Hamiltonian Fluid Dynamics

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Introduction

The ideal fluid description is one in which viscosity or other phenomenological terms are neglected. Thus, as is the case for systems governed by Newton's second law without dissipation, such fluid descriptions possess Lagrangian and Hamiltonian descriptions. In fact in the 18th century, Lagrange himself discussed what is in essence the action principle for the incompressible fluid. The subsequent history of action functional and Hamiltonian formulations of the ideal fluid is long and convoluted with contributions from Clebsch in the 19th century, and the likes of L. Landau and V. Arnold in the mid 20th century. In the early 1980's there was a flurry of activity on the noncanonical Poisson bracket formulation, and this formulation is the focus of the present treatment, which is motivated by the work of the author, D. Holm, J. Marsden, T. Ratiu, A. Weinstein, and others.

Noncanonical Hamiltonian Structure

The traditional arena for Hamiltonian dynamics is the cotangent bundle $\mathcal{M} := T^*\mathcal{Q}$, the phase space, which is naturally a symplectic manifold with a closed nondegenerate two-form. In coordinates, the two-form is given by $\omega_c = dq \wedge dp$, where q denotes the configuration coordinate for the base space manifold $\mathcal{Q}$ and p denotes the corresponding canonical momenta that arise from Legendre (convex) transformation. The two-form ω_c provides a natural identification at a point $z = (q,p) \in \mathcal{M}$ of $T_z\mathcal{M}$ with $T_z^*\mathcal{M}$, and because of nondegeneracy its inverse, the cosymplectic form, provides the map $J_c\colon T_z^*\mathcal{M} \to T_z\mathcal{M}$. Thus, for a Hamiltonian $H\colon \mathcal{M} \to \mathbb{R}$ we have the Hamiltonian system of ordinary differential equations $\dot z = J_c dH$, which in canonical coordinates has the familar form

$$\dot q^i = \partial H/\partial p_i\,, \qquad \dot p_i = -\partial H/\partial q^i\,, \tag{1}$$

with $i = 1, 2, \ldots N$, where N is the number of degrees of freedom.

Hamilton's equations can also be written in terms of the Poisson bracket $[f,g] := \omega_c(J_c df, J_c dg)$, where $f,g\colon \mathcal{M} \to \mathbb{R}$ are smooth phase space functions. In terms of $z=(q,p)$, Hamilton's equations are

$$\dot{z}^\alpha = J_c^{\alpha\beta}\frac{\partial H}{\partial z^\beta} = [z^\alpha, H]\,, \tag{2}$$

where the Poisson bracket is

$$[f,g] = \frac{\partial f}{\partial z^\alpha} J_c^{\alpha\beta} \frac{\partial g}{\partial z^\beta}\,, \tag{3}$$

with

$$(J_c^{\alpha\beta}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}\,. \tag{4}$$

Note, repeated indices are to be summed with $\alpha, \beta = 1, 2, \ldots 2N$. In (4), 0_N is an $N\times N$ matrix of zeros and I_N is the $N \times N$ unit matrix.

Noncanonical Poisson Brackets

The canonical Poisson bracket description of (2), (3), and (4) suggests a generalization, with antecedents to S. Lie and others, that was termed noncanonical Hamiltonian form in the fluid mechanics context by P. Morrison and J. Greene (1980):

A system has noncanonical Hamiltonian form if it can be written as $\dot{z} = [z, H]$, where the noncanonical Poisson bracket $[\ ,\ ]$ is a Lie product for a realization of a Lie enveloping algebra on phase space functions.

Recall a Lie enveloping algebra $\mathfrak{a}$ is a Lie algebra, with the usual product $[\ ,\ ]$ that is bilinear, antisymmetric, and satisfies the Jacobi identity, which in addition has a product $\mathfrak{a}\times\mathfrak{a}\to\mathfrak{a}$ that satisfies the Leibniz identity $[fg,h] = f[g,h] + [f,h]g$, for all $f,g,h \in \mathfrak{a}$.

The geometric description of noncanonical Hamiltonian form has evolved into a structure called the Poisson manifold, a differentiable manifold $\mathcal{Z}$ endowed with the binary bracket operation $[\ ,\ ]$ defined on smooth functions, say, $f,g\colon \mathcal{Z} \to \mathbb{R}$. Poisson manifolds differ from symplectic manifolds because the nondegeneracy condition is removed. In coordinates, $[\ ,\ ]$ is given by

$$[f,g] = \frac{\partial f}{\partial z^\alpha} J^{\alpha\beta} \frac{\partial g}{\partial z^\beta}\,, \qquad \alpha,\beta = 1,2,\ldots,M\,, \tag{5}$$

where $M = \dim \mathcal{Z}$. Note that J need not have the form of (3), may depend upon the coordinate z, and may have vanishing determinant. Bilinearity, $[f,g] = -[g,f]$ for all f,g, and the Jacobi identity, $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] \equiv 0$, for all f,g,h, imply that the cosymplectic matrix satisfies $J^{\alpha\beta} = -J^{\beta\alpha}$ and

$$J^{\alpha\delta}\frac{\partial J^{\beta\gamma}}{\partial z^\delta} + J^{\beta\delta}\frac{\partial J^{\gamma\alpha}}{\partial z^\delta} + J^{\gamma\delta}\frac{\partial J^{\alpha\beta}}{\partial z^\delta} \equiv 0\,, \tag{6}$$

respectively, for $\alpha,\beta,\gamma,\delta = 1,2,\ldots M$.

