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Two-fluid magnetic island dynamics in slab geometry. I. Isolated islands

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A set of reduced, two-dimensional, two-fluid, drift-MHD (magnetohydrodynamical) equations is derived. Using these equations, a complete and fully self-consistent solution is obtained for an isolated magnetic island propagating through a slab plasma with uniform but different ion and electron fluid velocities. The ion and electron fluid flow profiles around the island are uniquely determined, and are everywhere continuous. Moreover, the island phase velocity is uniquely specified by the condition that the island-induced modifications to the ion and electron velocity profiles remain localized in the vicinity of the island. Finally, the ion polarization current correction to the Rutherford island width evolution equation is evaluated and found to be stabilizing provided that the anomalous perpendicular ion viscosity significantly exceeds the anomalous perpendicular electron viscosity. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1833375]

I. INTRODUCTION

Tearing modes are magnetohydrodynamical (MHD) instabilities which often limit fusion plasma performance in magnetic confinement devices relying on nested toroidal magnetic flux surfaces.[1] As the name suggests, "tearing" modes tear and reconnect magnetic field lines, in the process converting nested toroidal flux surfaces into helical magnetic islands. Such islands degrade plasma confinement because heat and particles are able to travel radially from one side of an island to another by flowing along magnetic field lines, which is a relatively fast process, instead of having to diffuse across magnetic flux surfaces, which is a relatively slow process.[2]

Magnetic island physics is very well understood within the context of *single-fluid* MHD theory. According to this theory, the island width is governed by the well-known nonlinear evolution equation due to Rutherford.[3] Moreover, the island is required to propagate at the local flow velocity of the MHD fluid, since fluid flow across the island separatrix is effectively prohibited.

Magnetic island physics is less completely understood within the context of *two-fluid*, drift-MHD theory,[4–18] which is far more relevant to present-day magnetic confinement devices than single-fluid theory. In two-fluid theory, the island is generally embedded within ion and electron fluids which flow at *different* velocities. The island itself usually propagates at some intermediate velocity. For a sufficiently wide island, both fluids are required to flow at the island propagation velocity in the region lying within the magnetic separatrix (since neither fluid can easily cross the separatrix). However, the region immediately outside the separatrix is characterized by strongly sheared ion and electron fluid flow profiles, as the velocities of both fluids adjust to their unperturbed values far away from the island. The polarization current generated by the strongly sheared ion flow around the island separatrix gives rise to an additional term in the Rutherford island width evolution equation, which is stabilizing or destabilizing, depending on the island propagation velocity relative to the unperturbed (i.e., in the absence of an island) local flow velocities of the ion and MHD fluids. The key problems in two-fluid island theory are the unambiguous determination of the island phase velocity, and the calculation of the ion and electron fluid flow profiles around the island separatrix. As yet, no consensus has emerged within the magnetic fusion community regarding the solution of these problems.

In this paper, we first develop a set of reduced, two-dimensional (2D), two-fluid, drift-MHD equations. These equations contain both electron and ion diamagnetic effects (including the contribution of the ion gyroviscous tensor), as well as the Hall effect and parallel electron compressibility. However, they do not contain electron inertia or the compressible Alfvén wave (which play negligible roles in conventional magnetic island physics). Our set of equations consists of four coupled partial differential equations, and is both analytically tractable and easy to solve numerically. We employ our equations to study the evolution of an isolated magnetic island in *slab geometry*. Using a particular ordering scheme, we are able to calculate the island phase velocity and uniquely determine the ion and electron fluid flow profiles outside the island separatrix.

II. DERIVATION OF REDUCED EQUATIONS

A. Introduction

In this section, we shall generalize the analysis of Refs. 19 and 20 to obtain a set of reduced, 2D, two-fluid, drift-MHD equations which take ion diamagnetic flows into account.

B. Basic equations

Standard right-handed Cartesian coordinates (x,y,z) are adopted. Consider a quasineutral plasma with singly charged ions. The ion/electron number density n_0 is assumed to be

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uniform and *constant*. Suppose that $T_i = \tau T_e$, where $T_{i,e}$ is the ion/electron temperature, and τ is uniform and constant.

Broadly following Hazeltine and Meiss,[21] we adopt the following set of two-fluid, drift-MHD equations:

$$\boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{B} + \frac{1}{en_0}\left(\nabla P - \frac{\tau}{1+\tau}(\boldsymbol{b}\cdot\nabla P)\boldsymbol{b} - \boldsymbol{J}\times\boldsymbol{B} - \mu_e\nabla^2\boldsymbol{V}_e\right) = \eta\boldsymbol{J}, \quad (1)$$

$$m_i n_0\left[\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla + \frac{\tau}{1+\tau}\boldsymbol{V}_*\cdot\nabla\right)\boldsymbol{V} - \frac{\tau}{1+\tau}\boldsymbol{V}_*\cdot\nabla([\boldsymbol{b}\cdot\boldsymbol{V}]\boldsymbol{b})\right] = \boldsymbol{J}\times\boldsymbol{B} - \nabla P + \mu_i\nabla^2\boldsymbol{V}_i + \mu_e\nabla^2\boldsymbol{V}_e, \quad (2)$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla\right)P = -\Gamma P\,\nabla\cdot\boldsymbol{V} + \kappa\nabla^2 P. \quad (3)$$

