Two-fluid magnetic island dynamics in slab geometry.

I. Isolated islands

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A set of reduced, two-dimensional, two-fluid, drift-MHD (magnetohydrodynamical) equations is derived. Using these equations, a complete and fully self-consistent solution is obtained for an isolated magnetic island propagating through a slab plasma with uniform but different ion and electron fluid velocities. The ion and electron fluid flow profiles around the island are uniquely determined, and are everywhere continuous. Moreover, the island phase velocity is uniquely specified by the condition that the island-induced modifications to the ion and electron velocity profiles remain localized in the vicinity of the island. Finally, the ion polarization current correction to the Rutherford island width evolution equation is evaluated and found to be stabilizing provided that the anomalous perpendicular ion viscosity significantly exceeds the anomalous perpendicular electron viscosity. © 2005 American Institute of Physics. [DOI: 10.1063/1.1833375]

I. INTRODUCTION

Tearing modes are magnetohydrodynamical (MHD) instabilities which often limit fusion plasma performance in magnetic confinement devices relying on nested toroidal magnetic flux surfaces.1 As the name suggests, “tearing” modes tear and reconnect magnetic field lines, in the process converting nested toroidal flux surfaces into helical magnetic islands. Such islands degrade plasma confinement because islands. The ion/electron number density is generally embedded within ion and electron fluids which play negligible roles in conventional magnetic island physics. However, they do not contain electron inertia or the compressible Alfvén wave (which play negligible roles in conventional magnetic island physics). Our set of equations consists of four coupled partial differential equations, and is both analytically tractable and easy to solve numerically. We employ our equations to study the evolution of an isolated magnetic island in slab geometry. Using a particular ordering scheme, we are able to calculate the island phase velocity and uniquely determine the ion and electron fluid flow profiles outside the island separatrix.

II. DERIVATION OF REDUCED EQUATIONS

A. Introduction

In this section, we shall generalize the analysis of Refs. 19 and 20 to obtain a set of reduced, 2D, two-fluid, drift-MHD equations which take ion diamagnetic flows into account.

B. Basic equations

Standard right-handed Cartesian coordinates (x,y,z) are adopted. Consider a quasineutral plasma with singly charged ions. The ion/electron number density $n_0$ is assumed to be

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uniform and constant. Suppose that $T_i = \tau T_e$, where $T_{i,e}$ is the ion/electron temperature, and $\tau$ is uniform and constant.

Broadly following Hazeltine and Meiss, we adopt the following set of two-fluid, drift-MHD equations:

$$ E + V \times B + \frac{1}{en_0} \left( \nabla P - \frac{\tau}{1 + \tau} (b \cdot \nabla P)b - J \times B \right) $$

$$ - \mu_e \nabla^2 V_e = \eta J, $$

$$ m_i n_0 \left[ \frac{\partial}{\partial t} + V \cdot \nabla + \frac{\tau}{1 + \tau} V_s \cdot \nabla \right] V $$

$$ - \frac{\tau}{1 + \tau} V_s \cdot \nabla (|b \cdot V|b) $$

$$ = J \times B - \nabla P + \mu_i \nabla^2 V_i + \mu_e \nabla^2 V_e, $$

$$ \left( \frac{\partial}{\partial t} + V \cdot \nabla \right) P = -\Gamma P \nabla \cdot V + \kappa \nabla^2 P. $$

Here, $E$ is the electric field, $B$ is the magnetic field, $J$ is the electric current density, $V$ is the plasma guiding-center velocity, $P$ is the total plasma pressure, $e$ is the magnitude of the electron charge, $m_i$ is the ion mass, $\eta$ is the (uniform) plasma resistivity, $\mu_e$ is the (uniform) electron viscosity, $\mu_i$ is the (uniform) ion viscosity, $\kappa$ is the (uniform) plasma thermal conductivity, and $\Gamma = 5/3$ is the plasma ratio of specific heats. Furthermore, $b = B/B_z$, $V_i = b \times \nabla P/en_0B$, $V_s = \nu_0 (1 + \tau) V_s$, and $V_e = \nu_0 [ (1 + \tau) V_s - J/n_0 e]$. The above equations take into account the anisotropic ion gyroviscous tensor, but neglect electron inertia. Our system of equations is completed by Maxwell's equations: $\nabla \cdot B = 0$, $\nabla \times E = -\partial B/\partial t$, and $\nabla \times B = \mu_0 J$. Note that the transport coefficients, $\mu_i$, $\mu_e$, and $\kappa$, appearing in the above equations, are phenomenological in nature, and are supposed to represent the anomalous diffusive transport of energy and momentum across magnetic flux surfaces due to small-scale plasma turbulence.

