Scale hierarchy created in plasma flow

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The cooperation of nonlinearity (producing collapsed characteristics) and dispersion (unfolding singularities) underlies a robust mechanism that imparts two distinct scales (L measuring the system size, and δ_i typically of the order of the ion skin depth) to the double Beltrami states of a two-fluid plasma. It is shown that the conventional single-fluid model [magnetohydrodynamic (MHD)] seemingly valid for a large system ($\delta_i/L \approx 0$), fails to capture the small scale that is created by the singular perturbation of the two-fluid effect (dispersion). The small-scale component plays an important role in various plasma phenomena, such as coronal heating. The double Beltrami model is compared and contrasted with the standard MHD pathway (Parker's model of current sheet, for instance). © *2004 American Institute of Physics.* [DOI: 10.1063/1.1762877]

I. INTRODUCTION

It is physically reasonable to infer that processes at microscopic scales (associated with individual particle motion) influence the evolution of observed macroscopic structures even though we may not find explicit expressions of these scales in the observable of the final state. Reconnection of flux tubes, flares or accretion discs are a few examples where the "invisible" microdynamics is successfully invoked to understand the macro phenomena. The overall dynamics of all these systems are governed by (almost) ideal fluid equations. Some ideal invariants, such as flux or circulation conservation, must be therefore broken to enable a topological evolution. No known dissipation mechanism, operating at large scales, can account for the observed fast rates at which such phenomena occur. Dissipation processes acting on short scales could, however, cause rapid changes and it is quite natural to posit that the observable state, perhaps, came into existence through the mediation of short scale structures. Understanding the creation and working of such "mesoscopic" structure has already become one of the more "universal" pursuits in our exploration and study of physical systems with interacting hierarchies of scales that are vastly separated.

A minimal model for this pursuit is provided by Hall magnetohydrodynamics (Hall MHD) characterized by two disparate, interacting scales. The macroscopic scale of the system is much larger than the ion skin depth, the intrinsic scale of Hall MHD at which ion kinetic (inertia) effects become important. We will derive an analytical solution of a self-organization model for Hall MHD. The solution is expressed by a combination of two eigenstates of vortices (eigenfunctions of the curl operator)—one has a large scale defining the macroscopic structure (which may be regarded as the universality class of the system), while the other is in the scale hierarchy of the ion skin depth. It is the singular perturbation due to the hall term that connects the two scales and produces the interacting paired vortex states.

II. SINGULAR PERTURBATION

Generally the physics controlling the microscopic scale appears as a singular perturbation to the macroscopic equations of motion, i.e., it enters through a term containing higher order derivatives multiplying a small coefficient (ε). The measure of our success in understanding the system will be determined by our ability to understand the changes brought about by the singular perturbation.

We begin with somewhat formal considerations. Let us call the purely macroscopic model as the "0-model" and the more comprehensive microscopic model as the "ε-model." If a solution f_ε of our ε-model converges to f_0 that satisfies the 0-model when $\lim \varepsilon \to 0$, the singular perturbation has negligible effect on this solution and the 0-model would be quite adequate. The "ε-model" may still be useful in weeding out any unphysical f_0 that does not follow as a limit of f_ε. The "entropy solution" in the theory of shocks is a well-known example.

The situation changes drastically if f_ε is singular as $\lim \varepsilon \to 0$. Now the 0-model is not aware of important "physical" solutions that can be constructed in the ε-model. The ε-model may have a far richer physical content; the singular perturbation, then, is the harbinger of new complexity. And we are likely to run into phenomena which defy understanding in a purely macroscopic model.

We choose Hall MHD (two-fluid magneto-plasma with flows) as our ε-model; it has both the macroscopic and microscopic scales, the latter introduced as a singular perturbation. For negligible (or a specific kind of) flows, the system degenerates into standard magnetohydrodynamics (MHD) with a single relevant scale (macroscopic) corresponding to the 0-model of our argument.

In the present paper, we examine a particular solution f_ε of the ε-model (Hall MHD) that is called a double Beltrami field.[1] This solution is "decomposed" as a sum of the universality class and the singular (divergent) part, i.e., $f_\varepsilon = f_\varepsilon^- + f_\varepsilon^+$, where $f_\varepsilon^- \to f_0$ (macroscopic solution of MHD) and f_ε^+ is singular in the limit of $\varepsilon \to 0$. Within the framework of this model we will compare and contrast the virtues of two contending mechanisms for the creation of short-scale fields. Both these models, the relative recent double Beltrami relaxation model,[2] and the well-known "current sheet" model of Parker,[3] have been invoked, for instance, to explain the heating of the solar corona. The former relies on the dissipation of the short scale during the relaxation process.[4] It describes a "pattern" generated by the cooperation of the nonlinearity (convective type) and the dispersion (singular perturbation due to the two-fluid effect).