The local structure of $\mathcal{Z}$ is elucidated by the Darboux-Lie theorem, which states that in a neighborhood of a point $z \in \mathcal{Z}$, for which $\mathrm{rank} J = M$, there exist coordinates in which J has the following form:

$$(J) = \begin{pmatrix} 0_N & I_N & 0 \\ -I_N & 0_N & 0 \\ 0 & 0 & 0_{M-2N} \end{pmatrix} . \tag{7}$$

From (7) it is clear that in the right coordinates, the system looks like a canonical N degree-of-freedom Hamiltonian system with some extraneous coordinates, $M-2N$ in fact. Through any point of the M dimensional phase space $\mathcal{Z}$ there exists a local foliation by symplectic leaves of dimension $2N$.

A consequence of the degeneracy is that there exists a special class of invariants called Casimir invariants that is built into the phase space. Since the rank of J is $2N$, there exist possibly $M-2N$ independent null eigenvectors. A consequence of the Darboux-Lie theorem is that the independent null eigenvectors exist and, moreover, the null space can in fact be spanned by the gradients of the Casimir invariants, which satisfy $J^{\alpha\beta}\partial C^{(a)}/\partial z^\beta = 0$, where $a = 1, 2, 3, ..., M-2N$. That the Casimir invariants are constants of motion, follows from

$$\dot{C}^{(a)} = \frac{\partial C^{(a)}}{\partial z^\alpha} J^{\alpha\beta} \frac{\partial H}{\partial z^\beta} = 0 \, . \tag{8}$$

Thus Casimir invariants are constants of motion for any Hamiltonian. The symplectic leaves of dimension $2N$ are the intersections of the $M-2N$ surfaces defined by $C^{(a)} =$ constant. Dynamics generated by any H that begins on a particular symplectic leaf remains there. The structure of Poisson manifolds have now been widely studied, but we will not pursue this further here.

Let us turn to infinite-dimensional systems, field theories such as those that govern ideal fluids, where the governing equations are partial differential equations. Although the level of rigor does not match that achieved for the finite systems described above, formally one can parody most of the steps and, consequently, the finite theory provides cogent imagery and serves as a beacon for shedding light. In infinite dimensions an analogue of (5) is given by

$$\{F, G\} = \int_\Omega d\mu \, \frac{\delta F}{\delta \psi^i} \mathcal{J}^{ij} \frac{\delta G}{\delta \psi^j} =: \left\langle \frac{\delta F}{\delta \psi}, \mathcal{J} \frac{\delta G}{\delta \psi} \right\rangle , \tag{9}$$

where F and G are functionals of the functions $\psi^i(\mu, t)$, which are functions of $\mu = (\mu_1, ..., \mu_n)$, independent variables of some kind, $\delta F/\delta\psi^i$ denotes the functional (variational) derivative, and $\langle \, , \, \rangle$ is a pairing between a vector (function) space and its dual. The ψ^i, $i = 1, ..., n$, are n field components, and now $\mathcal{J}$ is a cosymplectic operator. To be noncanonically Hamiltonian requires antisymmetry, $\{F, G\} = -\{G, F\}$, and the Jacobi identity, $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} \equiv 0$, for all *functionals* F, G, and H. Antisymmetry requires $\mathcal{J}$ to be skew-symmetric, i.e. $\langle f, \mathcal{J}\, g\rangle = \left\langle \mathcal{J}^\dagger f, g \right\rangle = -\langle g, \mathcal{J}\, f\rangle$. The Jacobi identity for infinite-dimensional systems has a condition analogous to (6); it can be shown that one need only consider variations of $\mathcal{J}$ when calculating e.g. $\{F, \{G, H\}\}$.

Lie-Poisson Brackets

As noted in the Introduction, the usual variables of fluid mechanics are not a set of canonical variables, and, consequently, the Hamiltonian description in terms of these variables is noncanonical.

There is a special general form that the Poisson bracket takes for equations that describe media in terms of Eulerian-like variables, the so-called Lie-Poisson brackets, a special form of noncanonical Poisson bracket. Lie-Poisson brackets describe essentially every fundamental equation that describes classical media. In addition to the equations for the ideal fluid, they describe Liouville's equation for the dynamics of the phase space density of a collection of particles, the BBGKY hierarchy of kinetic theory, the Vlasov equation of plasma physics, and various approximations thereof, and magnetized and other more complicated fluids.

