Nonlinear finite-Larmor-radius drift-kinetic equation

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An efficient method is described for deriving the drift-kinetic equation. A maximal ordering is invoked: the ordering parameter $\epsilon \ll 1$ is formally taken to be proportional to m/e, subject to the proviso that the parallel electric field $E_{\parallel} \sim \epsilon$. Electric drifts can be of the order of particle thermal velocities. The drift-kinetic equation is derived up to second order in ϵ, and is in a form such that the phase-space volume following the particle phase-space trajectories is preserved. The mean density, mean velocity, momentum flow tensor, and the presure tensor are evaluated in terms of the electromagnetic fields and the velocity moments of the drift-kinetic distribution function $\bar{G}$. The moments of the drift-kinetic equation reproduce the corresponding moments of the Vlasov equation up to order ϵ^2. A consistent set of fluid-kinetic equations is formulated, with the fluid-like perpendicular motion described by the perpendicular component of the momentum equation. The drift-kinetic equation describes the parallel motion, and the solution $\bar{G}$ is required to evaluate the velocity moments necessary to close the set of equations.
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I. INTRODUCTION

A plasma is an ionized gas of charge particles. Nevertheless, the motion of plasmas across the magnetic field is fluid-like in character, and the low-frequency behavior of magnetized plasmas is often described by fluid equations in which the plasma state is characterized by a small number of macroscopic variables: mean density, mean velocity, parallel pressure, and perpendicular pressure. These fluid equations are the low-order velocity moments of the particle kinetic equation,[1–3] supplemented by a closure ansatz to express the higher-order velocity moments of the particle distribution function in terms of low-order moments. For collisionless magnetized plasmas, however, motion along the magnetic-field line is not fluid-like, and it is arguable whether a satisfactory scheme can be devised to close the parallel dynamical equations.

In place of a closure ansatz, an alternative tactic is to solve the drift-kinetic equation (or the gyrokinetic equation) to obtain the gyrophase-averaged distribution function. Thereafter, two procedures are possible: (1) the particle distribution function is used to evaluate directly the plasma charge and current densities (eliminating the need for fluid equations), and the electromagnetic fields are time advanced using Maxwell's equations; (2) a set of fluid-kinetic equations is derived to determine the plasma behavior (with the fluid-like motion perpendicular to the magnetic field described by the perpendicular component of the plasma momentum equation), and the distribution function is used to evaluate the velocity moments necessary to close the set of equations.

The principal challenge in formulating a consistent set of low-frequency fluid-kinetic equations is the derivation of the drift-kinetic equation with sufficient accuracy, particularly when it is necessary to include second-order Larmor radius corrections. Formulations of the gyrokinetic equations, which include finite-Larmor-radius effects of arbitrary magnitude, rely on the representation of the perturbed electromagnetic fields in an eikonal form and are restricted to small amplitude fluctuations.[4–6] Formulations of second-order effects in differential form are more difficult to derive due to the algebraic complexities involved in carrying out the required analysis. Existing formulations typically reduce the degree of algebraic complexity by imposing additional restrictions (electrostatic perturbations, low plasma beta)[7] or by invoking subsidiary orderings.[8]

In this paper, we discuss a novel and efficient method of deriving the nonlinear electromagnetic drift-kinetic equation from the collisionless Vlasov equation, up to second order in the Larmor radius. The analysis is carried out after transforming the velocity variable to the velocity relative to the perpendicular electric drift velocity, and electric drifts can be of the order of thermal velocities. A maximal ordering is invoked in which the term proportional to the cyclotron frequency $\Omega = eB/mc$ is of order $1/\epsilon$ while all other terms are of order unity. Thus the formal ordering parameter is effectively m/e, with the proviso that $E_{\parallel} \sim \epsilon$, where $E_{\parallel}$ is the parallel electric-field component.

In Sec. II, we describe the analysis of the zeroth-order, first-order, and second-order equations, and we derive the drift-kinetic equations up to second order in ϵ. The phase-space coordinates of the drift-kinetic distribution function, denoted by $\bar{G}(v_{\parallel}, \mu, \boldsymbol{r}, t)$, are taken to be the magnitude $v_{\parallel}$ of the parallel velocity component, the magnetic moment $\mu = mv_{\perp}^2/2B$, and the position vector $\boldsymbol{r}$. These coordinates are not canonical variables. Nevertheless, the drift-kinetic equation which determines $\bar{G}$ is derived in a form in which the phase-space volume is manifestly preserved following the particle phase-space trajectories. This form of the drift-kinetic equation is particularly desirable if it is to be used to simulate the long-time behavior of magnetized plasmas.

In Sec. III, we use the Larmor series expansion of the

Vlasov distribution function F to write the mean density, the mean velocity, the momentum flow tensor, and the pressure tensor in terms of velocity moments of the drift-kinetic distribution function $\bar{G}$

In Sec. IV, we evaluate the velocity moments of the drift-kinetic equation. We derive the continuity equation, the parallel momentum equation, and the time evolution equations for the parallel and perpendicular pressure. We verify that the continuity equation and the parallel momentum equation are equivalent to the corresponding moment equations of the Vlasov equation, expressed as a Larmor series in ϵ.

In Sec. V, we discuss the formulation of a consistent set of fluid-kinetic equations, with the fluid-like perpendicular plasma motion described by the perpendicular component of the plasma momentum equation. The parallel motion is described by the drift-kinetic equation, and the solution $\bar{G}$ is required to evaluate the velocity moments necessary to close the set of equations.

In Sec. VI, we derive the energy conservation equation, and in Sec. VI, we present a summary and discussion of our results.

II. DRIFT-KINETIC ORDERING AND ANALYSIS

The time evolution of the particle distribution function $F^{\bullet}(\boldsymbol{r}^{\bullet},\boldsymbol{v}^{\bullet},t)$ is determined by the collisionless Vlasov equation

$$\frac{\partial F^{\bullet}}{\partial t}+\boldsymbol{v}\cdot\frac{\partial F^{\bullet}}{\partial \boldsymbol{r}^{\bullet}}+\frac{e}{m}\left(\boldsymbol{E}+\frac{\boldsymbol{v}^{\bullet}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial F^{\bullet}}{\partial \boldsymbol{v}^{\bullet}}=0, \tag{1}$$

where $\boldsymbol{r}^{\bullet}$ is the position vector, $\boldsymbol{v}^{\bullet}$ the particle velocity vector, $\boldsymbol{E}(\boldsymbol{r}^{\bullet},t)$ the electric field, and $\boldsymbol{B}(\boldsymbol{r}^{\bullet},t)$ the magnetic field.

We find it convenient to transform to new coordinate variables $\boldsymbol{r}$ and $\boldsymbol{v}$, defined by the transformation equations:

$$\boldsymbol{r}^{\bullet}=\boldsymbol{r}, \quad \boldsymbol{v}^{\bullet}=\boldsymbol{v}+\boldsymbol{v}_E, \tag{2}$$

where $\boldsymbol{v}_E$ is the electric drift velocity

$$\boldsymbol{v}_E=\frac{c}{B^2}\boldsymbol{E}\times\boldsymbol{B}. \tag{3}$$

Let $F(\boldsymbol{r},\boldsymbol{v},t)=F^{\bullet}(\boldsymbol{r}^{\bullet},\boldsymbol{v}^{\bullet},t)$ be the distribution function written in terms of $\boldsymbol{r}$, $\boldsymbol{v}$, and t, and let the magnetic-field magnitude be denoted by $B=|\boldsymbol{B}|$, the unit magnetic-field vector by $\boldsymbol{b}=\boldsymbol{B}/B$, and the cyclotron frequency by $\Omega=eB/mc$. Then Eq. (1) can be expressed as follows:

$$\frac{\Omega}{\epsilon}\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial F}{\partial \boldsymbol{v}}+\hat{\mathcal{L}}F=0, \tag{4}$$

where the operator $\hat{\mathcal{L}}$ acting on $F(\boldsymbol{r},\boldsymbol{v},t)$ is defined by

$$\hat{\mathcal{L}}F\equiv\frac{\partial F}{\partial t}+(\boldsymbol{v}+\boldsymbol{v}_E)\cdot\nabla F+\frac{\partial F}{\partial \boldsymbol{v}}\cdot\left[\frac{e}{m}(\boldsymbol{b}\cdot\boldsymbol{E})\boldsymbol{b}-\frac{\partial \boldsymbol{v}_E}{\partial t}-(\boldsymbol{v}+\boldsymbol{v}_E)\cdot\nabla\boldsymbol{v}_E\right]. \tag{5}$$

We introduced a smallness parameter $\epsilon<1$ which is used to order the relative magnitude of terms. We are interested in low-frequency and long-wavelength plasma phenomena in which the characteristic frequency ω is less than the cyclotron frequency $|\Omega|$ and the characteristic wavelength λ is longer than the Larmor radius $r_L\sim|v/\Omega|$, that is, $|\omega/\Omega|\sim r_L/\lambda\sim\epsilon$. Note that $\boldsymbol{E}\cdot\boldsymbol{b}$ is considered to be of order ϵ. In this limit, Eq. (4) is reducible to the drift-kinetic equation by an appropriate averaging over the fast Larmor gyration of charged particle motion about the magnetic field.

In the standard method for averaging Eq. (4) over the particle Larmor gyration,[9] the velocity vector $\boldsymbol{v}$ is resolved into parallel ($v_\parallel\boldsymbol{b}$) and perpendicular ($\boldsymbol{v}_\perp$) components:

$$v_\parallel\equiv\boldsymbol{v}\cdot\boldsymbol{b}, \quad \boldsymbol{v}_\perp\equiv\boldsymbol{b}\times(\boldsymbol{v}\times\boldsymbol{b})=\boldsymbol{v}-\boldsymbol{b}\cdot\boldsymbol{v}\boldsymbol{b},$$

and the perpendicular velocity vector is expressed as follows, in order to exhibit explicitly the dependence of $\boldsymbol{v}_\perp$ on the Larmor phase angle Θ:

$$\boldsymbol{v}_\perp=-v_\perp\sin\Theta\boldsymbol{e}+v_\perp\cos\Theta\boldsymbol{e}^*, \tag{6}$$

where $\boldsymbol{b}(\boldsymbol{r},t),\boldsymbol{e}(\boldsymbol{r},t),\boldsymbol{e}^*(\boldsymbol{r},t)=\boldsymbol{b}\times\boldsymbol{e}$ are local unit orthogonal vectors and $v_\perp$ is the magnitude of the vector $\boldsymbol{v}_\perp$.

In a transformation of velocity variables, from $\boldsymbol{v}$ to $v_\parallel$, $v_\perp$, and Θ, we have

$$\boldsymbol{v}\cdot\boldsymbol{b}\times\frac{\partial}{\partial \boldsymbol{v}}=-\frac{\partial}{\partial\Theta}.$$

Let the distribution function F be expressed in the form of the following power series expansion in the smallness parameter ϵ:

$$F=F_0+\epsilon F_1+\epsilon^2F_2+\epsilon^3F_3+\cdots. \tag{7}$$

Substituting this series expansion for F in Eq. (4) and equating terms of the same order in ϵ, we obtain to lowest order in ϵ,

$$\Omega\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial F_0}{\partial \boldsymbol{v}}=0. \tag{8}$$

Thus F_0 must be independent of the Larmor phase angle. We consider F_0 to be a function of the position vector $\boldsymbol{r}$, the parallel velocity component $v_\parallel$, the magnetic moment variable $\mu=m\boldsymbol{v}_\perp\cdot\boldsymbol{v}_\perp/2B$, and the time variable t:

$$F_0=\bar{F}_0(\boldsymbol{r},v_\parallel,\mu,t). \tag{9}$$

We will use the overline $\bar{F}_n=\bar{F}_n(\boldsymbol{r},v_\parallel,\mu,t)$ to imply Larmor phase averaging as well as Larmor phase independence, and the tilde $\tilde{F}_n=\tilde{F}_n(\boldsymbol{r},v_\parallel,\mu,\Theta,t)$ to imply dependence on the Larmor phase angle. Let $F_n=\bar{F}_n+\tilde{F}_n$, $n=0,1,2,\ldots$, be separated into phase-independent and phase-dependent terms. Then equating terms of order ϵ^n, $n=0,1,2,\ldots$, we have

$$\Omega\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial\tilde{F}_{n+1}}{\partial \boldsymbol{v}}+\hat{\mathcal{L}}(\bar{F}_n+\tilde{F}_n)=0, \tag{10}$$

where $\tilde{F}_0=0$.

Note that we can express $\hat{\mathcal{L}}F_n$ in the form of a periodic function of Θ. It is now a straightforward exercise to carry out the Larmor phase average and Larmor phase integration of these equations order by order.

This method is, however, algebraically tedious to execute beyond first order in ϵ. The number of terms increases rapidly at higher orders in ϵ. Furthermore, the individual terms involve the arbitrary unit vectors $\boldsymbol{e}$ and $\boldsymbol{e}^*$, and there remains the additional task of collecting appropriate terms so that the final expressions involve only the unit magnetic vector $\boldsymbol{b}$.

We will adopt an alternative method of analyzing these equations. We will not carry out an explicit transformation of the velocity vector $\boldsymbol{v}$ to new variables $v_\parallel$, $v_\perp$, and Θ. Instead, we find it convenient to introduce the Cartesian tensors $(x_i, v_i, u_i, u_i^*, b_i, v_{Ei}, E_i, B_i)$ to denote the set of vectors $(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{v}_\perp, \boldsymbol{b}\times\boldsymbol{v}, \boldsymbol{b}, \boldsymbol{v}_E, \boldsymbol{E}, \boldsymbol{B})$, respectively, with the integer subscript $i=1,2,3$. Thus $\{v_1, v_2, v_3\}$ are the Cartesian components of the vector $\boldsymbol{v}$, etc. The perpendicular velocity tensors u_i and u_i^* are defined by

$$u_i \equiv v_k I_{ki}, \quad u_i^* \equiv \epsilon_{ilk} b_l v_k, \tag{11}$$

where I_{ij} is the perpendicular projection tensor,

$$I_{ij} \equiv \delta_{ij} - b_i b_j, \tag{12}$$

and δ_{ij} is the Kronecker delta. The invariant third-rank tensor ϵ_{ijk} is used to express vector products. Summation over repeated indices is implied.

It is useful to list a number of identities, involving I_{ij} and ϵ_{ijk}, which will be used throughout our analysis:

$$I_{ij} = I_{ji}, \quad I_{ij} I_{jk} = I_{ik}, \quad \delta_{ij} I_{ij} = 2, \tag{13}$$

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \quad \epsilon_{ijk} b_j \epsilon_{pqr} b_q = I_{ip} I_{kr} - I_{ir} I_{kp}, \tag{14}$$

$$\epsilon_{ijk} b_j I_{pq} = \epsilon_{pjk} b_j I_{iq} + \epsilon_{ijp} b_j I_{kq}. \tag{15}$$

The parallel velocity is $v_\parallel = b_1 v_1 + b_2 v_2 + b_3 v_3 \equiv b_i v_i$ and the magnetic moment is $\mu = I_{ij}\, m v_i v_j / 2B$. Any dependence on the Larmor phase angle is expressed through a dependence on the perpendicular velocity tensors u_i and u_i^*. It is then appropriate to consider $\tilde{F}_n$ to be a function of r_i, $v_\parallel$, μ, u_i, u_i^*, and t. However, it is sufficient to take $\tilde{F}_n = \tilde{F}_n(x_i, v_\parallel, \mu, v_i, t)$ to be a function of x_i, $v_\parallel$, μ, v_i, and t, where the dependence on v_i accounts for the additional dependence on the Larmor phase angle through u_i and u_i^*.

The operator $\hat{\mathcal{L}}$ acting on an arbitrary function $\mathcal{G}(r_i, v_\parallel, \mu, v_i, t)$ is given by

$$\begin{aligned}\hat{\mathcal{L}}\mathcal{G} = {} & \frac{D\mathcal{G}}{Dt} + u_i \frac{\partial' \mathcal{G}}{\partial x_i} + \left(\frac{e}{m} b_i E_i - b_i \frac{D v_{Ei}}{Dt} - b_i u_j \frac{\partial v_{Ei}}{\partial x_j} \right. \\ & \left. + u_i \frac{D b_i}{Dt} + u_i u_j \frac{\partial b_i}{\partial x_j} \right) \frac{\partial \mathcal{G}}{\partial v_\parallel} - \left[\mu \frac{DB}{Dt} + \mu u_j \frac{\partial B}{\partial x_j} \right. \\ & \left. + m u_i \left(v_\parallel \frac{D b_i}{Dt} + \frac{D v_{Ei}}{Dt} \right) + m u_i u_j \left(v_\parallel \frac{\partial b_i}{\partial x_j} + \frac{\partial v_{Ei}}{\partial x_j} \right) \right] \\ & \times \frac{1}{B} \frac{\partial \mathcal{G}}{\partial \mu} + \left(\frac{e}{m} b_j E_j b_i - \frac{D v_{Ei}}{Dt} - u_j \frac{\partial v_{Ei}}{\partial x_j} \right) \frac{\partial' \mathcal{G}}{\partial v_i},\end{aligned} \tag{16}$$

where the operator D/Dt is defined by

$$\frac{D}{Dt} \equiv \left[\frac{\partial'}{\partial t} + (v_\parallel b_j + v_{Ej}) \frac{\partial'}{\partial x_j} \right] \tag{17}$$

and $\partial'/\partial t$ is the time derivative holding x_i, $v_\parallel$, μ, and v_i constant, $\partial'/\partial x_i$ is the space coordinate derivative holding $v_\parallel$, μ, v_i, and t constant, and $\partial'/\partial v_i$ is the velocity derivative holding x_i, $v_\parallel$, μ, and t constant.

We will write our equations in vector and tensor notations interchangeably throughout our subsequent analysis, and for convenience we will hereafter delete the superscripts on the differential operators.

We anticipate that, in the ordered set of equations given by Eq. (10), the evaluation of $\hat{\mathcal{L}}F_n$ will produce terms which are polynomials in the perpendicular velocity vectors u_i and u_i^*. The analysis of these ordered equations will then involve the average and integration over the Larmor phase angle of these velocity polynomials.