Here, $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{J}$ the electric current density, $\boldsymbol{V}$ the plasma *guiding-center* velocity, P the total plasma pressure, e the magnitude of the electron charge, m_i the ion mass, η the (uniform) plasma resistivity, μ_e the (uniform) electron viscosity, μ_i the (uniform) ion viscosity, κ the (uniform) plasma thermal conductivity, and $\Gamma = 5/3$ the plasma ratio of specific heats. Furthermore, $\boldsymbol{b} = \boldsymbol{B}/B$, $\boldsymbol{V}_* = \boldsymbol{b}\times\nabla P/en_0 B$, $\boldsymbol{V}_i = \boldsymbol{V} + [\tau/(1+\tau)]\boldsymbol{V}_*$, and $\boldsymbol{V}_e = \boldsymbol{V} + [\tau/(1+\tau)]\boldsymbol{V}_* - \boldsymbol{J}/n_0 e$. The above equations take into account the anisotropic ion gyroviscous tensor, but neglect electron inertia. Our system of equations is completed by Maxwell's equations: $\nabla\cdot\boldsymbol{B} = 0$, $\nabla\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$, and $\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J}$. Note that the transport coefficients, μ_i, μ_e, and κ, appearing in the above equations, are *phenomenological* in nature, and are supposed to represent the anomalous *diffusive* transport of energy and momentum across magnetic flux surfaces due to small-scale plasma turbulence.

C. Normalized equations

Let $\hat{\nabla} = a\nabla$, $\hat{t} = t/(a/V_a)$, $\hat{\boldsymbol{B}} = \boldsymbol{B}/B_a$, $\hat{\boldsymbol{E}} = \boldsymbol{E}/(B_a V_a)$, $\hat{\boldsymbol{J}} = \boldsymbol{J}/(B_a/\mu_0 a)$, $\hat{\boldsymbol{V}} = \boldsymbol{V}/V_a$, $\hat{\boldsymbol{V}}_{*,i,e} = \boldsymbol{V}_{*,i,e}/V_a$, $\hat{P} = P/(B_a^2/\mu_0)$, $\hat{\eta} = \eta/(\mu_0 V_a a)$, $\hat{\mu}_{i,e} = \mu_{i,e}/(n_0 m_i V_a a)$, and $\hat{\kappa} = \kappa/(V_a a)$, where $V_a = B_a/\sqrt{\mu_0 n_0 m_i}$. Here, a is a convenient scale length and B_a a convenient scale magnetic field strength.

Neglecting hats, our normalized two-fluid equations take the form

$$\boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{B} + d_i\left(\nabla P - \frac{\tau}{1+\tau}(\boldsymbol{b}\cdot\nabla P)\boldsymbol{b} - \boldsymbol{J}\times\boldsymbol{B} - \mu_e\nabla^2\boldsymbol{V}_e\right) = \eta\boldsymbol{J}, \quad (4)$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla + \frac{\tau}{1+\tau}\boldsymbol{V}_*\cdot\nabla\right)\boldsymbol{V} - \frac{\tau}{1+\tau}\boldsymbol{V}_*\cdot\nabla([\boldsymbol{b}\cdot\boldsymbol{V}]\boldsymbol{b}) = \boldsymbol{J}\times\boldsymbol{B} - \nabla P + \mu_i\nabla^2\boldsymbol{V}_i + \mu_e\nabla^2\boldsymbol{V}_e, \quad (5)$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla\right)P = -\Gamma P\,\nabla\cdot\boldsymbol{V} + \kappa\nabla^2 P. \quad (6)$$

Here, $\boldsymbol{V}_* = d_i\boldsymbol{b}\times\nabla P/B$, $\boldsymbol{V}_i = \boldsymbol{V} + [\tau/(1+\tau)]\boldsymbol{V}_*$, $\boldsymbol{V}_e = \boldsymbol{V} + [\tau/(1+\tau)]\boldsymbol{V}_* - d_i\boldsymbol{J}$, and $d_i = (m_i/n_0 e^2\mu_0)^{1/2}/a$ is the normalized collisionless ion skin depth. Maxwell's equations are written as $\nabla\cdot\boldsymbol{B} = 0$, $\nabla\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$, and $\nabla\times\boldsymbol{B} = \boldsymbol{J}$.

D. 2D assumption

Let us make the simplifying assumption that there is no variation of quantities in the z direction, i.e., $\partial/\partial z \equiv 0$. It immediately follows that $\boldsymbol{B} = \nabla\psi\times\hat{\boldsymbol{z}} + B_z\hat{\boldsymbol{z}}$ and $E_z = -\partial\psi/\partial t$.

E. Reduction process

Let us adopt the following ordering, which is designed to decouple the compressional Alfvén wave from all the other waves in the system:

$$P = P_0 + B_0 p_1 + p_2, \quad (7)$$

$$B_z = B_0 + b_z. \quad (8)$$

Here, P_0 and B_0 are uniform *and* constant, and

$$P_0 \gg B_0 \gg 1. \quad (9)$$

Furthermore, p_1, p_2, b_z, ψ, $\boldsymbol{V}$, ∇, and $\partial/\partial t$ are all assumed to be $O(1)$, and $\nabla\cdot\boldsymbol{V}$ is assumed to be much less than $O(1)$.

Now, to lowest order, the z component of Ohm's law, Eq. (4), gives

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla\right)\psi = -d_i[b_z + \tau p_1/(1+\tau), \psi] + \eta\nabla^2\psi - d_i\mu_e\nabla^2(V_z + d_i\nabla^2\psi). \quad (10)$$

Here, $[A,B] \equiv \nabla A\times\nabla B\cdot\hat{\boldsymbol{z}}$. Likewise, the z component of the curl of Eq. (4) reduces to

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla\right)b_z = [V_z + d_i\nabla^2\psi, \psi] - B_0\nabla\cdot\boldsymbol{V} + \eta\nabla^2 b_z + d_i\mu_e\nabla^2\left[U - d_i\nabla^2\left(b_z + \frac{\tau}{1+\tau}p_1\right)\right]. \quad (11)$$

Here, $U = -\nabla\times\boldsymbol{V}\cdot\hat{\boldsymbol{z}}$.