Both finite and infinite dimensional Lie-Poisson brackets are intimately associated with a Lie group $\mathfrak{G}$. We use the pairing between a vector space and its dual, $\langle\, ,\, \rangle$, where the second slot is reserved for elements of the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$ and the first slot for elements of its dual $\mathfrak{g}^*$. Thus, $\langle\, ,\, \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$. In terms of the pairing, noncanonical Lie-Poisson brackets have the following compact form:

$$\{F, G\} = \langle \chi, [F_\chi, G_\chi] \rangle \,, \tag{10}$$

where we suppose the dynamical variable $\chi \in \mathfrak{g}^*$, $[\, ,\,]$ is the Lie algebra product, which takes $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, and we have introduced the shorthand $F_\chi := \delta F/\delta\chi$. The quantities F_χ and G_χ are, of course, in $\mathfrak{g}$. We refer to $\{\, ,\, \}$ as the "outer" bracket of the realization enveloping algebra and $[\, ,\,]$ as the "inner" bracket of the Lie algebra $\mathfrak{g}$. The binary operator $[\, ,\,]^\dagger$ is defined as follows:

$$\langle \chi, [f, g] \rangle =: \left\langle [\chi, g]^\dagger, f \right\rangle \,, \tag{11}$$

where evidently $\chi \in \mathfrak{g}^*$, $g, f \in \mathfrak{g}$, and $[\, ,\,]^\dagger \colon \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathfrak{g}^*$. The operator $[\, ,\,]^\dagger$, which defines the coadjoint orbit, is necessary for obtaining the equations of motion from a Lie-Poisson bracket.

For finite-dimensional systems, the group $\mathfrak{G}$ must be a finite-parameter Lie group, the variable ψ corresponds to w, and the cosymplectic form in coordinates is given by $J_{ab} = c_{ab}^c w_c$, where the c_{ab}^c are the structure constants for the Lie algebra $\mathfrak{g}$, which satisfy

$$c_{ab}^c = -c_{ba}^c \,, \qquad c_{ab}^e c_{ec}^d + c_{bc}^e c_{ea}^d + c_{ca}^e c_{eb}^d = 0 \,, \tag{12}$$

relations that imply (10) satisfies the antisymmetry condition and the Jacobi identity.

For infinite-dimensional systems, the group $\mathfrak{G}$ must be an infinite-parameter Lie group and the cosymplectic operator has the form $\mathcal{J}_{ij} = \mathcal{C}_{ij}^k \chi_k$, where $\mathcal{C}_{ij}^k$ are structure operators. The meaning of these structure operators will be clarified when we consider brackets for fluid mechanics.

The Fluid State

Fluid mechanics has a long history, and thus it comes as no surprise that the fluid state has been described in many ways. Because the Hamiltonian structure depends on the state variables, some of these ways are described below, beginning with Lagrangian variable description.

Lagrange Variables

The description of a fluid that is most like that of particle mechanics occurs in terms of variables usually referred to as Lagrangian variables. This description dates to the 18th century. The idea behind the use of these variables is a simple one: If a fluid is described as a continuum collection of fluid particles, also called fluid parcels or elements, then its motion is governed by an equation that is an infinite-dimensional version of Newton's second law and, consequently, as we will see, both the Hamiltonian and the Lagrangian descriptions are infinite degree-of-freedom generalizations of those of ordinary particle mechanics.

The position of a fluid element, referred to a fixed rectangular coordinate systems, is given by $q = q(a,t)$, where $q = (q_1, q_2, q_3)$ and $a = (a_1, a_2, a_3)$ is a continuum label that replaces the index i of (1). In practice the label can be any quantity that identifies a fluid particle, but it is often taken to be the position of the fluid particle at time $t = 0$ in rectangular coordinates. The quantities $q^i(a,t)$ are coordinates for the configuration space $\mathcal{Q}$, which is in fact a function space because in addition to the three indices "i" there is the continuum label a. We assume that a varies over a fixed domain, $\Omega \subset \mathbb{R}^3$, which is completely filled with fluid, and that the function $q\colon \Omega \to \Omega$ is 1-1 and onto. We will assume that as many derivatives of q with respect to a as needed exist, but we won't say more about $\mathcal{Q}$; in fact, not that much is known about the solution function space for the 3-D fluid equations in Lagrangian variables. Often in the Hamiltonian context the functions $q = q(a,t)$ are assumed to be diffeomorphisms and their collection is referred to as the diffeomorphism group.

In the sequel several manipulations are needed and so we record here some identities for later use. Viewing the map $a \mapsto q$ at fixed t as a coordinate change, the Jacobian matrix $\partial q^k/\partial a^i =: q^k_{,i}$ has an inverse given by

$$\frac{\partial q^k}{\partial a^j}\frac{A^i_k}{\mathfrak{J}} = \delta^i_j \tag{13}$$

where A^i_k is the cofactor of $q^k_{,i}$ and $\mathfrak{J}$ is its determinant. A convenient expression for A^i_k is given by

$$A^i_k = \tfrac{1}{2}\, \epsilon_{kjl}\epsilon^{imn}\frac{\partial q^j}{\partial a^m}\frac{\partial q^l}{\partial a^n}\,, \tag{14}$$

where $\epsilon_{ijk}(= \epsilon^{ijk})$ is the skew symmetric tensor (density). Evidently, $\partial \mathfrak{J}/\partial q^i_{,j} = A^j_i$, follows from (13).

