Two-fluid magnetic island dynamics in slab geometry. II. Islands interacting with resistive walls or resonant magnetic perturbations

Richard Fitzpatrick and François L. Waelbroeck
Institute for Fusion Studies, Department of Physics, University of Texas at Austin, Austin, Texas 78712

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The dynamics of a propagating magnetic island interacting with a resistive wall or an externally generated, resonant magnetic perturbation is investigated using two-fluid, drift-(magnetohydrodynamical) (MHD) theory in *slab geometry*. In both cases, the island equation of motion is found to take exactly the same form as that predicted by single-fluid MHD theory. *Three* ion polarization terms are found in the Rutherford island width evolution equation. The first is the drift-MHD polarization term for an isolated island, and is unaffected by the interaction with a wall or magnetic perturbation. Next, there is the polarization term due to interaction with a wall or magnetic perturbation which is predicted by *single-fluid* MHD theory. This term is always *destabilizing*. Finally, there is a hybrid of the other two polarization terms. The sign of this term depends on many factors. However, under normal circumstances, it is stabilizing if the noninteracting island propagates in the *ion diamagnetic direction* (with respect to the wall or magnetic perturbation) and *destabilizing* if it propagates in the *electron diamagnetic direction.*
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I. INTRODUCTION

Tearing modes are magnetohydrodynamical (MHD) instabilities which often limit fusion plasma performance in magnetic confinement devices relying on nested toroidal magnetic flux surfaces.[1] As the name suggests, "tearing" modes tear and reconnect magnetic field lines, in the process converting nested toroidal flux surfaces into helical magnetic islands. Such islands degrade plasma confinement because heat and particles are able to travel radially from one side of an island to another by flowing along magnetic field lines, which is a relatively fast process, instead of having to diffuse across magnetic flux surfaces, which is a relatively slow process.[2]

The interaction of rotating magnetic islands with resistive walls[3–11] or externally generated, resonant magnetic perturbations[5,7,12–14] has been the subject of a great deal of research in the magnetic fusion community. This paper focuses on the *ion polarization* corrections to the Rutherford island width evolution equation[15] which arise from the highly sheared ion flow profiles generated around magnetic islands whose propagation velocities are modified by interaction with either resistive walls or externally generated, magnetic perturbations. According to *single-fluid* MHD theory,[9,14] such polarization corrections are always *destabilizing*. The aim of this paper is to evaluate the ion polarization corrections using *two-fluid*, drift-MHD theory, which is far more relevant to present-day magnetic confinement devices than single-fluid theory. This goal is achieved by extending the analysis of the companion paper,[16] which investigates the dynamics of an *isolated* magnetic island in slab geometry using two-fluid, drift-MHD theory. For the sake of simplicity, we shall restrict our investigation to *slab geometry*.

II. REDUCED EQUATIONS

A. Basic equations

Standard right-handed Cartesian coordinates (x, y, z) are adopted. Consider a quasineutral plasma with singly charged ions of mass m_i. The ion/electron number density n_0 is assumed to be *uniform* and *constant*. Suppose that $T_i = \tau T_e$, where $T_{i,e}$ is the ion/electron temperature, and τ is uniform and constant. Let there be no variation of quantities in the z direction, i.e., $\partial/\partial z \equiv 0$. Finally, let all lengths be normalized to some convenient scale length a, all magnetic field strengths to some convenient scale field strength B_a, and all times to a/V_a, where $V_a = B_a/\sqrt{\mu_0\, n_0\, m_i}$.

We can write $\boldsymbol{B} = \boldsymbol{\nabla}\psi \times \hat{\boldsymbol{z}} + (B_0 + b_z)\,\hat{\boldsymbol{z}}$ and $P = P_0 - B_0\, b_z + O(1)$, where $\boldsymbol{B}$ is the magnetic field and P the total plasma pressure. Here, we are assuming that P_0 and B_0 are uniform, and $P_0 \gg B_0 \gg 1$, with ψ and b_z both $O(1)$.[16] Let $\beta = \Gamma\, P_0/B_0^2$ be (Γ times) the plasma β calculated with the "guide-field" B_0, where $\Gamma = 5/3$ is the plasma ratio of specific heats. Note that the above ordering scheme does not constrain β to be either much less than or much greater than unity.

We adopt the reduced, two-dimensional, two-fluid, drift-MHD equations derived in the companion paper,[16]

$$\frac{\partial \psi}{\partial t} = [\phi - d_\beta\, Z, \psi] + \eta\,(J - J_0) - \frac{\mu_e\, d_\beta\,(1+\tau)}{c_\beta}\,\boldsymbol{\nabla}^2[V_z + (d_\beta/c_\beta)\,J], \tag{1}$$

$$\frac{\partial Z}{\partial t} = [\phi, Z] + c_\beta\,[V_z + (d_\beta/c_\beta)\,J, \psi] + D\,Y + \mu_e\, d_\beta\,\boldsymbol{\nabla}^2(U - d_\beta\, Y), \tag{2}$$

$$\frac{\partial U}{\partial t} = [\phi,U] - \frac{d_\beta\,\tau}{2}\{\nabla^2[\phi,Z] + [U,Z] + [Y,\phi]\} + [J,\psi] + \mu_i\,\nabla^2(U + d_\beta\,\tau\,Y) + \mu_e\,\nabla^2(U - d_\beta Y), \tag{3}$$

$$\frac{\partial V_z}{\partial t} = [\phi,V_z] + c_\beta\,[Z,\psi] + \mu_i\,\nabla^2 V_z + \mu_e\,\nabla^2[V_z + (d_\beta/c_\beta)\,J], \tag{4}$$

where $D = c_\beta^{\,2}\,\eta + (1-c_\beta^{\,2})\,\kappa$, $U = \nabla^2\phi$, $J = \nabla^2\psi$, and $Y = \nabla^2 Z$. Here, $c_\beta = \sqrt{\beta/(1+\beta)}$, $d_\beta = c_\beta\,d_i/\sqrt{1+\tau}$, $Z = b_z/c_\beta\sqrt{1+\tau}$, $d_i = (m_i/n_0\,e^2\,\mu_0)^{1/2}/a$, and $[A,B] = \nabla A \times \nabla B \cdot \hat{\mathbf{z}}$. The *guiding-center* velocity is written as $\mathbf{V} = \nabla\phi \times \hat{\mathbf{z}} + \sqrt{1+\tau}\,V_z\,\hat{\mathbf{z}}$. Furthermore, η is the (uniform) plasma resistivity, $\mu_{i,e}$ the (uniform) ion/electron viscosity, κ the (uniform) plasma thermal conductivity, and $J_0(x)$ (minus) the inductively maintained, equilibrium plasma current in the z direction. The above equations contain both electron and ion diamagnetic effects, including the contribution of the anisotropic ion gyroviscous tensor, but neglect electron inertia. Our equations are "reduced" in the sense that they do not contain the compressible Alfvén wave. However, they do contain the shear-Alfvén wave, the magnetoacoustic wave, the whistler wave, and the kinetic-Alfvén wave.