\[\text{C. Normalized equations}\]

Let $\hat{V} = a \nabla \psi$, $\hat{J} = J/(B_0 \mu_0)$, $\hat{B} = B_0 B$, $\hat{E} = E_0 B_0 V_a$, $\hat{J} = J/(B_0 M_{i,e})$, $\hat{V} = V/V_a$, $\hat{V}_{i,e} = V_{i,e}/V_a$, $\hat{P} = P/(B_0^2 \mu_0)$, $\hat{\eta} = \eta/(\mu_0 V_a)$, $\hat{\mu}_i = \mu_i/(n_0 V_a a)$, and $\hat{\kappa} = \kappa/(V_a a)$, where $V_a = B_0/\sqrt{n_0 a}$. Here, $a$ is a convenient scale length and $B_0$ a convenient scale magnetic field strength.

Neglecting hats, our normalized two-fluid equations take the form

$$ E + V \times B + d_i \left( \nabla P - \frac{\tau}{1 + \tau} (b \cdot \nabla P)b - J \times B \right) $$

$$ - \mu_e \nabla^2 V_e = \eta J, $$

$$ \left( \frac{\partial}{\partial t} + V \cdot \nabla + \frac{\tau}{1 + \tau} V_s \cdot \nabla \right) V - \frac{\tau}{1 + \tau} V_s \cdot \nabla (|b \cdot V|b) $$

$$ = J \times B - \nabla P + \mu_i \nabla^2 V_i + \mu_e \nabla^2 V_e. $$

D. 2D assumption

Let us make the simplifying assumption that there is no variation of quantities in the $z$ direction, i.e., $\partial/\partial z = 0$. It immediately follows that $B = \nabla \psi \times \hat{z} + B \hat{z}$ and $E = -\partial \psi/\partial t$.

E. Reduction process

Let us adopt the following ordering, which is designed to decouple the compressional Alfvén wave from all the other waves in the system:

$$ P = P_0 + B_0 \rho_1 + p_2, $$

$$ B_z = B_0 + b_z. $$

Here, $P_0$ and $B_0$ are uniform and constant, and

$$ p_2 \gg b_0 \gg 1. $$

Furthermore, $p_1, p_2, b_z, \psi, V, \nabla, \text{and } \partial/\partial t$ are all assumed to be $O(1)$, and $\nabla \times V$ is assumed to be much less than $O(1)$.

Now, to lowest order, the $z$ component of Ohm's law, Eq. (4), gives

$$ \left( \frac{\partial}{\partial t} + V \cdot \nabla \right) \psi = -d_i [\hat{b}_z + \tau \rho_1/(1 + \tau), \psi] $$

$$ + \eta \nabla^2 \psi - d_i \mu_e \nabla^2 (V_z + d_i \nabla^2 \psi). $$

Here, $[A, B] = \nabla A \times \nabla B \cdot \hat{z}$. Likewise, the $z$ component of the curl of Eq. (4) reduces to

$$ \left( \frac{\partial}{\partial t} + V \cdot \nabla \right) \hat{b}_z = [\hat{V}_z + d_i \nabla^2 \psi, \psi] - B_0 \nabla \cdot V + \eta \nabla^2 \hat{b}_z $$

$$ + d_i \mu_e \nabla^2 \left[ U - d_i \nabla^2 \left( \hat{b}_z + \frac{\tau}{1 + \tau} \rho_1 \right) \right]. $$

Here, $U = -\nabla \times V \cdot \hat{z}$.

To lowest order, the equation of motion, Eq. (5), implies that

$$ p_1 = -b_z. $$

Furthermore, the $z$ component of this equation yields

$$ \left( \frac{\partial}{\partial t} + V \cdot \nabla \right) \hat{V}_z = [\hat{b}_z, \psi] + \mu_i \nabla^2 \hat{V}_z + \mu_e \nabla^2 (V_z + d_i \nabla^2 \psi), $$

whereas the $z$ component of its curl reduces to

$$ \left( \frac{\partial}{\partial t} + V \cdot \nabla \right) \hat{b}_z = \hat{V}_z [\hat{b}_z, \psi] + \mu_i \nabla^2 \hat{b}_z + \mu_e \nabla^2 (V_z + d_i \nabla^2 \psi). $$
\[
\frac{\partial}{\partial t}(\nabla \cdot \mathbf{U}) = \frac{d_i \tau}{2(1 + \tau)} \{ \nabla^2 \phi_b + [U, b_z] \} + \{ \nabla^2 b_z, \phi \} + \mu_s \nabla^2 (U + \frac{d_i \tau}{1 + \tau} \nabla^2 b_z) + \mu_e \nabla^2 (U - \frac{d_i \tau}{1 + \tau} \nabla^2 b_z).
\]