Eulerian Variables

In the Lagrangian variable description one picks out a particular particle, labeled by a, and keeps track in time t of where it goes. However, in the Eulerian variable description, one stays at a spatial observation point $r = (x_1, x_2, x_3) \in \Omega$ and monitors the nature of the fluid at r at time t.

The most important Eulerian variable is the Eulerian velocity field $v(r,t)$. This quantity is the velocity of the particular fluid element that is located at the spatial point r a time t. The label of that particular fluid element is given by $a = q^{-1}(r,t)$, and so

$$v(r,t) = \dot{q}(a,t)|_{a=q^{-1}(r,t)} := \dot{q} \circ q^{-1}(r,t)\,, \tag{15}$$

where $\cdot$ denotes differentiation with respect to time at fixed label a. Attached to a fluid element is a certain amount of mass described by a density function $\rho_0(a)$. As the fluid moves so that $a \mapsto q$, the volume of an infinitesimal region will change, but its mass must remain fixed. The statement of local mass conservation is $\rho d^3r = \rho_0 d^3a$, where d^3a is an initial infinitesimal volume element that maps to d^3q at time t, and $d^3r = \mathfrak{J}\, d^3a$. (When integrating over Ω we will replace d^3q by d^3r.) Thus we obtain

$$\rho(r,t) = \left.\frac{\rho_0(a)}{\mathfrak{J}(a,t)}\right|_{a=q^{-1}(r,t)} = \frac{\rho_0}{\mathfrak{J}} \circ q^{-1}(r,t)\,, \tag{16}$$

where recall the Jacobian $\mathfrak{J} = \det(q^i{}_{,j})$. Besides the density, for the ideal fluid, one attaches an entropy per unit mass, $s = s_0(a)$, to a fluid element, and this quantity remains fixed in time. In the Eulerian description this gives rise to the entropy field

$$s(r,t) = s_0(a)|_{a=q^{-1}(r,t)} = s_0 \circ q^{-1}(r,t)\,. \tag{17}$$

One could attach other scalar, vector, etc. quantities to the fluid element, but we will not pursue this. In the usual ideal fluid closure only the above variables are considered.

Equations (15), (16), and (17) express the Euler-Lagrange map. There is a natural representation of this map in terms of the Eulerian density variables, $M := \rho v$, ρ, and $\sigma := \rho s$, the momentum, mass, and entropy densities, respectively, which, as will be seen, are variables in which the noncanonical Poisson bracket has Lie-Poisson from.

Other Variables

Fluid mechanics is rife with variables that have been used for its description. For example, Euler, Monge, Clebsch, and others introduced potential representations, of varying generality, for the Eulerian velocity field, an example being

$$v(r,t) = \alpha \nabla \beta + \nabla \phi\,, \tag{18}$$

where the three components of v are replaced by the functions α, β, and ϕ, all of which depend on (r,t).

Often reduced variables that are tailored to specific ideal flows with less generality that those described by ρ, s, and v are considered. Examples include incompressible flow with $\nabla \cdot v = 0$, vortex dynamics, including contour dynamics and point vortex dynamics, flow governed by the shallow water equations, quasigeostrophy, etc. The Hamiltonian structure in terms of these reduced variables derives from that of the parent model in terms of Lagrangian variables. Specific variables may embody constraints, and understanding these constraints, although tractable, can be a cause of confusion. Pursuing this further is beyond the scope here.

Hamilton's Principle for Fluid

Lagrange, in his famous work of 1788, Mécanique Analytique, produced in essence a variational principle for incompressible fluid flow in terms of Lagrangian variables. The generalization to compressible flow awaited the discovery of thermodynamics, and that is what we describe here. In traditional mechanics nomenclature this variational principle is an infinite-dimensional generalization of what is known variously as the action principle, the principle of least action, or Hamilton's principle, whereby one constructs, on physical grounds, a Lagrangian function on $T\mathcal{Q}$ used in the action principle, where here $\mathcal{Q}$ is the function space of the $q(a,t)$.

Construction of the Lagrangian requires identification of the potential energy, and this requires thermodynamics, because potential energy is stored in terms of pressure and temperature. A basic assumption of the fluid approximation is that of local thermodynamic equilibrium. In the energy representation of thermodynamics the extensive energy is treated as a function of the entropy and the volume. For a fluid it is convenient to consider the energy per unit mass, denoted by U, to be a function of the entropy per unit mass, s, and the mass density, ρ, a measure of the volume. The intensive quantities, pressure and temperature, are given by $T=\partial U/\partial s$ and $p=\rho^2\partial U/\partial\rho$. Choices for U produce equations of state. For barotropic or isentropic flow, U depends only on ρ. For an ideal monatomic gas $U(\rho,s)=c\,\rho^{\gamma-1}\exp{(\alpha s)}$, where c, γ, and α are constants. The function U could also depend on additional scalar quantities, such as a quantity known as spice that has been considered in oceanography.