B. Plasma equilibrium

The plasma equilibrium satisfies $\partial/\partial y \equiv 0$. Suppose that the plasma is bounded by rigid walls at $x = \pm x_w$ and that the region beyond the walls is a vacuum. The equilibrium magnetic flux is written $\psi^{(0)}(x)$, where $\psi^{(0)}(-x) = \psi^{(0)}(x)$ and $d^2\psi^{(0)}(x)/dx^2 = J_0(x)$. The scale magnetic field strength B_a is chosen such that $\psi^{(0)}(x) \to -x^2/2$ as $|x| \to 0$. The equilibrium value of the field Z takes the form $Z^{(0)}(x) = -[V^{(0)}_{*\,y}/d_\beta\,(1+\tau)]\,x$, where $V^{(0)}_{*y}$ is the (uniform) total diamagnetic velocity in the y direction. The equilibrium value of the guiding-center stream-function is written $\phi^{(0)}(x) = -V^{(0)}_{EB_y}\,x$, where $V^{(0)}_{EB_y}$ is the (uniform) equilibrium $\boldsymbol{E}\times\boldsymbol{B}$ velocity in the y direction. Finally, the equilibrium value of the field V_z is simply $V_z^{(0)} = 0$.

C. Asymptotic matching

Consider a tearing perturbation which is periodic in the y direction with periodicity length l. According to conventional analysis, the plasma is conveniently split into two regions.[17] The "outer region" comprises most of the plasma, and is governed by the equations of linearized, ideal MHD. On the other hand, the "inner region" is localized in the vicinity of the magnetic resonance $x = 0$ (where $B_y^{(0)} = 0$). Nonlinear, dissipative, and drift-MHD effects all become important in the inner region.

In the outer region, we can write $\psi(x,y,t) = \psi^{(0)}(x) + \psi^{(1)}(x,t)\exp(iky)$, where $k = 2\pi/l$ and $|\psi^{(1)}| \ll |\psi^{(0)}|$. Linearized ideal MHD yields $[\psi^{(1)},J^{(0)}] + [\psi^{(0)},J^{(1)}] = 0$, where $J = \nabla^2\psi$. It follows that

$$\left(\frac{\partial^2}{\partial x^2} - k^2\right)\psi^{(1)} - \left(\frac{d^3\psi^{(0)}/dx^3}{d\psi^{(0)}/dx}\right)\psi^{(1)} = 0. \tag{5}$$

The solution to the above equation must be asymptotically matched to the full, nonlinear, dissipative, drift-MHD solution in the inner region.

III. INTERACTION WITH A RESISTIVE WALL

A. Introduction

Suppose that the walls bounding the plasma at $x = \pm x_w$ are thin and resistive, with time-constant τ_w. We can define the perfect-wall tearing eigenfunction $\psi_{pw}(x)$ as the continuous even (in x) solution to Eq. (6) which satisfies $\psi_{pw}(0) = 1$ and $\psi_{pw}(\pm x_w) = 0$. Likewise, the no-wall tearing eigenfunction $\psi_{nw}(x)$ is the continuous even solution to Eq. (6) which satisfies $\psi_{pw}(0) = 1$ and $\psi_{pw}(\pm\infty) = 0$. In general, both $\psi_{pw}(x)$, and $\psi_{nw}(x)$ have gradient discontinuities at $x = 0$. The quantity $\Delta_{pw} = [d\psi_{pw}/dx]_{0-}^{0+}$ is the conventional tearing stability index[17] in the presence of a perfectly conducting wall (i.e., $\tau_w \to \infty$), whereas $\Delta_{nw} = [d\psi_{nw}/dx]_{0-}^{0+} > \Delta_{pw}$ is the tearing stability index in the presence of no wall (i.e., $\tau_w \to 0$). Finally, the wall eigenfunction $\psi_w(x)$ is defined as the continuous even solution to Eq. (5) which satisfies $\psi_w(0) = 0$, $\psi_w(\pm x_w) = 1$, and $\psi_w(\pm\infty) = 0$. This eigenfunction has additional gradient discontinuities at $x = \pm x_w$. The wall stability index, $\Delta_w < 0$, is defined $\Delta_w = [d\psi_w/dx]_{x_w-}^{x_w+}$.

According to standard analysis,[7] the effective tearing stability index, $\Delta' = [d\ln\psi/dx]_{0-}^{0+}$, in the presence of a resistive wall is written as

$$\Delta' = \frac{V^2\,\Delta_{pw} + V_w^{\,2}\,\Delta_{nw}}{V^2 + V_w^{\,2}}, \tag{6}$$

where V is the phase velocity of the tearing mode in the lab frame and $V_w = (-\Delta_w)/(k\tau_w)$. Also, the net y-directed electromagnetic force acting on the inner region takes the form

$$f_y = -\frac{k}{2}(\Delta_{nw} - \Delta_{pw})\frac{VV_w}{V^2 + V_w^2}\Psi^2, \tag{7}$$

where $\Psi(t) = |\psi^{(1)}(0,t)|$ is the reconnected magnetic flux, which is assumed to have a very weak time dependence.

B. Island geometry

In the inner region, we can write

$$\psi(x,\theta,t) = -\frac{x^2}{2} + \Psi(t)\cos\theta, \tag{8}$$

where $\theta = ky$. As is well-known, the above expression for ψ describes a constant ψ magnetic island of full-width (in the x direction) $W = 4w$, where $w = \sqrt{\Psi}$. The region inside the magnetic separatrix corresponds to $\Psi \geqslant \psi \geqslant -\Psi$, whereas the region outside the separatrix corresponds to $\psi < -\Psi$. It is convenient to work in the *island rest frame*, in which $\partial/\partial t \simeq 0$.

