Anomalous skin effect for anisotropic electron velocity distribution function

Igor Kaganovich and Edward Startsev
Princeton Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08543

Gennady Shvets
University of Texas at Austin, Institute for Fusion Studies, Austin, Texas 78712

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The anomalous skin effect in a plasma with a highly anisotropic electron velocity distribution function (EVDF) is very different from the skin effect in a plasma with isotropic EVDF. An analytical solution was derived for the electric field penetrated into plasma with the EVDF described as a Maxwellian with two temperatures $T_x \gg T_z$, where x is the direction along the plasma boundary and z is the direction perpendicular to the plasma boundary. The skin layer was found to consist of two distinct regions of width of order v_{T_x}/ω and v_{T_z}/ω, where $v_{T_{x,z}} = \sqrt{T_{x,z}/m}$ is the thermal electron velocity and ω is the incident wave frequency. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1723461]

In a recent letter,[1] it was shown that a highly anisotropic electron velocity distribution function (EVDF) yields a large skin-layer depth compared with the isotropic EVDF. The EVDF was described as a Maxwellian with two temperatures $T_x \gg T_z$, where x is the direction along plasma boundary and z is the direction perpendicular plasma boundary. The electromagnetic wave is assumed to propagate also along z axis in vacuum. The skin layer was found to be much longer than the skin layer in a plasma with isotropic EVDF. The authors of Ref. 1 showed that under conditions

$$T_x \gg T_z, \quad \frac{v_{T_x}}{\omega} \gg \frac{c}{\omega_p}, \quad \omega_p \gg \omega, \tag{1}$$

where ω is the incident wave frequency, $\omega_p = \sqrt{4\pi e^2 n/m}$ is the plasma frequency, n is the electron density, $v_{T_x} = \sqrt{T_x/m}$, the electric field profile is exponential $E(z) \sim \exp(-z/l_s)$ where

$$l_s = \frac{v_{T_x}}{\omega}. \tag{2}$$

In their analysis the authors of Ref. 1 assumed from the outset that the skin depth is much longer than v_{T_z}/ω, where $v_{T_z} = \sqrt{T_z/m}$, T_z is the electron temperature along z axis perpendicular to the plasma boundary. We show that the skin layer actually consists of two distinctive regions of widths of order v_{T_x}/ω and v_{T_z}/ω. The latter short region was missed in Ref. 1.

In contrast to Ref. 1, we solve Maxwell's equation

$$\frac{d^2}{dz^2}E(z) + \frac{\omega^2}{c^2}E(z) = -\frac{4\pi i\omega}{c^2} j_x, \tag{3}$$

for x component of electric field without making any assumptions. For semi-infinite geometry, the electric field can be calculated by making use of the Fourier transform in the infinite plane by continuing the electric field symmetrically around plasma boundary $[E(-z) = E(z)]$. Following Ref. 2, the Fourier transform of the electric field is given by

$$E(k) = -\frac{2i\omega}{c}B(0)\frac{1}{k^2 - \varepsilon_t(\omega,k)\omega^2/c^2}, \tag{4}$$

where $B(0)$ is the magnetic field at plasma boundary and $\varepsilon_t(\omega,k)$ is the transverse plasma dielectric constant[2]

$$\varepsilon_t(\omega,k) = 1 - \frac{4\pi i}{\omega E(k)} e \int v_x \delta f \, d\mathbf{v}, \tag{5}$$

where δf is the perturbation of electron velocity distribution function due to a planar x-polarized electromagnetic wave with frequency ω and wavenumber $\vec{k} = k\vec{e}_z$. To determine δf and consequently ε_t, we perform the Fourier transform of the Vlasov equation:[1]

$$\delta f(k) = -\frac{e}{im}\left[\frac{E(k) - v_z B(k)/c}{\omega - v_z k}\frac{\partial f_0}{\partial v_x} + \frac{v_x B(k)/c}{\omega - v_z k}\frac{\partial f_0}{\partial v_z}\right]. \tag{6}$$