III. SINGULAR PERTURBATION INDUCED BY THE HALL EFFECT

The flow velocity $\boldsymbol{V}$ and the electric current $\boldsymbol{J}$ emerge as the natural dynamical variables for Hall MHD. For simplicity, we consider a quasi-neutral plasma with singly charged ions. Neglecting its small inertia, the electron equation of motion is

$$\boldsymbol{E}+\boldsymbol{V}_e\times\boldsymbol{B}+\frac{1}{en}\nabla p_e=0, \tag{1}$$

where $\boldsymbol{V}_e$ and p_e are, respectively, the electron flow velocity and pressure, $\boldsymbol{E}$ ($\boldsymbol{B}$) is the electric (magnetic) field, $-e$ is the electron charge, and n is the number density. These are related to the ion flow velocity $\boldsymbol{V}_i$ by $\boldsymbol{V}=(M\boldsymbol{V}_i+m\boldsymbol{V}_e)/(M+m)\approx\boldsymbol{V}_i$, where M (m) is the ion (electron) mass ($M \gg m$). The ion velocity $\boldsymbol{V}$ obeys

$$\frac{\partial}{\partial t}\boldsymbol{V}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}=\frac{e}{M}(\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B})-\frac{1}{Mn}\nabla p_i, \tag{2}$$

where p_i is the ion pressure. We can eliminate $\boldsymbol{E}$ and $\boldsymbol{V}_e$ using $\boldsymbol{V}_e=\boldsymbol{V}-\boldsymbol{j}/(en)$, $\boldsymbol{j}=\mu_0^{-1}\nabla\times\boldsymbol{B}$ ($\boldsymbol{j}$ is the electric current), and $\boldsymbol{E}=-\partial\boldsymbol{A}/\partial t-\nabla\phi$, where $\boldsymbol{A}$ (ϕ) is the vector (scalar) potential. We assume barotropic relations to write $n^{-1}\nabla p_j=\nabla\Pi_j$ ($j=e,i$).

Choosing an arbitrary length scale L_0 and representative magnetic field B_0 and density n_0, we normalize variables as

$$\boldsymbol{x}=L_0\hat{\boldsymbol{x}},\quad \boldsymbol{B}=B_0\hat{\boldsymbol{B}},\quad n=n_0\hat{n},\quad t=(L_0/V_A)\hat{t},$$
$$p=(B_0^2/\mu_0)\hat{p},\quad \phi=(L_0B_0V_A)\hat{\phi},\quad \boldsymbol{V}=V_A\hat{\boldsymbol{V}},$$

where $V_A=B_0/\sqrt{\mu_0 M n_0}$ is the Alfvén speed. Equations (1) and (2) transform to

$$\frac{\partial}{\partial\hat{t}}\hat{\boldsymbol{A}}=\left(\hat{\boldsymbol{V}}-\frac{\varepsilon}{\hat{n}}\hat{\nabla}\times\hat{\boldsymbol{B}}\right)\times\hat{\boldsymbol{B}}-\hat{\nabla}(\hat{\phi}-\varepsilon\hat{\Pi}_e), \tag{3}$$

$$\frac{\partial}{\partial\hat{t}}(\varepsilon\hat{\boldsymbol{V}}+\hat{\boldsymbol{A}})=\hat{\boldsymbol{V}}\times(\hat{\boldsymbol{B}}+\varepsilon\hat{\nabla}\times\hat{\boldsymbol{V}})-\hat{\nabla}(\varepsilon\hat{V}^2/2+\varepsilon\hat{\Pi}_i+\hat{\phi}), \tag{4}$$

where the scaling coefficient $\varepsilon=\delta_i/L_0$ is the ratio of the intrinsic scale, the ion skin depth

$$\delta_i=\frac{c}{\omega_{pi}}=\sqrt{\frac{M}{\mu_0 n e^2}}$$

to the macroscopic scale.