Finally, to lowest order, the energy equation, Eq. (6), gives

\[
\frac{\partial}{\partial t}(\mathbf{U} \cdot \mathbf{V})|_{p_1} = -\frac{\Gamma P_0}{B_0} \mathbf{\nabla} \cdot \mathbf{V} + \kappa \nabla^2 p_1.
\]

Eliminating \(\nabla \cdot \mathbf{V}\) between Eqs. (11) and (15), making use of Eq. (12), we obtain

\[
c^2 \left[ \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla) \right] b_z = \left[ V_z + d_i \nabla^2 \phi_b, \psi_b \right] + \left( \eta + \frac{\kappa}{\beta} \right) \nabla^2 b_z + d_i \mu_s \nabla^2 (U - \frac{d_i \tau}{1 + \tau} \nabla^2 b_z).
\]

Here, \(\beta = \Gamma P_0 / B_0^2\) is \((\Gamma\text{ times})\) the plasma \(\beta\) calculated with the “guide field,” \(B_0\), and \(c_p = \sqrt{\beta/(1 + \beta)}\). Note that our ordering scheme does not constrain \(\beta\) to be either much less than or much greater than unity. In tokamak terminology, the inequality (9) implies a high poloidal \(\beta\) value (i.e., \(\beta_p \gg P_0 \gg 1\)), but does not necessarily imply a high toroidal \(\beta\) value (i.e., \(\beta_t \gg P_0 / B_0^2\) is not necessarily \(\gg 1\)).

Equation (15) implies that \(\nabla \cdot \mathbf{V} \sim O(B_0^{-1})\): i.e., that the flow is almost incompressible. Hence, to lowest order, we can write

\[\mathbf{V} = \nabla \phi \times \mathbf{\hat{z}} + V_z \mathbf{\hat{z}}.\]

\section*{III. ISLAND PHYSICS}

\subsection*{A. Introduction}

The aim of this section is to derive expressions determining the phase velocity and width of an isolated magnetic island (representing the final, nonlinear stage of a tearing instability) from the previously derived set of reduced, 2D, two-fluid, drift-MHD equations.

Consider a slab plasma which is periodic in the y direction with periodicity length \(l\). Let the system be symmetric about \(x = 0\): i.e., \(\psi(x, y, t) = \psi(-x, y, t), Z(-x, y, t) = -Z(x, y, t), \phi(-x, y, t) = -\phi(x, y, t), \) and \(V_z(-x, y, t) = V_z(x, y, t)\). Consider a quasistatic, constant-\(\psi\) magnetic island, centered on \(x = 0\). It is convenient to transform to the island rest frame, in which \(\partial / \partial t = 0\). Suppose that the island is embedded in a plasma with uniform (but different) \(y\)-directed ion and electron fluid velocities. We are searching for an island solution in which the ion/electron fluid velocities asymptote to these uniform velocities far from the magnetic separatrix.

\subsection*{B. Island geometry}

In the immediate vicinity of the island, we can write

\[\psi(x, \theta, t) = -\frac{x^2}{2} + \Psi(t) \cos \theta,\]

where \(\theta = k y, k = 2 \pi / l\), and \(\Psi(t) > 0\) is the reconnected magnetic flux (which is assumed to have a very weak time dependence). As is well known, the above expression for \(\psi\) describes a “cat’s eye” magnetic island of full width (in the \(x\) direction) \(W = 4 w\), where \(w = \sqrt{\psi}\). The region inside the magnetic separatrix corresponds to \(\Psi > \psi > -\Psi\), the region outside the separatrix corresponds to \(\psi < -\Psi\), and the separatrix itself corresponds to \(\psi = -\Psi\). The island \(O\)- and \(X\)-points are located at \((x, \theta) = (0, 0)\), and \((x, \theta) = (0, \pi)\), respectively.

It is helpful to define a flux-surface average operator:
\[
\langle f(s, \psi, \theta) \rangle = \int f(s, \psi, \theta) \frac{d\theta}{2\pi} \quad (23)
\]
for \( \psi < -\Psi \), and
\[
\langle f(s, \psi, \theta) \rangle = \int_{\theta_0}^{\theta_1} f(s, \psi, \theta) + f(-s, \psi, \theta) \frac{d\theta}{2\pi} \quad (24)
\]
for \( \Psi \geq \psi \geq -\Psi \). Here, \( s = \text{sgn}(x) \) and \( x(s, \psi, \theta_0) = 0 \) (with \( \pi > \theta_0 > 0 \)). The most important property of this operator is that
\[
\langle [A, \psi] \rangle = 0 \quad (25)
\]
for any field \( A(s, \psi, \theta) \).