It is helpful to define a flux-surface average operator,

$$\langle f(s,\psi,\theta)\rangle = \oint \frac{f(s,\psi,\theta)}{|x|}\,\frac{d\theta}{2\pi} \tag{9}$$

for $\psi < -\Psi$, and

$$\langle f(s,\psi,\theta)\rangle = \int_{-\theta_0}^{\theta_0} \frac{f(s,\psi,\theta)+f(-s,\psi,\theta)}{2\,|x|}\,\frac{d\theta}{2\pi} \tag{10}$$

for $\Psi \geqslant \psi \geqslant -\Psi$. Here, $s=\mathrm{sgn}(x)$ and $x(s,\psi,\theta_0)=0$ (with $\pi > \theta_0 > 0$). The most important property of this operator is that $\langle [A,\psi]\rangle \equiv 0$, for any field $A(s,\psi,\theta)$.

C. Ordering scheme

For the purpose of our ordering scheme, we require both ∇ and ψ to be $O(1)$ in the vicinity of the island. This implies that our scale length a is $O(W)$ and our scale field strength B_a is $O(\Psi/W)$, where W and Ψ are the unnormalized island width and reconnected flux, respectively.

In the inner region, we adopt the following ordering of terms appearing in Eqs. (1)–(4): $d_\beta = d_\beta^{[1]}$, $\psi=\psi^{[0]}$, $\phi=\phi^{[1]}(s,\psi)+\phi^{[5]}(s,\psi,\theta)$, $Z=Z^{[0]}(s,\psi)+Z^{[4]}(s,\psi,\theta)$, $V_z = V_z^{[3]}(s,\psi,\theta)$, $\delta J \equiv 1+\nabla^2\psi = \delta J^{[2]}(s,\psi,\theta)$. Moreover, $\nabla = \nabla^{[0]}$, $\tau=\tau^{[0]}$, $c_\beta = c_\beta^{[0]}$, $\mu_{i,e}=\mu_{i,e}^{[3]}$, $\kappa=\kappa^{[3]}$, $\eta=\eta^{[3]}$, $D=D^{[3]}$, and $d\Psi/dt = d\Psi^{[5]}/dt$. Here, the superscript $[i]$ indicates a quantity which is order $(d_\beta)^i$, where it is assumed that $d_\beta \ll 1$. This ordering, which [together with Eqs. (11)–(14)] is completely self-consistent, implies weak (i.e., strongly sub-Alfvénic and sub-magnetoacoustic) diamagnetic flows, and very long (i.e., very much longer than the Alfvén time) transport evolution time scales.

Equations (1)–(4) yield

$$\frac{d\Psi^{[5]}}{dt}\cos\theta = [\phi^{[5]} - d_\beta^{[1]}\,Z^{[4]},\psi] + \eta^{[3]}\,\delta J^{[2]} - \frac{\mu_e^{[3]}\,d_\beta^{[1]}\,(1+\tau)}{c_\beta}\,\nabla^{2^{[-2]}}[V_z^{[3]} + (d_\beta^{[1]}/c_\beta)\,\delta J^{[2]}] + O(d_\beta^6), \tag{11}$$

$$0 = c_\beta\,[V_z^{[3]} + (d_\beta^{[1]}/c_\beta)\,\delta J^{[2]},\psi] + D^{[3]}\,Y^{[0]} + \mu_e^{[3]}\,d_\beta^{[1]}\,\nabla^{2^{[-2]}}(U^{[1]} - d_\beta^{[1]}\,Y^{[0]}) + O(d_\beta^4), \tag{12}$$

$$0 = -M^{[1]}\,[U^{[1]},\psi] - \frac{d_\beta^{[1]}\,\tau}{2}\{L^{[0]}\,[U^{[1]},\psi] + M^{[1]}\,[Y^{[0]},\psi]\} + [\delta J^{[2]},\psi] + \mu_i^{[3]}\,\nabla^2(U^{[1]} + d_\beta^{[1]}\,\tau\,Y^{[0]}) + \mu_e^{[3]}\,\nabla^2(U^{[1]} - d_\beta^{[1]}\,Y^{[0]}) + O(d_\beta^5), \tag{13}$$

$$0 = -M^{[1]}\,[V_z^{[3]},\psi] + c_\beta\,[Z^{[4]},\psi] + \mu_i^{[3]}\,\nabla^{2^{[-2]}} V_z^{[3]} + \mu_e^{[3]}\,\nabla^{2^{[-2]}}[V_z^{[3]} + (d_\beta^{[1]}/c_\beta)\,\delta J^{[2]}] + O(d_\beta^5), \tag{14}$$

where $Y^{[0]} = \nabla^2 Z^{[0]}$, $U^{[1]}=\nabla^2\phi^{[1]}$, $M^{[1]}(s,\psi) = d\phi^{[1]}/d\psi$, and $L^{[0]}(s,\psi) = dZ^{[0]}/d\psi$. Here, we have neglected the superscripts on most zeroth-order quantities, for the sake of clarity. As indicated, some of the ∇^2 terms are $O(d_\beta^{-2})$, since they operate on quantities which are only important in thin boundary layers of width $O(d_\beta)$ located on the magnetic separatrix. In the following, we shall neglect all superscripts for ease of notation.

D. Determination of flow profiles

Flux-surface averaging Eqs. (12) and (13), we obtain

$$\langle \nabla^2 U\rangle + \frac{d_\beta\,(\mu_i\,\tau - \mu_e)}{(\mu_i+\mu_e)}\,\langle \nabla^2 Y\rangle = 0 \tag{15}$$

and

$$\delta^2\,w^2\,\langle \nabla^2 Y\rangle - \langle Y\rangle = 0, \tag{16}$$

where

$$\delta = \frac{d_\beta}{w}\,\sqrt{\frac{\mu_i\mu_e(1+\tau)}{D(\mu_i+\mu_e)}}. \tag{17}$$

Our ordering scheme implies that $\delta \sim d_\beta \ll 1$.