To facilitate our analysis, we note the following identities involving first-rank tensors u_i and u_i^*:

$$\boldsymbol{v}\cdot\boldsymbol{b}\times\frac{\partial}{\partial \boldsymbol{v}} u_j = -u_j^*, \quad \boldsymbol{v}\cdot\boldsymbol{b}\times\frac{\partial}{\partial \boldsymbol{v}} u_j^* = u_j. \tag{18}$$

Occasionally, $-\partial/\partial\Theta$ will be used as a convenient shorthand notation for the operator $\boldsymbol{v}\times\boldsymbol{b}\cdot\partial/\partial\boldsymbol{v}$, and $-\int^\Theta d\Theta$ will be used to denote the inverse of the operator $\boldsymbol{v}\times\boldsymbol{b}\cdot\partial/\partial\boldsymbol{v}$, that is, integration over the Larmor phase angle. Thus, we have

$$\int^\Theta d\Theta\, u_j^* = u_j, \quad \int^\Theta d\Theta\, u_j = -u_j^*.$$

We note also that the second-rank tensor $u_i u_j$ can be separated into Larmor phase-independent and Larmor phase-dependent expressions as follows:

$$u_i u_j = \frac{v_\perp^2}{2} I_{ij} + \frac{1}{2}(u_i u_j - u_i^* u_j^*). \tag{19}$$

Let us define the second-rank perpendicular velocity tensors u_{ij} and u_{ij}^* as follows:

$$u_{ij} \equiv u_i u_j - u_i^* u_j^*, \quad u_{ij}^* \equiv u_i u_j^* + u_i^* u_j. \tag{20}$$

Then, we have

$$\boldsymbol{v}\cdot\boldsymbol{b}\times\frac{\partial}{\partial \boldsymbol{v}} u_{ij} = -2u_{ij}^*, \quad \boldsymbol{v}\cdot\boldsymbol{b}\times\frac{\partial}{\partial \boldsymbol{v}} u_{ij}^* = 2u_{ij}, \tag{21}$$

and the integration over the Larmor phase angle of u_{ij} and u_{ij}^* are given by

$$\int^\Theta d\Theta\, u_{ij}^* = \frac{1}{2} u_{ij}, \quad \int^\Theta d\Theta\, u_{ij} = -\frac{1}{2} u_{ij}^*. \tag{22}$$

Note that u_{ij} and u_{ij}^* are equal to sums of terms proportional to either $\cos 2\Theta$ or $\sin 2\Theta$.

Additional identities for evaluating the Larmor phase average and Larmor phase integration of third-rank and fourth-rank perpendicular velocity tensors are discussed in Appendix A.

Let us now discuss the analysis of Eq. (10), order by order.

A. Zeroth-order equation

The lowest-order equation is

$$\Omega \boldsymbol{v} \times \boldsymbol{b} \cdot \frac{\partial \widetilde{F}_1}{\partial \boldsymbol{v}} + \hat{\mathcal{L}} \bar{F}_0 = 0. \tag{23}$$

Substituting Eq. (19) for $u_i u_j$, we use Eq. (16), which defines the operator $\hat{\mathcal{L}}$, to express $\hat{\mathcal{L}}\bar{F}_0$ as the following polynomial in u_i and u_i^*:

$$\hat{\mathcal{L}}\bar{F}_0 = \bar{\mathcal{D}}^{(0)}\bar{F}_0 + u_k \hat{d}_k^{(1)} \bar{F}_0 + (u_l u_k - u_l^* u_k^*) \hat{d}_{lk}^{(1)} \bar{F}_0. \tag{24}$$

The Larmor phase-independent operator $\hat{\mathcal{D}}^{(0)}$ is defined by

$$\hat{\mathcal{D}}^{(0)}\bar{F}_0 \equiv \frac{\partial \bar{F}_0}{\partial t} + \dot{X}_i^{(0)} \frac{\partial \bar{F}_0}{\partial x_i} + \dot{V}_\parallel^{(0)} \frac{\partial \bar{F}_0}{\partial v_\parallel} + \dot{\mu}^{(0)} \frac{\partial \bar{F}_0}{\partial \mu}$$

$$= \frac{\partial \bar{F}_0}{\partial t} + \dot{Z}_\nu^{(0)} \frac{\partial \bar{F}_0}{\partial z_\nu}, \tag{25}$$

where the phase-space functions $\dot{X}_i^{(0)}$, $\dot{V}_\parallel^{(0)}$, and $\dot{\mu}_i^{(0)}$, are given by

$$\dot{X}_i^{(0)} = v_\parallel b_i + v_{Ei}, \tag{26}$$

$$\dot{V}_\parallel^{(0)} = \frac{e}{m} b_i E_i - \frac{\mu}{m} b_i \frac{\partial B}{\partial x_i} - b_i \frac{D v_{Ei}}{Dt}, \tag{27}$$

$$\dot{\mu}^{(0)} = -\frac{\mu}{B}\left[\frac{DB}{Dt} + B I_{ij}\left(v_\parallel \frac{\partial b_i}{\partial x_j} + \frac{\partial v_{Ei}}{\partial x_j}\right)\right]$$

$$= \frac{c\mu}{B} b_i E_i (\boldsymbol{b} \cdot \nabla \times \boldsymbol{b}), \tag{28}$$

and the five-dimensional phase-space variable z_ν and phase-space function $\dot{Z}_\nu^{(0)}$, with $\nu=1,2,3,4,5$, are defined by

$$z_i = x_i, \quad z_4 = v_\parallel, \quad z_5 = \mu, \tag{29}$$

$$\dot{Z}_i^{(0)} = \dot{X}_i^{(0)}, \quad \dot{Z}_4^{(0)} = \dot{V}_\parallel^{(0)}, \quad \dot{Z}_5^{(0)} = \dot{\mu}^{(0)}. \tag{30}$$

We use Greek subscripts to imply integer values ranging from 1 to 5 and Roman subscripts to imply integer values ranging from 1 to 3.

The velocity $\dot{X}_i^{(0)}$ is the vector sum of parallel motion along the magnetic-field line and electric drift across the field line. The parallel acceleration $\dot{V}_\parallel^{(0)}$ is due to the parallel electric field, the "mirroring" force, and the parallel component of the inertial force arising from the electric drift. The time derivative of the magnetic moment $\dot{\mu}^{(0)}$ is typically small and is zero in the limit of $b_i E_i = 0$ or $\boldsymbol{b} \cdot \nabla \times \boldsymbol{b} = 0$ (zero parallel current).

The operators $\hat{d}_k^{(1)}$ and $\hat{d}_{lk}^{(1)}$, with $\hat{d}_{lk}^{(1)} = \hat{d}_{kl}^{(1)}$ symmetric in an interchange of the subscript indices, are defined by

$$\hat{d}_i^{(1)} \bar{F}_0 \equiv \frac{\partial' \bar{F}_0}{\partial x_i} + \widetilde{V}_{\parallel i}^{(1)} \frac{\partial \bar{F}_0}{\partial v_\parallel} + \widetilde{\mu}_i^{(1)} \frac{\partial \bar{F}_0}{\partial \mu} = \Lambda_{i,\nu}^{(1)} \frac{\partial \bar{F}_0}{\partial z_\nu}, \tag{31}$$

$$\hat{d}_{ij}^{(1)} \bar{F}_0 \equiv \widetilde{V}_{\parallel ij}^{(1)} \frac{\partial \bar{F}_0}{\partial v_\parallel} + \widetilde{\mu}_{ij}^{(1)} \frac{\partial \bar{F}_0}{\partial \mu} = \Lambda_{ij,\nu}^{(1)} \frac{\partial \bar{F}_0}{\partial z_\nu}, \tag{32}$$

where the phase-space functions $\widetilde{V}_{\parallel i}^{(1)}$, $\widetilde{\mu}_i^{(1)}$, $\widetilde{V}_{\parallel ij}^{(1)}$, and $\widetilde{\mu}_{ij}^{(1)}$ are given by

$$\widetilde{V}_{\parallel i}^{(1)} = \frac{Db_i}{Dt} - b_l \frac{\partial v_{El}}{\partial x_i}, \tag{33}$$

$$\widetilde{\mu}_i^{(1)} = -\frac{1}{B}\left[\mu \frac{\partial B}{\partial x_i} + m\left(v_\parallel \frac{Db_i}{Dt} + \frac{Dv_{Ei}}{Dt}\right)\right], \tag{34}$$

$$\widetilde{V}_{\parallel ij}^{(1)} = \frac{1}{4}\left(\frac{\partial b_i}{\partial x_j} + \frac{\partial b_j}{\partial x_i}\right), \tag{35}$$

$$\widetilde{\mu}_{ij}^{(1)} = -\frac{m}{4B}\left[v_\parallel\left(\frac{\partial b_i}{\partial x_j} + \frac{\partial b_j}{\partial x_i}\right) + \frac{\partial v_{Ei}}{\partial x_j} + \frac{\partial v_{Ej}}{\partial x_i}\right], \tag{36}$$

and $\Lambda_{i,\nu}^{(1)}$ and $\Lambda_{ij,\nu}^{(1)}$, are defined by

$$\Lambda_{i,j}^{(1)} = \delta_{ij}, \quad \Lambda_{i,4}^{(1)} = \widetilde{V}_{\parallel i}^{(1)}, \quad \Lambda_{i,5}^{(1)} = \widetilde{\mu}_i^{(1)}, \tag{37}$$

$$\Lambda_{ij,k}^{(1)} = 0, \quad \Lambda_{ij,4}^{(1)} = \widetilde{V}_{\parallel ij}^{(1)}, \quad \Lambda_{ij,5}^{(1)} = \widetilde{\mu}_{ij}^{(1)}. \tag{38}$$

Equating the Larmor phase average of Eq. (23) to zero, we obtain the lowest-order drift-kinetic equation:

$$\frac{\partial \bar{F}_0}{\partial t} + \dot{Z}_\nu^{(0)} \frac{\partial \bar{F}_0}{\partial z_\nu} = 0, \tag{39}$$

where $\dot{Z}_\nu^{(0)}$ is the "guiding-center" phase-space velocity. The phase-space coordinates x_i, $v_\parallel$, and μ are not exact guiding-center coordinates. Nevertheless, it will be convenient to refer to $\dot{X}_i^{(n)}$, $\dot{V}_\parallel^{(n)}$, and $\dot{\mu}_i^{(n)}$ as the nth-order guiding-center velocity, parallel acceleration, and time derivative of the magnetic moment.

Integrating Eq. (23) over the Larmor phase angle, we obtain the first-order Larmor phase-dependent distribution function $\widetilde{F}_1$, oscillatory in the Larmor phase angle due to the finite Larmor radius gyrations of the particles about the magnetic field:

$$\widetilde{F}_1 = -\frac{u_i^*}{\Omega} \hat{d}_i^{(1)} \bar{F}_0 - \frac{u_{ij}^*}{2\Omega} \hat{d}_{ij}^{(1)} \bar{F}_0 = \widetilde{Z}_\nu^{(1)} \frac{\partial \bar{F}_0}{\partial z_\nu}, \tag{40}$$

where $\widetilde{Z}_\nu^{(1)}$ is defined by

$$\widetilde{Z}_\nu^{(1)} = -\frac{u_k^*}{\Omega} \Lambda_{k,\nu}^{(1)} - \frac{u_k^* u_l}{\Omega} \Lambda_{kl,\nu}^{(1)}. \tag{41}$$

Note that the operators $\{\hat{\mathcal{L}}, \hat{d}_i^{(1)}, \hat{d}_{ij}^{(1)}\}$ can be expressed as follows:

$$\hat{\mathcal{L}} = \hat{D}^{(0)} + u_i \hat{d}_i^{(1)} + u_{ij} \hat{d}_{ij}^{(1)} + \left(\frac{e}{m} b_j E_j b_i - \frac{Dv_{Ei}}{Dt} - u_j \frac{\partial v_{Ei}}{\partial x_j}\right) \frac{\partial}{\partial v_i}$$

$$= \hat{D}^{(0)} - \Omega \boldsymbol{v} \times \boldsymbol{b} \cdot \frac{\partial \widetilde{Z}_\nu^{(1)}}{\partial \boldsymbol{v}} \frac{\partial}{\partial z_\nu}$$

$$+ \left(\frac{e}{m} b_j E_j b_i - \frac{Dv_{Ei}}{Dt} - u_j \frac{\partial v_{Ei}}{\partial x_j}\right) \frac{\partial}{\partial v_i}, \tag{42}$$

$$\hat{d}_i^{(1)} = \Lambda_{i,\nu}^{(1)} \frac{\partial}{\partial z_\nu}, \quad \hat{d}_{ij}^{(1)} = \Lambda_{ij,\nu}^{(1)} \frac{\partial}{\partial z_\nu}. \tag{43}$$

Note also that $(\partial/\partial x_i) B\dot{X}_i^{(0)} = B(\partial v_{Ei}/\partial x_i) + v_{Ei}(\partial B/\partial x_i)$, $(\partial/\partial v_\parallel) B\dot{V}_\parallel^{(0)} = -Bb_i b_j(\partial v_{Ei}/\partial x_j)$, $(\partial/\partial\mu) B\dot{\mu}_i^{(0)} = -\partial B/\partial t - v_{Ei}(\partial B/\partial x_i) - BI_{ij}(\partial v_{Ei}/\partial x_j)$, and therefore

$$\frac{1}{B}\frac{\partial}{\partial t} B + \frac{1}{B}\frac{\partial}{\partial z_\nu} B\dot{Z}_\nu^{(0)} = 0.$$

Thus Eq. (39) can also be written in Liouville form:

$$\frac{1}{B}\frac{\partial}{\partial t} B\bar{F}_0 + \frac{1}{B}\frac{\partial}{\partial z_\nu} B\dot{Z}_\nu^{(0)}\bar{F}_0 = 0. \tag{44}$$

The element of volume in phase space is

$$d^3r d^3v = 2\pi\, dx_1\, dx_2\, dx_3\, B\, dv_\parallel\, d\mu,$$

and the zeroth-order drift-kinetic equation conserves not only the total number of particles but also the phase-space volume element.

B. First-order equation

The first-order equation is

$$\Omega \boldsymbol{v} \times \boldsymbol{b} \cdot \frac{\partial \tilde{F}_2}{\partial \boldsymbol{v}} + \hat{\mathcal{L}}(\bar{F}_1 + \tilde{F}_1) = 0, \tag{45}$$

where

$$\hat{\mathcal{L}}\bar{F}_1 = \hat{D}^{(0)}\bar{F}_1 - \Omega \boldsymbol{v} \times \boldsymbol{b} \cdot \frac{\partial \tilde{Z}_\nu^{(1)}}{\partial \boldsymbol{v}} \frac{\partial \bar{F}_1}{\partial z_\nu}.$$

Substituting Eq. (40) for $\tilde{F}_1$, we have

$$\hat{\mathcal{L}}\tilde{F}_1 = -\hat{\mathcal{L}}\left(\frac{u_k^*}{\Omega}\hat{d}_k^{(1)}\bar{F}_0 + \frac{u_{kl}^*}{2\Omega}\hat{d}_{kl}^{(1)}\bar{F}_0 \right).$$

In Appendix B, we evaluated terms of the form $-\hat{\mathcal{L}}(u_k^*/\Omega)\chi_k(x_j, v_\parallel, \mu, t)$ and $-\hat{\mathcal{L}}(u_{kl}^*/2\Omega)\chi_{kl}(x_j, v_\parallel, \mu, t)$. Substituting $\chi_k = \hat{d}_k^{(1)}\bar{F}_0 = \Lambda_{k,\nu}^{(1)}(\partial\bar{F}_0/\partial z_\nu)$ in Eq. (B3) and $\chi_{kl} = \hat{d}_{kl}^{(1)}\bar{F}_0 = \Lambda_{kl,\nu}^{(1)}(\partial\bar{F}_0/\partial z_\nu)$ in Eq. (B8), we obtain [Eq. (C1) of Appendix C] the following polynomial in the perpendicular velocity tensors u_i and u_i^*:

$$\hat{\mathcal{L}}\tilde{F}_1 = \hat{D}^{(1)}\bar{F}_0 + u_j\hat{d}_j^{(2)}\bar{F}_0 + u_{ij}\hat{d}_{ij}^{(2)}\bar{F}_0 + u_{ijk}\hat{d}_{ij}^{(1)}\frac{\epsilon_{kpq}b_p}{2\Omega}\hat{d}_q^{(1)}\bar{F}_0 + u_{ikl}\hat{d}_i^{(1)}\frac{\epsilon_{kpq}b_p I_{lr}}{4\Omega}\hat{d}_{qr}^{(1)}\bar{F}_0 - \frac{u_{ijkl}}{4\Omega}\hat{d}_{ij}^{(1)}\hat{d}_{kl}^{(1)}\bar{F}_0, \tag{46}$$

where u_{ijk} is a third-rank perpendicular velocity tensor equal to a sum of terms proportional to either $\sin 3\Theta$ or $\cos 3\Theta$, and u_{ijkl} is a fourth-rank perpendicular velocity tensor equal to a sum of terms proportional to either $\sin 4\Theta$ or $\cos 4\Theta$.

The operators $\hat{D}^{(1)}$, $\hat{d}_i^{(2)}$, and $\hat{d}_{ij}^{(2)}$ are defined in Appendix C by Eqs. (C2), (C3), and (C4), respectively.