C. Island equations

The equations governing the quasistatic island \( \text{which follow from Eqs. (18)-(21)} \) are
\[
\frac{d\Psi}{dt} \cos \theta = \phi - d_\beta Z, \hat{\psi} + \eta \delta I - \frac{\mu_0 d_\beta [1 + \tau]}{c_\beta} \nabla^2 [V_z + (d_\beta/c_\beta) \delta I], \quad (26)
\]
\[
0 = \left[ \phi, Z \right] + c_\beta [V_z + (d_\beta/c_\beta) \delta I, \psi] + D Y - \mu_0 d_\beta \nabla^2 (U - d_\beta Y), \quad (27)
\]
\[
0 = \left[ \phi, U \right] - \frac{d_\beta \tau}{2} \nabla^2 [\phi, Z] + [U, Z] + [Y, \phi] + [\delta I, \psi] + \mu_0 \nabla^2 (U - d_\beta Y), \quad (28)
\]
\[
0 = \left[ \phi, V_z \right] + c_\beta [Z, \psi] + \mu_0 \nabla^2 V_z + \mu_0 \nabla^2 [V_z + (d_\beta/c_\beta) \delta I], \quad (29)
\]
where \( \delta I = 1 + \nabla^2 \psi \) (the 1 represents an externally applied, inductive electric field maintaining the equilibrium plasma current), \( Y = \nabla^2 Z \), and \( U = \nabla^2 \phi \).

D. Ordering scheme

For the purpose of our ordering scheme, we require both \( \nabla \) and \( \psi \) to be \( O(1) \) in the vicinity of the island. This implies that our scale length \( a \) is \( O(W) \), and our scale field-strength \( B_a \) is \( O(\Psi/W) \), where \( W \) and \( \Psi \) are the unnormalized island width and reconnected flux, respectively.

We adopt the following ordering of terms appearing in Eqs. (26)-(29): \( d_\beta = \delta [1], \phi = \phi^{[0]} \), \( \phi = \phi^{[1]}(s, \psi, \theta) + \phi^{[2]}(s, \psi, \theta) + \phi^{[3]}(s, \psi, \theta) + \phi^{[4]}(s, \psi, \theta), z = Z^{[0]}(s, \psi, \theta) + Z^{[1]}(s, \psi, \theta), V_z = V_z^{[3]}(s, \psi, \theta) \), \( \delta I = \delta I^{[2]}(s, \psi, \theta) \). Moreover, \( \nabla = \nabla^{[0]}, \tau = \tau^{[0]}, c_\beta = c_\beta^{[1]}, \mu_0 = \mu_0^{[1]} \), \( k = k^{[2]}, \eta = \eta^{[3]} \), \( D = D^{[3]} \), and \( dN/dt = dN^{[5]}/dt \). Here, the superscript \( [i] \) indicates a quantity which is order \( (d_\beta)^i \), where it is assumed that \( d_\beta \ll 1 \). This ordering, which [together with Eqs. (30)-(33)] is completely selfconsistent, implies weak (i.e., strongly sub-Alfvénic and submagnetosacoustic) diamagnetic flows, and very long (i.e., very much longer than the Alfvén time) transport evolution timescales. Note, in particular, that our ordering scheme implies \( V_{\psi, \psi} = V_{\psi} \Psi \), which permits the magnetohydrostatic wave to flatten the plasma pressure profile within the island separatrix. According to our scheme, both \( Z \) and \( \phi \) are flux-surface functions, to lowest order. In other words, the lowest order electron and ion streamfunctions, \( \phi_e = -d_\beta \Psi \) and \( \phi_i = -d_\beta \tau Z \), respectively, are flux-surface functions.