We note that (3) and (4) have the gauge freedom with respect to the electromagnetic potentials $\boldsymbol{A}$ and ϕ. The system we will later analyze (in Sec. IV) is, however, expressed in terms of $\boldsymbol{B}=\nabla\times\boldsymbol{A}$, and hence, the gauge ambiguity will be removed.

In what follows, we remove $\hat{}$ to simplify notation. In the present purpose, constant density incompressible model is sufficient, so we assume $n=1$ and $\Pi_j=p_j$ ($j=e,i$).

Subtracting (3) from (4), and taking the curl of (3), we obtain the familiar form of Hall MHD equations:

$$\partial_t\boldsymbol{V}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}=(\nabla\times\boldsymbol{B})\times\boldsymbol{B}-\nabla p, \tag{5}$$

$$\partial_t\boldsymbol{B}=\nabla\times[(\boldsymbol{V}-\varepsilon\nabla\times\boldsymbol{B})\times\boldsymbol{B}], \tag{6}$$

where $p=p_i+p_e$. For ε much less than unity, the system of Hall MHD equations (5) and (6) constitutes our ε-model. Notice that in the $\lim\varepsilon\to 0$, the Hall MHD equations (5) and (6) reproduce the Lorentz force and the induction equations of MHD. The MHD system is clearly scale-less (that is it has no scale apart from the macroscopic system size) and represents the 0-model for the current investigation. The "Hall term" $\varepsilon(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$ in (6) is the singular perturbation included in the ε-model. Just as the dissipative viscosity term (multiplied by the reciprocal Reynolds number) provides the singular perturbation in the well-known Navier–Stokes equations, the "dispersive" Hall term does it for Hall MHD.

IV. BELTRAMI CONDITIONS

The equivalence of the two-fluid model to a system of vortex transport equations is easily revealed on taking the curl of (3) and (4) leading to

$$\partial_t\boldsymbol{\Omega}_j-\nabla\times(\boldsymbol{U}_j\times\boldsymbol{\Omega}_j)=0 \quad (j=1,2) \tag{7}$$

for a pair of "vorticities" and their corresponding flows:

$$\begin{cases}\boldsymbol{\Omega}_1=\boldsymbol{B},\\ \boldsymbol{U}_1=\boldsymbol{V}-\varepsilon\nabla\times\boldsymbol{B},\end{cases}\quad \begin{cases}\boldsymbol{\Omega}_2=\boldsymbol{B}+\varepsilon\nabla\times\boldsymbol{V},\\ \boldsymbol{U}_2=\boldsymbol{V}.\end{cases}$$

We note that the distinctness as well as coupling of the two vorticities ($\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_2$) is induced by the singular perturbation scaled by ε. For $\varepsilon=0$, the sets ($\boldsymbol{\Omega}_j$, $\boldsymbol{U}_j$) become degenerate, and the system (7) reduces into a single vortex dynamics, the "0-model."

The simplest stationary solution to the ε-model (7) is given by the "Beltrami conditions"

$$\boldsymbol{U}_j=\mu_j\boldsymbol{\Omega}_j \quad (j=1,2), \tag{8}$$

implying the alignment of the vorticities with the corresponding flows.[1] Writing $a=1/\mu_1$ and $b=1/\mu_2$, and assuming that a and b are constants, the Beltrami conditions (8) read as

$$\boldsymbol{B}=a(\boldsymbol{V}-\varepsilon\nabla\times\boldsymbol{B}), \tag{9}$$

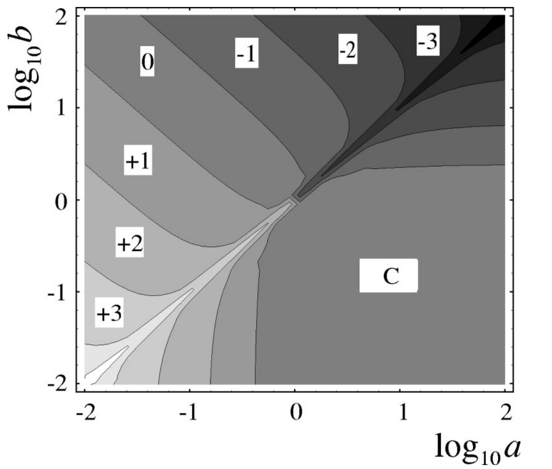

FIG. 1. Separation of the two length scales of double Beltrami fields. The contour of $\log_{10}(|\lambda_-|/|\lambda_+|)$ as the function of the two Beltrami parameters a and b are shown. In the region "C," $\lambda_\pm$ are the complex conjugates, and hence, $|\lambda_-|=|\lambda_+|$.