To lowest order, the equation of motion, Eq. (5), implies that

$$p_1 \simeq -b_z. \quad (12)$$

Furthermore, the z component of this equation yields

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla\right)V_z = [b_z, \psi] + \mu_i\nabla^2 V_z + \mu_e\nabla^2(V_z + d_i\nabla^2\psi), \quad (13)$$

whereas the z component of its curl reduces to

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}\right)U = -\frac{d_i}{2}\frac{\tau}{1+\tau}\{\nabla^2[\phi,b_z] + [U,b_z] + [\nabla^2 b_z,\phi]\} + [\nabla^2\psi,\psi] + \mu_i\nabla^2\left(U + \frac{d_i\tau}{1+\tau}\nabla^2 b_z\right) + \mu_e\nabla^2\left(U - \frac{d_i}{1+\tau}\nabla^2 b_z\right). \tag{14}$$

Finally, to lowest order, the energy equation, Eq. (6), gives

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}\right)p_1 = -\frac{\Gamma P_0}{B_0}\boldsymbol{\nabla}\cdot\boldsymbol{V} + \kappa\nabla^2 p_1. \tag{15}$$

Eliminating $\boldsymbol{\nabla}\cdot\boldsymbol{V}$ between Eqs. (11) and (15), making use of Eq. (12), we obtain

$$c_\beta^{-2}\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}\right)b_z = [V_z + d_i\nabla^2\psi,\psi] + \left(\eta + \frac{\kappa}{\beta}\right)\nabla^2 b_z + d_i\mu_e\nabla^2\left(U - \frac{d_i}{1+\tau}\nabla^2 b_z\right). \tag{16}$$

Here, $\beta = \Gamma P_0/B_0^2$ is (Γ times) the plasma β calculated with the "guide field," B_0, and $c_\beta = \sqrt{\beta/(1+\beta)}$. Note that our ordering scheme does not constrain β to be either much less than or much greater than unity. In tokamak terminology, the inequality (9) implies a high *poloidal* β value (i.e., $\beta_p \sim P_0 \gg 1$), but does not necessarily imply a high *toroidal* β value (i.e., $\beta_t \sim P_0/B_0^2$ is not necessarily $\gg 1$).

Equation (15) implies that $\boldsymbol{\nabla}\cdot\boldsymbol{V} \sim O(B_0^{-1})$: i.e., that the flow is *almost* incompressible. Hence, to lowest order, we can write

$$\boldsymbol{V} = \boldsymbol{\nabla}\phi \times \hat{\boldsymbol{z}} + V_z\,\hat{\boldsymbol{z}}. \tag{17}$$

F. Final equations

Let $d_\beta = c_\beta d_i/\sqrt{1+\tau}$, $Z = b_z/c_\beta\sqrt{1+\tau}$, and $\overline{V}_z = V_z/\sqrt{1+\tau}$. Neglecting the bar over $\overline{V}_z$, our final set of reduced, 2D, two-fluid, drift-MHD equations takes the form

$$\frac{\partial\psi}{\partial t} = [\phi - d_\beta Z,\psi] + \eta J - \frac{\mu_e d_\beta(1+\tau)}{c_\beta}\nabla^2[V_z + (d_\beta/c_\beta)J], \tag{18}$$

$$\frac{\partial Z}{\partial t} = [\phi,Z] + c_\beta[V_z + (d_\beta/c_\beta)J,\psi] + DY + \mu_e d_\beta\nabla^2(U - d_\beta Y), \tag{19}$$

$$\frac{\partial U}{\partial t} = [\phi,U] - \frac{d_\beta\tau}{2}\{\nabla^2[\phi,Z] + [U,Z] + [Y,\phi]\} + [J,\psi] + \mu_i\nabla^2(U + d_\beta\tau Y) + \mu_e\nabla^2(U - d_\beta Y), \tag{20}$$

$$\frac{\partial V_z}{\partial t} = [\phi,V_z] + c_\beta[Z,\psi] + \mu_i\nabla^2 V_z + \mu_e\nabla^2[V_z + (d_\beta/c_\beta)J]. \tag{21}$$

Here, $D = c_\beta^2\eta + (1-c_\beta^2)\kappa$, $U = \nabla^2\phi$, $J = \nabla^2\psi$, and $Y = \nabla^2 Z$. The four fields which are evolved in the above equations are the magnetic flux function ψ, the (normalized) perturbed z-directed magnetic field $Z(=b_z/c_\beta\sqrt{1+\tau})$, the z-directed guiding-center vorticity U, and the (normalized) z-directed guiding-center (and ion) fluid velocity $V_z(=\boldsymbol{V}\cdot\hat{\boldsymbol{z}}/\sqrt{1+\tau})$. The (normalized) z-directed electron fluid velocity is $V_z + (d_\beta/c_\beta)J$. The quantity ϕ is the guiding-center stream function. The ion stream function takes the form $\phi_i = \phi + d_\beta\tau Z$, whereas the electron stream function is written $\phi_e = \phi - d_\beta Z$. The above equations are "reduced" in the sense that they do not contain the compressible Alfvén wave. However, they do contain the shear-Alfvén wave, the magnetoacoustic wave, the whistler wave, and the kinetic-Alfvén wave. Our equations are similar to the "four-field" equations of Hazeltine, Kotschenreuther, and Morrison,[22] except that they are not limited to small values of β.

III. ISLAND PHYSICS

A. Introduction

The aim of this section is to derive expressions determining the phase velocity and width of an *isolated* magnetic island (representing the final, nonlinear stage of a tearing instability) from the previously derived set of reduced, 2D, two-fluid, drift-MHD equations.