Conventional thermodynamic variables can be viewed as Eulerian variables with a static velocity field. Thus we write $U(\rho,s)$, where ρ and s are spatially independent, or if the system has only locally relaxed, these variables can be functions of r. For the ideal fluid each fluid element can be viewed as a self contained isentropic thermodynamic system that moves with the fluid. Thus the total fluid potential energy functional is given by $V[q]=\int_\Omega d^3a\,\rho_0 U(s_0,\rho_0/\mathfrak{J})$, which is a functional of q that depends only upon $\mathfrak{J}$ and hence only upon $\partial q/\partial a$.

The next item required for constructing Hamilton's principle is the kinetic energy functional, which is given by $T[q,\dot q]=\int_\Omega d^3a\,\rho_0\,\dot q^2/2$, where $\dot q^2:=\eta_{ij}\dot q^i\dot q^j$, with the Cartesian metric $\eta_{ij}:=\delta_{ij}$. This metric and its inverse can be used to raise and lower indices.

The Lagrangian functional is $L[q,\dot q]:=T-V$, where $L[q,\dot q]=\int_\Omega d^3a\,\mathcal{L}(q,\dot q,\partial q/\partial a)$ and $\mathcal{L}$ is the Lagrangian density, in terms of which the action functional of Hamilton's principle is given by

$$S[q]=\int_{t_0}^{t_1} dt\,L[q,\dot q]=\int_{t_0}^{t_1} dt\int_\Omega d^3a\,\left[\tfrac{1}{2}\,\rho_0\,\dot q^2-\rho_0 U\right]\,. \tag{19}$$

The end conditions for Hamilton's principle for the fluid are the same as those of mechanics, viz. $\delta q(a,t_0)=\delta q(a,t_1)=0$. The nonpenetration condition, $\delta q\cdot\hat n=0$ on $\partial\Omega$, where $\hat n$ is a unit normal vector is also assumed. Other boundary conditions, such as periodic and free boundary conditions, are also possibilities. Hamilton's principle amounts to $\delta S/\delta q(a,t)=0$, which with the end and boundary conditions implies the following equations of motion:

$$\rho_0\ddot q_i+A_i^j\frac{\partial}{\partial a^j}\left(\frac{\rho_0^2}{\mathfrak{J}^2}\frac{\partial U}{\partial\rho}\right)=0\,. \tag{20}$$

Here we have used $\partial A_i^j/\partial a^j=0$, which can be seen using (14). Equation (20) amounts to Newton's second law for the ideal fluid, which is made clearer by using the following useful identity:

$$\frac{\partial}{\partial q^k}=\frac{1}{\mathfrak{J}}A_k^i\frac{\partial}{\partial a^i}\,. \tag{21}$$

Alternatively, upon using (13), (20) is sometimes written in the form

$$\rho_0\ddot q_j\frac{\partial q^j}{\partial a^i}+\mathfrak{J}\frac{\partial}{\partial a^i}\left(\frac{\rho_0^2}{\mathfrak{J}^2}\frac{\partial U}{\partial\rho}\right)=0\,. \tag{22}$$

The Eulerian variable force law follows from (20) upon using (21)

$$\rho\left(\frac{\partial v}{\partial t} + v\cdot\nabla v\right) = -\nabla p\,, \tag{23}$$

where $v = v(r,t)$. The remaining Eulerian equations of mass conservation and entropy advection follow from the constraints that s_0 and ρ_0 are constant on fluid elements. Time differentiation and the transformations of (16) and (17) yield

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0\,, \tag{24}$$

$$\frac{\partial s}{\partial t} + v\cdot\nabla s = 0\,. \tag{25}$$

Equations (23), (24), and (25), together with a given function $U(\rho,s)$ and the relation $p = \rho^2\partial U/\partial\rho$ constitute the Eulerian description.

Variational principles similar to that described above exist for essentially all ideal fluid models, including incompressible flow, magnetohydrodynamics, the two-fluid equations of plasma physics, etc.

Eulerian Action Principles

Some early researchers sought variational principles that directly produce the ideal fluid equations in Eulerian form. Because the Eulerian form of the equations does not treat the fluid as a collection of particles, the resulting action principles possess a certain awkwardness. Below, we describe three approaches to such action principles.

Clebsch Action

The action principle for electromagnetism proceeds by introducing the four-vector potential. In a similar way, the Clebsch action principle anticipates this idea by using a potential representation of the velocity field, an example being that of (18).