Equations (26)-(29) yield
\[
\frac{d\Psi^{[5]}}{dt} \cos \theta = \left[ \phi^{[5]}(s, \psi) - d_\beta^{[1]} Z^{[4]}(s, \psi) \right] + \eta^{[3]} \delta I^{[2]} - \frac{\mu_0^{[3]} d_\beta^{[1]} (1 + \tau)}{c_\beta} \nabla^2 \left[ V_z^{[3]} + (d_\beta^{[1]}/c_\beta) \delta I^{[2]} \right] + O(d_\beta^{[3]}), \quad (30)
\]
\[
0 = c_\beta [V_z^{[3]} + (d_\beta^{[1]}/c_\beta) \delta I^{[2]}, \psi] + D^{[3]} Y^{[0]} + \mu_0^{[3]} d_\beta^{[1]} \nabla^2 \left[ (U^{[1]} - d_\beta^{[1]} Y^{[0]}) + O(d_\beta^{[3]}), \quad (31)
\]
\[
0 = -M^{[1]} \left[ U^{[1]}, \psi \right] - \frac{d_\beta^{[1]} \tau}{2} \left[ U^{[0]} [U^{[1]}, \psi] + M^{[1]} [Y^{[0]}, \psi] \right] + \delta I^{[2]}, \quad (32)
\]
\[
0 = -M^{[1]} [V_z^{[3]}, \psi] + c_\beta [Z^{[1]}, \psi] + \mu_0^{[3]} \nabla^2 V_z^{[3]} + \mu_0^{[3]} \nabla^2 \left[ V_z^{[3]} + (d_\beta^{[1]}/c_\beta) \delta I^{[2]} \right] + O(d_\beta^{[3]}), \quad (33)
\]
where \( Y^{[0]} = \nabla^2 Z^{[0]}, \quad U^{[1]} = \nabla^2 \phi^{[1]}, \quad M^{[1]}(s, \psi) = d\phi^{[1]} / d\psi, \quad \text{and} \quad L^{[0]}(s, \psi) = \nabla^2 \phi^{[0]} / d\psi. \) Here, we have neglected the superscripts on most zeroth-order quantities for the sake of clarity. As indicated, some of the \( \nabla^2 \) terms are \( O(d_\beta^{[3]} \), since they operate on quantities which are only important in thin boundary layers of width \( O(d_\beta) \) located on the magnetic separatrix. In the following, we shall neglect all superscripts for ease of notation.

E. Boundary conditions

It is easily demonstrated that the \( y \) components of the (lowest order) electron and ion fluid velocities (in the island rest frame) take the form \( V_{\psi \psi} = \chi (M - d_\beta L), \quad \text{and} \quad V_{\psi \psi} = \chi (M + d_\beta L), \) respectively. Incidentally, since \( V_{\psi \psi} \) and \( V_{\psi \psi} \) are even functions of \( x \), it follows that \( M(s, \psi) \) and \( L(s, \psi) \) are odd functions. We immediately conclude that \( M(s, \psi) \) and \( L(s, \psi) \) are both zero inside the island separatrix (since it is impossible to have a nonzero, odd flux-surface function in this region). Now, we are searching for island solutions for which \( xM \rightarrow M_0 \) and \( xL \rightarrow L_0 \) as \( |x| \rightarrow \infty \). In other words, we desire solutions which match to an unperturbed plasma far from the island. If \( \phi^{[0]} \) and \( V_y^{[0]} \) are the unperturbed (i.e., in the absence of an island) \( y \)-directed electron and ion fluid velocities in the laboratory frame, then \( V_y^{[0]} = -V = M_0 - d_\beta L_0, \quad \text{and} \quad V_y^{[0]} = V - M_0 + d_\beta L_0 \), where \( V \) is the island phase velocity in the laboratory frame. It follows that \( L_0^{[0]} = (V_y^{[0]} - V^{[0]}) / (1 + \tau) \) and \( M_0^{[0]} = M^{[0]} - V^{[0]} \), where \( V_{\psi \psi}^{[0]} = V_{\psi \psi}^{[0]} = \chi (M^{[0]} - d_\beta L_0) \). Hence, determining the island phase
velocity is equivalent to determining the value of \( M_0 \).

### F. Determination of flow profiles

Flux-surface averaging Eqs. (31) and (32), we obtain

\[
\langle \nabla^2 U \rangle + \frac{d_{\beta}^\phi \langle \mu_r^2, \tau \rangle \langle \mu_e \rangle}{(\mu_r + \mu_e)} \langle \nabla^2 Y \rangle = 0 \tag{34}
\]

and

\[
\delta^2 w^2 \langle \nabla^2 Y \rangle - \langle Y \rangle = 0, \tag{35}
\]

where

\[
\delta = \frac{d_{\beta}^\phi}{w} \sqrt{\frac{\mu_r \mu_e (1 + \tau)}{D(\mu_r + \mu_e)}}. \tag{36}
\]

Assuming that the island is “thin” (i.e., \( w \ll l \)), we can write \( \nabla^2 \approx \tilde{\delta}^2 / \partial \psi^2 \). Hence, Eqs. (34) and (35) yield

\[
M(s, \psi) = -\frac{d_{\beta}^\phi \langle \mu_r^2, \tau \rangle \langle \mu_e \rangle}{(\mu_r + \mu_e)} L(s, \psi) + F(s, \psi), \tag{37}
\]

where

\[
\frac{d}{d\psi} \left[ \frac{d}{d\psi} \left( \delta^2 w^2 \langle x^4 \rangle \frac{dL}{d\psi} \right) - \langle x^2 \rangle L \right] = 0 \tag{38}
\]

and

\[
\frac{d^2}{d\psi^2} \left( \langle x^4 \rangle \frac{dF}{d\psi} \right) = 0. \tag{39}
\]

