c(\boldsymbol{x},\boldsymbol{x}') = [\gamma(\boldsymbol{x})\gamma(\boldsymbol{x}')] = \int d^3x''\; c_0(\boldsymbol{x}'')h(\boldsymbol{x},\boldsymbol{x}'')h(\boldsymbol{x}',\boldsymbol{x}''). \tag{14}$$

We consider the simplest case of a plasma consisting of electrons and a single-ion species in approximate thermal equilibrium. By some means a small fluctuation, Δn_i, in the constant density distribution of the ions arises. We will assume that for each species the imposed ion fluctuation induces a response that follows the Maxwell-Boltzmann law, determined by the perturbed electrostatic potential, $\mathbf{\Delta\Phi}$.

In thermal equilibrium the ions follow an approximately Maxwellian distribution,

$$f = n_0\left(\frac{m}{2\pi T}\right)^{3/2} e^{-mv^2/2T} e^{e\Phi/T},$$

where $n = n_0 e^{e\Phi/T}$; here we are taking $q_{e^-} = +e$. Now we let $n_i \rightarrow n_0 + \Delta n_i$ and $\Phi \rightarrow \Phi_0 + \Delta\Phi$, we Taylor expand the exponent as in Eq. (10), and neglect second-order perturbation terms to find

$$f_i = f_{M0}\left(1 + \frac{e\Delta\Phi}{T} + \frac{\Delta n_i}{n_0}\right) = f_{M0}(1 + \phi + \alpha), \tag{15}$$

consistent with Eq. (8), where the ion response is given by the term $\phi = e\,\Delta\Phi/T$. Clearly then

$$n_i = n_0(1 + \alpha + \phi). \tag{16}$$

All that the electron distribution sees is the perturbed electrostatic potential, so the electron response is the same as for the ion response, with an oppositely signed charge:

$$n_e = n_0(1 - \phi). \tag{17}$$

Beginning with Poisson's equation in one dimension for a single ion species, we have

$$\lambda^2 \frac{\partial^2 \phi}{\partial x^2} = \frac{n_i - n_e}{n_0},$$

where λ is the Debye length and $\phi = e\,\Delta\Phi/T$, as above. Using Eqs. (16) and (17), in Poisson's equation, gives the following linear relation:

$$\lambda^2 \frac{\partial^2 \phi}{\partial x^2} - \phi = \alpha + \phi. \tag{18}$$

We consider here the one-dimensional problem with slow spatial variation of n_0. Identifying $\gamma_0 = \alpha + \phi$ and $\gamma = \phi$, we take the Fourier transform of Eq. (18), and compare with Eq. (13), to find that

$$\tilde{\gamma} = \left(\frac{1}{-(1 + \lambda^2 k^2)} \right) \tilde{\gamma}_0 = \tilde{h}(\boldsymbol{k}) \tilde{\gamma}_0.$$

Noting that Eq. (14), for a constant c_0, involves the convolution of h with itself, the Fourier transform of Eq. (14) is

$$\tilde{c}(\boldsymbol{k}) = \frac{1}{n_0} \tilde{h}(\boldsymbol{k})^2,$$

so that the inverse transform is

$$c(\boldsymbol{x}) = \frac{1}{(2\pi)^3 n_0} \int d^3k \frac{e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}}{(1 + \lambda^2 k^2)^2}. \tag{19}$$

The integral, Eq. (19), is elementary, and we find that

$$c(\boldsymbol{x}) = \frac{e^{-|x|/\lambda}}{2\pi n_0 \lambda^3}. \tag{20}$$

This result is probably not new, although we have not found it in the literature. What is striking, as in previous work,[5] is the simplicity and directness of the Einstein method.

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