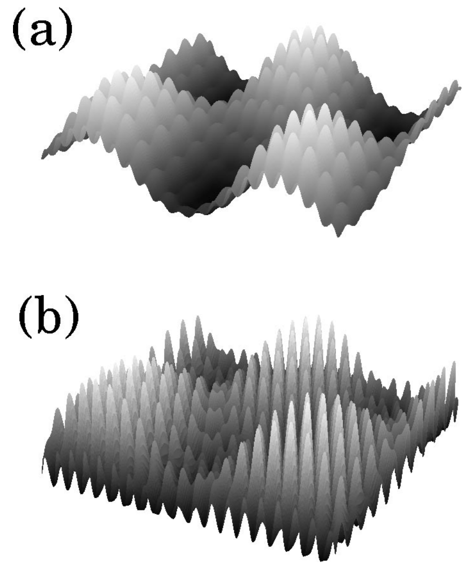

FIG. 2. Distribution of a double Beltrami field ($\lambda_-=0.8$, $\lambda_+=10$); (a) plot of $|B|$, (b) plot of $|\nabla\times B|$.

$$\boldsymbol{B}+\varepsilon\nabla\times\boldsymbol{V}=b\boldsymbol{V}. \tag{10}$$

Combining (9) and (10) yields ($\boldsymbol{u}=\boldsymbol{B}$ or $\boldsymbol{V}$)[1]

$$\varepsilon^2\nabla\times(\nabla\times\boldsymbol{u})+\varepsilon(a^{-1}-b)\nabla\times\boldsymbol{u}+(1-b/a)\boldsymbol{u}=0. \tag{11}$$

Defining "Beltrami fields" by

$$\nabla\times\boldsymbol{G}_\pm=\lambda_\pm\boldsymbol{G}_\pm,$$

$$\lambda_\pm=\frac{1}{2\varepsilon}\left[(b-a^{-1})\pm\sqrt{(b-a^{-1})^2-4(1-b/a)}\right],$$

the general solution of (11) is given by the linear combination $\boldsymbol{u}=c_+\boldsymbol{G}_++c_-\boldsymbol{G}_-$. In view of (9), we obtain

$$\boldsymbol{B}=C_+\boldsymbol{G}_++C_-\boldsymbol{G}_-, \tag{12}$$

$$\boldsymbol{V}=(a^{-1}+\varepsilon\lambda_+)C_+\boldsymbol{G}_++(a^{-1}+\varepsilon\lambda_-)C_-\boldsymbol{G}_-, \tag{13}$$

where $C_\pm$ are arbitrary constants.

The parameter λ_+ (λ_-), being the eigenvalue of the curl operator, characterizes the reciprocal of the length scale on which $\boldsymbol{G}_+$ ($\boldsymbol{G}_-$) changes significantly. As the "Beltrami parameters" a and b vary, $\lambda_\pm$ can range from real to complex values of arbitrary magnitude (see Figs. 1 and 2).

To view the solution and the associated scale lengths as explicit functions of the small parameter ε, let $|\lambda_-|=O(1)$ so that $\boldsymbol{G}_-$ varies on the system size. The terms of order ε in (9) and (10) must, then, be negligible for the $\boldsymbol{G}_-$–parts of $\boldsymbol{B}$ and $\boldsymbol{V}$ dictating $a\approx b$ to have a significant large-scale component in the solution. Consequently the inverse of the second scale, $\lambda_+\approx(a-a^{-1})/\varepsilon$. Barring the case $a\approx b\approx 1$ (Alfvénic flows—the normalized flow speed of order unity), we observe $\lim_{\varepsilon\to0}|\lambda_+|=\infty$, i.e, the short scale shrinks to zero.[5] Writing $b/a=1+\delta$ [$\delta=O(\varepsilon)$], we can approximate

$$\lambda_-\approx\frac{\delta}{\varepsilon}\left(\frac{1}{a}-a\right)^{-1},\quad \lambda_+\approx-\frac{1}{\varepsilon}\left(\frac{1}{a}-a\right). \tag{14}$$