Consider a *slab* plasma which is periodic in the y direction with periodicity length l. Let the system be *symmetric* about $x=0$: i.e., $\psi(-x,y,t) = \psi(x,y,t)$, $Z(-x,y,t) = -Z(x,y,t)$, $\phi(-x,y,t) = -\phi(x,y,t)$, and $V_z(-x,y,t) = V_z(x,y,t)$. Consider a quasistatic, constant-ψ magnetic island, centered on $x=0$. It is convenient to transform to the *island rest frame*, in which $\partial/\partial t \simeq 0$. Suppose that the island is embedded in a plasma with uniform (but different) y-directed ion and electron fluid velocities. We are searching for an island solution in which the ion/electron fluid velocities asymptote to these uniform velocities far from the magnetic separatrix.

B. Island geometry

In the immediate vicinity of the island, we can write

$$\psi(x,\theta,t) = -\frac{x^2}{2} + \Psi(t)\cos\theta, \tag{22}$$

where $\theta = ky$, $k = 2\pi/l$, and $\Psi(t) > 0$ is the reconnected magnetic flux (which is assumed to have a very weak time dependence). As is well known, the above expression for ψ describes a "cat's eye" magnetic island of full width (in the x direction) $W = 4w$, where $w = \sqrt{\Psi}$. The region inside the magnetic separatrix corresponds to $\Psi \geq \psi > -\Psi$, the region outside the separatrix corresponds to $\psi < -\Psi$, and the separatrix itself corresponds to $\psi = -\Psi$. The island *O*- and *X*-points are located at $(x,\theta) = (0,0)$, and $(x,\theta) = (0,\pi)$, respectively.

It is helpful to define a flux-surface average operator:

$$\langle f(s,\psi,\theta)\rangle = \oint \frac{f(s,\psi,\theta)}{|x|}\frac{d\theta}{2\pi} \tag{23}$$

for $\psi < -\Psi$, and

$$\langle f(s,\psi,\theta)\rangle = \int_{-\theta_0}^{\theta_0} \frac{f(s,\psi,\theta)+f(-s,\psi,\theta)}{2|x|}\frac{d\theta}{2\pi} \tag{24}$$

for $\Psi \geqslant \psi \geqslant -\Psi$. Here, $s=\text{sgn}(x)$ and $x(s,\psi,\theta_0)=0$ (with $\pi > \theta_0 > 0$). The most important property of this operator is that

$$\langle [A,\psi]\rangle \equiv 0 \tag{25}$$

for any field $A(s,\psi,\theta)$.

C. Island equations

The equations governing the quasistatic island [which follow from Eqs. (18)–(21)] are

$$\frac{d\Psi}{dt}\cos\theta = [\phi - d_\beta Z,\psi] + \eta\delta J - \frac{\mu_e d_\beta(1+\tau)}{c_\beta}\nabla^2[V_z + (d_\beta/c_\beta)\delta J], \tag{26}$$

$$0 = [\phi,Z] + c_\beta[V_z + (d_\beta/c_\beta)\delta J,\psi] + DY + \mu_e d_\beta \nabla^2(U - d_\beta Y), \tag{27}$$

$$0 = [\phi,U] - \frac{d_\beta\tau}{2}\{\nabla^2[\phi,Z] + [U,Z] + [Y,\phi]\} + [\delta J,\psi] + \mu_i\nabla^2(U + d_\beta\tau Y) + \mu_e\nabla^2(U - d_\beta Y), \tag{28}$$

$$0 = [\phi,V_z] + c_\beta[Z,\psi] + \mu_i\nabla^2 V_z + \mu_e\nabla^2[V_z + (d_\beta/c_\beta)\delta J], \tag{29}$$

where $\delta J = 1 + \nabla^2\psi$ (the 1 represents an externally applied, inductive electric field maintaining the equilibrium plasma current), $Y=\nabla^2 Z$, and $U=\nabla^2\phi$.

D. Ordering scheme

For the purpose of our ordering scheme, we require both $\boldsymbol{\nabla}$ and ψ to be $O(1)$ in the vicinity of the island. This implies that our scale length a is $O(W)$, and our scale field-strength B_a is $O(\Psi/W)$, where W and Ψ are the unnormalized island width and reconnected flux, respectively.

We adopt the following ordering of terms appearing in Eqs. (26)–(29): $d_\beta = d_\beta^{[1]}$, $\psi=\psi^{[0]}$, $\phi=\phi^{[1]}(s,\psi)+\phi^{[5]}(s,\psi,\theta)$, $Z=Z^{[0]}(s,\psi)+Z^{[4]}(s,\psi,\theta)$, $V_z = V_z^{[3]}(s,\psi,\theta)$, and $\delta J = \delta J^{[2]}\times(s,\psi,\theta)$. Moreover, $\boldsymbol{\nabla}=\boldsymbol{\nabla}^{[0]}$, $\tau=\tau^{[0]}$, $c_\beta = c_\beta^{[0]}$, $\mu_{i,e}=\mu_{i,e}^{[3]}$, $\kappa=\kappa^{[3]}$, $\eta=\eta^{[3]}$, $D=D^{[3]}$, and $d\Psi/dt = d\Psi^{[5]}/dt$. Here, the superscript $[i]$ indicates a quantity which is order $(d_\beta)^i$, where it is assumed that $d_\beta \ll 1$. This ordering, which [together with Eqs. (30)–(33)] is completely selfconsistent, implies weak (i.e., strongly sub-Alfvénic and submagnetoacoustic) diamagnetic flows, and very long (i.e., very much longer than the Alfvén time) transport evolution timescales. Note, in particular, that our ordering scheme implies $V_{iy}, V_{ey} \ll c_\beta$, which permits the magnetoacoustic wave to flatten the plasma pressure profile within the island separatrix. According to our scheme, both Z and ϕ are *flux-surface functions*, to lowest order. In other words, the lowest order electron and ion streamfunctions, $\phi_e = \phi - d_\beta Z$ and $\phi_i = \phi + d_\beta\tau Z$, respectively, are flux-surface functions.