Because in the planar electromagnetic wave $B(k) = ckE(k)/\omega$, Eq. (5) simplifies to

$$\varepsilon_t(\omega,k) = 1 + \frac{4\pi e^2}{m\omega^2}\int d\mathbf{v}\left[v_x\frac{\partial f_0}{\partial v_x} + \frac{v_x^2 k}{(\omega - v_z k)}\frac{\partial f_0}{\partial v_z}\right]. \tag{7}$$

Substituting f_0 as a Maxwellian with two different temperatures T_x and T_z into Eq. (7) and making use of an algebraic identity

$$\frac{v_x^2 k}{(\omega - v_z k)}\frac{\partial f_0}{\partial v_z} = \frac{m v_x^2}{T_z} f_0 \left(1 + \frac{\omega/v_{T_z}k}{v_z/v_{T_z} - \omega/v_{T_z}k}\right) \tag{8}$$

gives

$$\varepsilon_t(\omega,k)=1-\frac{\omega_p^2}{\omega^2}\left\{1-\frac{T_x}{T_z}\left[1+\frac{\omega}{\sqrt{2}v_{T_z}k}Z\left(\frac{\omega}{\sqrt{2}v_{T_z}k}\right)\right]\right\}, \tag{9}$$

where $Z(s)$ is the plasma dielectric function.[2]

The spatial profile of the electric field $E(z)$ is given by the inverse Fourier transform of Eq. (4),

$$E(z)=-\frac{i\omega}{\pi c}B(0)\int_{-\infty}^{\infty}\frac{e^{ikz}}{k^2-\varepsilon_t(\omega,|k|)\omega^2/c^2}dk. \tag{10}$$

The $|k|$ denotes the fact that $E(z)$ is continued symmetrically to the semi-plane $z<0$ and $E(z)=E(-z)$, which is satisfied by setting $E(k)=E(-k)$.[2] Note that despite $E(z)=E(-z)$, the derivative of $E(z)$ is not continuous at $z=0$.

The contour of integration in Eq. (10) can be shifted into the complex k-plane. Because $|k|=\sqrt{k^2}$, the contour of integration has to enclose the branch point $k=0$ with the cut along the imaginary k axis.[3] As a result, Eq. (10) can be represented as a sum of contributions from poles and an integral along the imaginary axis of the complex k-plane

$$E(z)=-\frac{i\omega}{\pi c}B(0)\left\{\sum_n e^{ik_{pn}z}2\pi i\,\mathrm{Res}\left(\frac{1}{k_{pn}^2-\varepsilon_t(\omega,|k_{pn}|)\omega^2/c^2}\right)\right.$$
$$\left.+\int_0^{\infty}\frac{\mathrm{Im}\,\varepsilon_t(\omega,is)\omega^2/c^2e^{-sz}}{[s^2+\mathrm{Re}\,\varepsilon_t(\omega,is)\omega^2/c^2]^2+[\mathrm{Im}\,\varepsilon_t(\omega,is)\omega^2/c^2]^2}\,ds\right\}. \tag{11}$$

Here, k_{pn} are the poles of denominator in Eq. (10) in the complex k-plane given by

$$k_{pn}^2-\omega^2/c^2+\omega_p^2/c^2\times\left\{1-\frac{T_x}{T_z}\left[1+\frac{\omega}{\sqrt{2}v_{T_z}k_{pn}}Z\left(\frac{\omega}{\sqrt{2}v_{T_z}k_{pn}}\right)\right]\right\}=0. \tag{12}$$

In the limit of small k, $\omega/\sqrt{2}|k|v_{T_z}\equiv s\gg 1$ and $1+sZ(s)\to -1/2s^2$,

$$\varepsilon_t(\omega,k)=1-\frac{\omega_p^2}{\omega^2}\left(1+\frac{v_{T_x}^2k^2}{\omega^2}\right) \tag{13}$$

and the pole is

$$k_{p1}^2=-\frac{(\omega_p^2-\omega^2)}{c^2+\omega_p^2v_{T_x}^2/\omega^2}. \tag{14}$$

Under the conditions of Eq. (1), Eq. (14) simplifies to become

$$k_{p1}=i\frac{\omega}{v_{T_x}}. \tag{15}$$

Calculations of the residual in Eq. (15) give the electric field profile from this pole

$$E_{p1}(z)=-\frac{i\omega^2c}{\omega_p^2v_{T_x}}B(0)e^{ik_{p1}z}. \tag{16}$$

This corresponds to the exponential decay of the electric field with the scale $l_s=v_{T_x}/\omega$ described in Ref. 1.