The first-order drift-kinetic operator $\hat{D}^{(1)}$ is

$$\begin{aligned}\hat{D}^{(1)}\bar{F}_0 &\equiv -\overline{\hat{\mathcal{L}}\frac{u_k^*}{\Omega}\hat{d}_k^{(1)}\bar{F}_0} - \overline{\hat{\mathcal{L}}\frac{u_{kl}^*}{2\Omega}\hat{d}_{kl}^{(1)}\bar{F}_0} \\ &= \frac{1}{B}\frac{\partial}{\partial z_\sigma}\frac{\mu B^2}{m\Omega}\Lambda_{q,\sigma}^{(1)}\epsilon_{qpk}b_p\hat{d}_k^{(1)}\bar{F}_0 \\ &\quad + \frac{1}{B}\frac{\partial}{\partial z_\sigma}\frac{2\mu^2 B^3}{m^2\Omega}\Lambda_{ij,\sigma}^{(1)}\epsilon_{ipk}b_p I_{jl}\hat{d}_{kl}^{(1)}\bar{F}_0 \\ &= \frac{1}{B}\frac{\partial}{\partial z_\sigma}B\Gamma_{\sigma,\nu}^{(1)}\frac{\partial\bar{F}_0}{\partial z_\nu},\end{aligned} \tag{47}$$

where the phase-space function $\Gamma_{\tau,\nu}^{(1)}$ is given by

$$\Gamma_{\sigma,\nu}^{(1)} = \frac{\mu B}{m\Omega}\Lambda_{q,\sigma}^{(1)}\epsilon_{qpk}b_p\Lambda_{k,\nu}^{(1)} + \frac{2\mu^2 B^2}{m^2\Omega}\Lambda_{ij,\sigma}^{(1)}\epsilon_{ipk}b_p I_{jl}\Lambda_{kl,\nu}^{(1)}. \tag{48}$$

Note that $\Gamma_{\sigma,\nu}^{(1)} = -\Gamma_{\nu,\sigma}^{(1)}$ is antisymmetric in an interchange of indices, and that $(\partial/\partial z_\sigma)(\partial/\partial z_\nu)B\Gamma_{\sigma,\nu}^{(1)} = 0$. We can therefore express $\hat{D}^{(1)}\bar{F}_0$ as follows:

$$\begin{aligned}\hat{D}^{(1)}\bar{F}_0 &= \frac{1}{B}\frac{\partial}{\partial z_\nu}B\bar{F}_0\frac{1}{B}\frac{\partial}{\partial z_\sigma}B\Gamma_{\sigma,\nu}^{(1)} \\ &\equiv \frac{1}{B}\frac{\partial}{\partial z_\nu}B\dot{Z}_\nu^{(1)}\bar{F}_0 = \dot{Z}_\nu^{(1)}\frac{\partial\bar{F}_0}{\partial z_\nu} \\ &= \dot{X}_i^{(1)}\frac{\partial\bar{F}_0}{\partial x_i} + \dot{V}_\parallel^{(1)}\frac{\partial\bar{F}_0}{\partial v_\parallel} + \dot{\mu}^{(1)}\frac{\partial\bar{F}_0}{\partial\mu},\end{aligned} \tag{49}$$

where $\dot{Z}_\nu^{(1)}$, which represents the first-order correction to the guiding-center motion, is given by

$$\dot{Z}_\nu^{(1)} = \frac{1}{B}\frac{\partial}{\partial z_\sigma}B\Gamma_{\sigma,\nu}^{(1)}. \tag{50}$$

The first-order guiding-center velocity $\dot{X}_i^{(1)}$, parallel acceleration $\dot{V}_\parallel^{(1)}$, and time derivative of the magnetic moment $\dot{\mu}^{(1)}$ are given by

$$\begin{aligned}\dot{X}_i^{(1)} = \dot{Z}_i^{(1)} &= \frac{1}{B}\frac{\partial}{\partial z_\sigma}\frac{\mu B^2}{m\Omega}\Lambda_{q,\sigma}^{(1)}\epsilon_{qpi}b_p \\ &= v_{Di} + \frac{\mu B}{m\Omega}(\boldsymbol{b}\cdot\nabla\times\boldsymbol{b})b_i,\end{aligned} \tag{51}$$

$$\begin{aligned}\dot{V}_\parallel^{(1)} = \dot{Z}_4^{(1)} &= \frac{1}{B}\frac{\partial}{\partial z_\sigma}\frac{\mu B^2}{m\Omega}\Lambda_{q,\sigma}^{(1)}\epsilon_{qpk}b_p\tilde{V}_{\parallel k}^{(1)} \\ &\quad + \frac{1}{B}\frac{\partial}{\partial z_\sigma}\frac{2\mu^2 B^3}{m^2\Omega}\Lambda_{rs,\sigma}^{(1)}\epsilon_{rpk}b_p I_{sl}\tilde{V}_{\parallel kl}^{(1)} \\ &= v_{Di}\tilde{V}_{\parallel i}^{(1)} + \frac{\mu B}{m\Omega}(\boldsymbol{b}\cdot\nabla\times\boldsymbol{b})b_i\tilde{V}_{\parallel i}^{(1)} \\ &\quad - \frac{\mu B\epsilon_{kpq}b_p}{m\Omega}\left(\frac{\partial'\tilde{V}_{\parallel k}^{(1)}}{\partial x_q} + \kappa_k\tilde{V}_{\parallel q}^{(1)}\right) + \frac{\mu B\epsilon_{kpq}b_p}{4m\Omega} \\ &\quad \times\left(-\kappa_k b_l + \frac{\partial b_l}{\partial x_k} + \frac{\partial b_k}{\partial x_l}\right)\left(\frac{\partial v_{El}}{\partial x_q} + \frac{\partial v_{Eq}}{\partial x_l}\right),\end{aligned} \tag{52}$$

$$\dot{\mu}^{(1)} = \dot{Z}_5^{(1)} = \frac{1}{B}\frac{\partial}{\partial z_\sigma}\frac{\mu B^2}{m\Omega}\Lambda_{q,\sigma}^{(1)}\epsilon_{qpk}b_p\tilde{\mu}_k^{(1)} + \frac{1}{B}\frac{\partial}{\partial z_\sigma}\frac{2\mu^2 B^3}{m^2\Omega}\Lambda_{rs,\sigma}^{(1)}\epsilon_{rpk}b_p I_{sl}\tilde{\mu}_{kl}^{(1)} = -\mu\left(\frac{\partial' v_{Di}}{\partial x_i} + \kappa_i v_{Di} + V_{\parallel i}^{(1)}\frac{\partial v_{Di}}{\partial v_\parallel} + \frac{v_{Di}}{B}\frac{\partial B}{\partial x_i}\right), \tag{53}$$

where v_{Dk} is the drift velocity equal to the vector sum of the magnetic-field gradient drift and the inertial drifts:

$$v_{Di} = -\frac{B}{m\Omega}\epsilon_{ipq}b_p\tilde{\mu}_q^{(1)} = \frac{\epsilon_{ipq}b_p}{m\Omega}\left[\mu\frac{\partial B}{\partial x_q} + m\left(v_\parallel\frac{Db_q}{Dt} + \frac{Dv_{Eq}}{Dt}\right)\right]. \tag{54}$$

Similar expressions for the first-order phase-space "velocity" $\dot{Z}_\nu^{(1)}$ have also been derived by Hazeltine and Hinton,[10] and the above expressions reduce to those derived by Sivukhin[11] in the limit where the electric drift velocity $\boldsymbol{v}_E$ is first order (rather than zero order) in $\boldsymbol{\epsilon}$.

Averaging Eq. (45) over the Larmor phase angle, we obtain the first-order drift-kinetic equation:

$$\frac{1}{B}\frac{\partial}{\partial t}B\bar{F}_1 + \frac{1}{B}\frac{\partial}{\partial z_\nu}B(\dot{Z}_\nu^{(0)}\bar{F}_1 + \dot{Z}_\nu^{(1)}\bar{F}_0) = \frac{\partial}{\partial t}\bar{F}_1 + \dot{Z}_\nu^{(0)}\frac{\partial\bar{F}_1}{\partial z_\nu} + \dot{Z}_\nu^{(1)}\frac{\partial F_0}{\partial z_\nu} = 0. \tag{55}$$

Integrating Eq. (45) over the Larmor phase angle, we obtain the second-order phase-dependent distribution function (see Appendix A):

$$\tilde{F}_2 = \tilde{Z}_\nu^{(1)}\frac{\partial\bar{F}_1}{\partial z_\nu} - \frac{\overset{*}{u}_j}{\Omega}\hat{d}_j^{(2)}\bar{F}_0 - \frac{\overset{*}{u}_{ij}}{2\Omega}\hat{d}_{ij}^{(2)}\bar{F}_0 - \frac{\overset{*}{u}_{ijk}}{3\Omega}\hat{d}_{ij}^{(1)}\frac{\epsilon_{kpq}b_p}{2\Omega}\hat{d}_q^{(1)}\bar{F}_0 - \frac{\overset{*}{u}_{ikl}}{3\Omega}\hat{d}_i^{(1)}\frac{\epsilon_{kpq}b_p I_{lr}}{4\Omega}\hat{d}_{qr}^{(1)}\bar{F}_0 + \frac{\overset{*}{u}_{ijkl}}{16\Omega}\hat{d}_{ij}^{(1)}\hat{d}_{kl}^{(1)}\bar{F}_0. \tag{56}$$

An alternative expression for $\tilde{F}_2$ is obtained by evaluating $\hat{\mathcal{L}}\tilde{F}_1 = \hat{\mathcal{L}}\tilde{Z}_\nu^{(1)}(\partial\bar{F}_0/\partial z_\nu)$ as follows:

$$\hat{\mathcal{L}}\tilde{F}_1 = \frac{\partial\bar{F}_0}{\partial z_\nu}\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)} + \tilde{Z}_\nu^{(1)}\left(\hat{D}^{(0)} - \Omega\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial\tilde{Z}_\sigma^{(1)}}{\partial\boldsymbol{v}}\frac{\partial}{\partial z_\sigma}\right)\frac{\partial\bar{F}_0}{\partial z_\nu} = \frac{\partial\bar{F}_0}{\partial z_\nu}\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)} + \tilde{Z}_\nu^{(1)}\left(\hat{D}^{(0)}\frac{\partial\bar{F}_0}{\partial z_\nu}\right) - \frac{1}{2}\frac{\partial^2\bar{F}_0}{\partial z_\sigma\partial z_\nu}\Omega\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{v}}(\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)} - \overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}).$$

Substituting this expression for $\hat{\mathcal{L}}\tilde{F}_1$ in Eq. (45) and integrating over the Larmor phase angle, we obtain the following alternative equivalent expression for $\tilde{F}_2$:

$$\tilde{F}_2 = \tilde{Z}_\nu^{(1)}\frac{\partial\bar{F}_1}{\partial z_\nu} + \frac{\partial\bar{F}_0}{\partial z_\nu}\int^\Theta\frac{d\Theta}{\Omega}(\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)} - \overline{\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)}}) + \left(\hat{D}^{(0)}\frac{\partial\bar{F}_0}{\partial z_\nu}\right)\int^\Theta\frac{d\Theta}{\Omega}\tilde{Z}_\nu^{(1)} + \frac{1}{2}\frac{\partial^2\bar{F}_0}{\partial z_\sigma\partial z_\nu}(\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)} - \overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}). \tag{57}$$

In Appendix C we verify that Eqs. (56) and (57) are identical.

C. Second-order equation

The second-order equation is

$$\Omega\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial\tilde{F}_3}{\partial\boldsymbol{v}} + \hat{\mathcal{L}}(\bar{F}_2 + \tilde{F}_2) = 0. \tag{58}$$

We will not require an explicit solution for $\tilde{F}_3$. We will evaluate only the Larmor phase average of $\hat{\mathcal{L}}(\bar{F}_2 + \tilde{F}_2)$ and thereby derive the second-order drift-kinetic equation.

Substituting Eq. (56) for $\tilde{F}_2$, we have

$$\overline{\hat{\mathcal{L}}\tilde{F}_2} = \oint\frac{d\Theta}{2\pi}\hat{\mathcal{L}}\tilde{F}_2 = \hat{D}^{(1)}\bar{F}_1 + \hat{D}^{(2)}\bar{F}_0, \tag{59}$$

where the second-order drift-kinetic operator $\hat{D}^{(2)}$ is defined by [compare with Eq. (47)]

$$\hat{D}^{(2)}\bar{F}_0 \equiv -\overline{\hat{\mathcal{L}}\frac{\overset{*}{u}_k}{\Omega}}\hat{d}_k^{(2)}\bar{F}_0 - \overline{\hat{\mathcal{L}}\frac{\overset{*}{u}_{kl}}{2\Omega}}\hat{d}_{kl}^{(2)}\bar{F}_0 = \frac{1}{B}\frac{\partial}{\partial z_\sigma}\frac{\mu B^2}{m\Omega}\Lambda_{q,\sigma}^{(1)}\epsilon_{qpk}b_p\hat{d}_k^{(2)}\bar{F}_0 + \frac{1}{B}\frac{\partial}{\partial z_\sigma}\frac{2\mu^2 B^3}{m^2\Omega}\Lambda_{ij,\sigma}^{(1)}\epsilon_{ipk}b_p I_{jl}\hat{d}_{kl}^{(2)}\bar{F}_0. \tag{60}$$

Note that the Larmor phase average of terms involving $\overset{*}{u}_{ijk}$ and $\overset{*}{u}_{ijkl}$ are equal to zero (see Appendix B).

We can express $\hat{\mathcal{D}}^{(2)}\bar{F}_0$ in terms of the functions $\Lambda_{p,\sigma}^{(1)}$ and $\Lambda_{rs,\sigma}^{(1)}$ by substituting for $\hat{d}_k^{(2)}\bar{F}_0$ and $\hat{d}_{kl}^{(2)}\bar{F}_0$, given by Eqs. (C3) and (C4), respectively. However, this is more readily achieved by evaluating $\overline{\hat{\mathcal{L}}\tilde{F}_2}$ using Eq. (57) for $\tilde{F}_2$:

$$\hat{D}^{(2)}\bar{F}_0 = \frac{\partial\bar{F}_0}{\partial z_\nu}\oint\frac{d\Theta}{2\pi}\hat{\mathcal{L}}\int^\Theta\frac{d\Theta}{\Omega}(\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)} - \overline{\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)}}) + \left(\hat{D}^{(0)}\frac{\partial\bar{F}_0}{\partial z_\nu}\right)\oint\frac{d\Theta}{2\pi}\hat{\mathcal{L}}\int^\Theta\frac{d\Theta}{\Omega}\tilde{Z}_\nu^{(1)} - \overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}} \times\frac{\partial}{\partial z_\sigma}\left(\hat{D}^{(0)}\frac{\partial\bar{F}_0}{\partial z_\nu}\right) - \frac{1}{2}\frac{\partial^2\bar{F}_0}{\partial z_\sigma\partial z_\nu}\hat{D}^{(0)}(\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}), \tag{61}$$

where $\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}$ is given by

$$\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}} = \frac{\mu B}{m\Omega^2}I_{ik}\Lambda_{i,\sigma}^{(1)}\Lambda_{k,\nu}^{(1)} + \frac{\mu^2 B^2}{2m^2\Omega^2}(I_{ik}I_{jl} + I_{il}I_{jk} - I_{ij}I_{kl})\Lambda_{ij,\sigma}^{(1)}\Lambda_{kl,\nu}^{(1)}. \tag{62}$$

In Appendix E we evaluate the Larmor phase averages in Eq. (61), and we obtain $\hat{\mathcal{D}}^{(2)}\bar{F}_0$ in the following form:

$$\hat{D}^{(2)}\bar{F}_0 = -\hat{D}^{(0)}\frac{1}{B}\frac{\partial}{\partial z_\sigma}B\frac{\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}}{2}\frac{\partial \bar{F}_0}{\partial z_\nu} + \frac{1}{B}\frac{\partial}{\partial z_\nu}B\dot{Z}_\nu^{(2)}\bar{F}_0$$
$$= -\hat{D}^{(0)}\frac{1}{B}\frac{\partial}{\partial z_\sigma}B\frac{\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}}{2}\frac{\partial \bar{F}_0}{\partial z_\nu} + \dot{Z}_\nu^{(2)}\frac{\partial \bar{F}_0}{\partial z_\nu}, \tag{63}$$

where $\dot{Z}_\nu^{(2)}$, which represents the second-order corrections to the guiding-center motion, is given by

$$\dot{Z}_\nu^{(2)} = \frac{1}{B}\frac{\partial}{\partial z_\sigma}B\Gamma_{\sigma,\nu}^{(2)}, \tag{64}$$

and the phase-space function function $\Gamma_{\sigma,\nu}^{(2)}$ is a sum of two terms:

$$\Gamma_{\sigma,\nu}^{(2)} = \Delta_{\sigma,\nu} + \Upsilon_{\sigma,\nu}, \tag{65}$$

with $\Delta_{\sigma,\nu}$ and $\Upsilon_{\sigma,\nu}$, given by Eqs. (E8) and (E4), respectively, having the following properties:

$$\frac{\partial}{\partial z_\sigma}\frac{\partial}{\partial z_\nu}B\Delta_{\sigma,\nu} = 0, \quad \frac{\partial}{\partial z_\sigma}\frac{\partial}{\partial z_\nu}B\Upsilon_{\sigma,\nu} = 0. \tag{66}$$

The second-order guiding-center velocity $\dot{X}_i^{(2)}$, parallel acceleration $\dot{V}_\parallel^{(2)}$, and time derivative of the magnetic moment $\dot{\mu}^{(2)}$ are given by

$$\dot{X}_i^{(2)} = \dot{Z}_i^{(2)} = \frac{1}{B}\frac{\partial}{\partial z_\sigma}B(\Delta_{\sigma,i} + \Upsilon_{\sigma,i}), \tag{67}$$

$$\dot{V}_\parallel^{(2)} = \dot{Z}_4^{(2)} = \frac{1}{B}\frac{\partial}{\partial z_\sigma}B(\Delta_{\sigma,4} + \Upsilon_{\sigma,4}), \tag{68}$$

$$\dot{\mu}^{(2)} = \dot{Z}_5^{(2)} = \frac{1}{B}\frac{\partial}{\partial z_\sigma}B(\Delta_{\sigma,5} + \Upsilon_{\sigma,5}). \tag{69}$$

Substituting $\Lambda_{k,i}^{(1)} = \delta_{ki}$ and $\Lambda_{kl,i}^{(1)} = 0$ in Eqs. (E8) and (E4), we obtain

$$\Delta_{\sigma,i} = \frac{\overline{\tilde{Z}_\sigma^{(0)}\tilde{Z}_\tau^{(0)}}}{2}\frac{\partial \dot{X}_i^{(0)}}{\partial z_\tau} - I_{iq}\frac{\mu B}{2m\Omega^2}\Lambda_{q,\tau}^{(1)}\frac{\partial \dot{Z}_\sigma}{\partial z_\tau}$$
$$+ \frac{\mu B}{2m\Omega^2}(\epsilon_{qti}b_t\hat{\mathcal{D}}^{(0)}\Lambda_{q,\sigma}^{(1)*} - \Lambda_{q,\sigma}^{(1)*}\hat{\mathcal{D}}^{(0)}\epsilon_{qti}b_t),$$
$$\tag{70}$$
$$\Upsilon_{\sigma,i} = -(2I_{ps}I_{ir} - I_{pi}I_{rs})\frac{\mu B}{2m\Omega^2}\Lambda_{p,\sigma}^{(1)}\left(v_\parallel\frac{\partial b_s}{\partial x_r} + \frac{\partial v_{Es}}{\partial x_r}\right)$$
$$- I_{pq,rs}\frac{\mu^2 B^2}{2m\Omega}\Lambda_{rs,\sigma}^{(1)}\left(I_{ip}\frac{\partial}{\partial x_q}\frac{1}{m\Omega} - \frac{b_i}{m\Omega}\frac{\partial b_p}{\partial x_q}\right).$$

Expressions for $\Delta_{\sigma,4}, \Upsilon_{\sigma,4}$ and $\Delta_{\sigma,5}, \Upsilon_{\sigma,5}$ can similarly be obtained by substituting $\Lambda_{k,4}^{(1)} = \tilde{V}_{\parallel k}^{(1)}$, $\Lambda_{kl,4}^{(1)} = \tilde{V}_{\parallel kl}^{(1)}$, and $\Lambda_{k,5}^{(1)} = \tilde{\mu}_k^{(1)}$, $\Lambda_{kl,5}^{(1)} = \tilde{\mu}_{kl}^{(1)}$, respectively, in Eqs. (E8) and (E4).

The second-order corrections are separable into two parts, determined by the functions $\Delta_{\sigma,\nu}$ and $\Upsilon_{\sigma,\nu}$. The function $\Delta_{\sigma,\nu}$, involving the zeroth-order drift-kinetic operator $\hat{\mathcal{D}}^{(0)}$, describes the finite-Larmor-radius modifications of the zeroth-order guiding-center phase-space trajectory and the coupling of the zeroth-order trajectory with the first-order Larmor phase-dependent oscillations, while the function $\Upsilon_{\sigma,\nu}$ describes modifications which do not involve the operator $\hat{\mathcal{D}}^{(0)}$.

Explicit expressions for $\dot{X}_i^{(2)}$, $\dot{V}_\parallel^{(2)}$, and $\dot{\mu}^{(2)}$ can readily be obtained by substituting Eqs. (E8) and (E4) for $\Delta_{\sigma,\nu}$ and $\Upsilon_{\sigma,\nu}$, respectively. We will not, however, attempt to discuss second-order effects in general. Second-order effects are formally small, of order ϵ^2, and there are many terms to be considered. It is therefore appropriate to postpone a more detailed analysis of second-order effects to those special circumstances where second-order effects become significant and the relevant second-order terms can be identified. Such circumstances can arise, for example, due to subsidiary orderings which render lower-order effects small, or due to plasma "singularities" which result in the presence of structures with relatively large spatial gradients.