Equations (26)–(29) yield

$$\frac{d\Psi^{[5]}}{dt}\cos\theta = [\phi^{[5]} - d_\beta^{[1]}Z^{[4]},\psi] + \eta^{[3]}\delta J^{[2]} - \frac{\mu_e^{[3]}d_\beta^{[1]}(1+\tau)}{c_\beta}\nabla^{2^{[-2]}}[V_z^{[3]} + (d_\beta^{[1]}/c_\beta)\delta J^{[2]}] + O(d_\beta^6), \tag{30}$$

$$0 = c_\beta[V_z^{[3]} + (d_\beta^{[1]}/C_\beta)\delta J^{[2]},\psi] + D^{[3]}Y^{[0]} + \mu_e^{[3]}d_\beta^{[1]}\nabla^{2^{[-2]}}(U^{[1]} - d_\beta^{[1]}Y^{[0]}) + O(d_\beta^4), \tag{31}$$

$$0 = -M^{[1]}[U^{[1]},\psi] - \frac{d_\beta^{[1]}\tau}{2}\{L^{[0]}[U^{[1]},\psi] + M^{[1]}[Y^{[0]},\psi]\} + [\delta J^{[2]},\psi] + \mu_i^{[3]}\nabla^2(U^{[1]} + d_\beta^{[1]}\tau Y^{[0]}) + \mu_e^{[3]}\nabla^2(U^{[1]} - d_\beta^{[1]}Y^{[0]}) + O(d_\beta^5), \tag{32}$$

$$0 = -M^{[1]}[V_z^{[3]},\psi] + c_\beta[Z^{[4]},\psi] + \mu_i^{[3]}\nabla^{2^{[-2]}}V_z^{[3]} + \mu_e^{[3]}\nabla^{2^{[-2]}}[V_z^{[3]} + (d_\beta^{[1]}/c_\beta)\delta J^{[2]}] + O(d_\beta^5), \tag{33}$$

where $Y^{[0]}=\nabla^2 Z^{[0]}$, $U^{[1]}=\nabla^2\phi^{[1]}$, $M^{[1]}(s,\psi)=d\phi^{[1]}/d\psi$, and $L^{[0]}(s,\psi)=dZ^{[0]}/d\psi$. Here, we have neglected the superscripts on most zeroth-order quantities for the sake of clarity. As indicated, some of the ∇^2 terms are $O(d_\beta^{-2})$, since they operate on quantities which are only important in thin boundary layers of width $O(d_\beta)$ located on the magnetic separatrix. In the following, we shall neglect all superscripts for ease of notation.

E. Boundary conditions

It is easily demonstrated that the y components of the (lowest order) electron and ion fluid velocities (in the island rest frame) take the form $V_{ey}=x(M-d_\beta L)$ and $V_{iy}=x(M+d_\beta\tau L)$, respectively. Incidentally, since V_{ey} and V_{iy} are even functions of x, it follows that $M(s,\psi)$ and $L(s,\psi)$ are odd functions. We immediately conclude that $M(s,\psi)$ and $L(s,\psi)$ are both *zero* inside the island separatrix (since it is impossible to have a nonzero, odd flux-surface function in this region). Now, we are searching for island solutions for which $xM \to M_0$ and $xL \to L_0$ as $|x|/w \to \infty$. In other words, we desire solutions which match to an unperturbed plasma far from the island. If $V_{ey}^{(0)}$ and $V_{iy}^{(0)}$ are the unperturbed (i.e., in the absence of an island) y-directed electron and ion fluid velocities in the *laboratory frame*, then $V_{ey}^{(0)} - V = M_0 - d_\beta L_0$ and $V_{iy}^{(0)} - V = M_0 + d_\beta\tau L_0$, where V is the island phase velocity in the laboratory frame. It follows that $L_0 = (V_{iy}^{(0)} - V_{ey}^{(0)})/d_\beta(1+\tau)$ and $M_0 = V_{EB_y}^{(0)} - V$, where $V_{EB_y}^{(0)} = (V_{iy}^{(0)} + \tau V_{ey}^{(0)})/(1+\tau)$ is the unperturbed plasma $\boldsymbol{E}\times\boldsymbol{B}$ velocity in the laboratory frame. Hence, determining the island phase velocity is equivalent to determining the value of M_0.

F. Determination of flow profiles

Flux-surface averaging Eqs. (31) and (32), we obtain

$$\langle\nabla^2 U\rangle + \frac{d_\beta(\mu_i\tau-\mu_e)}{(\mu_i+\mu_e)}\langle\nabla^2 Y\rangle = 0 \tag{34}$$

and

$$\delta^2 w^2\langle\nabla^2 Y\rangle - \langle Y\rangle = 0, \tag{35}$$

where

$$\delta = \frac{d_\beta}{w}\sqrt{\frac{\mu_i\mu_e(1+\tau)}{D(\mu_i+\mu_e)}}. \tag{36}$$