We can integrate Eq. (38) once to give

\[
\delta^2 w^2 \frac{d}{d\psi} \left( \langle x^4 \rangle \frac{dL}{d\psi} \right) - \langle x^2 \rangle L = -sL_0. \tag{40}
\]

We can solve Eq. (39), subject to the constraints that \( F \) be continuous, and \( F = 0 \) inside the separatrix, to give

\[
F(s, \psi) = sF_0 \int_{-\psi}^\psi \frac{d\psi}{\langle x^4 \rangle} \left/ \int_{-\psi}^\psi \frac{d\psi}{\langle x^4 \rangle} \right. \tag{41}
\]

outside the separatrix. (We reject the solution in which \( F \) blows up as \( |x|/w \to \infty \). Note that \( xF \to |x|/F_0 \) as \( |x|/w \to \infty \), which is only consistent with our requirement that \( xM \to M_0 \) as \( |x|/w \to 0 \) if \( F_0 \to 0 \). However, for the moment, we shall continue to regard \( F_0 \) as a nonzero quantity.

In order to solve Eq. (40), we write \( \tilde{\psi} = -\psi/\Psi, \langle \cdot \cdot \cdot \rangle = \langle \cdot \cdot \cdot \rangle_w, X = x/w, \) and \( \tilde{L} = L/\langle L \rangle_w \). It follows that

\[
\delta^2 \frac{d}{d\tilde{\psi}} \left( \langle X^4 \rangle \frac{d\tilde{L}}{d\tilde{\psi}} \right) - \langle X^2 \rangle \tilde{L} = -s. \tag{42}
\]

According to our ordering scheme, \( \delta - d_{\beta}^\phi \ll 1 \). Thus, \( \hat{L}(s, \tilde{\psi}) \) takes the value \( s/\langle X^2 \rangle \) in the region outside the magnetic separatrix, apart from a thin boundary layer on the separatrix itself of width \( \delta \). In this layer, the function \( \hat{L}(s, \tilde{\psi}) \) makes a smooth transition from its exterior value (which is \( s\pi/4 \) immediately outside the separatrix) to its interior value 0. We can write

\[
\hat{L}(s, \tilde{\psi}) = s \left( \frac{1}{\langle X^2 \rangle} + l(y) \right), \tag{43}
\]

where \( y = (\tilde{\psi} - 1)/\delta \). It follows that

\[
\frac{d^2 l}{dy^2} - \frac{3}{8} l = 0, \tag{44}
\]

since \( \langle X^2 \rangle \tilde{\psi}_0 = 4/\pi \) and \( \langle X^4 \rangle \tilde{\psi}_0 = 32/3\pi \). Hence, the continuous solution to Eq. (40) which satisfies the appropriate boundary conditions is

\[
\hat{L}(s, \tilde{\psi}) = s \left[ \frac{1}{\langle X^2 \rangle} - \pi \frac{\sin \left( \beta \frac{\tilde{\psi} - 1}{\delta} \right)}{3 \delta} \right] \tag{45}
\]

in the region outside the separatrix (i.e., \( \tilde{\psi} \approx 1 \)). Of course, \( \hat{L}(s, \tilde{\psi}) = 0 \) in the region inside the separatrix (i.e., \( \tilde{\psi} < 1 \)).

### G. Determination of island phase velocity

Let \( \partial I = \partial I_r + \partial I_s \), where \( \partial I_r \) has the symmetry of \( \cos \theta \), whereas \( \partial I_s \) has the symmetry of \( \sin \theta \). Now, it is easily demonstrated that

\[
\langle \partial I_s \sin \theta \rangle = \frac{1}{k\Psi} \langle \tilde{x}[\partial I_s \tilde{\psi}] \rangle. \tag{46}
\]

Hence, it follows from Eqs. (32) and (37) that

\[
\langle \partial I_s \sin \theta \rangle = -\frac{\langle \mu_r + \mu_e \rangle}{k\Psi} \frac{d}{d\psi} \langle \tilde{x} \rangle \frac{d^2 F}{d\tilde{\psi}^2} - 2\langle \tilde{x} \rangle \frac{dF}{d\psi} - \langle \tilde{x} \rangle F. \tag{47}
\]