However, there is another pole $k_{p2}\gg k_{p1}$. In the limit of large k, so that $\omega/\sqrt{2}|k|v_{T_z}\equiv s\ll 1$ and $sZ(s)\ll 1$, under the conditions in Eq. (1), Eq. (12) simplifies to become

$$\mathrm{Re}\,k_{p2}=\frac{\omega_p}{c}\sqrt{T_x/T_z}. \tag{17}$$

Note that according to Eq. (1)

$$\frac{\omega}{k_{p2}v_{T_z}}=\frac{\omega c}{\omega_p v_{T_x}}\ll 1. \tag{18}$$

Imaginary part of k_{p2} can be determined by taking into account imaginary part of $Z(0)=i\sqrt{\pi}$, which yields

$$\mathrm{Im}\,k_{p2}=\frac{\sqrt{\pi}\omega}{2\sqrt{2}v_{T_z}}. \tag{19}$$

The pole at k_{p2} gives rise to the rapidly oscillating field in the plasma

$$E_{p2}(z)=\frac{\omega}{ck_{p2}}B(0)e^{ik_{p2}z}. \tag{20}$$

Under the conditions of Eq. (1), the contribution of the branch point [last integral in Eq. (11)] is small. Indeed, the width of the integral is determined by the dispersion function and it is equal to ω/v_{T_z} while the amplitude of the function under the integral is of order $c^2T_z/\omega_p^2T_x$. This gives an estimate for the contribution from the branch point $E_b(z)$,

$$E_b(z)\sim B(0)\frac{\omega^2c\sqrt{T_z/T_x}}{\pi v_{T_x}\omega_p^2}, \tag{21}$$

which is $2\pi\sqrt{T_x/T_z}$ times smaller than $E_{p1}(z)$ in Eq. (16). Note that it is in contrast to the "classical" anomalous skin effect in plasma with an isotropic EDVF, where the contribution of the branch point is comparable to the pole contribution.[3]

Exact numerical integration of the inverse Fourier transform of Eq. (10) confirms the importance of the oscillating solution, as shown in Fig. 1. Therefore, the prediction of Ref.

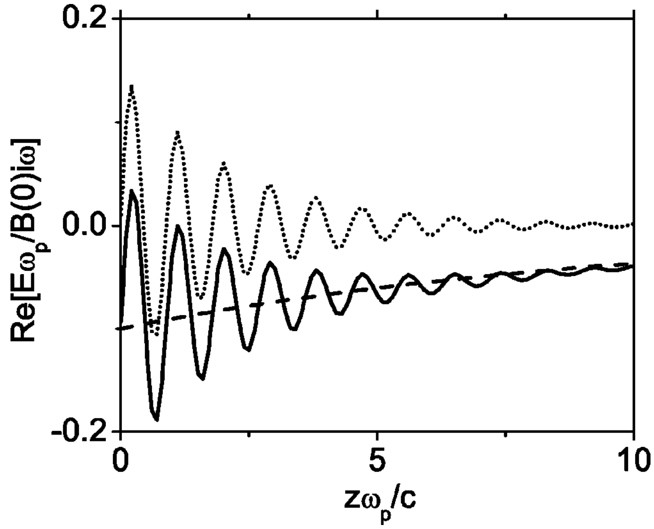

FIG. 1. The electric field in plasma with $v_{T_x}=0.1c$, $\omega=0.01\omega_p$, $T_x/T_z=50$. Solid line shows the real part of the electric field profile obtained from the full solution making use of Eq. (10). The dashed line corresponds to the solution of Ref. 1 $E_{p1}e^{ik_{p1}z}$ given by Eq. (16). The dotted line corresponds to $E_{p2}e^{ik_{p2}z}$ given by Eq. (20).