Now, we can write $\nabla^2 \simeq \partial^2/\partial x^2$, provided that the island is "thin" (i.e., $w \ll l$). It follows that

$$M(s,\psi) = -\frac{d_\beta(\mu_i\tau - \mu_e)}{(\mu_i+\mu_e)} L(s,\psi) + F(s,\psi), \tag{18}$$

where

$$\frac{d}{d\psi}\left[\frac{d}{d\psi}\left(\delta^2 w^2 \langle x^4\rangle \frac{dL}{d\psi}\right) - \langle x^2\rangle L\right] = 0 \tag{19}$$

and

$$\frac{d^2}{d\psi^2}\left(\langle x^4\rangle\,\frac{dF}{d\psi}\right) = 0. \tag{20}$$

Note that $L(s,\psi)$ and $F(s,\psi)$ are *odd* functions of x. We immediately conclude that $L(s,\psi)$ and $F(s,\psi)$ are both *zero* inside the island separatrix (since it is impossible to have a nonzero, odd flux-surface function in this region). The function $L(s,\psi)$ satisfies the additional boundary condition $xL \to V_{*y}^{(0)}/d_\beta(1+\tau)$ as $|x|/w \to \infty$. Here, we are assuming that $w \ll x_w$. Moreover, the function $F(s,\psi)$ satisfies the additional boundary condition $x\,F \to (|x|/x_w)(V^{(0)} - V)$ as $|x|/w \to 0$, where $V^{(0)}$ is the unperturbed island phase velocity (i.e., the phase velocity in the absence of a resistive wall or an external magnetic perturbation) in the lab frame.

It is helpful to define the following quantities: $\hat{\psi} = -\psi/\Psi$, $\langle\langle \cdots\rangle\rangle = \langle \cdots\rangle\,w$, and $X = x/w$. The solutions to Eqs. (19) and (20), subject to the above mentioned boundary conditions, are

$$L(s,\hat{\psi}) = \frac{sV_{*y}^{(0)}}{wd_\beta(1+\tau)}\,\frac{1}{\langle\langle X^2\rangle\rangle} \tag{21}$$

and

$$F(s,\hat{\psi}) = \frac{s(V^{(0)} - V)}{x_w}\int_1^{\hat{\psi}} \frac{d\hat{\psi}}{\langle\langle X^4\rangle\rangle} \Bigg/ \int_1^{\infty} \frac{d\hat{\psi}}{\langle\langle X^4\rangle\rangle}, \tag{22}$$

respectively. Of course, both $L(s,\hat{\psi})$ and $F(s,\hat{\psi})$ are zero inside the island separatrix (i.e., $\hat{\psi} < 1$). In writing Eq. (21), we have neglected the thin boundary layer (width, δw) which resolves the apparent discontinuity in $L(s,\hat{\psi})$ across the island separatrix. This boundary layer, which need not be resolved in any of our calculations, is described in the companion paper.[16] Note that the function $L(s,\hat{\psi})$ corresponds to a velocity profile which is *localized* in the vicinity of the island, whereas the function $F(s,\hat{\psi})$ corresponds to a *nonlocalized* profile which extends over the whole plasma.

E. Force balance

The net electromagnetic force acting on the island region can be written as[14]

$$f_y = -2k\Psi \int_{\Psi}^{-\infty} \langle \delta J_s \sin\theta \rangle d\psi, \tag{23}$$

where δJ_s is the component of δJ with the symmetry of $\sin\theta$. Now, it is easily demonstrated that

$$\langle \delta J_s \sin\theta \rangle = \frac{1}{k\Psi}\langle x[\delta J_s, \psi]\rangle, \tag{24}$$

so it follows from Eq. (13) that

$$\langle \delta J_s \sin\theta \rangle = -\frac{(\mu_i+\mu_e)}{k\Psi}\frac{d}{d\psi}\left(\langle x^5\rangle \frac{d^2F}{d\psi^2} - 2\langle x^3\rangle\frac{dF}{d\psi} - \langle x\rangle F\right). \tag{25}$$

Hence,

$$\begin{aligned} f_y &= 2(\mu_i+\mu_e)\lim_{x/w\to\infty}\left(\langle x^5\rangle \frac{d^2F}{d\psi^2} - 2\langle x^3\rangle\frac{dF}{d\psi} - \langle x\rangle F\right) \\ &= 2s(\mu_i+\mu_e)\lim_{x/w\to\infty}\left[x^2\frac{d}{dx}\left(\frac{1}{x}\frac{d(xF)}{dx}\right)\right]. \end{aligned} \tag{26}$$

Finally, Eq. (22) yields

$$f_y = -\frac{2(\mu_i+\mu_e)(V^{(0)}-V)}{x_w}. \tag{27}$$

Equating Eqs. (7) and (27), we obtain the island force balance equation:

$$\frac{2(\mu_i+\mu_e)(V^{(0)}-V)}{x_w} = \frac{k}{2}(\Delta_{nw}-\Delta_{pw})\frac{V\,V_w}{V^2+V_w^2}(W/4)^4. \tag{28}$$

This equation describes the competition between the viscous restoring force (left-hand side) and the electromagnetic wall drag (right-hand side) acting on the island, and determines the island phase velocity V as a function of the island width W. Note that the above force balance equation is identical to that obtained from single-fluid MHD theory.[7]

F. Determination of ion polarization correction

It follows from Eqs. (11), (13), and (14) that

$$\begin{aligned} \delta J_c = &-\frac{1}{2}\left(X^2 - \frac{\langle\langle X^2\rangle\rangle}{\langle\langle 1\rangle\rangle}\right)\frac{d}{d\hat{\psi}}[M\,(M + d_\beta\,\tau L)] \\ &+ \eta^{-1}\frac{d\Psi}{dt}\frac{\langle\langle \cos\theta\rangle\rangle}{\langle\langle 1\rangle\rangle}, \end{aligned} \tag{29}$$

where δJ_c is the component of δJ with the symmetry of $\cos\theta$. In writing the above expression, we have neglected any boundary layers on the island separatrix, since these are either unimportant or need not be resolved in our calculations (see Ref. 16). Now, making use of Eqs. (18), (21), and (22), we can write

$$M(s,\hat{\psi}) = -\frac{s\,(V^{(0)}-V^{(0)}_{EBy})}{w}\,\mathcal{L}(\hat{\psi}) + \frac{s\,(V^{(0)}-V)}{x_w}\,\mathcal{F}(\hat{\psi}) \tag{30}$$

and

$$\begin{aligned} M(s,\hat{\psi}) + d_\beta\,\tau L(x,\hat{\psi}) = &-\frac{s\,(V^{(0)}-V^{(0)}_{i\,y})}{w}\,L(\hat{\psi}) \\ &+ \frac{s\,(V^{(0)}-V)}{x_w}\,\mathcal{F}(\hat{\psi}). \end{aligned} \tag{31}$$