Although compressible flow with an arbitrary equation of state can be treated in full generality, for simplicity and variety we will restrict to incompressible flow and set $\nabla\cdot v \equiv 0$. This constraint is enforced by requiring ϕ to be dependent on α and β according to $\phi[\alpha,\beta] := -\triangle^{-1}(\alpha\nabla\beta)$, where $\triangle^{-1}$ is the inverse Laplacian. The Clebsch action is then written as follows:

$$S_C[\alpha,\beta] := \int_{t_0}^{t_1} dt \int_\Omega d^3r\,\left[\beta\alpha_t - \tfrac{1}{2}\,v^2\right]\,, \tag{26}$$

where the subscript t denotes differentiation at fixed r, we have set $\rho\equiv 1$, and v is a shorthand for the expression of (18) with $\phi = \phi[\alpha,\beta]$. The form of S_C is that of the phase space action that produces Hamilton's equations upon independent variation of the configuration space coordinate and its conjugate momentum, which are here α and β, respectively. Thus we require $\delta\alpha(r,t_0) =$

$\delta\alpha(r, t_1) = 0$, but no condition is needed for $\delta\beta$ at $t_{0,1}$. We also require $\hat{n} \cdot v = 0$ on $\partial\Omega$. The variations $\delta S_C/\delta\beta = 0$ and $\delta S_C/\delta\alpha = 0$ imply

$$\alpha_t = \frac{\delta H}{\delta \beta} = -v \cdot \nabla\alpha\,, \qquad \beta_t = -\frac{\delta H}{\delta \alpha} = -v \cdot \nabla\beta = 0\,, \tag{27}$$

an infinite-dimensional version of (1) with $H := \int_\Omega d^3r\, v^2/2$. Evidently both α and β are advected by the flow.

Because the vorticity, $\zeta := \nabla \times v = \nabla\alpha \times \nabla\beta$, knowledge of α and β determines ζ and one can invert the curl operator to obtain v in the usual way. The intersection of level sets of α and β define vortex lines, and, evidently, these quantities, like the entropy for compressible dynamics, are constant on fluid elements. It is not difficult to show that the advection of α and β implies the correct dynamical equation for incompressible v.

Herivel-Lin Action

The Herivel-Lin action incorporates (24) and (25) as constraints with Lagrange multipliers, φ and $\rho\beta$. (Here β is not the Clebsch β and the factor of ρ is included for convenience.) It was discovered early on that these constraints were not enough to achieve complete generality and so a new one, known as the Lin constraint, was added. The Lin constraint corresponds to constancy of the fluid particle label. One defines an Eulerian label field by setting $q(a, t) = r$ and solving for the label $a = q^{-1}(r, t) =: a(r, t)$. Conservation of particle identity is thus given by $a_t + v \cdot \nabla a = 0$, and this constraint is associated with a Lagrange multiplier $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. The Herivel-Lin action is thus given by

$$\begin{aligned} S_{HL}[v, \rho, s, a; \varphi, \beta, \gamma] \;&=\; \int_{t_0}^{t_1} dt \int_\Omega d^3r \,\Big(\tfrac{1}{2}\,\rho v^2 - \rho U(\rho, s) + \varphi\left[\rho_t + \nabla\cdot(\rho v)\right] \\ &\quad -\; \rho\beta\left[s_t + v\cdot\nabla s\right] - \rho\gamma\cdot\left[a_t + v\cdot\nabla a\right]\Big)\,. \end{aligned} \tag{28}$$

Variation of (28) with respect to the Lagrange multipliers just reproduces the constraints; however, variation with respect to v, ρ, s, and a produces equations that imply (23). Moreover, every flow can be shown to be an extremal of S_{HL}.

Euler-Poincaré-Hamel Action

Another approach is to use directly constrained variations. The essential idea is to only consider Eulerian variable variations that are induced by underlying Lagrangian variable variations δq, the so-called dynamically accessible variations. Explicitly, a basic Eulerian variation $\eta = (\eta_1, \eta_2, \eta_3)$ is given by $\eta(r, t) = \delta q(a, t)|_{a=q^{-1}(r,t)}$. In terms of this quantity, the dynamically accessible variations of the Eulerian velocity field, density, and entropy are given, respectively, by $\delta v = \eta_t + v\cdot\nabla\eta - \eta\cdot\nabla v$, $\delta\rho = -\nabla\cdot(\rho\eta)$, and $\delta s = -\eta\cdot\nabla s$. Upon inserting them into the variation of

$$S_{EPH}[\eta] = \int_{t_0}^{t_1} dt \int_\Omega d^3r\, \left[\tfrac{1}{2}\,\rho v^2 - \rho U(\rho, s)\right] \tag{29}$$

and integrating by parts gives

$$\delta S_{EPH} = \int_{t_0}^{t_1} dt \int_\Omega d^3r \, [\, \dots \,] \cdot \eta = 0 \,,$$

where $[\, \dots \,]$ is equivalent to (23). Thus, assuming η is arbitrary, we obtain directly the equation of motion.

There is a version of this kind of constrained variational principle for all ideal fluid and plasma equations. Also, it possesses a geometric interpretation. In a more practical vein, constrained variations can be used to derive reduced models, and dynamically accessible variations can also be used for stability calculations. Exploring these ideas is outside the present scope.