In the limit of low-frequency perturbations with long parallel wavelengths, but short perpendicular wavelengths, finite-Larmor-radius modifications of the zeroth-order phase-space trajectories can become significant. Assuming negligible second-order compressional effects, we ignore the contributions of $\Upsilon_{\sigma,\nu}$, and we retain in $\Delta_{\sigma,\nu}$ only those terms which involve the perpendicular spatial derivative of $\dot{X}_i^{(0)}$, $\dot{V}_\parallel^{(0)}$, and $\dot{\mu}^{(0)}$. We may then approximate $\Gamma_{\sigma,i}^{(2)}$ as follows:

$$\Gamma_{\sigma,\nu}^{(2)} = \Delta_{\sigma,\nu} + \cdots = \frac{\mu B}{2m\Omega^2}\left(I_{pq}\Lambda_{p,\sigma}^{(1)}\frac{\partial \dot{Z}_\nu^{(0)}}{\partial x_q} - I_{pq}\Lambda_{p,\nu}^{(1)}\frac{\partial \dot{Z}_\sigma^{(0)}}{\partial x_q}\right)$$
$$+ \cdots \tag{71}$$

and the second-order phase-space velocity $\{\dot{X}_i^{(2)}, \dot{V}_\parallel^{(2)}, \dot{\mu}^{(2)}\}$ can be approximated by

$$\dot{X}_i^{(2)} = \frac{1}{B}\frac{\partial}{\partial x_p}I_{pq}\frac{\mu B^2}{2m\Omega^2}\frac{\partial \dot{X}_i^{(0)}}{\partial x_q} - \frac{1}{B}\frac{\partial}{\partial x_p}I_{iq}\frac{\mu B^2}{2m\Omega^2}\frac{\partial \dot{X}_p^{(0)}}{\partial x_q}$$
$$- \frac{1}{B}\frac{\partial}{\partial v_\parallel}I_{iq}\frac{\mu B^2}{2m\Omega^2}\frac{\partial \dot{V}_\parallel^{(0)}}{\partial x_q} - \frac{1}{B}\frac{\partial}{\partial \mu}I_{iq}\frac{\mu B^2}{2m\Omega^2}\frac{\partial \dot{\mu}^{(0)}}{\partial x_q}\cdots, \tag{72}$$

$$\dot{V}_\parallel^{(2)} = \frac{1}{B}\frac{\partial}{\partial x_p}I_{pq}\frac{\mu B^2}{2m\Omega^2}\frac{\partial \dot{V}_\parallel^{(0)}}{\partial x_q} + \cdots, \tag{73}$$

$$\dot{\mu}^{(2)} = \frac{1}{B}\frac{\partial}{\partial x_p}I_{pq}\frac{\mu B^2}{2m\Omega^2}\frac{\partial \dot{\mu}^{(0)}}{\partial x_q} + \cdots. \tag{74}$$

Equations (72)–(74) describe the finite-Larmor-radius modifications of the zeroth-order phase-space trajectories, and it is these modifications that are typically included in the gyrokinetic equations.

Finally, averaging Eq. (58) over the Larmor phase angle, and introducing a modified Larmor phase averaged distribution function $\bar{F}_2'$, where $\bar{F}_2$ is related to $\bar{F}_2'$ by

$$\bar{F}_2 = \bar{F}_2' + \Delta\bar{F}_0, \tag{75}$$

$$\Delta\bar{F}_0 \equiv \frac{1}{B}\frac{\partial}{\partial z_\sigma}B\frac{\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}}{2}\frac{\partial \bar{F}_0}{\partial z_\nu}, \tag{76}$$

we obtain the following second-order drift-kinetic equation for $\bar{F}_2'$:

$$\frac{1}{B}\frac{\partial}{\partial t}B\bar{F}_2' + \frac{1}{B}\frac{\partial}{\partial z_\nu}B(\dot{Z}_\nu^{(0)}\bar{F}_2' + \dot{Z}_\nu^{(1)}\bar{F}_1 + \dot{Z}_\nu^{(2)}\bar{F}_0)$$
$$= \frac{\partial \bar{F}_2'}{\partial t} + \dot{Z}_\nu^{(0)}\frac{\partial \bar{F}_2'}{\partial z_\nu} + \dot{Z}_\nu^{(1)}\frac{\partial \bar{F}_1}{\partial z_\nu} + \dot{Z}_\nu^{(2)}\frac{\partial \bar{F}_0}{\partial z_\nu} = 0. \tag{77}$$

Note that the phase-space variables $\{x_i, v_\parallel, \mu\}$ are not exact guiding-center variables, and the second-order correction $\Delta\bar{F}_0$ is a consequence of this fact.

In summary, let $\bar{G}$ denote the Larmor phase averaged distribution function defined by

$$\bar{G} = \bar{F}_0 + \epsilon\bar{F}_1 + \epsilon^2\bar{F}_2 - \epsilon^2\Delta\bar{F}_0 + \cdots. \tag{78}$$

The Vlasov distribution function F, up to second order in the Larmor radius, is given by

$$F = \bar{G} + \epsilon\tilde{G}_1 + \epsilon^2\tilde{G}_2 + \epsilon^2\Delta\bar{G}, \tag{79}$$

where

$$\tilde{G}_1 = -\left(\frac{u_j^*}{\Omega}\hat{d}_j^{(1)}\bar{G} + \frac{u_{ij}^*}{2\Omega}\hat{d}_{ij}^{(1)}\bar{G}\right) = \tilde{Z}_\nu^{(1)}\frac{\partial \bar{G}}{\partial z_\nu}, \tag{80}$$

$$\tilde{G}_2 = -\left(\frac{u_j^*}{\Omega}\hat{d}_j^{(2)}\bar{G} + \frac{u_{ij}^*}{2\Omega}\hat{d}_{ij}^{(2)}\bar{G}\right) - \frac{u_{ijk}^*}{3\Omega}\hat{d}_{ij}^{(1)}\frac{\epsilon_{kpq}b_p}{2\Omega}\hat{d}_q^{(1)}\bar{G}$$
$$- \frac{u_{ikl}^*}{3\Omega}\hat{d}_i^{(1)}\frac{I_{lr}\epsilon_{kpq}b_p}{4\Omega}\hat{d}_{qr}^{(1)}\bar{G} + \frac{u_{ijkl}^*}{16\Omega}\hat{d}_{ij}^{(1)}\hat{d}_{kl}^{(1)}\bar{G}, \tag{81}$$

and the second-order Larmor phase-independent correction $\Delta\bar{G}$ is given by

$$\Delta\bar{G} \equiv \frac{1}{B}\frac{\partial}{\partial z_\sigma}B\frac{\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}}{2}\frac{\partial \bar{G}}{\partial z_\nu}. \tag{82}$$

The equation which determines $\bar{G}$, up to second order in ϵ, is expressed compactly as follows:

$$\frac{1}{B}\frac{\partial}{\partial t}B\bar{G} + \frac{1}{B}\frac{\partial}{\partial z_\nu}B\dot{Z}_\nu\bar{G} = \frac{\partial \bar{G}}{\partial t} + \dot{Z}_\nu\frac{\partial \bar{G}}{\partial z_\nu} = 0, \tag{83}$$

with

$$\dot{Z}_\nu = \dot{Z}_\nu^{(0)} + \epsilon\dot{Z}_\nu^{(1)} + \epsilon^2\dot{Z}_\nu^{(2)} + \cdots. \tag{84}$$

Note that $\tilde{G}_1$ and $\tilde{G}_2$ satisfy the following equations, analogous to Eqs. (23) and (45):

$$\Omega\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{v}}\tilde{G}_1 = \hat{\mathcal{L}}\bar{G} - \overline{\hat{\mathcal{L}}\bar{G}}, \tag{85}$$

$$\Omega\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{v}}\tilde{G}_2 = \hat{\mathcal{L}}\tilde{G}_1 - \overline{\hat{\mathcal{L}}\tilde{G}_1}. \tag{86}$$

III. MEAN PERPENDICULAR VELOCITY AND PRESSURE TENSOR

The particle density $\mathcal{N}(\boldsymbol{r},t)$ and mean particle velocity $\mathcal{V}_i(\boldsymbol{r},t)$, in terms of the particle distribution function $F(\boldsymbol{r},\boldsymbol{v},t)$, are given by

$$\mathcal{N} = \int d^3v\, F(\boldsymbol{r},\boldsymbol{v},t) = N + \epsilon^2\int d^3v\,\Delta\bar{G}, \tag{87}$$

$$\mathcal{N}(\boldsymbol{r},t)\mathcal{V}_i(\boldsymbol{r},t) = \int d^3v\, F(\boldsymbol{r},\boldsymbol{v},t)(v_i + v_{Ei})$$
$$= N(V_\parallel b_i + v_{Ei} + \epsilon U_i) + \epsilon^2$$
$$\times \int d^3v(v_\parallel b_i + v_{Ei})\Delta\bar{G}, \tag{88}$$

where we substituted Eq. (79) for F, and $N, V_\parallel, U_i$ are defined in terms of moments of $\bar{G}(v_\parallel,\mu,\boldsymbol{r},t)$ by

$$N = \int d^3v\,\bar{G}, \tag{89}$$

$$NV_\parallel = \int d^3v\,\bar{G}v_\parallel, \tag{90}$$

$$NU_i = \int d^3v\, u_i(\tilde{G}_1 + \epsilon\tilde{G}_2 + \cdots) \equiv U_i^{(1)} + \epsilon U_i^{(2)} + \cdots. \tag{91}$$

Let $\mathcal{P}_{ij}(\boldsymbol{r},t)$ denote the momentum flow tensor, given in terms of moments of the particle distribution function $F(\boldsymbol{r},\boldsymbol{v},t)$ by

$$\mathcal{P}_{ij} = m\int d^3v\, F(v_i + v_{Ei})(v_j + v_{Ej}). \tag{92}$$

Substituting Eq. (79) for F, we express $\mathcal{P}_{ij}$ as follows:

$$\mathcal{P}_{ij} = m\int d^3v\, F(v_i + v_{Ei})(v_j + v_{Ej})$$
$$= mN(V_i + v_{Ei})(V_j + v_{Ej}) + P_{ij},$$

where V_i denotes the vector sum of the parallel velocity component $V_\parallel b_i$ and the first-order perpendicular velocity U_i,

$$V_i = V_\parallel b_i + \epsilon U_i, \tag{93}$$

and P_{ij} is the pressure tensor, expressed as a power series in ϵ by

$$P_{ij} \equiv P_{ij}^{(0)} + \epsilon P_{ij}^{(1)} + \epsilon^2 P_{ij}^{(2)} + \cdots, \tag{94}$$

with

$$P_{ij}^{(0)} = P_\perp I_{ij} + P_\parallel b_i b_j, \tag{95}$$

$$P_{ij}^{(1)} = m\int d^3v\left(u_i\delta v_\parallel b_j + u_j\delta v_\parallel b_i + \frac{u_{ij}}{2}\right)\tilde{G}_1, \tag{96}$$

$$P_{ij}^{(2)} = m\int d^3v\left(u_i\delta v_\| b_j + u_j\delta v_\| b_i + \frac{u_{ij}}{2}\right)\widetilde{G}_2 - mNU_iU_j + \int d^3v\,\Delta\bar{G}[\mu B I_{ij} + m(v_\| b_i + v_{Ei})(v_\| b_j + v_{Ej})], \tag{97}$$

$$\delta v_\| \equiv v_\| - V_\|. \tag{98}$$

The perpendicular and parallel pressures $P_\perp$ and $P_\|$ are defined by

$$P_\perp = \int d^3v\,\bar{G}\,\mu B, \quad P_\| \int d^3v\,\bar{G}\,m(\delta v_\|)^2. \tag{99}$$

A. Mean perpendicular velocity: $U_i = U_i^{(1)} + \epsilon U_i^{(2)}$

Substituting the Larmor phase-dependent distribution functions $\widetilde{G}_1$ and $\widetilde{G}_2$, given by Eqs. (80) and (81), we obtain the mean perpendicular velocities $U_i^{(1)}$ and $U_i^{(2)}$:

$$NU_i^{(1)} = \int d^3v\,\widetilde{G}_1 u_i = \frac{\epsilon_{ilj}b_l}{m\Omega}\int d^3v\,\mu B\hat{d}_j^{(1)}\bar{G} = \frac{\epsilon_{ilj}b_l N}{\Omega}\left(\frac{1}{mN}\frac{\partial P_{jk}^{(0)}}{\partial x_k} + V_\|\frac{D_0 b_j}{Dt} + \frac{D_0 v_{Ej}}{Dt}\right), \tag{100}$$

$$NU_i^{(2)} = \int d^3v\,\widetilde{G}_2\,u_i = \frac{\epsilon_{ilj}b_l}{m\Omega}\int d^3v\,\mu B\hat{d}_j^{(2)}\bar{G}, \tag{101}$$

where the operator D_0/Dt is defined by

$$\frac{D_0 b_j}{Dt} \equiv \left[\frac{\partial}{\partial t} + (V_\| b_l + v_{El})\frac{\partial}{\partial x_l}\right]b_j. \tag{102}$$

The velocity integral $\int d^3v\,\mu B\hat{d}_j^{(2)}\bar{G}$ in Eq. (101) can be expressed in terms of moments of $\widetilde{G}_1$ by substituting Eq. (C3) for $\hat{d}_j^{(2)}\bar{G}$. However, an identical expression for $U_i^{(2)}$ is more conveniently obtained by making use of Eq. (86):

$$\begin{aligned} NU_i^{(2)} &= \int d^3v\,\widetilde{G}_2 \boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{v}}u_i^* = \int d^3v\frac{u_i^*}{\Omega}\hat{\mathcal{L}}\widetilde{G}_1 \\ &= \frac{\epsilon_{ilj}b_l}{\Omega}\left[\frac{\partial}{\partial t}\int d^3v\,v_j\widetilde{G}_1 + \frac{\partial}{\partial x_k}\int d^3v\,\widetilde{G}_1(v_k + v_{Ek})v_j + \int d^3v\,\widetilde{G}_1\left(\frac{Dv_{Ej}}{Dt} + u_r\frac{\partial v_{Ej}}{\partial x_r}\right)\right] \\ &= \frac{\epsilon_{ilj}b_l}{\Omega}\left[\frac{1}{m}\frac{\partial}{\partial x_k}P_{jk}^{(1)} + \frac{\partial}{\partial t}NU_j^{(1)} + \frac{\partial}{\partial x_k}NU_j^{(1)}(V_\| b_k + v_{Ek}) + NU_k^{(1)}\frac{\partial}{\partial x_k}(V_\| b_j + v_{Ej})\right]. \end{aligned}$$

B. First-order pressure tensor: $P_{ij}^{(1)}$

Substituting for $\widetilde{G}_1$, we obtain the first-order pressure tensor $P_{ij}^{(1)}$:

$$P_{ij}^{(1)} = I_{ij,k}m\int d^3v\,\widetilde{G}_1 u_k\delta v_\| + \frac{m}{2}\int d^3v\,\widetilde{G}_1 u_{ij} = I_{ij,k}^* P_{\|k}^{(1)} + I_{ij,kl}^* P_{\perp kl}^{(1)}, \tag{103}$$

$$P_{\|k}^{(1)} \equiv \frac{1}{\Omega}\int d^3v\,\delta v_\|\mu B\hat{d}_k^{(1)}\bar{G}, \tag{104}$$

$$P_{\perp kl}^{(1)} \equiv \frac{1}{2m\Omega}\int d^3v\,\mu^2B^2\hat{d}_{kl}^{(1)}\bar{G}, \tag{105}$$

where the projection tensors $I_{ij,k}$, $I_{ij,k}^*$, and $I_{ij.kl}^*$ are defined by

$$I_{ij,k} \equiv I_{ik}b_j + I_{jk}b_i; \quad I_{ij,k}^* \equiv I_{ij,q}\epsilon_{qpk}b_p; \quad b_ib_jI_{ij,k}^* = 0, \quad \delta_{ij}I_{ij,k}^* = 0, \tag{106}$$

$$I_{ij.kl}^* \equiv \frac{1}{2}(I_{ij,ql}\epsilon_{qpk}b_p + I_{ij,kq}\epsilon_{qpl}b_p); \quad b_ib_jI_{ij.kl}^* = 0, \quad \delta_{ij}I_{ij.kl}^* = 0, \tag{107}$$

and the velocity integrals are given by

$$\begin{aligned} \int d^3v\,\delta v_\|\mu B\hat{d}_k^{(1)}\bar{G} &= (P_\| - P_\perp)\frac{\hat{D}_0 b_k}{Dt} + P_\perp b_l\frac{\partial}{\partial x_k}(V_\| b_l + v_{El}) + P_\| b_l\frac{\partial}{\partial x_l}(V_\| b_k + v_{Ek}) + \left(\frac{\partial}{\partial x_k} - 2\kappa_k\right)\int d^3v\,\mu B\,\delta v_\|\bar{G} + \kappa_j m\int d^3v(\delta v_\|)^3\bar{G}, \\ \frac{1}{2m}\int d^3v\,\mu^2B^2\hat{d}_{kl}^{(1)}\bar{G} &= \frac{P_\perp}{4}\left[V_\|\left(\frac{\partial b_k}{\partial x_l} + \frac{\partial b_l}{\partial x_k}\right) + \frac{\partial v_{Ek}}{\partial x_l} + \frac{\partial v_{El}}{\partial x_k}\right] + \frac{1}{4}\left(\frac{\partial b_k}{\partial x_l} + \frac{\partial b_l}{\partial x_k}\right)\int d^3v\,\mu B\,\delta v_\|\bar{G}. \end{aligned} \tag{108}$$

Identical expressions for $I_{ij,k}^* P_{\|k}^{(1)}$ and $I_{ij,kl}^* P_{\perp kl}^{(1)}$ are obtained by making use of Eq. (85) and proceeding as described below for the second-order pressure tensor.