Assuming that the island is "thin" (i.e., $w \ll l$), we can write $\nabla^2 \simeq \partial^2/\partial x^2$. Hence, Eqs. (34) and (35) yield

$$M(s,\psi) = -\frac{d_\beta(\mu_i\tau-\mu_e)}{(\mu_i+\mu_e)}L(s,\psi) + F(s,\psi), \tag{37}$$

where

$$\frac{d}{d\psi}\left[\frac{d}{d\psi}\left(\delta^2 w^2\langle x^4\rangle\frac{dL}{d\psi}\right) - \langle x^2\rangle L\right] = 0 \tag{38}$$

and

$$\frac{d^2}{d\psi^2}\left(\langle x^4\rangle\frac{dF}{d\psi}\right) = 0. \tag{39}$$

We can integrate Eq. (38) once to give

$$\delta^2 w^2\frac{d}{d\psi}\left(\langle x^4\rangle\frac{dL}{d\psi}\right) - \langle x^2\rangle L = -sL_0. \tag{40}$$

We can solve Eq. (39), subject to the constraints that F be continuous, and $F=0$ inside the separatrix, to give

$$F(s,\psi) = sF_0\int_{-\psi}^{\psi}\frac{d\psi}{\langle x^4\rangle}\Bigg/\int_{-\psi}^{-\infty}\frac{d\psi}{\langle x^4\rangle} \tag{41}$$

outside the separatrix. (We reject the solution in which F blows up as $|x|/w\to\infty$.) Note that $xF\to|x|F_0$ as $|x|/w\to\infty$, which is only consistent with our requirement that $xM \to M_0$ as $|x|/w\to 0$ if $F_0=0$. However, for the moment, we shall continue to regard F_0 as a nonzero quantity,

In order to solve Eq. (40), we write $\hat\psi=-\psi/\Psi$, $\langle\langle\cdots\rangle\rangle = \langle\cdots\rangle w$, $X=x/w$, and $\hat L = L/(L_0/w)$. It follows that

$$\delta^2\frac{d}{d\hat\psi}\left(\langle\langle X^4\rangle\rangle\frac{d\hat L}{d\hat\psi}\right) - \langle\langle X^2\rangle\rangle\hat L = -s. \tag{42}$$

According to our ordering scheme, $\delta\sim d_\beta\ll 1$. Thus, $\hat L(s,\hat\psi)$ takes the value $s/\langle\langle X^2\rangle\rangle$ in the region outside the magnetic separatrix, apart from a thin boundary layer on the separatrix itself of width δw. In this layer, the function $\hat L(s,\hat\psi)$ makes a smooth transition from its exterior value (which is $s\pi/4$ immediately outside the separatrix) to its interior value 0. We can write

$$\hat L(s,\hat\psi) = s\left(\frac{1}{\langle\langle X^2\rangle\rangle} + l(y)\right), \tag{43}$$

where $y=(\hat\psi-1)/\delta$. It follows that

$$\frac{d^2 l}{dy^2} - \frac{3}{8}l \simeq 0, \tag{44}$$

since $\langle\langle X^2\rangle\rangle_{\hat\psi=1}=4/\pi$ and $\langle\langle X^4\rangle\rangle_{\hat\psi=1}=32/3\pi$. Hence, the continuous solution to Eq. (40) which satisfies the appropriate boundary conditions is

$$\hat L(s,\hat\psi) \simeq s\left[\frac{1}{\langle\langle X^2\rangle\rangle} - \frac{\pi}{4}\exp\left(-\sqrt{\frac{3}{8}}\frac{\hat\psi-1}{\delta}\right)\right] \tag{45}$$

in the region outside the separatrix (i.e., $\hat\psi\geq 1$). Of course, $\hat L(s,\hat\psi)=0$ in the region inside the separatrix (i.e., $\hat\psi<1$).

G. Determination of island phase velocity

Let $\delta J=\delta J_c+\delta J_s$, where δJ_c has the symmetry of $\cos\theta$, whereas δJ_s has the symmetry of $\sin\theta$. Now, it is easily demonstrated that

$$\langle\delta J_s\sin\theta\rangle = \frac{1}{k\Psi}\langle x[\delta J,\psi]\rangle. \tag{46}$$

Hence, it follows from Eqs. (32) and (37) that

$$\langle\delta J_s\sin\theta\rangle = -\frac{(\mu_i+\mu_e)}{k\Psi}\frac{d}{d\psi}\left(\langle x^5\rangle\frac{d^2F}{d\psi^2} - 2\langle x^3\rangle\frac{dF}{d\psi} - \langle x\rangle F\right). \tag{47}$$

Now, for an isolated magnetic island which is not interacting electromagnetically with any external structure, such as a resistive wall, the net electromagnetic force acting on the island must be zero. This constraint translates to the well-known requirement that

$$\int_{-\Psi}^{\infty}\langle\delta J_s\sin\theta\rangle d\psi = 0. \tag{48}$$

Using Eq. (47), this requirement reduces to the condition

$$\lim_{x/w\to\infty}\left(\langle x^5\rangle\frac{d^2F}{d\psi^2} - 2\langle x^3\rangle\frac{dF}{d\psi} - \langle x\rangle F\right) \propto \lim_{x/w\to\infty}\left[sx^2\frac{d}{dx}\left(\frac{1}{x}\frac{d(xF)}{dx}\right)\right] = -F_0 = 0, \tag{49}$$

since $xF\to|x|F_0$ as $|x|/w\to\infty$. Hence, we conclude that $F_0 = 0$ [i.e., $F(\psi)=0$, everywhere] for an isolated magnetic island. Indeed, it follows from the analysis of Sec. III F that Eq. (48) is *automatically* satisfied when the island-induced modifications to the ion and electron velocity profiles are *localized* in the vicinity of the island, as must be the case for an isolated island.