Fluid Hamiltonian Description

Having described variational principles we turn to the associated canonical and noncanonical Hamiltonian descriptions.

Canonical Description

Because the action of (19) is of standard form, it is convex in $\dot{q}$ and the Legendre transform follows easily: the canonical momentum density is $\pi_i(a,t) := \delta L/\delta \dot{q}^i(a) = \rho_0 \dot{q}_i$ and $H[q,\pi] = \int_\Omega d^3a \, [\pi \cdot \dot{q} - \mathcal{L}] = \int_\Omega d^3a \, [\pi^2/(2\rho_0) + \rho_0 U]$. Hamilton's equations are then

$$\dot{q}^i = \frac{\delta H}{\delta \pi_i} = \{q^i, H\} \,, \qquad \dot{\pi}_i = -\frac{\delta H}{\delta q^i} = \{\pi_i, H\} \,, \tag{30}$$

an infinite-dimensional version of (1), with the canonical Poisson bracket

$$\{F, G\} = \int_\Omega \left[\frac{\delta F}{\delta q} \cdot \frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta q} \cdot \frac{\delta F}{\delta \pi} \right] d^3a \,. \tag{31}$$

(Note, $\delta q^i(a)/\delta q^j(a') = \delta^i_j \delta(a - a')$, a relation analogous to $\partial q^j/\partial q^i = \delta^j_i$ for finite systems.)

Reduction to Noncanonical Poisson Brackets

Reduction is a procedure for reducing the size of a Hamiltonian system. Given constants of motion in involution, i.e., with pair wise vanishing Poisson brackets, the dimension of a Hamiltonian system can be reduced by two for each such constant of motion. However, when constants do not commute the situation is more complicated and one must invoke a theory due to Lie, Poincaré, Cartan, and others. Associated with invariants are symmetries, and so a complete discussion of this theory requires examination of symmetry groups and associated geometry. For the ideal fluid the map from the Lagrangian to the Eulerian descriptions is an example of reduction, whereby the Poisson bracket of (31) is mapped into a noncanonical Poisson bracket. En route to describing this example, a brief discussion of reduction of finite systems is considered first.

Reduction of Finite-Dimensional Systems

Consider a canonical system with the phase space $\mathcal{M}$, a $2N$-dimensional symplectic manifold. In a coordinate patch with coordinates $z = (q, p)$ the system has the canonical description of (2), (3), and (4). Suppose we have a map $P : \mathcal{M} \to \mathfrak{m}^*$, where $\mathfrak{m}^*$ is some $M < 2N$-dimensional space described by coordinates $w = (w_1, w_2, \ldots w_M)$. In coordinates this map is represented in terms of functions $w_a = w_a(z)$, with $a = 1, 2, \cdots, M$, which, because $M < 2N$, is always noninvertible. Suppose $f, g : \mathcal{M} \to \mathbb{R}$ obtain their z-dependence through the functions w, i.e., $f(z) = \bar{f}(w(z)) = \bar{f} \circ w$. Making use of the chain rule yields

$$[f, g] = \frac{\partial \bar{f}}{\partial w_a} J_{ab} \frac{\partial \bar{g}}{\partial w_b} \tag{32}$$

where the quantity

$$J_{ab} := \frac{\partial w_a}{\partial z^\alpha} J_c^{\alpha\beta} \frac{\partial w_b}{\partial z^\beta} \tag{33}$$

is in general a function of z. However, it is possible that J_{ab} may only depend on w. When this happens, we have a reduction of the phase space $\mathcal{M}$. In the language of S. Lie, reduction amounts to the existence of a function group as a subgroup.

If the original dynamics of interest has the Hamiltonian vector field generated by $H(z)$, and if it is possible that $H(z)$ can be expressed solely in terms of the w's, i.e., $H(z) = \bar{H}(w)$, then the system has been reduced. Clearly this is a statement of symmetry, since the function $H(z)$ in reality depends on a fewer number of variables, the w's.

A beautiful form of reduction occurs when the map P has a special form $w_a = L_a^i(q)\, p_i$, where the quantity L is associated with a symmetry group. An identity for what is required of L_a^i in order for the transformed bracket to be expressible in terms of the w's can be worked out, but this is explained in terms of Lie groups. If the space $\mathfrak{m}$ is a Lie algebra $\mathfrak{g}$, then the functions $\bar{f}, \bar{g}$ are real valued functions on $\mathfrak{g}^*$ that can be extended by left or right translation to functions f, g on $T^*\mathfrak{G}$. Thus f restricted to $T^*\mathfrak{G}$ at the identity, $T_e^*\mathfrak{G} = \mathfrak{g}^*$, is $\bar{f}$. Because $T^*\mathfrak{G}$ is a cotangent bundle, it carries the canonical Poisson bracket and we get a natural map P, called a momentum map, into the dual of a Lie algebra. This geometrical description of obtaining brackets on $\mathfrak{g}^*$ from brackets on $T^*\mathfrak{G}$ is a case of Marsden-Weinstein reduction. In the early 1980's these authors and others developed the geometrical interpretation of the noncanonical Poisson brackets for the ideal fluid.