C. Second-order pressure tensor: $P_{ij}^{(2)}$

Substituting for $\widetilde{G}_2$, we express the first term in Eq. (97) for the second-order pressure tensor as follows:

$$m\int d^3v\left(u_i\delta v_\| b_j + u_j\delta v_\| b_i + \frac{u_{ij}}{2}\right)\widetilde{G}_2 = I^*_{ij,k}P^{(2)}_{\|k} + I^*_{ij,kl}P^{(2)}_{\perp kl}, \tag{109}$$

where

$$\Omega P^{(2)}_{\|k} = \int d^3v\,\delta v_\|\mu B\hat{d}^{(2)}_k\bar{G}, \tag{110}$$

$$\Omega P^{(2)}_{\perp kl} = \frac{1}{2m}\int d^3v\,\mu^2B^2\hat{d}^{(2)}_{kl}\bar{G}. \tag{111}$$

The velocity integrals $\int d^3v\,\delta v_\|\mu B\hat{d}^{(2)}_k\bar{G}$ and $\int d^3v\,\mu^2B^2\hat{d}^{(2)}_{kl}\bar{G}$ can be written in terms of moments of $\widetilde{G}_1$ by substituting Eq. (C3) for $\hat{d}^{(2)}_k\bar{G}$ and Eq. (C4) for $\hat{d}^{(2)}_{kl}\bar{G}$. However, identical expressions are more conveniently obtained by making use of Eq. (86). We note that

$$\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{v}}\left(\delta v_\| b_iu^*_j + \delta v_\| b_ju^*_i + \frac{u^*_{ij}}{4}\right) = \left(\delta v_\| b_iu_j + \delta v_\| b_ju_i + \frac{u_{ij}}{2}\right),$$

and we can therefore express $I^*_{ij,k}P^{(2)}_{\|k}$ and $I^*_{ij,kl}P^{(2)}_{\perp kl}$ as follows:

$$I^*_{ij,k}P^{(2)}_{\|k} = I^*_{ij,k}\frac{m}{\Omega}\int d^3v\,\delta v_\| v_k\hat{\mathcal{L}}\widetilde{G}_1, \tag{112}$$

$$I^*_{ij.kl}P^{(2)}_{\perp kl} = I^*_{ij,kl}\frac{m}{4\Omega}\int d^3v\; v_k\,v_l\hat{\mathcal{L}}\widetilde{G}_1. \tag{113}$$

Evaluating the velocity integrals, we obtain

$$\begin{aligned}\Omega P^{(2)}_{\|k} = m\int d^3v\,\delta v_\| v_k\hat{\mathcal{L}}\widetilde{G}_1 = &\left\{\delta_{kr}\frac{D_0}{Dt} + \delta_{kr}\left[\frac{\partial}{\partial x_l}(V_\| b_l + v_{El})\right] + \delta_{lr}\left(V_\|\frac{\partial b_k}{\partial x_l} + \frac{\partial v_{Ek}}{\partial x_l}\right)\right\}\epsilon_{rpq}b_pP^{(1)}_{\|q}\\ &+\left(m\frac{D_0V_\|}{Dt} + mb_s\frac{D_0v_{Es}}{Dt} - eE_\|\right)NU^{(1)}_k + \left(\frac{\partial V_\|}{\partial x_l} + b_s\frac{\partial v_{Es}}{\partial x_l}\right)P^{(1)}_{kl} - \frac{D_0b_l}{Dt}I^*_{kl,pq}P^{(1)}_{\perp pq} - (I_{kr}I_{ls} + I_{ks}I_{lr} + I_{rs}I_{kl})\frac{\partial b_s}{\partial x_l}\frac{\epsilon_{rpq}b_p}{2m\Omega}\\ &\times\int d^3v\,\mu^2B^2\hat{d}^{(1)}_q\bar{G} + \frac{1}{2}\left(\frac{\partial}{\partial x_l} - \kappa_l\right)m\int d^3v\;\widetilde{G}_1u_{kl}\delta v_\| + \left(\delta_{ks}b_l\frac{\partial}{\partial x_l} + \delta_{ks}\frac{\partial b_l}{\partial x_l} + \delta_{ls}\frac{\partial b_k}{\partial x_l}\right)m\int d^3v\;\widetilde{G}_1u_s(\delta v_\|)^2\end{aligned} \tag{114}$$

and

$$\begin{aligned}4\Omega P^{(2)}_{\perp kl} = m\int d^3v\;v_k\,v_l\hat{\mathcal{L}}\widetilde{G}_1 = &\frac{\partial}{\partial t}P^{(1)}_{kl} + \frac{\partial}{\partial x_r}(V_\| b_r + v_{Er})P^{(1)}_{kl} + \frac{\partial}{\partial x_r}(I_{kr}I_{ls} + I_{ks}I_{lr})\frac{\epsilon_{spq}b_p}{2m\Omega}\int d^3v\,\mu^2B^2\hat{d}^{(1)}_q\bar{G} + (\delta_{ks}\delta_{lr} + \delta_{ls}\delta_{kr})\\ &\times\left[\left(V_\|\frac{D_0b_s}{Dt} + \frac{D_0v_{Es}}{Dt}\right)mNU^{(1)}_r + P^{(1)}_{rq}\left(V_\|\frac{\partial b_s}{\partial x_q} + \frac{\partial v_{Es}}{\partial x_q}\right)\right] + \frac{1}{2}\frac{\partial}{\partial x_r}m\int d^3v\;\widetilde{G}_1(b_ru_{kl} + b_ku_{lr} + b_lu_{kr})\delta v_\|\\ &+ m\int d^3v\;\widetilde{G}_1(u_k\kappa_l + u_l\kappa_k)(\delta v_\|)^2.\end{aligned} \tag{115}$$

We display on the right-hand side of Eqs. (114) and (115) only those terms which are nonzero when multiplied by $I^*_{ij,k}$ and $I^*_{ij,kl}$, respectively.

Note that

$$\int d^3v\;\widetilde{G}_1u_l(\cdots) = \frac{\epsilon_{ltk}b_t}{m\Omega}\int d^3v\,\mu B(\cdots)\hat{d}^{(1)}_k\bar{G},$$

$$\int d^3v\;\widetilde{G}_1u_{lr}(\cdots) = \frac{I^*_{lr,kl}}{m^2\Omega}\int d^3v\,\mu^2B^2(\cdots)\hat{d}^{(1)}_{kl}\bar{G},$$

where the bracket $(\cdots)$ encloses terms independent of the Larmor phase angle.

The second-order pressure tensor is therefore given in terms of moments of $\bar{G}$ by

$$P^{(2)}_{ij} = I^*_{ij,k}P^{(2)}_{\|k} + I^*_{ij,kl}P^{(2)}_{\perp kl} - mNU_iU_j + \int d^3v\,\Delta\bar{G}[\mu BI_{ij} + m(v_\| b_i + v_{Ei})(v_\| b_j + v_{Ej})]. \tag{116}$$

IV. MOMENT EQUATIONS

The description of plasma behavior in terms of the particle distribution function emphasizes the particle characteristics of plasmas. Many plasma phenomena, however, can usefully be described macroscopically in terms of the continuity and momentum equation for each particle species.

In this section, we describe the low-order moments of the drift-kinetic equation, and we verify that they are equivalent to the corresponding moments of the Vlasov equation when expressed as a Larmor series expansion in ϵ.

The continuity equation, obtained from the zeroth moment of the Vlasov equation [Eq. (4)], is

$$\frac{\partial \mathcal{N}}{\partial t}+\frac{\partial}{\partial x_i}\mathcal{N}\mathcal{V}_i=\frac{\partial N}{\partial t}+\frac{\partial}{\partial x_i}N(V_\| b_i+v_{Ei}+\epsilon U_i^{(1)}+\epsilon^2 U_i^{(2)}) + \epsilon^2\frac{\partial}{\partial t}\int d^3v\,\Delta\bar{G}+\epsilon^2\frac{\partial}{\partial x_i}\int d^3v\,\dot{X}_i^{(0)}\Delta\bar{G} + O(\epsilon^3)=0, \tag{117}$$

where we substituted Eq. (79) for F to obtain a Larmor series expansion in ϵ.

The momentum equation, obtained from the velocity moment of Eq. (4), is

$$\frac{\partial}{\partial t}m\mathcal{N}\mathcal{V}_i+\frac{\partial}{\partial x_j}\mathcal{P}_{ij}-e\mathcal{N}(\boldsymbol{b}\cdot\boldsymbol{E})b_i-\Omega mN\epsilon_{ijk}U_jb_k=0. \tag{118}$$

Substituting Eq. (79) for F, we obtain the parallel component of the momentum equation,

$$\frac{\partial}{\partial t}mNV_\|+\frac{\partial}{\partial x_j}mNV_\|(V_\| b_j+v_{Ej}+\epsilon U_j) + b_i\frac{\partial}{\partial x_j}P_{ij}-eNE_\|+b_imN\frac{D_0v_{Ei}}{Dt}-\epsilon mNU_i\frac{D_0b_i}{Dt} + \epsilon b_imNU_j\frac{\partial v_{Ei}}{\partial x_j}+\epsilon^2b_imNU_j\frac{\partial U_i}{\partial x_j} + \epsilon^2\left[b_i\frac{\partial}{\partial t}\int d^3v(v_\| b_i+v_{Ei})\Delta\bar{G}-eE_\|\int d^3v\,\Delta\bar{G}\right] + O(\epsilon^3)=0. \tag{119}$$

From the vector product of $\boldsymbol{b}$ and Eq. (118), we obtain the following expression for the mean perpendicular velocity U_i:

$$U_i=\frac{\epsilon_{ilk}b_l}{mN\Omega}\left(m\frac{\partial}{\partial t}\mathcal{N}\mathcal{V}_i+\frac{\partial}{\partial x_j}\mathcal{P}_{kj}\right). \tag{120}$$

This equation expresses the mean perpendicular velocity U_i in terms of the inertial drifts and the gradient of the pressure tensor, and when iterated, produces a Larmor series representation of $U_i=U_i^{(1)}+\epsilon U_i^{(2)}+\cdots$, equivalent to Eq. (91).

Let us now consider the moments of the drift-kinetic equation, Eq. (83).

A. Continuity equation

Integrating Eq. (83) over velocity space, we obtain

$$\frac{\partial}{\partial t}N+\frac{\partial}{\partial x_i}N(V_\| b_i+v_{Ei})+\epsilon\frac{\partial}{\partial x_i}\int d^3v\,\bar{G}(\dot{X}_i^{(1)}+\epsilon\dot{X}_i^{(2)})=0, \tag{121}$$

where the mean first-order guiding-center velocity is

$$\int d^3v\,\bar{G}\dot{X}_i^{(1)}=\frac{\epsilon_{ipq}b_p}{m\Omega}\left(mN\frac{D_0v_{Eq}}{Dt}+mNV_\|\frac{D_0b_q}{Dt}+\frac{P_\perp}{B}\frac{\partial B}{\partial x_q} + P_\|\kappa_q\right)+\frac{P_\perp}{m\Omega}(\boldsymbol{b}\cdot\nabla\times\boldsymbol{b})b_i = NU_i^{(1)}+\frac{\partial}{\partial x_j}\frac{\epsilon_{jli}b_l}{m\Omega}P_\perp \tag{122}$$

and $\dot{X}_i^{(2)}$ is given by Eq. (67).

Equation (121) is equivalent to Eq. (117). To verify this equivalence, we evaluate the velocity integral $\int d^3v\,\hat{D}^{(1)}\bar{G}$ using the two expressions for the operator $\hat{D}^{(1)}$ given by Eqs. (47) and (49). Equating the two results [see also Eq. (122)]:

$$\frac{\partial}{\partial x_i}NU_i^{(1)}=\frac{\partial}{\partial x_i}\int d^3v\,\bar{G}\dot{X}_i^{(1)}. \tag{123}$$

Similarly, for the operator $\hat{D}^{(2)}$ given by Eqs. (60) and (63), we have

$$\frac{\partial}{\partial x_i}NU_i^{(2)}=\frac{\partial}{\partial x_i}\int d^3v\,\bar{G}\dot{X}_i^{(2)}-\frac{\partial}{\partial x_i}\int d^3v\,\dot{X}_i^{(0)}\Delta\bar{G} - \frac{\partial}{\partial t}\int d^3v\,\Delta\bar{G}. \tag{124}$$

Substituting for $(\partial/\partial x_i)NU_i^{(1)}$ and $(\partial/\partial x_i)NU_i^{(2)}$ in Eq. (117), we obtain Eq. (121).

B. Parallel momentum equation

Multiplying Eq. (83) by $mv_\|$, and integrating over velocity space, we obtain

$$\frac{\partial}{\partial t}mNV_\|+\frac{\partial}{\partial x_i}[mNV_\|(V_\| b_i+v_{Ei})+P_\| b_i]-eNb_iE_i + \frac{P_\perp}{B}b_i\frac{\partial B}{\partial x_i}+mNb_i\frac{D_0v_{Ei}}{Dt} + \epsilon\frac{\partial}{\partial x_i}m\int d^3v\bar{G}v_\|(\dot{X}_i^{(1)}+\epsilon\dot{X}_i^{(2)}) - \epsilon m\int d^3v\bar{G}(\dot{V}_\|^{(1)}+\epsilon\dot{V}_\|^{(2)})=0, \tag{125}$$

where $\dot{X}_i^{(1)}$, $\dot{X}_i^{(2)}$, $\dot{V}_\|^{(1)}$, and $\dot{V}_\|^{(2)}$ are given by Eqs. (51), (67), (52), and (68).

Equation (125) is equivalent to Eq. (119). To verify this equivalence, we evaluate the velocity integral $\int d^3v\,v_\|\hat{D}^{(1)}\bar{G}$ using the two expressions for the operator $\hat{D}^{(1)}$ given by Eqs. (47) and (49). Equating the two results:

$$\frac{\partial}{\partial x_i}NV_\| U_i^{(1)}-\left(\frac{\hat{D}_0b_i}{Dt}-b_l\frac{\partial v_{El}}{\partial x_i}\right)NU_i^{(1)}+b_i\frac{\partial}{\partial x_j}P_{ij}^{(1)} = \frac{\partial}{\partial x_i}\int d^3v\,v_\|\dot{X}_i^{(1)}\bar{G}-\int d^3v\,\dot{V}_\|^{(1)}\bar{G}.$$

Similarly, for the operator $\hat{D}^{(2)}$ given by Eqs. (60) and (63), we have

$$\frac{\partial}{\partial x_i}NV_\parallel U_i^{(2)} - \left(\frac{\hat{D}_0 b_i}{Dt} - b_l\frac{\partial v_{El}}{\partial x_i}\right)NU_i^{(2)} + \frac{b_i}{m}\frac{\partial}{\partial x_j}P_{ij}^{(2)}$$
$$+ b_i\frac{\partial}{\partial x_j}NU_i^{(1)}U_j^{(1)} = -\frac{\partial}{\partial t}\int d^3v\ v_\parallel \Delta\bar{G}$$
$$- b_i\frac{\partial v_{Ei}}{\partial t}\int d^3v\ \Delta\bar{G} + \frac{e}{m}E_\parallel\int d^3v\ \Delta\bar{G}$$
$$+ \frac{\partial}{\partial x_i}\int d^3v\ v_\parallel \dot{X}_i^{(2)}\bar{G} - \int d^3\ \dot{V}_\parallel^{(2)}\bar{G}.$$

Substituting for $(\partial/\partial x_i)NV_\parallel U_i^{(1)}$ and $(\partial/\partial x_i)NV_\parallel U_i^{(2)}$ in Eq. (119), we obtain Eq. (125).

C. Perpendicular component of the plasma momentum equation

The perpendicular component of the plasma momentum equation is obtained by summing Eq. (118) over particle species.

Note that

$$\sum eN(\boldsymbol{b}\cdot\boldsymbol{E})b_i + \sum\frac{eB}{c}N\epsilon_{ijk}U_j b_k$$
$$= \rho(\boldsymbol{b}\cdot\boldsymbol{E})b_i + \frac{1}{c}\epsilon_{ijk}(J_j B_k - \rho v_{Ej}B_k)$$
$$= -\frac{\partial}{\partial x_k}\Pi_{ik} - \frac{1}{4\pi c}\frac{\partial}{\partial t}\epsilon_{ijk}E_j B_k, \tag{126}$$

where Π_{ik} is the electromagnetic stress tensor,

$$\Pi_{ik} = \frac{1}{4\pi}\left(\frac{\boldsymbol{B}\cdot\boldsymbol{B}}{2}\delta_{ik} - B_i B_k + \frac{\boldsymbol{E}\cdot\boldsymbol{E}}{2}\delta_{ik} - E_i E_k\right). \tag{127}$$

Thus, the perpendicular component of Eq. (118), summed over particle species, can be expressed as follows:

$$I_{ij}\left\{\sum mN\left[\frac{\partial}{\partial t}(v_{Ej}+V_j) + (v_{Ek}+V_k)\frac{\partial}{\partial x_k}(v_{Ej}+V_j)\right]\right.$$
$$\left.+ \frac{\partial}{\partial x_k}\left(\sum P_{jk} + \Pi_{jk}\right) + \frac{1}{4\pi c}\frac{\partial}{\partial t}\epsilon_{jlk}E_l B_k\right\}$$
$$+ \epsilon^2 I_{ij}\sum\left[\frac{\partial}{\partial t}m\int d^3v(v_\parallel b_j + v_{Ej})\Delta G - m(V_\parallel b_j\right.$$
$$\left.+ v_{Ej})\int d^3v\ \hat{\mathcal{D}}^{(0)}\Delta G\right] = 0. \tag{128}$$

D. Parallel and perpendicular pressure equations

The equation for the parallel pressure $P_\parallel$, up to second order in ϵ, is

$$\frac{\partial P_\parallel}{\partial t} + \frac{\partial}{\partial x_i}P_\parallel(V_\parallel b_i + v_{Ei}) + 2P_\parallel\left(b_i\frac{\partial V_\parallel}{\partial x_i} - v_{Ei}\kappa_i\right)$$
$$+ 2b_i\frac{\partial B}{\partial x_i}\int d^2v\ \bar{G}\mu\delta v_\parallel + \frac{\partial}{\partial x_i}mb_i\int d^3v\ \bar{G}(\delta v_\parallel)^3$$
$$+ \epsilon 2m\frac{\partial V_\parallel}{\partial x_i}\int d^3v\ \bar{G}\delta v_\parallel(\dot{X}_i^{(1)} + \epsilon\dot{X}_i^{(2)})$$
$$+ \epsilon m\frac{\partial}{\partial x_i}\int d^3v\ \bar{G}(\delta v_\parallel)^2(\dot{X}_i^{(1)} + \epsilon\dot{X}_i^{(2)})$$
$$- \epsilon 2m\int d^3v\ \bar{G}\delta v_\parallel(\dot{V}_\parallel^{(1)} + \epsilon\dot{V}_\parallel^{(2)}) = 0. \tag{129}$$

The equation for the perpendicular pressure $P_\perp$, up to second order in ϵ, is

$$\frac{\partial P_\perp}{\partial t} + \frac{\partial}{\partial x_i}P_\perp(V_\parallel b_i + v_{Ei}) + P_\perp\left(\frac{\partial v_{Ei}}{\partial x_i} + V_\parallel\frac{\partial b_i}{\partial x_i} + v_{Ei}\kappa_i\right)$$
$$+ B\frac{\partial}{\partial x_i}b_i\int d^3v\ \bar{G}\mu\delta v_\parallel + \epsilon B\frac{\partial}{\partial x_i}\int d^3v\ \bar{G}\mu(\dot{X}_i^{(1)}$$
$$+ \epsilon\dot{X}_i^{(2)}) - \epsilon B\int d^3v\ \bar{G}(\dot{\mu}^{(1)} + \epsilon\dot{\mu}^{(2)}) = 0. \tag{130}$$

The moment equations involve higher-order moments of $\bar{G}$, and these equations are not closed. The usual procedure is to invoke a closure scheme which relates the higher-order moments to the lower-order moments. However, it is not obvious that a satisfactory closure scheme can be achieved to account for the parallel plasma dynamics along the magnetic field.

An alternative strategy is to solve the drift-kinetic equation and to use the solution for the distribution function $\bar{G}$ to evaluate the velocity moments required to obtain a close set of equations.