1 of monotonically decaying electric field is inaccurate.

Finally, the profile of the electric field is a sum of the two complex exponents given by Eqs. (16) and (20),

$$E(z)=E_{p1}\exp(-ik_{p1}z)+E_{p2}\exp(-ik_{p2}z), \tag{22}$$

with k_{p1} given by Eq. (15) and k_{p2} given by Eqs. (17) and (19). The first pole in Eq. (22) produces a slowly decaying electric field, while the second pole produces a faster decaying electric field ($\mathrm{Re}\,k_{p2} \gg \mathrm{Re}\,k_{p1}$). Note that, in contrast to anomalous skin effect in plasma with an isotropic EVDF, the skin layer in a plasma with an anisotropic EVDF consists of two distinct layers with very different lengths. The electric field amplitude in the short layer, E_{p2}, is larger in most cases than the amplitude of long layer, E_{p1}. It follows from Eqs. (16) and (20) that

$$\frac{|E_{p2}|}{|E_{p1}|}\sim\frac{\omega_p v_{T_x}}{\omega c}\frac{1}{\sqrt{T_x/T_z}}, \tag{23}$$

and under conditions in Eq. (1) amplitude of the electric field E_{p2} is large compared with E_{p1} for modest anisotropy ($\sqrt{T_x/T_z}\sim 1$), whereas amplitudes are comparable for very large anisotropy ($\sqrt{T_x/T_z}\gg 1$), as can be seen in Fig. 1.

The surface impedance—the ratio of the electric and magnetic fields at the boundary—characterizes the absorption coefficient and the phase of reflected wave.[2,4] Substituting Eqs. (16) and (20) [together with Eqs. (17) and (19)] yields

$$\zeta=\frac{E(0)}{B(0)}=-\frac{i\omega^2 c}{\omega_p^2 v_{T_x}}+\frac{\omega}{\omega_p\sqrt{T_x/T_z}+ic\sqrt{\pi}\omega/2\sqrt{2}v_{T_z}}. \tag{24}$$

The energy dissipation in the plasma and, correspondingly, the absorption coefficient are determined by the real part of the surface impedance. Under the conditions (1), it follows from Eq. (24) that the real part of the surface impedance can be expressed as

$$\mathrm{Re}(\zeta)=\frac{\omega}{\omega_p\sqrt{T_x/T_z}}. \tag{25}$$

Therefore, the absorption coefficient in semi-infinite plasma is entirely governed by the short scale region of width of order v_{T_z}/ω. Equation (25) recovers the result previously obtained in Ref. 5.

Generally speaking, the anisotropic EVDF is the subject to the Weibel instability.[6] The growth rate can be obtained analyzing the poles of Eq. (12) with real k, but complex ω. The maximum growth rate is given by $\gamma=\omega_p v_{T_x}/c$.[6] Instability develops quickly during the penetration of the electric field into the plasma on a time scale which is shorter than the wave period. Indeed, $\gamma/\omega=\omega_p v_{T_x}/c\omega=\omega_p/\omega(v_{T_x}/c)$. The γ/ω ratio is large according to the assumption in Eq. (1) and, therefore, the instability has time to develop. However, particle-in-cell simulations carried out in Ref. 7 show that the Weibel instability may saturate at relatively low levels where the EVDF remains very anisotropic.

In summary, we have discovered that the electric field structure in the skin layer differs from a monotonic exponentially decaying profile predicted in Ref. 1. In fact, the skin layer contains multiple oscillations of the electric field. The nonmonotonic nature of the electric field decay is responsible for the finite dissipation neglected in Ref. 1. The anisotropic EVDF is subject to the Weibel instability, which develops quickly during the penetration of the electric field into the plasma. However, the Weibel instability may saturate at relatively low levels where the EVDF remains very anisotropic. The exact estimates of the saturation level are difficult analytically and, therefore, self consistent particle-in-cell simulations are necessary for making further progress.

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