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NONLINEAR KINETIC THEORY OF A SINGLE HELICITY TEARING INSTABILITY

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Abstract

The evolution of a single helicity tearing mode is analyzed by kinetic theory, including modification of the electron orbits by the islands themselves. Using flux coordinates appropriate to island geometry, analytic solutions for the time-dependent island width are obtained. These describe evolution from linear growth to nonlinear saturation in the collisionless limit, and transition from such saturation to slow Rutherford-type growth for increasing finite collisionality. Previous work is thus unified and generalized.

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I. INTRODUCTION

The evolution of a single tearing mode island is of importance to both disruptions (low m numbers) and anomalous transport (high m numbers) in tokamaks.[1,2] In this paper we attempt to unify and extend previous kinetic theories of the nonlinear evolution of such islands. We employ flux coordinates appropriate to the island configuration, which are the natural generalization of conventional flux coordinates. This simplifies the analysis sufficiently that approximate solutions for the evolution of the island width are found analytically. In regimes where previous results have been obtained by partly analytic and partly numerical calculation[1,2], our analytic solutions show good agreement. Specifically, our solution in the collisionless limit describes evolution from linear growth to nonlinear saturation. With finite collisionality, an analytic solution is possible only under more restricted conditions, but nevertheless smoothly joins collisionless behavior with Rutherford-type growth[3] as the collisionality is varied. Our formalism may also simplify the solution of problems more complicated than the ones solved here.

To summarize the remainder of the paper, in Sec. II island flux coordinates are developed and used to formulate our problem. In Sec. III the collisionless problem is solved. In Sec. IV the finite collisionality problem is solved. In Sec. V the restrictions and possible extensions of our work are discussed.

II. FORMULATION IN ISLAND FLUX COORDINATES

Flux variables appropriate to the magnetic field topology in the presence of a single, finite amplitude tearing mode are constructed in this section. These variables greatly simplify the solution of the electron drift-kinetic equation. In conventional $(\underset{\sim}{x}\ ,\ \mu\ ,\ \varepsilon)$ variables this equation is[4]

$$\frac{\partial f_T}{\partial t} + \frac{v_\parallel}{B}\,\underset{\sim}{B}\cdot\underset{\sim}{\nabla} f_T + \underset{\sim}{v}_D\cdot\underset{\sim}{\nabla} f_T - \frac{e}{m_e}\,v_\parallel E_\parallel\,\frac{\partial f_T}{\partial \varepsilon} = C\left(f_T\ ,\ f_T\right)\ . \qquad (1)$$

Here $f_T = f_0 + f$ is the total electron distribution, where f_0 is the equilibrium and f is the perturbation, $-e$ is the negative electron charge, m_e is the electron mass, $\underset{\sim}{v}_D$ is the guiding center drift velocity, including magnetic field gradient, curvature and $\underset{\sim}{E}\times\underset{\sim}{B}$ drifts, $\varepsilon \equiv v^2/2$, $E_\parallel$ is the parallel component of the perturbed electric field, and $C(f_T\ ,\ f_T)$ is the collision integral. The quantity $v_\parallel$ stands for $v_\parallel = \sigma\left[2(\varepsilon-\mu B)\right]^{1/2}$, where $\sigma \equiv \mathrm{sgn}(\underset{\sim}{v}\cdot\underset{\sim}{B}/B)$ and $\mu \equiv (v_\perp^2/2B)$ is the magnetic moment. The magnetic field is $\underset{\sim}{B} = \underset{\sim}{B}_0 + \underset{\sim}{\tilde{B}}$, where $\underset{\sim}{B}_0$ is the equilibrium field and $\underset{\sim}{\tilde{B}}$ corresponds to the tearing mode. In conventional flux variables[5], $\underset{\sim}{B}_0$ is given by

$$\underset{\sim}{B}_0 = \underset{\sim}{\nabla}\alpha \times \underset{\sim}{\nabla}\left(\theta - \frac{\zeta}{q}\right)\ , \qquad (2)$$

where the radial variable $\alpha \equiv \left(\psi_T/2\pi\right)$, ψ_T is the toroidal magnetic flux, θ and ζ are the poloidal and toroidal angles respectively, and $q(\alpha)$ is the safety factor. An appropriate tearing mode disturbance (radial and poloidal components only) can then be written as

$$\tilde{\underset{\sim}{B}} = \underset{\sim}{\nabla}\zeta \times \underset{\sim}{\nabla}\tilde{A} \quad , \tag{3}$$

where $\tilde{A}$ is an appropriate flux. ($\tilde{A}$ is related to the parallel component of the vector potential $A_{\parallel}$ by $\tilde{A} = -RA_{\parallel}$, where R is the major radial coordinate.) We take $\tilde{A} = A(\alpha\ ,\ \theta - \zeta/q_0) \equiv A(\alpha,\beta)$, where $\beta \equiv (\theta - \zeta/q_0)$, corresponding to a mode with helicity q_0 .