Here, $V^{(0)}_{EBy} = (V^{(0)}_{i\,y} + \tau\,V^{(0)}_{e\,y})/(1+\tau)$ is the unperturbed $\boldsymbol{E}\times\boldsymbol{B}$ velocity (i.e., the $\boldsymbol{E}\times\boldsymbol{B}$ velocity in the absence of an island), $V^{(0)}_{iy}$ is the unperturbed ion fluid velocity (i.e., the ion fluid velocity in the absence of an island), and $V^{(0)}_{ey}$ is the unperturbed electron fluid velocity (i.e., the electron fluid velocity in the absence of an island). [Note that $V^{(0)}_{*y} = V^{(0)}_{iy} - V^{(0)}_{ey}$.] Furthermore, $V^{(0)} = (\mu_i V^{(0)}_{iy} + \mu_e V^{(0)}_{ey})/(\mu_i+\mu_e)$ (see Ref. 16) is the unperturbed island phase velocity (i.e., the phase velocity in the absence of a resistive wall), and V the actual phase velocity. All of these velocities are measured in the lab frame. Finally, both $\mathcal{L}(\hat{\psi})$ and $\mathcal{F}(\hat{\psi})$ are zero for $\hat{\psi}<1$, whereas

$$\mathcal{L}(\hat{\psi}) = \frac{1}{\langle\langle X^2\rangle\rangle} \tag{32}$$

and

$$F(\hat{\psi}) = \int_1^{\hat{\psi}} \frac{d\hat{\psi}}{\langle\langle X^4\rangle\rangle} \bigg/ \int_1^{\infty} \frac{d\hat{\psi}}{\langle\langle X^4\rangle\rangle} \tag{33}$$

in the region $\hat{\psi}\geqslant 1$.

Now

$$\Delta'(V) = \frac{4}{w}\int_{-1}^{\infty} \langle\langle \delta J_c\cos\theta\rangle\rangle\, d\hat{\psi} \tag{34}$$

(see Ref. 14), where $\Delta'(V)$, which is specified in Eq. (6), is the effective tearing stability index in the presence of the resistive wall. Hence, it follows from Eqs. (29)–(31) and (34) that

$$\frac{I_1}{\eta}\frac{dW}{dt} = \Delta'(V) + I_2 \frac{(V^{(0)} - V^{(0)}_{EBy})(V^{(0)} - V^{(0)}_{iy})}{(W/4)^3} - I_3 \frac{2(V^{(0)} - [V^{(0)}_{EBy} + V^{(0)}_{iy}]/2)(V^{(0)} - V)}{x_w (W/4)^2} + I_4 \frac{(V^{(0)} - V)^2}{x_w^2 (W/4)}, \tag{35}$$

where

$$I_1 = 2\int_{-1}^{\infty} \frac{\langle\langle \cos\theta\rangle\rangle^2}{\langle\langle 1\rangle\rangle}\, d\hat{\psi} = 0.823, \tag{36}$$

$$I_2 = \int_{-1}^{\infty} \left(\langle\langle X^4\rangle\rangle - \frac{\langle\langle X^2\rangle\rangle^2}{\langle\langle 1\rangle\rangle}\right) \frac{d(\mathcal{L}^2)}{d\hat{\psi}}\, d\hat{\psi} = 1.38, \tag{37}$$

$$I_3 = \int_{-1}^{\infty} \left(\langle\langle X^4\rangle\rangle - \frac{\langle\langle X^2\rangle\rangle^2}{\langle\langle 1\rangle\rangle}\right) \frac{d(\mathcal{L}\mathcal{F})}{d\hat{\psi}}\, d\hat{\psi} = 0.195, \tag{38}$$

$$I_4 = \int_{-1}^{\infty} \left(\langle\langle X^4\rangle\rangle - \frac{\langle\langle X^2\rangle\rangle^2}{\langle\langle 1\rangle\rangle}\right) \frac{d(\mathcal{F}^2)}{d\hat{\psi}}\, d\hat{\psi} = 0.469. \tag{39}$$

Equation (35) is the Rutherford island width evolution equation[15] for a propagating magnetic island interacting with a resistive wall. There are *three* separate ion polarization terms on the right-hand side (RHS) of this equation. The first (second term on RHS) is the drift-MHD polarization term for an isolated island (see Ref. 16) and is unaffected by wall braking. This term, which varies as W^{-3}, is stabilizing provided that the unperturbed island phase velocity lies between the unperturbed local ion fluid velocity and the unperturbed local $\boldsymbol{E}\times\boldsymbol{B}$ velocity, and is destabilizing otherwise. The third (fourth term on RHS) is the single-fluid MHD polarization term due to the island velocity shift induced by wall braking (see Ref. 9). This term is *always destabilizing*, and varies as W^{-1} and the square of the wall-induced velocity shift. The second (third term on RHS) is a hybrid of the other two polarization terms. The sign of this term depends on many factors. However, in the limit of small electron viscosity (compared to the ion viscosity), when the unperturbed island phase velocity lies close to the unperturbed velocity of the ion fluid,[16] the hybrid term is stabilizing provided $V^{(0)}_{*y} V^{(0)} > 0$, and destabilizing otherwise. In other words, the hybrid term is stabilizing if the noninteracting island propagates in the *ion diamagnetic direction* (with respect to the wall), and destabilizing if it propagates in the *electron diamagnetic direction*. The hybrid polarization term varies as W^{-2} and is directly proportional to the wall-induced island velocity shift.