It follows from Eq. (37) that

$$M(s,\psi) = -\frac{d_\beta(\mu_i\tau-\mu_e)}{(\mu_i+\mu_e)}L(s,\psi). \tag{50}$$

Hence, $M_0=-[d_\beta(\mu_i\tau-\mu_e)/(\mu_i+\mu_e)]L_0$. Recalling that $M_0 = V_{EBy}^{(0)}-V$, $d_\beta L_0=(V_{iy}^{(0)}-V_{ey}^{(0)})/(1+\tau)$, $V_{iy}^{(0)}=V_{EBy}^{(0)}+d_\beta\tau L_0$, and

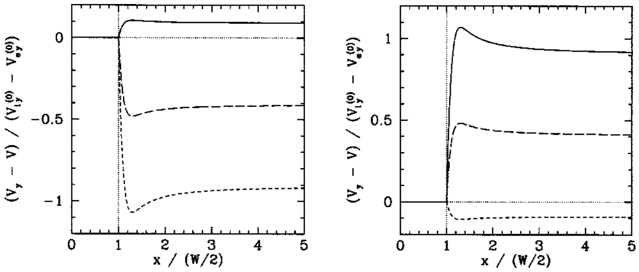

FIG. 1. Velocity profiles as functions of x, at constant θ, evaluated on a line passing through the island O point (i.e., at $\theta=0$) in the island rest frame. The O point lies at $x=0$. The island separatrix is indicated by a vertical dotted line. The solid curves show the normalized ion fluid velocity profile: $(V_{iy}-V)/(V_{iy}^{(0)}-V_{ey}^{(0)})$. The short-dashed curves show the normalized electron fluid velocity profile: $(V_{ey}-V)/(V_{iy}^{(0)}-V_{ey}^{(0)})$. The long-dashed curves show the normalized $\boldsymbol{E}\times\boldsymbol{B}$ velocity profile: $(V_{EB_y}-V)/(V_{iy}^{(0)}-V_{ey}^{(0)})$. The left-hand panel shows the case of viscous ions: $\mu_e/\mu_i=0.1$, $\tau=1.0$, and $\delta=0.2$. The right-hand panel shows the case of viscous electrons: $\mu_i/\mu_e=0.1$, $\tau=1.0$, and $\delta=0.2$.

$V_{ey}^{(0)}=V_{EBy}^{(0)}-d_\beta L_0$, we obtain the following expression for the island phase velocity:

$$V=\frac{\mu_i V_{iy}^{(0)}+\mu_e V_{ey}^{(0)}}{\mu_i+\mu_e}. \tag{51}$$

In other words, the island phase velocity is the *viscosity weighted mean* of the unperturbed (i.e., in the absence of an island) local ion and electron fluid velocities. Hence, if the ions are far more viscous then the electrons, then the island propagates with the ion fluid. In this case, the ion fluid velocity profile remains largely unaffected by the island, but the electron fluid velocity profile is highly sheared just outside the island separatrix. The opposite is true if the electrons are far more viscous than the ions. This is illustrated in Fig. 1.

We have now fully specified the ion and electron stream functions ϕ_i and ϕ_e, respectively, in the island rest frame. In fact, $\phi_i=0$ inside the separatrix, and

$$\frac{d\phi_i(s,\hat{\psi})}{d\psi}=(V_{iy}^{(0)}-V_{ey}^{(0)})\frac{\mu_i}{\mu_i+\mu_e}\frac{\hat{L}(s,\hat{\psi})}{w} \tag{52}$$

outside the separatrix, where the function $\hat{L}(s,\hat{\psi})$ is specified in Eq. (45). Likewise, ϕ_e is zero inside the separatrix, and

$$\frac{d\phi_e(s,\hat{\psi})}{d\psi}=-(V_{iy}^{(0)}-V_{ey}^{(0)})\frac{\mu_e}{\mu_i+\mu_e}\frac{\hat{L}(s,\hat{\psi})}{w} \tag{53}$$

outside the separatrix. Note that the stream functions and their first derivatives are everywhere continuous, which implies that the ion and electron fluid velocities are everywhere continuous.

H. Determination of ion polarization correction

It follows from Eq. (32) that

$$\delta J_c=\frac{(V-V_{EBy}^{(0)})(V-V_{iy}^{(0)})}{2}\left(x^2-\frac{\langle x^2\rangle}{\langle 1\rangle}\right)\frac{d}{d\psi}\left[\frac{H(\hat{\psi}-1)}{\langle x^2\rangle^2}\right]+I(s,\psi), \tag{54}$$

where $I(s,\psi)$ is as yet undetermined. The function $H(\vartheta)$ is zero for $\vartheta<0$, and unity for $\vartheta\geqslant 0$. Here, we have made use of the fact that outside the separatrix $L(s,\psi)\simeq sL_0/\langle x^2\rangle$, and $M(s,\psi)\simeq sM_0/\langle x^2\rangle$, apart from a thin boundary layer on the separatrix itself. It turns out that we do not need to resolve this boundary layer in order to calculate the total ion polarization current. However, we do have to include the net current flowing in this layer in our calculation of the total current.[12,14] Flux-surface averaging Eqs. (30) and (33), we obtain

$$\epsilon^2 w^2\langle\nabla^2\delta J_c\rangle-\langle\delta J_c\rangle=-\eta^{-1}\frac{d\Psi}{dt}\langle\cos\theta\rangle, \tag{55}$$

where

$$\epsilon=\frac{d_i}{w}\sqrt{\frac{\mu_i\mu_e}{\eta(\mu_i+\mu_e)}}. \tag{56}$$

Note that $\epsilon\sim d_\beta\ll 1$, according to our ordering scheme.

Equation (55) implies that

$$\langle\delta J_c\rangle\simeq\eta^{-1}\frac{d\Psi}{dt}\langle\cos\theta\rangle, \tag{57}$$

apart from in a thin boundary layer on the separatrix of width ϵw. It is easily demonstrated that the deviation of $\langle\delta J_c\rangle$ in the boundary layer from the value given in Eq. (57) makes a negligible contribution to the total ion polarization current. Hence, we shall treat Eq. (57) as if it applied everywhere.