V. FIELD EQUATIONS

The plasma charge density ρ and current density $\boldsymbol{J}$ are given by

$$\rho = \sum e\int d^3v\ F = \sum Ne + \epsilon^2\sum e\int d^3v\ \Delta\bar{G} + \cdots, \tag{131}$$

$$J_i = \sum e\int d^3v\ F(v_i + v_{Ei}) = \sum eN(V_\parallel b_i + v_E i + \epsilon U_i)$$
$$+ \sum e\int d^3v\ \Delta\bar{G}(v_\parallel b_i + v_{Ei}) + \cdots, \tag{132}$$

where the summation is over the particle species of the plasma.

The electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, expressed in terms of the electrostatic scalar potential ϕ and the magnetic potential $\boldsymbol{A}$, are given by:

$$\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}, \quad \boldsymbol{B} = \nabla\times\boldsymbol{A}. \tag{133}$$

The electromagnetic fields are determined by Maxwell's equations:

$$\nabla \cdot \boldsymbol{E} = -\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = 4\pi\rho, \tag{134}$$

$$\nabla \times \boldsymbol{B} = -\nabla \cdot \nabla \boldsymbol{A} + \nabla(\nabla \cdot \boldsymbol{A}) = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}. \tag{135}$$

It is often convenient to use the low-order moment equations to relate the charge density and current density to the electromagnetic fields and to other plasma macroscopic variables: plasma pressure and higher-order velocity moments of $\bar{G}$. Alternative forms of the field equations can then be derived by combining these macroscopic equations with Maxwell's equations.

In particular, from the perpendicular component of Eq. (135), we have

$$\boldsymbol{B} \times (\nabla \times \boldsymbol{B}) = \frac{4\pi}{c}\boldsymbol{B} \times \boldsymbol{J} + \frac{1}{c}\boldsymbol{B} \times \frac{\partial \boldsymbol{E}}{\partial t}. \tag{136}$$

Using Eq. (118) to express $\boldsymbol{B}\times\boldsymbol{J}$ in terms of v_{Ei}, V_i, and P_{ij}, we obtain from Eq. (136) the perpendicular component of the plasma momentum equation [Eq. (128)], an equation which provides an adequate description of fluid-like perpendicular motion across the magnetic field.

The electric drift velocity v_{Ei} is a sum of two terms:

$$v_{Ei} = V_{\phi i} + V_{\xi i}, \tag{137}$$

$$V_{\phi i} = \frac{c\epsilon_{ilk}b_l}{B}\frac{\partial \phi}{\partial x_k}, \tag{138}$$

$$V_{\xi i} = \frac{\epsilon_{ilk}b_l}{B}\frac{\partial A_k}{\partial t}, \tag{139}$$

where the first term, denoted by $V_{\phi i}$, is due to the gradient of the electrostatic potential, and the second term, denoted by $V_{\xi i}$, is due to the perpendicular component of the inductive electric field.

The inductive component of the electric drift $V_{\xi i}$ is, in fact, an alternative field variable which can be used to replace the time-dependent perpendicular vector potential. The time variation of the magnetic field $\boldsymbol{B}$ is then determined by

$$\frac{\partial}{\partial t}\boldsymbol{B} = \nabla \times (\boldsymbol{V}_\xi \times \boldsymbol{B}) - c\nabla_\wedge(E_\parallel + \boldsymbol{b}\cdot\nabla\phi)\boldsymbol{b}. \tag{140}$$

Equation (128) can therefore be used to time advance the perturbed magnetic fields. Note that, although Eqs. (128) and (135) are equivalent, it is necessary to evaluate U_i with an accuracy one order higher in the case of Eq. (135) than in the case of Eq. (128) in order to achieve the same accuracy in ϵ.

Note also that Eq. (128) reduces to the low-frequency magnetohydrodynamic fluid equations for perpendicular plasma dynamics when it is supplemented by an equation of state for the plasma pressure.

Additional fluid-kinetic equations for determining the electrostatic potential ϕ and the parallel component of the vector potential $A_\parallel$ are obtained by combining the continuity equation with Poisson's equation, Eq. (134), and the parallel momentum equation with the parallel component of Eq. (135).[12]

Thus, a consistent set of fluid-kinetic equations can be formulated, with the perpendicular plasma motion described by the perpendicular component of the plasma momentum equation. Parallel motion along the magnetic field is not fluid-like. The parallel plasma dynamics is described by the drift-kinetic equation, and the solution $\bar{G}$ is required to determine the velocity moments necessary to close the set of equations.

Finally, we note that the electromagnetic fields are invariant to transformations to new electromagnetic potentials ϕ' and $\boldsymbol{A}'$ of the form

$$\phi' = \phi + \frac{1}{c}\frac{\partial \chi}{\partial t}, \quad \boldsymbol{A}' = \boldsymbol{A} - \nabla\chi, \tag{141}$$

where χ is an arbitrary scalar gauge function.

Our drift-kinetic equation, given by Eq. (83), involves only the electric and magnetic fields ($\boldsymbol{E}$ and $\boldsymbol{B}$), and is therefore gauge invariant. This freedom to choose the gauge function χ can be exploited to simplify the set of fluid-kinetic equations for time advancing the perturbed electromagnetic potentials. We will explore this topic in a later publication.

VI. ENERGY CONSERVATION

In this section, we determine the equation for energy conservation by evaluating the work done by the electric field.

Let the angular bracket $\langle(\cdots)\rangle \equiv \int d^3r \int d^3v(\cdots)$ denote integration over the total phase-space volume.

To lowest order in ϵ, we use Eq. (23) to express the work done by the perpendicular electric field $\boldsymbol{E}_\perp$ as follows:

$$\sum e\langle \tilde{F}_1 \boldsymbol{E}_\perp \cdot \boldsymbol{v}\rangle = \sum \left\langle \tilde{F}_1 \Omega\left(\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{v}}\right)\frac{m}{2}(v_i + v_{Ei})^2\right\rangle, \tag{142}$$

$$= \sum \left\langle \frac{m}{2}(v_i + v_{Ei})^2 \hat{\mathcal{L}}\bar{F}_0\right\rangle, \tag{143}$$

$$= \frac{\partial}{\partial t}\sum \left\langle \frac{m}{2}(v_i + v_{Ei})^2 \bar{F}_0\right\rangle - \sum e\langle \bar{F}_0 E_\parallel v_\parallel\rangle, \tag{144}$$

where we assume the electromagnetic fields to be vanishingly small at the boundaries of configuration space.

Note that we ordered the parallel component of the electric field $E_\parallel \sim \epsilon|\boldsymbol{E}_\perp|$ to be of order ϵ compared to the perpendicular component. Thus, to zero order in ϵ, the work done by the electric field $\boldsymbol{E}$ is

$$\sum e\langle \tilde{F}_1 \boldsymbol{E}_\perp \cdot \boldsymbol{v}\rangle + \sum e\langle \bar{F}_0 E_\parallel v_\parallel\rangle = \frac{\partial}{\partial t}\sum \left\langle \left[\mu B + \frac{m}{2}(v_\parallel^2 + v_{Ei}^2)\right]\bar{F}_0\right\rangle, \tag{145}$$

and the equation for energy conservation is

$$\frac{\partial}{\partial t}\int d^3r\frac{(E_i^2+B_i^2)}{8\pi}+\frac{\partial}{\partial t}\int d^3r\, W_p=0, \tag{146}$$

where the plasma energy density W_p is

$$W_p=\sum\int d^3v\left[\mu B+\frac{m}{2}(v_\parallel^2+v_{Ei}^2)\right]\bar{F}_0. \tag{147}$$

From Eqs. (23) and (45), we obtain up to first order in ϵ

$$\begin{aligned}\sum e\langle(\tilde{F}_1+\epsilon\tilde{F}_2)\boldsymbol{E}_\perp\cdot\boldsymbol{v}\rangle+\sum e\langle(\bar{F}_0+\epsilon\bar{F}_1)E_\parallel v_\parallel\rangle\\ =\frac{\partial}{\partial t}\sum\left\langle\left[\mu B+\frac{m}{2}(v_\parallel^2+v_{Ei}^2)\right](\bar{F}_0+\epsilon\bar{F}_1)\right.\\ \left.+\epsilon m v_{Ei}u_i\tilde{F}_1\right\rangle,\end{aligned} \tag{148}$$

and the equation for energy conservation is given by Eq. (146) with plasma energy density W_p,

$$\begin{aligned}W_p=\sum\int d^3v\left\{\left[\mu B+\frac{m}{2}(v_\parallel^2+v_{Ei}^2)\right](\bar{F}_0+\epsilon\bar{F}_1)\right.\\ \left.+\epsilon m v_{Ei}u_i\tilde{F}_1\right\}.\end{aligned}$$

From Eqs. (23), (45), and (58), we obtain up to second order in ϵ

$$\begin{aligned}\sum e\langle(\tilde{F}_1+\epsilon\tilde{F}_2+\epsilon^2\tilde{F}_2)\boldsymbol{E}_\perp\cdot\boldsymbol{v}\rangle+\sum e\langle(\bar{F}_0+\epsilon\bar{F}_1\\ +\epsilon^2\bar{F}_2)E_\parallel v_\parallel\rangle\\ =\frac{\partial}{\partial t}\sum\left\langle\left[\mu B+\frac{m}{2}(v_\parallel^2+v_{Ei}^2)\right](\bar{F}_0+\epsilon\bar{F}_1+\epsilon^2\bar{F}_2)\right.\\ \left.+\epsilon m v_{Ei}u_i(\tilde{F}_1+\epsilon\tilde{F}_2)\right\rangle,\end{aligned}$$

and the equation for energy conservation is given by Eq. (146) with plasma energy density W_p,

$$\begin{aligned}W_p=\sum\int d^3v\left\{\left[\mu B+\frac{m}{2}(v_\parallel^2+v_{Ei}^2)\right](\bar{F}_0+\epsilon\bar{F}_1+\epsilon^2\bar{F}_2)\right.\\ \left.+\epsilon m v_{Ei}u_i(\tilde{F}_1+\epsilon\tilde{F}_2)\right\}\\ =\sum\left[P_\perp+\frac{1}{2}(mNV_\parallel^2+P_\parallel+mNv_{Ei}^2)+\epsilon mNv_{Ei}(U_i^{(1)}\right.\\ \left.+\epsilon U_i^{(2)})\right]+\epsilon^2\sum\int d^3v\,\Delta\bar{G}\left[\mu B+\frac{m}{2}(v_\parallel^2+v_{Ei}^2)\right].\end{aligned} \tag{149}$$

VII. SUMMARY

A new and efficient method is described for deriving the drift-kinetic equation by averaging the Vlasov kinetic equation over the Larmor phase angle. The analysis is carried out after first transforming the velocity variable to the velocity relative to the perpendicular electric drift velocity. Electric drifts can be of the order of particle thermal velocities. A maximal ordering is invoked, and all terms proportional to m/e are taken to be of order $\epsilon\ll 1$, subject to the proviso that $E_\parallel\sim\epsilon$. The drift-kinetic equation is derived up to second order in ϵ.

The phase-space variables of the drift-kinetic distribution function $\bar{G}(v_\parallel,\mu,x_i,t)$ are the parallel speed $v_\parallel$, the magnetic moment μ, and the position coordinate x_i. These are noncanonical variables. Nevertheless, the drift-kinetic equation is derived in Liouville form, and approximations to the particle trajectory can be made while preserving the Liouville structure.

The Vlasov distribution function F is given in terms of $\bar{G}$ as a Larmor series expansion up to second order in ϵ, and the mean perpendicular velocity, momentum flow tensor, pressure tensor are expressed in terms of the velocity moments of $\bar{G}$. The results are identical to that obtained by iterating the moments of the Vlasov equation to derive a Larmor series expansion for the mean perpendicular velocity and the "off-diagonal" components of the pressure tensor.

Perpendicular plasma dynamics is adequately described by the perpendicular component of the plasma momentum equation. The low-order moments of the drift-kinetic equation relate the plasma density and parallel component of the mean plasma velocity to the mean perpendicular velocity, the plasma pressure, and higher-order velocity moments of the particle distribution as well as the electromagnetic fields. Alternative forms of the field equations for advancing the perturbed electromagnetic fields are obtained by combining Maxwell's equations with low-order moment equations. However, parallel motion along the magnetic field is not fluid-like, and solutions of the drift-kinetic equation are required to describe accurately the parallel plasma dynamics and to determine the higher-order velocity moments of $\bar{G}$ which are present in the low-order moment equations.

These fluid-kinetic equations provide a basis for the development of hybrid numerical codes which can be used to simulate low-frequency electromagnetic perturbations of magnetized plasmas.

In a recent derivation of the drift-kinetic equation by Simakov and Catto,[8] a less than maximal ordering is invoked. Instead, the time variation is ordered to be on the diamagnetic drift time scale $\partial/\partial t\sim\epsilon^2\Omega$ and electric drift velocities are ordered to be ϵ compared to thermal velocities. In addition, the gyrophase-averaged distribution function is taken to be isotropic in velocity space in leading order. All these restrictions are removed in the derivation discussed in this paper.

In an alternative analysis of the Vlasov equation discussed by Ramos,[3] a set of finite-Larmor-radius moment equations has been derived for collisionless magnetized plasmas. It is based on perturbative but otherwise general solutions for the second- and third-rank fluid moments (the stress and stress flux tensors), and closure is achieved by specifying closure conditions on the fourth-rank moment.

Fluid equations are attractive in that they describe the time evolution of a finite number of macroscopic plasma variables. However, the uncertainties involved in specifying the closure conditions are a major defect.

Kinetic equations are solved by numerically integrating the particle phase-space trajectories. An accurate determination of the particle distribution function requires the integration of a very large number of particles. Thus, a satisfactory description of plasma behavior using fluid-kinetic equations is limited by the total number of particles which can be accommodated in numerical simulations.

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APPENDIX A: THIRD-RANK AND FOURTH-RANK VELOCITY TENSORS

In this appendix, we discuss the Larmor phase average and Larmor phase integration of third-rank and fourth-rank tensors formed by the tensor product of the perpendicular velocity vectors $u_i, u_i^* = \epsilon_{ilk} b_l u_k$.

Note that $\boldsymbol{v}\times\boldsymbol{b}\cdot\partial/\partial\boldsymbol{v}\to -\partial/\partial\Theta$, and that

$$\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{v}}u_i^* = u_i, \quad -\int d\Theta\, u_i = u_i^*, \tag{A1}$$

$$\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{v}}u_i = -u_i^*, \quad -\int d\Theta\, u_i^* = -u_i, \tag{A2}$$

where $-\int d\Theta$ denotes the inverse of the operator $\boldsymbol{v}\times\boldsymbol{b}\cdot\partial/\partial\boldsymbol{v}$.

1. Third-rank velocity tensor: $u_i u_{jk} = u_i(u_j u_k - u_j^* u_k^*)$

The third-rank velocity tensor $u_i u_{jk}$ can be rewritten as follows:

$$2u_i u_{jk} = v_\perp^2(u_j I_{ik} + u_k I_{ij} - u_i I_{jk}) + \{u_i u_{jk} - u_i^* u_{jk}^*\}. \tag{A3}$$

Let u_{ijk} and u_{ijk}^* denote the symmetric perpendicular velocity tensors:

$$u_{ijk} = u_i u_{jk} - u_i^* u_{jk}^*, \quad u_{ijk}^* = u_i^* u_{jk} + u_i u_{jk}^* = \epsilon_{ipq} b_p u_{qjk}, \tag{A4}$$

where u_{ijk} and u_{ijk}^* remain unchanged under all permutations of its indices, a property that can be verified by expanding to obtain a sum of cubic polynomials.

Using Eqs. (A1) and (A2), we establish the identities

$$\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{v}}u_{ijk}^* = 3u_{ijk}, \quad -\int d\Theta\, u_{ijk} = \frac{u_{ijk}^*}{3}, \tag{A5}$$

$$\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{v}}u_{ijk} = -3u_{ijk}^*, \quad -\int d\Theta\, u_{ijk}^* = -\frac{u_{ijk}}{3}. \tag{A6}$$

The third-rank tensors, u_{ijk} and u_{ijk}^*, consist of a sum of terms proportional to either $\cos 3\Theta$ or $\sin 3\Theta$.

2. Fourth-rank velocity tensor: $u_{ij}u_{kl}^* = (u_i u_j - u_i^* u_j^*)(u_k u_l^* + u_k^* u_l)$

The fourth-rank velocity tensor $u_{ij}u_{kl}^*$ can be rewritten as follows:

$$2u_{ij}u_{kl}^* = v_\perp^4(\epsilon_{kpq}b_p I_{iq} I_{jl} + \epsilon_{lpq} b_p I_{jq} I_{ik}) + (u_{ij}u_{kl}^* + u_{ij}^* u_{kl}). \tag{A7}$$

Let u_{ijkl} and u_{ijkl}^* denote the symmetric perpendicular velocity tensors:

$$u_{ijkl} = u_{ij}u_{kl}^* + u_{ij}^* u_{kl}, \quad u_{ijkl}^* = u_{ij}^* u_{kl}^* - u_{ij}u_{kl} = \epsilon_{ipq} b_p u_{qjkl}, \tag{A8}$$

where u_{ijkl} and u_{ijkl}^* remain unchanged under all permutations of its indices, a property that can be verified by expanding to obtain a sum of quartic polynomials.

Using Eqs. (A1) and (A2), we establish the identities

$$\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{v}}u_{ijkl}^* = 4u_{ijkl}, \quad -\int d\Theta\, u_{ijkl} = \frac{u_{ijkl}^*}{4}, \tag{A9}$$

$$\boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial}{\partial\boldsymbol{v}}u_{ijkl} = -4u_{ijkl}^*, \quad -\int d\Theta\, u_{ijkl}^* = -\frac{u_{ijkl}}{4}. \tag{A10}$$

The fourth-rank tensors, u_{ijkl} and u_{ijkl}^*, consist of a sum of terms proportional to either $\cos 4\Theta$ or $\sin 4\Theta$.

APPENDIX B: EVALUATION OF $\hat{\mathcal{L}}(u_k^*/\Omega)\chi_k(x_j, v_\parallel, \mu, t)$ AND $\hat{\mathcal{L}}(u_{kl}^*/2\Omega)\chi_{kl}(x_j, v_\parallel, \mu, t)$

In this appendix, we evaluate $\hat{\mathcal{L}}(u_k^*/\Omega)\chi_k(x_j, v_\parallel, \mu, t)$ and $\hat{\mathcal{L}}(u_{kl}^*/2\Omega)\chi_{kl}(x_j, v_\parallel, \mu, t)$, where the phase-space functions χ_k and χ_{kl} are functions of the variables x_j, $v_\parallel$, μ, and t. There is no loss of generality in assuming that χ_{kl} remains unchanged under permutations of its indices.