IV. INTERACTION WITH A RESONANT MAGNETIC PERTURBATION

A. Introduction

Let the walls bounding the plasma at $x = \pm x_w$ now be nonconducting (i.e., $\tau_w \to 0$). Suppose that an even (in x) propagating magnetic perturbation (with the same wavelength as the magnetic island in the plasma) is generated by currents flowing in field coils located in the vacuum region beyond the walls.

The no-wall tearing stability index Δ_{nw} is defined in Sec. III A. The coil eigenfunction $\psi_c(x)$ is the continuous even solution to Eq. (5) which satisfies $\psi_c(0)=0$ and $\psi_c(\pm x_w)=1$. In general, this eigenfunction has a gradient discontinuity at $x=0$. It is helpful to define $\Delta_c = [d\psi_c/dx]_{0-}^{0+}$.

According to standard analysis,[7] the effective tearing stability index, $\Delta' = [d \ln \psi/dx]_{0-}^{0+}$, in the presence of an externally generated, magnetic perturbation is

$$\Delta'(t) = \Delta_{nw} + \Delta_c \frac{\Psi_c}{\Psi} \cos\varphi(t), \tag{40}$$

where $\Psi(t) = |\psi^{(1)}(0,t)|$ is the reconnected magnetic flux, which is assumed to vary slowly in time, and Ψ_c the flux at the walls solely due to currents flowing in the external coils. Furthermore, $\varphi(t)$ is the phase of the island measured with respect to that of the externally generated perturbation. Let the phase velocity of the externally generated perturbation be V_c. It follows that

$$\frac{d\varphi}{dt} = k\, V'(t), \tag{41}$$

where $V' = V - V_c$, and $V(t)$ is the instantaneous island phase velocity. Also, the net y-directed electromagnetic force acting on the island takes the form

$$f_y(t) = -\frac{k}{2}\Delta_c \Psi \Psi_c \sin\varphi(t). \tag{42}$$

Note that, unlike the braking force due to a resistive wall, this force *oscillates* in sign as the island propagates.

B. Determination of flow profiles

We can reuse the analysis of Sec. III D, except that we must allow for *time dependence* of the function F to take into account the *oscillating* nature of the locking force exerted on the island by the external perturbation. Hence, we write

$$M(s,\psi,t) = -\frac{d_\beta(\mu_i\,\tau - \mu_e)}{(\mu_i + \mu_e)} L(s,\psi) + F(s,\psi,t), \tag{43}$$

where

$$L(s,\hat{\psi}) = \frac{s\, V^{(0)}_{*y}}{w\, d_\beta (1+\tau)} \frac{1}{\langle\langle X^2\rangle\rangle} \tag{44}$$

and

$$\frac{\partial}{\partial\psi}\left[(\mu_i + \mu_e) \frac{\partial}{\partial\psi}\left(\langle x^4\rangle \frac{\partial F}{\partial\psi}\right) - \langle x^2\rangle \frac{\partial F}{\partial t}\right] = 0. \tag{45}$$

In order to proceed further, we adopt the separable form approach to solve Eq. (45) which was introduced and justified in Ref. 14. In other words, we try the following solution:

$$F(s,\psi,t) = s\,F_1(\psi)\,\sin\left(\int_0^t k\,V'(t')\,dt'\right) + s\,F_2(\psi)\,\cos\left(\int_0^t k\,V'(t')\,dt'\right). \quad (46)$$

Of course, $F_1(\psi)$ and $F_2(\psi)$ are both zero within the island separatrix. Furthermore,

$$|x|\,F_1 \to F_0, \quad (47)$$

$$|x|\,F_2 \to 0, \quad (48)$$

as $|x|/w\to\infty$. Here, F_0 is a constant. The above boundary conditions imply that the function $F(s,\psi,t)$ corresponds to a velocity profile which is localized in the vicinity of the island.

Matching to the outer region yields

$$F_0\,\sin\left(\int_0^t k\,V'(t')\,dt'\right) = V^{(0)} - V(t). \quad (49)$$

Hence, differentiating with respect to t, we obtain

$$\frac{1}{k\,V'}\,\frac{dV}{dt} = -\,F_0\,\cos\left(\int_0^t k\,V'(t')\,dt'\right) \quad (50)$$

and

$$\frac{d}{dt}\left(\frac{1}{k\,V'}\,\frac{dV}{dt}\right) = k\,V'\,(V^{(0)} - V). \quad (51)$$

Substituting Eq. (46) into Eq. (45), and integrating once in ψ using the boundary conditions (47) and (48), we get

$$\mathrm{sgn}(V')\,\frac{\lambda^2}{2\,w^2}\,\frac{d}{d\hat{\psi}}\left(\langle\langle X^4\rangle\rangle\,\frac{dF_1}{d\hat{\psi}}\right) + \langle\langle X^2\rangle\rangle\,F_2 = 0, \quad (52)$$

$$\mathrm{sgn}(V')\,\frac{\lambda^2}{2\,w^2}\,\frac{d}{d\hat{\psi}}\left(\langle\langle X^4\rangle\rangle\,\frac{dF_2}{d\hat{\psi}}\right) - \langle\langle X^2\rangle\rangle\,F_1 = -\,\frac{F_0}{w}. \quad (53)$$

Here, $\lambda=\sqrt{2\,(\mu_i+\mu_e)/k\,|V'|}$ is the localization scale length of the velocity profile corresponding to the function F.

Suppose that $w\ll\lambda\ll x_w$. In other words, suppose that the localization scale length of the velocity profile associated with F is much larger than the island width, but much smaller than the extent of the plasma. In this limit (which corresponds to the "weakly localized" regime of Ref. 14), Eqs. (52) and (53) can be solved to give

$$|X|\,F_1 = \frac{F_0}{w}\left[1 - \exp\left(-\,\frac{w|X|}{\lambda}\right)\,\cos\left(\frac{w|X|}{\lambda}\right)\right]\mathcal{F}(\hat{\psi}), \quad (54)$$

$$|X|\,F_2 = \mathrm{sgn}(V')\,\frac{F_0}{w}\,\exp\left(-\,\frac{w|X|}{\lambda}\right)\,\sin\left(\frac{w|X|}{\lambda}\right)\,\mathcal{F}(\hat{\psi}). \quad (55)$$