A flux label ψ for the full magnetic field satisfies $(\underset{\sim}{B}_0 + \tilde{\underset{\sim}{B}})\cdot\underset{\sim}{\nabla}\psi = 0$. In conventional flux coordinates this condition is

$$\frac{\partial\psi}{\partial\zeta} - \left(\frac{\partial A}{\partial\theta}\right)\frac{\partial\psi}{\partial\alpha} + \left(\frac{1}{q} + \frac{\partial A}{\partial\alpha}\right)\frac{\partial\psi}{\partial\theta} = 0 \quad , \tag{4}$$

so that ψ has the same helical symmetry as A . Defining $Q(\alpha)$ such that

$$\frac{dQ}{d\alpha} = \frac{1}{q} - \frac{1}{q_0} \quad , \tag{5}$$

then $\psi(\alpha,\beta) = A(\alpha,\beta) + Q(\alpha)$ is the resulting solution of Eq. (4). To see the significance of ψ explicitly, $q(\alpha)$ in Eq. (5) may be expanded about α_s , where $q(\alpha_s) \equiv q_0$. (This expansion is valid when the scalelength of q in α is large compared to the island width W[Eq. (24)]. Then, defining $\tilde{\alpha} \equiv (\alpha - \alpha_s)$,

$$\psi \approx A - q_0' \frac{(\tilde{\alpha})^2}{2(q_0)^2} \quad , \tag{6}$$

where $q_0' \equiv (dq_0/d\alpha)$. A single tearing mode (Fourier component) with m islands of helicity q_0 is of the form

$$A = A_0(\tilde{\alpha}) \cos(m\beta) \quad . \tag{7}$$

From Eq. (6) the relative radial coordinate $\tilde{\alpha}$ is then

$$\tilde{\alpha} = \pm \left(\frac{2q_0^2}{q_0'}\right)^{1/2} \left(A_0 \cos m\beta - \psi\right)^{1/2} \quad . \tag{8}$$

Here, the "constant ψ " approximation[6] is implicit, assuming the radial scale-length for A_0 to be large compared to the island width [see comments after Eq. (22)]. From Eq. (8) we see that the magnetic surfaces are islands for $|\psi| < A_0$, and open surfaces for $\psi < -A_0$. The $\psi = -A_0$ surface is the separatrix whose half-width is conventionally taken as the island width $W = 2q_0(A_0/q_0')^{1/2}$. We are thus led to choose

$$\kappa^2 \equiv \left(\frac{1}{2}\right)\left(1 - \frac{\psi}{A_0}\right) \quad , \tag{9}$$

as the radial island flux variable, so that $0 < \kappa < 1$ and $\kappa > 1$ correspond to regions inside and outside the separatrix, respectively.

The island flux variables are completed by constructing an appropriate angle variable. By analogy to Eq. (2), the angle variable θ_* satisfies

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\alpha_*(\psi) \times \underset{\sim}{\nabla}\left(\theta_* - \frac{\zeta}{q_*}\right) \quad , \tag{10}$$

where, analogous to q , $q_* \equiv (d\alpha_*/d\psi)$, which is just $(d\zeta/d\theta_*)$ along the field line, as seen from Eq. (10). Noting that θ_* must be expressible as a function of ψ and β , and rewriting $\nabla\alpha_*$ in terms of q_* ,

$$\underset{\sim}{B} = q_* \frac{\partial\theta_*}{\partial\beta} \left(\nabla\psi\times\nabla\beta\right) + \left(\nabla\zeta\times\nabla\psi\right) \tag{11}$$

follows. Equation (11) is to be compared to

$$\underset{\sim}{B} = \nabla\alpha\times\nabla\beta + \nabla\zeta\times\nabla\psi \quad , \tag{12}$$

which follows from Eqs. (2) - (5). Then

$$q_*(\psi)\,\frac{\partial\theta_*(\beta,\psi)}{\partial\beta} = \frac{\partial\alpha(\beta,\psi)}{\partial\psi} \tag{13}$$

results.