Ideal Fluid Noncanonical Poisson Brackets

The Euler-Lagrange map of the fluid is of the form of the map P above. It maps the canonical bracket of (31) into a noncanonical Poisson bracket. If we use the Eulerian variabes $M := \rho v$, ρ, and $\sigma := \rho s$, then the resulting noncanonical bracket is of Lie-Poisson form. To effect this map, one must vary Eqs. (15), (16), and (17) to relate functional derivatives with respect to q and π to those with respect to M, ρ, and σ. This amounts to working out the chain rule for functionals. Upon doing this, one obtains the following noncanonical bracket:

$$\{F, G\} \;=\; -\int_\Omega \left[M_i \left(\frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i} \right) \right.$$

$$+ \quad \rho\bigg(\frac{\delta F}{\delta M}\cdot\nabla\frac{\delta G}{\delta\rho}-\frac{\delta G}{\delta M}\cdot\nabla\frac{\delta F}{\delta\rho}\bigg)+\sigma\bigg(\frac{\delta F}{\delta M}\cdot\nabla\frac{\delta G}{\delta\sigma}-\frac{\delta G}{\delta M}\cdot\nabla\frac{\delta F}{\delta\sigma}\bigg)\bigg]\, d^3r\,. \qquad (34)$$

This bracket, together with the Hamiltonian $\bar{H}[M,\rho,\sigma]=\int_\Omega d^3r\,[M^2/(2\rho)+\rho U(\rho,\sigma/\rho)]$ generates the ideal fluid equations. This Hamiltonian follows from $\bar{H}[M,\rho,\sigma]:=H[q,\pi]$ with $H[q,\pi]=\int_\Omega d^3a\,[\pi^2/(2\rho_0)+\rho_0 U]$. The bracket of (34) is clearly seen to be linear in the variables M, ρ, and σ, and the form of the cosymplectic operator and structure operators $\mathcal{C}_{ij}^k$ can be obtained by integration by parts. The Lie group in this case can be seen to be an extension by semi-direct product of the diffeomorphism group.

An alternative form of the noncanonical Poisson bracket is given in terms of the variables v, ρ, and s. Upon changing to these coordinates the noncanonical Poisson bracket transforms into

$$\begin{aligned}\{F,G\} &= -\int_\Omega \bigg[\bigg(\frac{\delta F}{\delta\rho}\nabla\cdot\frac{\delta G}{\delta v}-\frac{\delta G}{\delta\rho}\nabla\cdot\frac{\delta F}{\delta v}\bigg)\\ &+ \bigg(\frac{\nabla\times v}{\rho}\cdot\frac{\delta G}{\delta v}\times\frac{\delta F}{\delta v}\bigg)+\frac{\nabla s}{\rho}\cdot\bigg(\frac{\delta F}{\delta s}\frac{\delta G}{\delta v}-\frac{\delta G}{\delta s}\frac{\delta F}{\delta v}\bigg)\bigg]\, d^3r\,. \qquad (35)\end{aligned}$$

which with the Hamiltonian $H[v,\rho,s]=\int_\Omega d^3r\,[\rho v^2/2+\rho U(\rho,s)]$ produces the Eulerian fluid equations of (23), (24), and (25) directly as $v_t=\{v,H\}$, $\rho_t=\{\rho,H\}$, and $s_t=\{s,H\}$, respectively. Observe that in these variables, the bracket is no longer of Lie-Poisson form.

Conclusion

In a general sense, Hamiltonian dynamics is about coordinate changes, and it is clear from the above that there is no shortage of coordinates for describing the ideal fluid. The most intuitive from of fluid equations (at present) is the Eulerian form, and this possesses a noncanonical Hamiltonian description. Other noncanonical variables are also used for both less and more general fluid systems than those described above. Vortex dynamics, shallow water theory, and other equations of geophysical fluid dynamics are possibilities, as well as equations from plasma physics and other disciplines. The general story for these systems is much the same as above, although in some descriptions constraints are involved and they can complicate matters.

There are various motivations for pursuing an understanding of the Hamiltonian structure of fluids, but ultimately these motivations are the same as those for investigating the Hamiltonian dynamics of particle and other finite degree-of-freedom systems. Hamiltonian theory serves as an organizing framework, one that can be used for the derivation and approximation of systems. If one understands something about a particular Hamiltonian system, then often it can be said to be true of a general class of Hamiltonian systems. By now, many applications have been worked out, some of which can be accessed from the literature cited below.

See Also

Introductory articles: Classical mechanics. Introductory articles: Differential geometry. Infinite-dimensional Hamiltonian systems. Marsden-Weinstein reduction. Hamiltonian group actions. Poisson manifolds, Lie bialgebras, and classical r-matrices.

Further Reading

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