Here, $\mathcal{F}(\hat{\psi})$ is specified in Eqs. (33). It follows from Eqs. (46), (49), and (50) that

$$F(s,\hat{\psi},t) = \frac{s}{w}(V^{(0)} - V)\left[1 - \exp\left(-\,\frac{w|X|}{\lambda}\right)\,\cos\left(\frac{w|X|}{\lambda}\right)\right]\,\frac{\mathcal{F}(\hat{\psi})}{|X|} - \frac{s}{w}\,\frac{1}{k|V'|}\,\frac{dV}{dt}\,\exp\left(-\,\frac{w|X|}{\lambda}\right)\,\sin\left(\frac{w|X|}{\lambda}\right)\,\frac{\mathcal{F}|\hat{\psi}|}{|X|}. \quad (56)$$

C. Island equation of motion

Reusing the analysis of Sec. III E, taking into account the time dependence of F, we obtain

$$f_y = 2s(\mu_i + \mu_e)\,\lim_{x/w\to\infty}\left[x^2\frac{\partial}{\partial x}\left(\frac{1}{x}\,\frac{\partial\,(xF)}{\partial x}\right)\right] - 2\,\frac{\partial}{\partial t}\int_{-\psi}^{-\infty}\left(\langle x^3\rangle\frac{\partial F}{\partial \psi} - \langle x\rangle F\right)\,d\psi. \quad (57)$$

According to the boundary conditions (47) and (48), the first term on the right-hand side is identically zero. Transforming the second term on the right-hand side, using the fact that the integral is dominated by the region $|X|\gg 1$, we get

$$f_y = -\,2\,s\,\Psi\,\frac{\partial}{\partial t}\int_0^\infty X\,\frac{\partial\,(X\,F)}{\partial X}\,dX. \quad (58)$$

Finally, Eqs. (50), (51), and (56) yield

$$f_y = \lambda\left[\frac{dV}{dt} + k\,|V'|(V - V^{(0)})\right]. \quad (59)$$

Making use of Eq. (42), the island equation of motion takes the form

$$\sqrt{\frac{2(\mu_i + \mu_e)}{k|V'|}}\,\frac{dV}{dt} + \sqrt{2(\mu_i + \mu_e)k|V'|}(V - V^{(0)}) + \frac{k}{2}\left(\frac{W}{4}\right)^2\left(\frac{W_c}{4}\right)^2\,\sin\varphi = 0. \quad (60)$$

Here, $(W_c/4)^2=\Delta_c\,\Psi_c$. The first term on the left-hand side represents the inertia of the region of the plasma (of width $\sqrt{2\,(\mu_i+\mu_e)/k\,|V'|}$) which is viscously coupled to the island, the second term represents the viscous restoring force, and the third term represents the locking force due to the external perturbation. Note that the above equation is identical to that obtained from single-fluid MHD theory.[14] The above analysis is valid provided $w\ll\sqrt{2\,(\mu_i+\mu_e)/k\,|V'|}\ll x_w$.

D. Determination of ion polarization correction

Reusing the analysis of Sec. III F, we obtain

$$\delta J_c = -\,\frac{1}{2}\left(X^2 - \frac{\langle\langle X^2\rangle\rangle}{\langle\langle 1\rangle\rangle}\right)\frac{\partial}{\partial\,\hat{\psi}}[M(M + d_\beta\tau L)] + \eta^{-1}\frac{d\Psi}{dt}\,\frac{\langle\langle\cos\,\theta\rangle\rangle}{\langle\langle 1\rangle\rangle}, \quad (61)$$

where

$$M(s,\hat{\psi},t) = -\frac{s(V^{(0)} - V_{EBy}^{(0)})}{w}\mathcal{L}(\hat{\psi}) - \frac{s f_y(t)}{2(\mu_i + \mu_e)}\mathcal{F}(\hat{\psi}) \quad (62)$$

and

$$M(s,\hat{\psi},t) + d_\beta \tau L(x,\hat{\psi}) = -\frac{s(V^{(0)} - V_{iy}^{(0)})}{w}\mathcal{L}(\hat{\psi}) - \frac{s f_y(t)}{2(\mu_i + \mu_e)}\mathcal{F}(\hat{\psi}). \quad (63)$$

Here, use has been made of Eqs. (56) and (59), as well as the fact that the polarization term integral is dominated by the region $|X| \sim O(1)$. Finally, Eqs. (34), (40), and (42) yield

$$\frac{I_1}{\eta}\frac{dW}{dt} = \Delta_{nw} + \left(\frac{W_c}{W}\right)^2 \cos\varphi + I_2 \frac{(V^{(0)} - V_{EBy}^{(0)})(V^{(0)} - V_{iy}^{(0)})}{(W/4)^3} - I_3 \frac{k}{2}\frac{(V^{(0)} - [V_{EB}^{(0)} + V_{iy}^{(0)}]/2)}{(\mu_i + \mu_e)}\left(\frac{W_c}{4}\right)^2 \sin\varphi + I_4 \frac{k^2}{16(\mu_i + \mu_e)^2}\left(\frac{W}{4}\right)^3\left(\frac{W_c}{4}\right)^4 \sin^2\varphi, \quad (64)$$

where I_1, I_2, I_3, and I_4 are specified in Sec. III F.

Equation (64) is the Rutherford island width evolution equation for a propagating island interacting with an externally generated, resonant magnetic perturbation. There are three separate ion polarization terms on the right-hand side of this equation. The first (third term on RHS) is the drift-MHD polarization term for an isolated island (see Ref. 16), and is unaffected by the external perturbation. The third (fifth term on RHS) is the single-fluid MHD polarization term due to the oscillation in island phase velocity induced by the externally generated perturbation (see Ref. 14). This term modulates as the island propagates, but is always destabilizing. The second (fourth term on RHS) is a hybrid of the other two polarization terms.