Equations (54) and (57) give

$$\delta J_c=\frac{(V-V_{EBy}^{(0)})(V-V_{iy}^{(0)})}{2}\left(x^2-\frac{\langle x^2\rangle}{\langle 1\rangle}\right)\frac{d}{d\psi}\left[\frac{H(\hat{\psi}-1)}{\langle x^2\rangle^2}\right]+\eta^{-1}\frac{d\Psi}{dt}\frac{\langle\cos\theta\rangle}{\langle 1\rangle}. \tag{58}$$

The island width evolution equation is obtained by asymptotic matching to the region far from the island.[3] In fact,

$$\Delta'\Psi=-4\int_{\Psi}^{-\infty}\langle\delta J_c\cos\theta\rangle d\psi, \tag{59}$$

where Δ' is the conventional tearing stability index.[23] It follows from Eqs. (58) and (59) that

$$\Delta'=-\frac{(V-V_{EBy}^{(0)})(V-V_{iy}^{(0)})}{w^3}\int_{-1}^{\infty}\left(\langle\langle X^4\rangle\rangle-\frac{\langle\langle X^2\rangle\rangle^2}{\langle\langle 1\rangle\rangle}\right)\frac{d}{d\hat{\psi}}\left[\frac{H(\hat{\psi}-1)}{\langle\langle X^2\rangle\rangle^2}\right]d\hat{\psi}+\frac{8}{\eta}\frac{dw}{dt}\int_{-1}^{\infty}\frac{\langle\langle\cos\theta\rangle\rangle^2}{\langle\langle 1\rangle\rangle}d\hat{\psi}. \tag{60}$$

Performing the flux surface integrals, whose values are well known,[14] we obtain the following island width evolution equation:

$$\frac{0.823}{\eta}\frac{dW}{dt} = \Delta' + 1.38\frac{(V - V_{EBy}^{(0)})(V - V_{iy}^{(0)})}{(W/4)^3}. \tag{61}$$

Here, $W=4w$ is the full island width. The ion polarization current term (the second term on the rhs) is only stabilizing when the island phase velocity V lies between the unperturbed local $\boldsymbol{E}\times\boldsymbol{V}$ velocity $V_{EBy}^{(0)}$, and the unperturbed local velocity of the ion fluid $V_{iy}^{(0)}$.[24]

IV. SUMMARY

A set of reduced, 2D, two-fluid, drift-MHD equations has been developed. This set of equations takes into account both electron and ion diamagnetism (including the contribution of the ion gyroviscous tensor), as well as the Hall effect and parallel electron compressibility, but neglects electron inertia and the compressible Alfvén wave. For the sake of simplicity, the plasma density is assumed to be uniform, and the ion and electron temperatures constant multiples of one another.

Using our equations, we have found a complete and self-consistent solution for an isolated magnetic island propagating through a slab plasma with uniform but different ion and electron fluid velocities. Our solution is valid provided that the ordering scheme described in Sec. III D holds good, which implies that

$$d_\beta \ll c_\beta \tag{62}$$

and

$$\mu_{i,e}, \kappa, \eta \ll V_{*y}^{(0)} \ll c_\beta, \tag{63}$$

where $V_{*y}^{(0)}=V_{iy}^{(0)}-V_{ey}^{(0)}$. Here, all lengths and velocities are approximately normalized to the island width W and the shear-Alfvén speed calculated with $B_y^{(0)}(W)$, respectively. Note that the condition $V_{*y}^{(0)}\ll c_\beta$ permits the magnetoacoustic wave to flatten the plasma pressure profile within the island separatrix.

Our solution yields ion and electron fluid velocity profiles which are *uniquely determined* in the vicinity of the island (see Fig. 1). These profiles are everywhere continuous and asymptote to the unperturbed fluid velocities far from the island. Incidentally, the inclusion of *electron viscosity* in both the Ohm's law and the plasma equation of motion is key to the determination of continuous velocity profiles.[15]

The island phase velocity is uniquely specified by the condition that the island-induced modifications to the ion and electron velocity profiles remain *localized* in the vicinity of the island, as must be the case for an *isolated* island (i.e., an island which is not interacting electromagnetically with any external agency such as a resistive wall or an error field). Such velocity profiles automatically ensure that there is zero net electromagnetic force acting on the island.[16] It turns out that the phase velocity is the *viscosity weighted mean* of the unperturbed (i.e., in the absence of an island) local ion and electron fluid velocities. Note that, in this paper, we have adopted phenomenological diffusive ion and electron viscosity operators, which are supposed to represent anomalous perpendicular momentum transport due to small-scale plasma turbulence.

The ion polarization current correction to the Rutherford island width evolution equation is found to be stabilizing when the island phase velocity lies between the unperturbed local ion fluid velocity and the unperturbed local $\boldsymbol{E}\times\boldsymbol{B}$ velocity [see Eq. (61)].[16] It follows, from our result for the island phase velocity that the polarization term is *stabilizing* when the anomalous perpendicular ion viscosity significantly exceeds the anomalous perpendicular electron viscosity (see Fig. 1, left panel). Conversely, the polarization term is destabilizing when the electron viscosity significantly exceeds the ion viscosity [see Fig. 1, right panel].[25] Note, however, that in order for the electron viscosity to exceed the ion viscosity, the electron momentum confinement time would need to be at least a *mass ratio smaller* than the ion momentum confinement time, which does not seem very probable. Hence, we conclude that under normal circumstances the polarization term is stabilizing.

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