Now, for an isolated magnetic island which is not interacting electromagnetically with any external structure, such as a resistive wall, the net electromagnetic force acting on the island must be zero. This constraint translates to the well-known requirement that

\[
\int_{-\Psi}^\infty \langle \partial I_s \sin \theta \rangle d\psi = 0. \tag{48}
\]

Using Eq. (47), this requirement reduces to the condition

\[
\lim_{x \to -\infty} \langle \tilde{x} \rangle \frac{d^2 F}{d\tilde{\psi}^2} - 2\langle \tilde{x} \rangle \frac{dF}{d\psi} - \langle \tilde{x} \rangle F \approx \lim_{x \to -\infty} \left[ s \frac{x^2}{dx} \left( \frac{1}{s} \frac{d(xF)}{dx} \right) \right] = -F_0, \tag{49}
\]

since \( xF \to |x|/F_0 \) as \( |x|/w \to \infty \). Hence, we conclude that \( F_0 \to 0 \) [i.e., \( F(\psi) = 0 \), everywhere] for an isolated magnetic island. Indeed, it follows from the analysis of Sec. III F that Eq. (48) is automatically satisfied when the island-induced modifications to the ion and electron velocity profiles are localized in the vicinity of the island, as must be the case for an isolated island.

It follows from Eq. (37) that

\[
M(s, \psi) = -\frac{d_{\beta}^\phi \langle \mu_r^2, \tau \rangle \langle \mu_e \rangle}{(\mu_r + \mu_e)} L(s, \psi). \tag{50}
\]

Hence, \( M_0 = -\langle \mu_r^2, \tau \rangle \langle \mu_e \rangle \frac{1}{(\mu_r + \mu_e)} L_{\Psi} \). Recalling that \( M_0 = \frac{1}{\mu_r^2} - V, \frac{dM_{\Psi}}{d\Psi} = \frac{(V_{\Psi} - V_{\Psi})/1 + \tau, \Psi_{\Psi} = V_{\Psi} + d_{\beta}^\phi R_{\Psi}}{d\Psi} \), and
H. Determination of ion polarization correction

It follows from Eq. (32) that

\[ \delta I_c = \frac{(V - V_{EBy}^0)(V - V_{cy}^0)}{2} \left( x^2 - \frac{\langle x^2 \rangle}{\langle 1 \rangle} \right) \frac{d}{d\psi} \left[ \frac{H(\psi - 1)}{\langle x^2 \rangle^2} \right] \]

\[ + I(s, \psi), \]  

(54)

where \( I(s, \psi) \) is as yet undetermined. The function \( H(\psi) \) is zero for \( \psi < 0 \), and unity for \( \psi > 0 \). Here, we have made use of the fact that outside the separatrix \( L(s, \psi) = s L_0 / \langle x^2 \rangle \), and \( M(s, \psi) = s M_0 / \langle x^2 \rangle \), apart from a thin boundary layer on the separatrix itself. It turns out that we do not need to resolve this boundary layer in order to calculate the total ion polarization current. However, we do have to include the net current flowing in this layer in our calculation of the total current.\(^{12,14} \) Flux-surface averaging Eqs. (30) and (33), we obtain

\[ \varepsilon^2 \omega^2 \langle \nabla^2 \delta I_c \rangle - \langle \delta I_c \rangle = - \eta \frac{d\Psi}{dt} (\cos \theta), \]  

(55)

where

\[ \varepsilon = \frac{d_i}{w} \sqrt{\frac{\mu_i \mu_e}{\eta (\mu_i + \mu_e)}}, \]  

(56)

Note that \( \varepsilon \sim d_\beta \ll 1 \), according to our ordering scheme.

Equation (55) implies that

\[ \langle \delta I_c \rangle = \eta \frac{d\Psi}{dt} (\cos \theta), \]  

(57)

apart from in a thin boundary layer on the separatrix of width \( \omega \). It is easily demonstrated that the deviation of \( \langle \delta I_c \rangle \) in the boundary layer from the value given in Eq. (57) makes a negligible contribution to the total ion polarization current. Hence, we shall treat Eq. (57) as if it applied everywhere.