The operator $\hat{\mathcal{L}}$ is defined by Eq. (42):

$$\hat{\mathcal{L}} = \hat{D}^{(0)} + u_i \hat{d}_i^{(1)} + u_{ij}\hat{d}_{ij}^{(1)} + \left(\frac{e}{m}b_j E_j\, b_i - \frac{Dv_{Ei}}{Dt} - u_j\frac{\partial v_{Ei}}{\partial x_j}\right)\frac{\partial'}{\partial v_i}, \tag{B1}$$

with $\hat{d}_i^{(1)} = \Lambda_{i,\nu}^{(1)}(\partial/\partial z_\nu)$ and $\hat{d}_{ij}^{(1)} = \Lambda_{ij,\nu}^{(1)}(\partial/\partial z_\nu)$.

1. $-\hat{\mathcal{L}}(u_k^*/\Omega)\chi_k(x_j, v_\parallel, \mu, t)$

Let $(u_k^*/\Omega)\chi_k$ be expressed as follows:

$$\frac{u_k^*}{\Omega}\chi_k = \frac{\epsilon_{kpq}\, b_p\, v_q\, \chi_k}{\Omega} = -\frac{v_q \chi_q^*}{\Omega},$$

where we find it convenient to introduce the phase-space function χ_q^* defined by

$$\chi_q^* \equiv \epsilon_{qpk} b_p \chi_k. \tag{B2}$$

The functions χ_k and χ_q^* can be used interchangeably, but it will sometimes be more convenient to use one or the other.

Substituting Eq. (B1) for $\hat{\mathcal{L}}$, we have

$$-\hat{\mathcal{L}}\frac{u_k^*}{\Omega}\chi_k(x_j,v_\parallel,\mu)$$
$$=\hat{\mathcal{L}}\frac{v_k}{\Omega}\chi_k^*=\frac{\chi_k^*}{\Omega}\hat{\mathcal{L}}v_k+(v_\parallel b_k+u_k)\hat{\mathcal{L}}\frac{\chi_k^*}{\Omega}$$
$$=-\frac{\chi_k^*}{\Omega}\left(\frac{D}{Dt}+u_j\frac{\partial'}{\partial x_j}\right)(v_{Ek}+v_\parallel b_k)+u_k(\mathcal{D}^{(0)}+u_i\hat{d}_i^{(1)}$$
$$+u_{ij}\hat{d}_{ij}^{(1)})\frac{\chi_k^*}{\Omega}=\left[-\left(v_\parallel\frac{Db_k}{Dt}+\frac{Dv_{Ek}}{Dt}\right)+\frac{\mu B}{m}I_{kr}\hat{d}_r^{(1)}\right]\frac{\chi_k^*}{\Omega}$$
$$+u_i\left(\frac{\delta_{ik}}{\sqrt{\mu B}}\hat{\mathcal{D}}^{(0)}\sqrt{\mu B}\frac{\chi_k^*}{\Omega}+\hat{\mathcal{D}}_{i,k}^{(1)}\frac{\chi_k^*}{\Omega}\right)+u_{ij}\hat{\mathcal{D}}_{ij,k}^{(1)}\frac{\chi_k^*}{\Omega}$$
$$+u_{ijk}\hat{d}_{ij}^{(1)}\frac{\chi_k^*}{2\Omega},\tag{B3}$$

where we used the identitities

$$2u_qu_r=v_\perp^2I_{qr}+u_{qr},$$
$$2u_q(u_ru_s-u_r^*u_s^*)=v_\perp^2u_j(\delta_{js}I_{qr}+\delta_{jr}I_{qs}-\delta_{jq}I_{rs})+u_{qrs},$$

and the operators $\hat{\mathcal{D}}_{i,k}^{(1)}$ and $\hat{\mathcal{D}}_{ij,k}^{(1)}$ are defined by

$$\hat{\mathcal{D}}_{i,k}^{(1)}\frac{\chi_k^*}{\Omega}=(2\delta_{is}I_{kr}-\delta_{ik}I_{rs})\Lambda_{rs,\nu}^{(1)}\frac{\partial}{\partial z_\nu}\frac{\mu B}{m\Omega}\chi_k^*+(\delta_{is}I_{kr}-\delta_{ir}I_{ks})\times\left(v_\parallel\frac{\partial b_s}{\partial x_r}+\frac{\partial v_{Es}}{\partial x_r}\right)\frac{\chi_k^*}{2\Omega},\tag{B4}$$
$$\hat{\mathcal{D}}_{ij,k}^{(1)}\frac{\chi_k^*}{\Omega}=\frac{(\delta_{ik}\delta_{jr}+\delta_{ir}\delta_{jk})}{4}\Lambda_{r,\nu}^{(1)}\frac{\partial}{\partial z_\nu}\frac{\chi_k^*}{\Omega}.$$

The Larmor phase average of $-\hat{\mathcal{L}}(u_k^*/\Omega)\chi_k$ is

$$-\oint d\Theta\hat{\mathcal{L}}\frac{u_k^*\chi_k}{\Omega}=-\left(v_\parallel\frac{Db_q}{Dt}+\frac{Dv_{Eq}}{Dt}\right)\frac{\chi_q^*}{\Omega}+\frac{\mu}{m}\frac{\partial}{\partial z_\nu}\frac{B\Lambda_{r,\nu}^{(1)}I_{rq}\chi_q^*}{\Omega}=\frac{1}{B}\frac{\partial}{\partial z_\nu}\frac{\mu B^2}{m\Omega}\Lambda_{q,\nu}^{(1)}\chi_q^*,$$

where

$$\epsilon_{kpq}b_p\frac{\partial}{\partial z_\nu}B\Lambda_{r,\nu}I_{rq}=\epsilon_{kpq}b_p\left(\frac{\partial}{\partial x_r}BI_{rq}+\frac{\partial}{\partial v_\parallel}B\tilde{V}_{\parallel,r}^{(1)}+\frac{\partial}{\partial\mu}B\tilde{\mu}_r^{(1)}\right)=0,\tag{B5}$$

$$\Lambda_{r,\nu}\frac{\partial}{\partial z_\nu}\mu B=\mu\frac{\partial B}{\partial x_r}+B\tilde{\mu}_r^{(1)}=-m\left(v_\parallel\frac{Db_r}{Dt}+\frac{Dv_{Er}}{Dt}\right).\tag{B6}$$

2. $-\hat{\mathcal{L}}(u_{kl}^*/2\Omega)\chi_{kl}(x_j,v_\parallel,\mu,t)$

Let $(u_{kl}^*/2\Omega)\chi_{kl}$ be expressed as follows:

$$\frac{u_{kl}^*}{2\Omega}\chi_{kl}=\frac{v_qv_r}{2}(\epsilon_{kpq}b_pI_{lr}+\epsilon_{kpr}b_pI_{lq})\chi_{kl}=-v_qv_r\chi_{qr}^*,$$

where we find it convenient to introduce the phase-space function χ_{qr}^* defined by

$$\chi_{qr}^*\equiv\frac{1}{2}(\epsilon_{qpk}b_pI_{rl}+\epsilon_{rpk}b_pI_{ql})\chi_{kl}.\tag{B7}$$

The functions $\chi_{kl}=\chi_{lk}$ and $\chi_{qr}^*=\chi_{rq}^*$ are symmetric, with $\chi_{qq}^*=0$.

Substituting Eq. (42) for $\hat{\mathcal{L}}$, and using the identities given by (A3) and (A7), we proceed as above and we have

$$-\hat{\mathcal{L}}\frac{u_{kl}^*}{2\Omega}\chi_{kl}(x_j,v_\parallel,\mu)=\hat{\mathcal{L}}\frac{v_kv_l}{\Omega}\chi_{kl}^*=\frac{\chi_{kl}^*}{\Omega}\hat{\mathcal{L}}v_kv_l+v_kv_l\hat{\mathcal{L}}\frac{\chi_{qr}^*}{\Omega}$$
$$=-(I_{kr}\delta_{ls}+I_{lr}\delta_{ks})\left(v_\parallel\frac{\partial b_s}{\partial x_r}+\frac{\partial v_{Es}}{\partial x_r}\right)\frac{\mu B}{m\Omega}\chi_{kl}^*$$
$$+\frac{2\mu^2B^2\epsilon_{ipk}b_pI_{jl}}{m^2\Omega}\hat{d}_{ij}^{(1)}\chi_{kl}+u_i\hat{\mathcal{D}}_{i,kl}^{(1)}\frac{\chi_{kl}^*}{\Omega}$$
$$+u_{ij}\left[\frac{(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})}{\mu B}\hat{\mathcal{D}}^{(0)}\frac{\mu B}{4\Omega}\chi_{kl}^*+\hat{\mathcal{D}}_{ij,kl}^{(1)}\frac{\chi_{kl}^*}{\Omega}\right]$$
$$+u_{ikl}\hat{d}_i^{(1)}\frac{\chi_{kl}^*}{4\Omega}-\frac{u_{ijkl}}{4\Omega}\hat{d}_{ij}^{(1)}\chi_{kl},\tag{B8}$$

where the operators $\hat{\mathcal{D}}_{i,kl}^{(1)}$ and $\hat{\mathcal{D}}_{ij,kl}^{(1)}$ are defined by

$$\hat{\mathcal{D}}_{i,kl}^{(1)}\frac{\chi_{kl}^*}{\Omega}=\frac{(I_{sk}\delta_{il}+I_{sl}\delta_{ik})}{\mu B}\Lambda_{s,\nu}^{(1)}\frac{\partial}{\partial z_\nu}\frac{\mu^2B^2}{2m\Omega}\chi_{kl}^*,\tag{B9}$$

$$\hat{\mathcal{D}}_{ij,kl}^{(1)}\frac{\chi_{kl}^*}{\Omega}=-[(I_{ir}I_{jq}+I_{jr}I_{iq})I_{ls}\epsilon_{qpk}b_p-(I_{il}I_{jq}+I_{jl}I_{iq})I_{kr}\epsilon_{qps}b_p]\left(v_\parallel\frac{\partial b_s}{\partial x_r}+\frac{\partial v_{Es}}{\partial x_r}\right)\frac{\chi_{kl}}{4\Omega}.\tag{B10}$$

Note that $b_r\chi_{qr}^*=b_q\chi_{qr}^*=0$ and $\delta_{qr}\chi_{qr}^*=0$, and therefore

$$\left(\frac{\mu BI_{rq}}{m}+v_\parallel^2b_rb_q\right)\hat{\mathcal{L}}\frac{\chi_{qr}^*}{2\Omega}=\frac{\mu B}{m}\delta_{rq}\hat{\mathcal{L}}\frac{\chi_{qr}^*}{2\Omega}=0.$$

The Larmor phase average of $-\hat{\mathcal{L}}(u_{kl}^*/2\Omega)\chi_{kl}$ is

$$-\oint d\Theta\hat{\mathcal{L}}\frac{u_{kl}^*\chi_{kl}}{2\Omega}=\frac{4\mu B^2\epsilon_{ipk}b_pI_{lj}}{m^2\Omega}\tilde{\mu}_{ij}^{(1)}\chi_{kl}+\frac{2\mu^2B^2\epsilon_{ipk}b_pI_{lj}}{m^2\Omega}\Lambda_{ij,\nu}^{(1)}\frac{\partial}{\partial z_\nu}\chi_{kl}=\frac{1}{B}\frac{\partial}{\partial z_\nu}\frac{2\mu^2B^3}{m^2\Omega}\Lambda_{ij,\nu}^{(1)}\epsilon_{ipk}b_pI_{jl}\chi_{kl}.\tag{B11}$$

Proceeding as above, it can readily be established that the Larmor phase averages of $\hat{\mathcal{L}}(u_{klm}^*/3\Omega)\chi_{klm}(x_j,v_\parallel,\mu,t)$ and $\hat{\mathcal{L}}(u_{klmn}^*/4\Omega)\chi_{klmn}(x_j,v_\parallel,\mu,t)$ are equal to zero:

$$\oint d\Theta\,\hat{\mathcal{L}}\frac{u_{klm}^*}{3\Omega}\chi_{klm}=0,$$

$$\oint d\Theta\; \hat{\mathcal{L}} \frac{\overset{*}{u}_{klmn}}{4\Omega} \chi_{klmn} = 0.$$

APPENDIX C: EVALUATION OF $\hat{\mathcal{L}}\tilde{F}_1$

In this appendix, we express $\hat{\mathcal{L}}\tilde{F}_1$ as a polynomial in the perpendicular velocity vectors.

Substituting Eq. (40) for $\tilde{F}_1$, we have

$$\hat{\mathcal{L}}\tilde{F}_1 = -\hat{\mathcal{L}}\left(\frac{\overset{*}{u}_k}{\Omega}\hat{d}_k^{(1)}\bar{F}_0 + \frac{\overset{*}{u}_k u_l}{\Omega}\hat{d}_{kl}^{(1)}\bar{F}_0\right).$$

In Appendix B, we evaluated terms of the form $-\hat{\mathcal{L}}(u_k^*/\Omega)\chi_k(x_j,v_\parallel,\mu,t)$ and $-\hat{\mathcal{L}}(u_k^* u_l/\Omega)\chi_{kl}(x_j,v_\parallel,\mu,t)$. Substituting $\chi_k=\hat{d}_k^{(1)}\bar{F}_0=\Lambda_{k,\nu}^{(1)}(\partial/\partial z_\nu)\bar{F}_0$ in Eq. (B3) and $\chi_{kl}=\hat{d}_{kl}^{(1)}\bar{F}_0=\Lambda_{kl,\nu}^{(1)}(\partial/\partial z_\nu)\bar{F}_0$ in Eq. (B8), we obtain

$$\hat{\mathcal{L}}\tilde{F}_1 = \hat{D}^{(1)}\bar{F}_0 + u_i\hat{d}_i^{(2)}\bar{F}_0 + u_{ij}\hat{d}_{ij}^{(2)}\bar{F}_0 + u_{ijk}\hat{d}_{ij}^{(1)}\frac{\epsilon_{kpq}b_p}{2\Omega}\hat{d}_q^{(1)}\bar{F}_0 + u_{ikl}\hat{d}_i^{(1)}\frac{\epsilon_{kpq}b_p I_{lr}}{4\Omega}\hat{d}_{qr}^{(1)}\bar{F}_0 - \frac{u_{ijkl}}{4\Omega}\hat{d}_{ij}^{(1)}\hat{d}_{kl}^{(1)}\bar{F}_0, \tag{C1}$$

where the operator $\hat{D}^{(1)}$ is defined by

$$\hat{D}^{(1)}\bar{F}_0 = \frac{1}{B}\frac{\partial}{\partial z_\tau}\frac{\mu B^2}{m\Omega}\Lambda_{q,\tau}^{(1)}\epsilon_{qpk}b_p\Lambda_{k,\nu}^{(1)}\frac{\partial \bar{F}_0}{\partial z_\nu} + \frac{1}{B}\frac{\partial}{\partial z_\tau}\frac{2\mu^2B^3}{m^2\Omega}\Lambda_{ij,\tau}^{(1)}\epsilon_{ipk}b_p I_{jl}\Lambda_{kl,\nu}^{(1)}\frac{\partial \bar{F}_0}{\partial z_\nu}, \tag{C2}$$

and the operators $\hat{d}_i^{(2)}$ and $\hat{d}_{ij}^{(2)}$, with $\hat{d}_{ij}^{(2)}=\hat{d}_{ji}^{(2)}$, are defined by

$$\hat{d}_i^{(2)}\bar{F}_0 = \frac{\delta_{ik}}{\sqrt{\mu B}}\hat{\mathcal{D}}^{(0)}\frac{\sqrt{\mu B}}{\Omega}\Lambda_{k,\nu}^{(1)*}\frac{\partial \bar{F}_0}{\partial z_\nu} + \Lambda_{i,\nu}^{(2)}\frac{\partial \bar{F}_0}{\partial z_\nu} + \Lambda_{i,\sigma\nu}^{(2)}\frac{\partial^2 \bar{F}_0}{\partial z_\sigma\,\partial z_\nu}, \tag{C3}$$

$$\hat{d}_{ij}^{(2)}\bar{F}_0 = \frac{(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})}{\mu B}\hat{\mathcal{D}}^{(0)}\frac{\mu B}{4\Omega}\Lambda_{kl,\nu}^{(1)*}\frac{\partial \bar{F}_0}{\partial z_\nu} + \Lambda_{ij,\nu}^{(2)}\frac{\partial \bar{F}_0}{\partial z_\nu} + \Lambda_{ij,\sigma\nu}^{(2)}\frac{\partial^2 \bar{F}_0}{\partial z_\sigma\,\partial z_\nu}, \tag{C4}$$

with phase-space functions $\Lambda_{i,\nu}^{(2)}$, $\Lambda_{i,\sigma\nu}^{(2)}$, $\Lambda_{ij,\nu}^{(2)}$, and $\Lambda_{ij,\sigma\nu}^{(2)}$ given by

$$\Lambda_{i,\nu}^{(2)} = \hat{\mathcal{D}}_{i,k}^{(1)}\frac{\Lambda_{k,\nu}^{(1)*}}{\Omega} + \hat{\mathcal{D}}_{i,kl}^{(1)}\frac{\Lambda_{kl,\nu}^{(1)*}}{\Omega}, \tag{C5}$$

$$\Lambda_{i,\sigma\nu}^{(2)} = \frac{\mu B}{4m\Omega}(2\delta_{il}I_{qk} - \delta_{iq}I_{kl})\epsilon_{qps}b_p(\Lambda_{kl,\sigma}^{(1)}\Lambda_{s,\nu}^{(1)} + \Lambda_{kl,\nu}^{(1)}\Lambda_{s,\sigma}^{(1)}), \tag{C6}$$

$$\Lambda_{ij,\nu}^{(2)} = \hat{\mathcal{D}}_{ij,k}^{(1)}\frac{\Lambda_{k,\nu}^{(1)*}}{\Omega} + \hat{\mathcal{D}}_{ij,kl}^{(1)}\frac{\Lambda_{kl,\nu}^{(1)*}}{\Omega}, \tag{C7}$$

$$\Lambda_{ij,\sigma\nu}^{(2)} = \frac{(\delta_{ir}\delta_{jq}+\delta_{jr}\delta_{iq})}{8\Omega}(\Lambda_{r,\sigma}^{(1)}\Lambda_{q,\nu}^{(1)*} + \Lambda_{r,\nu}^{(1)}\Lambda_{q,\sigma}^{(1)*}). \tag{C8}$$

The functions $\Lambda_{q,\nu}^{(1)*}$ and $\Lambda_{qr,\nu}^{(1)*}$ are related to $\Lambda_{q,\nu}^{(1)}$ and $\Lambda_{kl,\nu}^{(1)}$ by

$$\Lambda_{q,\nu}^{(1)*} \equiv \epsilon_{qpk}b_p\Lambda_{q,\nu}^{(1)},$$

$$\Lambda_{qr,\nu}^{(1)*} \equiv \frac{1}{2}(\epsilon_{qpk}b_pI_{rl} + \epsilon_{rpk}b_pI_{ql})\Lambda_{kl,\nu}^{(1)}.$$

APPENDIX D: EQUIVALENCE OF TWO ALTERNATIVE EXPRESSIONS FOR $\tilde{F}_2$

In this appendix, we verify the equivalence of two alternative expressions for $\tilde{F}_2$, given by Eqs. (56) and (57).

Let us evaluate the Larmor phase integrals in Eq. (57).