E. Solution of island equations of motion

Let us solve the island equations of motion, (41) and (60), in the limit in which the externally generated magnetic perturbation is sufficiently weak that it does not significantly perturb the island phase velocity. Let us also assume that η is so small that the island width W does not vary appreciably with island phase. In this limit, we can write

$$\varphi(t) = kV'^{(0)}t + \alpha_s \sin(k\,V'^{(0)}t) + \alpha_c \cos(k\,V'^{(0)}t), \quad (65)$$

where $|\alpha_s|, |\alpha_c| \ll 1$ and $V'^{(0)} = V^{(0)} - V_c$. Substitution of the above expression into Eqs. (41) and (60) yields

$$\alpha_s \simeq \left(\frac{W}{4}\right)^2\left(\frac{W_c}{4}\right)^2 \bigg/ 4\lambda[V'^{(0)}]^2 \quad (66)$$

and $\alpha_e \simeq \mathrm{sgn}(V'^{(0)})\alpha_s$, where $\lambda = \sqrt{2(\mu_i+\mu_e)/k|V'^{(0)}|}$ is the velocity localization scale length. Averaging over island phase, using Eq. (65), we obtain

$$\overline{\cos\varphi} \simeq \frac{\alpha_s}{2}, \quad (67)$$

$$\overline{\sin\varphi} \simeq \mathrm{sgn}(V'^{(0)})\frac{\alpha_s}{2}, \quad (68)$$

$$\overline{\sin^2\varphi} \simeq \tfrac{1}{2}. \quad (69)$$

Hence, the average of the Rutherford island width evolution equation (64) over island phase takes the form

$$\frac{I_1}{\eta}\frac{dW}{dt} = \Delta_{nw} + I_2 \frac{(V^{(0)} - V_{EBy}^{(0)})(V^{(0)} - V_{iy}^{(0)})}{(W/4)^3} - \frac{\alpha_s}{2}\left(\frac{W_c}{W}\right)^2 \times \left\{1 + I_3 \frac{(V^{(0)} - [V_{EB}^{(0)} + V_{iy}^{(0)}]/2)}{V'^{(0)}}\left(\frac{w}{\lambda}\right)^2 - I_4\left(\frac{w}{\lambda}\right)^3\right\}. \quad (70)$$

The first two terms on the right-hand side of the above equation are the intrinsic tearing mode drive and the drift-MHD polarization term, respectively, and are unaffected by the external perturbation. The next three terms (within the curly braces) are the *phase-averaged* external perturbation drive, hybrid polarization term, and single-fluid MHD polarization term, respectively. It can be seen that the external perturbation drive is on average stabilizing, whereas the single-fluid MHD polarization term is destabilizing.[7] The sign of the hybrid term depends on many factors. However, in the limit of small electron viscosity (compared to the ion viscosity), when the unperturbed island phase velocity lies close to the unperturbed velocity of the ion fluid,[16] the hybrid term is on average stabilizing provided $V_{*y}^{(0)}\,V'^{(0)} > 0$, and destabilizing otherwise. In other words, the hybrid term is stabilizing if the noninteracting island propagates in the ion diamagnetic direction with respect to the external perturbation, and destabilizing if it propagates in the electron diamagnetic direction.

V. SUMMARY

We have investigated the dynamics of a propagating magnetic island interacting with a resistive wall or an externally generated, resonant magnetic perturbation using two-fluid, drift-MHD theory in slab geometry. In both cases, we find that the island equation of motion takes exactly the same form as that predicted by single-fluid MHD theory (see Secs. III E and IV C). However, two-fluid effects do give rise to additional ion polarization terms in the Rutherford island width evolution equation.

In general, we find that there are three separate ion polarization terms in the Rutherford equation (see Secs. III F and IV D). The first is the drift-MHD polarization term for an isolated island and is completely unaffected by interaction with a resistive wall or an externally generated magnetic perturbation. Next, there is the polarization term due to interaction with a resistive wall or magnetic perturbation which is predicted by single-fluid MHD theory. This term is always destabilizing. Finally, there is a hybrid of the other two polarization terms. The sign of this term depends on many factors. However, in the limit of small electron viscosity (compared to the ion viscosity), when the noninteracting (i.e., in the absence of a resistive wall or external magnetic perturbation) island phase velocity lies close to the unperturbed (i.e., in the absence of an island) velocity of the ion fluid,[16] the hybrid term is stabilizing if the noninteracting island propagates in the ion diamagnetic direction (with respect to the wall or external perturbation) and destabilizing if it propagates in the electron diamagnetic direction.

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[1] M. N. Rosenbluth, Plasma Phys. Controlled Fusion **41**, A99 (1999).
[2] Z. Chang and J. D. Callen, Nucl. Fusion **30**, 219 (1990).
[3] J. A. Snipes, D. J. Campbell, P. S. Haynes *et al.*, Nucl. Fusion **28**, 1085 (1988).
[4] T. C. Hender, C. G. Gimblett, and D. C. Robinson, Nucl. Fusion **29**, 1279 (1989).
[5] H. Zohm, A. Kallenbach, H. Bruhns, G. Fussmann, and O. Kluber, Europhys. Lett. **11**, 745 (1990).
[6] M. F. F. Nave and J. A. Wesson, Nucl. Fusion **30**, 2575 (1990).
[7] R. Fitzpatrick, Nucl. Fusion **33**, 1049 (1993).
[8] D. A. Gates and T. C. Hender, Nucl. Fusion **36**, 273 (1996).
[9] F. L. Waelbroeck and R. Fitzpatrick, Phys. Rev. Lett. **78**, 1703 (1997).
[10] R. Fitzpatrick, S. C. Guo, D. J. Den Hartog, and C. C. Hegna, Phys. Plasmas **6**, 3878 (1999).
[11] B. E. Chapman, R. Fitzpatrick, D. Craig, P. Martin, and G. Spizza, Phys. Plasmas **11**, 2156 (2004).
[12] A. W. Morris, T. C. Hender, J. Hugill *et al.*, Phys. Rev. Lett. **64**, 1254 (1990).
[13] G. A. Navratil, C. Cates, M. E. Mauel *et al.*, Phys. Plasmas **5**, 1855 (1998).
[14] R. Fitzpatrick and F. L. Waelbroeck, Phys. Plasmas **7**, 4983 (2000).
[15] P. H. Rutherford, Phys. Fluids **16**, 1903 (1973).
[16] R. Fitzpatrick and F. L. Waelbroeck, "Two-fluid magnetic island dynamics in slab geometry: I-isolated islands" Phys. Plasmas (submitted).
[17] H. P. Furth, J. Killeen, and M. N. Rosenbluth, Phys. Fluids **6**, 459 (1963).