Equations (54) and (57) give

\[ \delta I_c = \frac{(V - V_{EBy}^0)(V - V_{cy}^0)}{2} \left( x^2 - \frac{\langle x^2 \rangle}{\langle 1 \rangle} \right) \frac{d}{d\psi} \left[ \frac{H(\psi - 1)}{\langle x^2 \rangle^2} \right] \]

\[ + \eta \frac{d\Psi}{dt} (\cos \theta), \]  

(58)

The island width evolution equation is obtained by asymptotic matching to the region far from the island.\(^{3} \) In fact,

\[ \Delta' \Psi = -4 \int_{\psi}^{\infty} \left\langle \delta I_c \cos \theta \right\rangle d\psi, \]  

(59)

where \( \Delta' \) is the conventional tearing stability index.\(^{23} \) It follows from Eqs. (58) and (59) that

\[ \Delta' = \frac{(V - V_{EBy}^0)(V - V_{cy}^0)}{w^2} \int_{-1}^{\infty} \left( \frac{\langle X^2 \rangle}{\langle 1 \rangle} \right) \frac{d}{d\psi} \left[ \frac{H(\psi - 1)}{\langle X^2 \rangle^2} \right] d\psi \]

\[ - \frac{8\omega w}{\eta} \int_{-1}^{\infty} \left\langle \cos \theta \right\rangle \frac{d}{d\psi} \left( \frac{\langle X^2 \rangle}{\langle 1 \rangle} \right) d\psi. \]  

(60)

Performing the flux surface integrals, whose values are well
known, we obtain the following island width evolution equation:
\[
\frac{0.823 \, dW}{\eta \, dt} = \Delta' + 1.38 \frac{(V - V_{ix}^{(0)})(V - V_{iy}^{(0)})}{(W/4)^{3/2}}.
\] (61)

Here, \(W = 4W\) is the full island width. The ion polarization current term (the second term on the rhs) is only stabilizing when the island phase velocity \(V\) lies between the unperturbed local \(E \times V\) velocity \(V_{ix}^{(0)}\) and the unperturbed local velocity of the ion fluid \(V_{iy}^{(0)}\).

**IV. SUMMARY**

A set of reduced, 2D, two-fluid, drift-MHD equations has been developed. This set of equations takes into account both electron and ion diamagnetism (including the contributions of the ion gyroviscous tensor), as well as the Hall effect and parallel electron compressibility, but neglects electron inertia and the compressible Alfvén wave. For the sake of simplicity, the plasma density is assumed to be uniform, and the ion and electron temperatures constant multiples of one another.

Using our equations, we have found a complete and self-consistent solution for an isolated magnetic island propagating through a slab plasma with uniform but different ion and electron fluid velocities. Our solution is valid provided that the ordering scheme described in Sec. III D holds good, which implies that
\[
d_B \ll c_B
\] (62)
and
\[
\mu_{i,e} \kappa_{i,e} \eta \ll V_{iy}^{(0)} \ll c_B \eta
\] (63)
where \(V_{iy}^{(0)} = V_{iy}^{(0)} - V_{ey}^{(0)}\). Here, all lengths and velocities are approximately normalized to the island width \(W\) and the shear-Alfvén speed calculated with \(B_{0}^{(1)}(W)\), respectively. Note that the condition \(V_{iy}^{(0)} \ll c_B\) permits the magnetoacoustic wave to flatten the plasma pressure profile within the island separatrix. Our solution yields ion and electron fluid velocity profiles which are uniquely determined in the vicinity of the island (see Fig. 1). These profiles are everywhere continuous and asymptote to the unperturbed fluid velocities far from the island. Incidentally, the inclusion of electron viscosity in both the Ohm’s law and the plasma equation of motion is key to the determination of continuous velocity profiles.\(^{15}\)

The island phase velocity is uniquely specified by the condition that the island-induced modifications to the ion and electron velocity profiles remain localized in the vicinity of the island, as must be the case for an isolated island (i.e., an island which is not interacting electromagnetically with any external agency such as a resistive wall or an error field). Such velocity profiles automatically ensure that there is zero net electromagnetic force acting on the island.\(^{16}\) It turns out that the phase velocity is the viscosity weighted mean of the unperturbed (i.e., in the absence of an island) local ion and electron fluid velocities. Note that, in this paper, we have adopted phenomenological diffusive ion and electron viscosity operators, which are supposed to represent anomalous perpendicular momentum transport due to small-scale plasma turbulence.

The ion polarization current correction to the Rutherford island width evolution equation is found to be stabilizing when the island phase velocity lies between the unperturbed local ion fluid velocity and the unperturbed local \(E \times B\) velocity [see Eq. (61)].\(^{16}\) It follows, from our result for the island phase velocity that the polarization term is stabilizing when the anomalous perpendicular ion viscosity significantly exceeds the anomalous perpendicular electron viscosity (see Fig. 1, left panel). Conversely, the polarization term is destabilizing when the electron viscosity significantly exceeds the ion viscosity [see Fig. 1, right panel].\(^{25}\) Note, however, that in order for the electron viscosity to exceed the ion viscosity, the electron momentum confinement time would need to be at least a mass ratio smaller than the ion momentum confinement time, which does not seem very probable. Hence, we conclude that under normal circumstances the polarization term is stabilizing.

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