Since in the limit $\varepsilon\to0$, the current ($\nabla\times\boldsymbol{B}$) and the vorticity ($\nabla\times\boldsymbol{V}$) diverge in the small-scale ($\boldsymbol{G}_+$), the 0-model cannot ever capture the essence of the ε-model. The divergence of these small-scale components implies that the resistive and viscous dissipations can be very large even when resistivity and viscosity coefficient are relatively small. Moreover, the jittery magnetic fields in the length scale of the ion skin depth can produce a large chaos-induced dissipations.[6]

Let us now derive the condition for which the singular part of the solution vanishes (i.e., $C_+=0$). To do this we will relate $C_\pm$ with the energy E and the helicity H of the fields,[2] and express this condition as a relation between E and H. Here we assume that $\boldsymbol{B}$ and $\boldsymbol{V}$ are confined in a simply connected domain (normal components vanish on the boundary). The resulting orthogonality $\int\boldsymbol{G}_-\cdot\boldsymbol{G}_+\,dx=0$ (integral is taken over the total domain)[7] helps simplify the analysis. We can evaluate

$$E\equiv\int(B^2+V^2)dx=\alpha_-C_-^2+\alpha_+C_+^2,$$

$$H\equiv\int\boldsymbol{A}\cdot\boldsymbol{B}dx=\frac{C_-^2}{\lambda_-}+\frac{C_+^2}{\lambda_+},$$

where $\alpha_\pm=1+(a^{-1}+\varepsilon\lambda_\pm)^2$. Solving these equations for $C_\pm$, we observe

$$C_-^2=-\frac{\lambda_-}{D}(E-\alpha_+\lambda_+H)\to\Lambda_-H\quad(\varepsilon\to0),$$

$$C_+^2=\frac{\lambda_+}{D}(E-\alpha_-\lambda_-H)$$

$$\to\frac{1}{(1+a^2)}[E-(1+a^{-2})\Lambda_-H]\quad(\varepsilon\to0),$$

where $D=b(b+a^{-1})(\lambda_+-\lambda_-)$ and $\Lambda_-=\lim_{\varepsilon\to0}\lambda_-$ [see (14)]. If E and H satisfy the relation

$$E=(1+a^{-2})\Lambda_-H, \tag{15}$$

the singular component vanishes ($\lim_{\varepsilon\to 0} C_+ = 0$), and the solution converges to the "relaxed state"

$$\boldsymbol{B} = C_- \boldsymbol{G}_-, \quad \boldsymbol{V} = \boldsymbol{B}/a.$$

The energy E satisfying (15) is the "minimum energy" accessible for given helicity H and cross helicity $K \equiv \int \boldsymbol{V}\cdot\boldsymbol{B}\, dx$.[8] The single Beltrami parameter a is determined by $a + a^{-1} = E/K$.

These relations clearly show that the singular (small scale) part of the double Beltrami field, which can produce a large resistive and viscous dissipations,[4] disappears as the field relaxes into the final minimum energy state for given helicity and cross helicity.

V. PATTERNS GENERATED BY NONLINEAR DISPERSIVE INTERFERENCE

We now explore the mechanism that may create small scales in the double Beltrami field, and compare it with a well-known model for generating small scales—Parker's model for "current sheets."[3] The latter is based on single fluid (ideal) MHD equations, where singularities ("tangential discontinuities" of the magnetic field associated with current sheets) may be created along the Cauchy characteristics of the governing hyperbolic partial differential equations. The analysis becomes simple in two dimensional geometry ($\partial_z = 0$). Eliminating the short time scales of waves and possible instabilities ($\partial_t = 0$), we may analyze the structure that can persist for longer time scales. When $\boldsymbol{V} = 0$, we obtain the Grad–Shafranov equation; writing $\boldsymbol{B} = \nabla\psi\times\boldsymbol{e}_z + B_z\boldsymbol{e}_z$ ($\boldsymbol{e}_z = \nabla z$), the momentum balance equation (5) reduces into

$$-\Delta\psi = F'(\psi), \tag{16}$$

where $F(\psi) = B_z^2(\psi)/2 + p(\psi)$ is the Cauchy data to be specified as a function of the label ψ of the Cauchy characteristics (the integral surfaces of magnetic field lines).[9] Since $F(\psi)$ is a combination of "arbitrary" functions of ψ, it may contain any arbitrary small scale. Parker's model of current sheets can, then, be represented by "wrinkles" in $[B_z^2(\psi)/2]'$ (force-free current), which may be produced by merging flux tubes.