Using the results of Appendix B to evaluate $\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)}$, where $\tilde{Z}_\nu^{(1)}=-(u_k^*/\Omega)\Lambda_{k,\nu}^{(1)}-(u_k^*u_l/\Omega)\Lambda_{kl,\nu}^{(1)}$, we obtain the Larmor phase integral of $(\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)}-\overline{\hat{\mathcal{D}}\tilde{Z}_\nu^{(1)}})$ (see also Appendix A):

$$\begin{aligned}\int^\Theta \frac{d\Theta}{\Omega}(\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)} - \overline{\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)}}) &= -\frac{\overset{*}{u}_i}{\Omega}\left(\frac{\delta_{ik}}{\sqrt{\mu B}}\hat{\mathcal{D}}^{(0)}\frac{\sqrt{\mu B}}{\Omega}\Lambda_{k,\nu}^{(1)*} + \Lambda_{i,\nu}^{(2)}\right) - \frac{\overset{*}{u}_{ij}}{2\Omega}\left[\frac{(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})}{\mu B}\hat{\mathcal{D}}^{(0)}\frac{\mu B}{4\Omega}\Lambda_{kl,\nu}^{(1)*} + \Lambda_{ij,\nu}^{(2)}\right] \\ &\quad - \frac{\overset{*}{u}_{ijk}}{3\Omega}\left(\Lambda_{ij,\sigma}^{(1)}\frac{\partial}{\partial z_\sigma}\frac{\Lambda_{k,\nu}^{(1)*}}{2\Omega} + \Lambda_{k,\sigma}^{(1)}\frac{\partial}{\partial z_\sigma}\frac{\Lambda_{ij,\nu}^{(1)*}}{4\Omega}\right) + \frac{\overset{*}{u}_{ijkl}}{16\Omega}\Lambda_{ij,\sigma}^{(1)}\frac{\partial}{\partial z_\sigma}\Lambda_{kl,\nu}^{(1)}.\end{aligned} \tag{D1}$$

The Larmor phase integral of $\tilde{Z}_\nu^{(1)}$ is

$$\begin{aligned}\int^\Theta \frac{d\Theta}{\Omega}\tilde{Z}_\nu^{(1)} &= -\frac{u_i}{\Omega^2}\Lambda_{i,\nu}^{(1)} - \frac{u_{ij}}{4\Omega^2}\Lambda_{ij,\nu}^{(1)} \\ &= -\frac{\overset{*}{u}_i}{\Omega^2}\Lambda_{i,\nu}^{(1)*} - \frac{\overset{*}{u}_{ij}}{8\Omega^2}(I_{jl}I_{iq} + I_{il}I_{jq})\epsilon_{qpk}b_p\Lambda_{kl,\nu}^{(1)} \\ &= -\frac{\overset{*}{u}_i}{\Omega^2}\Lambda_{i,\nu}^{(1)*} - \frac{\overset{*}{u}_{ij}}{8\Omega^2}(I_{jl}I_{ik} + I_{il}I_{jk})\Lambda_{kl,\nu}^{(1)*},\end{aligned} \tag{D2}$$

and the product $\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}$ is

$$\begin{aligned}\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)} &= \frac{\overset{*}{u}_i\overset{*}{u}_k}{\Omega^2}\Lambda_{i,\sigma}^{(1)}\Lambda_{k,\nu}^{(1)} + \frac{\overset{*}{u}_i\overset{*}{u}_{kl}}{2\Omega^2}(\Lambda_{i,\sigma}^{(1)}\Lambda_{kl,\nu}^{(1)} + \Lambda_{i,\nu}^{(1)}\Lambda_{kl,\sigma}^{(1)}) + \frac{\overset{*}{u}_{ij}\overset{*}{u}_{kl}}{4\Omega^2}\Lambda_{ij,\sigma}^{(1)}\Lambda_{kl,\nu}^{(1)} \\ &= \overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}} - 2\frac{\overset{*}{u}_i}{\Omega}\Lambda_{i,\sigma\nu}^{(2)} - \frac{\overset{*}{u}_{ij}}{\Omega}\Lambda_{ij,\sigma\nu}^{(2)} - \frac{\overset{*}{u}_{ikl}}{4\Omega^2}(\Lambda_{i,\sigma}^{(1)}\Lambda_{kl,\nu}^{(1)*} + \Lambda_{i,\nu}^{(1)}\Lambda_{kl,\sigma}^{(1)*}) + \frac{\overset{*}{u}_{ijkl}}{8\Omega^2}\Lambda_{ij,\sigma}^{(1)}\Lambda_{kl,\nu}^{(1)},\end{aligned} \tag{D3}$$

where

$$\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}} = \frac{\mu B}{m\Omega^2} I_{ik}\Lambda_{i,\sigma}^{(1)}\Lambda_{k,\nu}^{(1)} + \frac{\mu^2B^2}{2m^2\Omega^2}(2I_{ik}I_{jl} - I_{ij}I_{kl})\Lambda_{ij,\sigma}^{(1)}\Lambda_{kl,\nu}^{(1)}. \tag{D4}$$

Substituting Eq. (D1) for $\int^\Theta (d\Theta/\Omega)(\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)} - \overline{\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)}})$, Eq. (D2) for $\int^\Theta (d\Theta/\Omega)\tilde{Z}_\nu^{(1)}$, and Eq. (D3) for $(\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)} - \overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}})$, we verify that Eq. (57) for $\tilde{F}_2$ is identical to Eq. (56).

APPENDIX E: LARMOR PHASE AVERAGE. $\oint(d\Theta/2\pi)\hat{\mathcal{L}}\tilde{F}_2$

In this appendix, we evaluate $\oint(d\Theta/2\pi)\hat{\mathcal{L}}\tilde{F}_2$ where $\tilde{F}_2$ is given by Eq. (57), and the operator $\hat{\mathcal{L}}$, given by Eq. (42), is

$$\hat{\mathcal{L}} = \hat{D}^{(0)} - \Omega \boldsymbol{v}\times\boldsymbol{b}\cdot\frac{\partial \tilde{Z}_\sigma^{(1)}}{\partial \boldsymbol{v}}\frac{\partial}{\partial z_\sigma} + \left(\frac{e}{m}b_jE_jb_i - \frac{Dv_{Ei}}{Dt} - u_j\frac{\partial v_{Ei}}{\partial x_j}\right)\frac{\partial'}{\partial v_i}. \tag{E1}$$

Averaging over the Larmor phase angle, we obtain

$$\overline{\hat{\mathcal{L}}\tilde{F}_2} = \oint\frac{d\Theta}{2\pi}\hat{\mathcal{L}}\tilde{F}_2 = \hat{\mathcal{D}}^{(1)}\bar{F}_1 + \hat{\mathcal{D}}^{(2)}\bar{F}_0,$$

where $\hat{\mathcal{D}}^{(2)}$, the second-order drift-kinetic operator, is given by

$$\begin{aligned}\hat{D}^{(2)}\bar{F}_0 ={}& \frac{\partial\bar{F}_0}{\partial z_\nu}\oint\frac{d\Theta}{2\pi}\hat{\mathcal{L}}\int^\Theta\frac{d\Theta}{\Omega}(\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)} - \overline{\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)}}) \\ &+ \left(\hat{D}^{(0)}\frac{\partial\bar{F}_0}{\partial z_\nu}\right)\oint\frac{d\Theta}{2\pi}\hat{\mathcal{L}}\int^\Theta\frac{d\Theta}{\Omega}\tilde{Z}_\nu^{(1)} \\ &- \overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}\frac{\partial}{\partial z_\sigma}\left(\hat{D}^{(0)}\frac{\partial\bar{F}_0}{\partial z_\nu}\right) - \frac{1}{2}\frac{\partial^2\bar{F}_0}{\partial z_\sigma\,\partial z_\nu}\hat{D}^{(0)} \\ &\times(\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}})\end{aligned} \tag{E2}$$

and $\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}$ is given by Eq. (D4).

Substituting Eq. (D1) for $\int^\Theta (d\Theta/\Omega)(\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)} - \overline{\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)}})$ and Eq. (D2) for $\int^\Theta (d\Theta/\Omega)\tilde{Z}_\sigma^{(1)}$, we evaluate the Larmor phase averages in Eq. (E2) (see Appendix B):

$$\begin{aligned}&\oint\frac{d\Theta}{2\pi}\hat{\mathcal{L}}\int^\Theta\frac{d\Theta}{\Omega}(\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)} - \overline{\hat{\mathcal{L}}\tilde{Z}_\nu^{(1)}}) \\ &\quad= -\frac{1}{B}\frac{\partial}{\partial z_\sigma}\frac{\mu B^2}{m\Omega}\Lambda_{i,\sigma}^{(1)*}\left(\frac{1}{\sqrt{\mu B}}\hat{\mathcal{D}}^{(0)}\frac{\sqrt{\mu B}}{\Omega}\Lambda_{i,\nu}^{(1)*} + \Lambda_{i,\nu}^{(2)}\right) \\ &\quad\quad - \frac{1}{B}\frac{\partial}{\partial z_\sigma}\frac{2\mu^2B^3}{m^2\Omega}\Lambda_{ij,\sigma}^{(1)*}\left[\frac{(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})}{\mu B}\hat{\mathcal{D}}^{(0)}\frac{\mu B}{4\Omega}\Lambda_{kl,\nu}^{(1)*}\right. \\ &\quad\quad \left. + \Lambda_{ij,\nu}^{(2)}\right],\end{aligned}$$

$$\oint\frac{d\Theta}{2\pi}\hat{\mathcal{L}}\int^\Theta\frac{d\Theta}{\Omega}\tilde{Z}_\nu^{(1)} = -\frac{1}{B}\frac{\partial}{\partial z_\sigma}B(\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}).$$

We can therefore express Eq. (E2) for $\hat{\mathcal{D}}^{(2)}$ as follows:

$$\begin{aligned}\hat{D}^{(2)}\bar{F}_0 ={}& -\frac{1}{B}\frac{\partial}{\partial z_\sigma}\frac{B\sqrt{\mu B}}{m\Omega}\Lambda_{j,\sigma}^{(1)*}\hat{\mathcal{D}}^{(0)}\frac{\sqrt{\mu B}}{\Omega}\Lambda_{j,\nu}^{(1)*}\frac{\partial\bar{F}_0}{\partial z_\nu} \\ &- \frac{1}{B}\frac{\partial}{\partial z_\sigma}\frac{2\mu B^2}{m^2\Omega}\Lambda_{ij,\sigma}^{(1)*}(\delta_{iq}\delta_{jr} + \delta_{jq}\delta_{ir}) \\ &\times\hat{\mathcal{D}}^{(0)}\frac{\mu B}{4\Omega}\Lambda_{qr,\nu}^{(1)*}\frac{\partial\bar{F}_0}{\partial z_\nu} \\ &+ \frac{\partial\bar{F}_0}{\partial z_\nu}\frac{1}{B}\frac{\partial}{\partial z_\sigma}B\Upsilon_{\sigma,\nu},\end{aligned} \tag{E3}$$

where the phase-space function $\Upsilon_{\sigma,\nu}$ is given by

$$\Upsilon_{\sigma,\nu} = -\frac{\mu B}{m\Omega}\Lambda_{i,\sigma}^{(1)*}\Lambda_{i,\nu}^{(2)} - \frac{2\mu^2B^2}{m^2\Omega}\Lambda_{ij,\sigma}^{(1)*}\Lambda_{ij,\nu}^{(2)}.$$

Substituting Eq. (C5) for $\Lambda_{i,\nu}^{(2)}$ and Eq. (C7) for $\Lambda_{ij,\nu}^{(2)}$:

$$\begin{aligned}\Upsilon_{\sigma,\nu} ={}& -I_{ik,rs}\left(\frac{\mu B}{m\Omega}\Lambda_{i,\sigma}^{(1)*}\Lambda_{rs,\tau}^{(1)}\frac{\partial}{\partial z_\tau}\frac{\mu B}{m\Omega}\Lambda_{k,\nu}^{(1)*}\right. \\ &\left. - \frac{\mu^2B^2}{2m\Omega}\Lambda_{rs,\sigma}^{(1)}\Lambda_{i,\tau}^{(1)*}\frac{\partial}{\partial z_\tau}\frac{1}{m\Omega}\Lambda_{k,\nu}^{(1)*}\right) \\ &+ \frac{1}{2Bm\Omega}\Lambda_{i,\sigma}^{(1)*}\frac{\partial}{\partial z_\tau}I_{ik,rs}\frac{\mu^2B^3}{m\Omega}\Lambda_{k,\tau}^{(1)*}\Lambda_{rs,\nu}^{(1)} \\ &+ I_{ir}I_{ks}\frac{\mu B}{2m\Omega^2}\left(v_\parallel\frac{\partial b_s}{\partial x_r} + \frac{\partial v_{Es}}{\partial x_r}\right)(\Lambda_{i,\sigma}^{(1)*}\Lambda_{k,\nu}^{(1)*} \\ &- \Lambda_{k,\sigma}^{(1)*}\Lambda_{i,\nu}^{(1)*}) + I_{ls}I_{ki}I_{rj}\frac{\mu^2B^2}{m^2\Omega^2}\left(v_\parallel\frac{\partial b_s}{\partial x_r} + \frac{\partial v_{Es}}{\partial x_r}\right) \\ &\times(\Lambda_{ij,\sigma}^{(1)}\Lambda_{kl,\nu}^{(1)} - \Lambda_{kl,\sigma}^{(1)}\Lambda_{ij,\nu}^{(1)}),\end{aligned} \tag{E4}$$

$$I_{ik,rs} \equiv I_{ir}I_{ks} + I_{is}I_{kr} - I_{ik}I_{rs}, \tag{E5}$$

and we note that

$$\frac{\partial^2}{\partial z_\sigma\,\partial z_\nu}B\Upsilon_{\sigma,\nu} = 0. \tag{E6}$$

We can also express Eq. (E2) for $\hat{\mathcal{D}}^{(2)}$ as follows:

$$\begin{aligned}\hat{D}^{(2)}\bar{F}_0 ={}& -\hat{\mathcal{D}}^{(0)}\frac{1}{B}\frac{\partial}{\partial z_\sigma}B\frac{\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}}{2}\frac{\partial\bar{F}_0}{\partial z_\nu} + \frac{\partial\bar{F}_0}{\partial z_\nu}\frac{1}{B}\frac{\partial}{\partial z_\sigma}B\Delta_{\sigma,\nu} \\ &+ \frac{\partial\bar{F}_0}{\partial z_\nu}\frac{1}{B}\frac{\partial}{\partial z_\sigma}B\Upsilon_{\sigma,\nu},\end{aligned} \tag{E7}$$

where the phase-space function $\Delta_{\sigma,\nu} = -\Delta_{\nu,\sigma}$, antisymmetric in an interchange of indices, is given by

$$\begin{aligned}\Delta_{\sigma,\nu} ={}& \frac{\partial\dot{Z}_\nu^{(0)}}{\partial z_\tau}\frac{\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\tau^{(1)}}}{2} - \frac{\partial\dot{Z}_\sigma^{(0)}}{\partial z_\tau}\frac{\overline{\tilde{Z}_\tau^{(1)}\tilde{Z}_\nu^{(1)}}}{2} \\ &+ \frac{\mu B}{2m\Omega^2}(\Lambda_{j,\nu}^{(1)*}\hat{\mathcal{D}}^{(0)}\Lambda_{j,\sigma}^{(1)*} - \Lambda_{j,\sigma}^{(1)*}\hat{\mathcal{D}}^{(0)}\Lambda_{j,\nu}^{(1)*}) \\ &+ (\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})\frac{\mu^2B^2}{4m^2}(\Lambda_{kl,\nu}^{(1)*}\hat{D}^{(0)}\Lambda_{ij,\sigma}^{(1)*}\end{aligned}$$

$$- \Lambda_{ij,\sigma}^{(1)*}\hat{D}^{(0)}\Lambda_{kl,\nu}^{(1)*}) \tag{E8}$$

and

$$\frac{\partial^2}{\partial z_\sigma \partial z_\nu} B\Delta_{\sigma,\nu} = 0. \tag{E9}$$

In deriving Eq. (E7), we recall that $\hat{\mathcal{D}}^{(0)}\bar{F}_0 = [\partial/\partial t + \dot{Z}_\tau^{(0)}(\partial/\partial z_\tau)]\bar{F}_0 = 0, (\partial/\partial t)B + (\partial/\partial z_\tau)B\dot{Z}_\tau^{(0)} = 0$, and we rewrite the following terms involving the zeroth-order drift-kinetic operator $\hat{\mathcal{D}}^{(0)}$ in the form

$$\begin{aligned}
&\frac{1}{B}\frac{\partial}{\partial z_\sigma} B\hat{\mathcal{D}}^{(0)} \frac{\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}}{2}\frac{\partial \bar{F}_0}{\partial z_\nu} + \frac{1}{B}\frac{\partial}{\partial z_\sigma} B\frac{\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}}{2}\hat{\mathcal{D}}^{(0)}\frac{\partial \bar{F}_0}{\partial z_\nu} \\
&= \hat{\mathcal{D}}^{(0)}\frac{1}{B}\frac{\partial}{\partial z_\sigma} B\frac{\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}}{2}\frac{\partial \bar{F}_0}{\partial z_\nu} + \frac{1}{B}\frac{\partial}{\partial z_\tau} B\frac{\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}}{2}\frac{\partial \dot{Z}_\tau^{(0)}}{\partial z_\sigma}\frac{\partial \bar{F}_0}{\partial z_\nu} \\
&\quad - \frac{1}{B}\frac{\partial}{\partial z_\sigma} B\frac{\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}}{2}\frac{\partial \dot{Z}_\tau^{(0)}}{\partial z_\nu}\frac{\partial \bar{F}_0}{\partial z_\tau}.
\end{aligned}$$

In summary, the second-order drift-kinetic operator $\hat{\mathcal{D}}^{(2)}$ is given by

$$\begin{aligned}
\hat{D}^{(2)}\bar{F}_0 &= -\hat{\mathcal{D}}^{(0)}\frac{1}{B}\frac{\partial}{\partial z_\sigma} B\frac{\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}}{2}\frac{\partial \bar{F}_0}{\partial z_\nu} + \frac{\partial \bar{F}_0}{\partial z_\nu}\frac{1}{B}\frac{\partial}{\partial z_\sigma} B\Gamma_{\sigma,\nu}^{(2)} \\
&= -\hat{\mathcal{D}}^{(0)}\frac{1}{B}\frac{\partial}{\partial z_\sigma} B\frac{\overline{\tilde{Z}_\sigma^{(1)}\tilde{Z}_\nu^{(1)}}}{2}\frac{\partial \bar{F}_0}{\partial z_\nu} \\
&\quad + \frac{1}{B}\frac{\partial}{\partial z_\nu} B\bar{F}_0\frac{1}{B}\frac{\partial}{\partial z_\sigma} B\Gamma_{\sigma,\nu}^{(2)},
\end{aligned} \tag{E10}$$

where

$$\Gamma_{\sigma,\nu}^{(2)} = \Delta_{\sigma,\nu} + \Upsilon_{\sigma,\nu}. \tag{E11}$$

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