Introducing a finite flow $\boldsymbol{V}$, we find that the ideal MHD system allows only special class of (stationary) flows; for $\boldsymbol{V} = \nabla\phi\times\boldsymbol{e}_z + V_z\boldsymbol{e}_z$, (6) with $\varepsilon = 0$ demands $\phi = \phi(\psi)$ and $V_z = V_z(\psi)$, implying that the poloidal components of $\boldsymbol{B}$ and $\boldsymbol{V}$ must be parallel. The self-consistent field is governed by a generalized Grad–Shafranov equation:

$$\nabla\cdot[(1-\phi'^2)\nabla\psi] + \left(\frac{\phi'^2}{2}\right)' |\nabla\psi|^2 + G'(\psi) = 0, \tag{17}$$

where $G(\psi) = B_z^2(\psi)/2 - V_z^2(\psi)/2 + P(\psi)$ and $P = p + V^2/2$. We observe the "Alfvén singularity" (poloidal Alfvén number$= \phi'^2 = 1$) as the vanishing of the elliptic operator.[10]

In contrast to these observations, the Hall MHD system, including the singular perturbations, can describe a much larger variety of flows, viz., ϕ is no longer a function of ψ (the flow and magnetic field characteristics can be different), and both of them are governed by a couple of regular elliptic equations (the Alfvén singularity is removed):

$$\begin{aligned} -\Delta\psi &= \partial_\psi G(\psi,\phi), \\ -\Delta\phi &= -\partial_\phi G(\psi,\phi), \end{aligned} \tag{18}$$

where $G(\psi,\phi) = B_z^2(\psi,\phi)/2 - V_z^2(\psi,\phi)/2 + P(\psi,\phi)$. When $G(\psi,\phi)$ is a quadratic form of ϕ and ψ, (18) becomes a system of linear equations representing a two dimensional double Beltrami structure. It is now clear that the small scale (λ_+^{-1}) in the double Beltrami field is not a product of "wrinkles" in arbitrary Cauchy data, but it is due to the essential interaction of the two characteristics ϕ and ψ. Indeed, a most smooth Cauchy data given of the form

$$G(\psi,\phi) = \frac{1}{\varepsilon^2}[(a^{-2}-1)\psi^2/2 + (b-a^{-1})\psi\phi - (b^2-1)\phi^2/2]$$

yields the double Beltrami field.[11] We, thus, see that the small scale captured by the double Beltrami solution has a different root from that found in Parker's current sheet solution.

It is remarkable that the double Beltrami solution exists in any three-dimensional geometry. This is due to the assumption that Beltrami parameters μ_j (a and b) are constant. The divergence-free conditions on $\boldsymbol{\Omega}_j$ and $\boldsymbol{U}_j$ demand $\boldsymbol{U}_j\cdot\nabla\mu_j = 0$ ($j = 1,2$) implying that the Beltrami parameters are the required Cauchy data—they are assumed to be constant, and hence, the double Beltrami fields are robust in chaotic characteristics. This is in marked contrast to Parker's model of current sheets that considers discontinuities in the Cauchy data as the origin of small scales. When we consider general nonintegrable characteristics in three-dimensional systems, inhomogeneous Cauchy data lead to pathology, and only homogeneous Cauchy data may be assumed. Then, the 0-model (ideal MHD equations) yields the relatively trivial Taylor relaxed state (with a parallel flow). The singular perturbation extends the scope of the ε-model to capture the diversity of structures created in nonlinear systems. The particular solution, the double Beltrami fields highlight the coupling of large-scale hierarchy (universality class) and the intrinsic small-scale hierarchy.

VI. SUMMARY

By analyzing a concrete example, we have demonstrated that physical effects, which translate as a singular perturbation, create a scale-hierarchy in the original system (with a single macroscopic scale). We have also shown that the relatively complex behavior created by the singular perturbations can be expressed in terms of well-behaved functions implying that the ensuing short scale structure can be delineated by familiar methods. The emergence of the short scale and its understanding can help us construct the hidden dynamics (not visible at the macro-level) which so effectively determines the nature of the macro structures. This is in sharp contrast to the original ideal MHD model which fails to yield classes of important solutions (equilibrium or slowly evolving states with perpendicular flows) without any artificial symmetry. With the singular perturbation (dispersive effects) contained in Hall MHD, one not only extends the diversity of structures, but also recovers the regularity of solutions.

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