Equation (13) allows us to compute θ_* and q_* explicitly. For $\kappa < 1$, the normalization condition $\left[\theta_*(\psi\ ,\ \beta_c/m) - \theta_*(\psi\ ,0)\right] = \pi/2$, where (β_c/m) is the angle corresponding to the island tip, and $\cos\,\beta_c \equiv (\psi/A_0)$, combined with Eqs. (7) - (9) yields

$$q_* = \frac{2K(\kappa)}{\pi}\,\frac{q_0}{m}\,\left(\frac{1}{q_0'A_0}\right)^{1/2} \quad (\kappa < 1) \ , \tag{14}$$

where K is the complete elliptic integral of the first kind.[7] Then

$$\theta_* = \frac{\pi}{2} \frac{F(\phi,\kappa)}{K(\kappa)} \qquad (\kappa < 1) \quad , \tag{15}$$

follows at once, where $\phi \equiv \sin^{-1}[\sin(m\beta/2)/\sin(\beta_c/2)]$ and F is the elliptic integral of the first kind.[7] For $\kappa > 1$, an appropriate normalization which we find convenient is that θ_* goes from 0 to $\pi/2$ as $(m\beta)$ goes from 0 to $-\pi$. This yields

$$q_* = \frac{2q_0 K(1/\kappa)}{\pi m \kappa (q_0' A_0)^{1/2}} \qquad (\kappa > 1) \quad , \tag{16}$$

and

$$\theta_* = \frac{-\pi}{2K(1/\kappa)} F\left(\frac{m\beta}{2}, \frac{1}{\kappa}\right) \qquad (\kappa > 1) \ . \tag{17}$$

The angle variable thus goes from 0 to 2π as we go once around an island inside the separatrix, and from 0 to 2π in the opposite direction as we go twice around a surface outside the separatrix. We also note that $q_* \sim (r/mW)$ is typically large compared to one for small values of mW, as is the case for small m's and realistic (saturated or slowing growing) W's.

Returning to the kinetic equation, the $\underset{\sim}{B}\cdot\underset{\sim}{\nabla}$ operator must be expressed in terms of the island flux coordinates. Recalling that α_* is a flux variable,

$$\underset{\sim}{B}\cdot\underset{\sim}{\nabla} = (\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\zeta)\frac{\partial}{\partial\zeta} + (\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\theta_*)\frac{\partial}{\partial\theta_*} \ .$$

Then, since $\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\zeta = \underset{\sim}{B}_0\cdot\underset{\sim}{\nabla}\zeta$, $\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\zeta \approx (B/R)$, where $B \approx B_0$ has been taken. Equation (10) implies $\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\theta_* = (1/q_*)(\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\zeta)$ and

$$\frac{\partial f_T}{\partial t} + \frac{v_\parallel}{R}\left(\frac{\partial}{\partial \zeta} + \frac{1}{q_*}\frac{\partial}{\partial \theta_*}\right)f_T + \underset{\sim}{v}_D\cdot\underset{\sim}{\nabla}f_T - \frac{e}{m_e}v_\parallel E_\parallel\frac{\partial f_T}{\partial \varepsilon} = C(f_T , f_T) \qquad (18)$$

follows immediately. In Eq. (18), f_T is, in general, a function of κ , θ_* , ζ , ε , μ and t .

We make several simplifications in Eq. (18) in order to formulate a nontrivial but soluble nonlinear problem. The equilibrium distribution f_0 is taken to be a Maxwellian. The guiding center drift term in Eq. (18) is neglected; the curvature and gradient contributions to this term appear to offer only small corrections to the terms which we retain. The electrostatic $\underset{\sim}{E} \times \underset{\sim}{B}$ contribution is likely to be important for the case of island widths substantially exceeding the ion gyroradius, but its effects, which are also omitted in previous studies[1-3], are sufficiently complicated to deserve separate consideration. Regarding the $E_\parallel$ term, we use $E_\parallel = -(1/c)\partial\tilde{A}_\parallel/\partial t$, neglecting the electrostatic contribution. This is appropriate for sufficiently slow island growth and low collisionality, because electron parallel streaming forces the perturbed magnetic surfaces to be equipotential surfaces. It is also permissible at large collisionality, essentially because whenever a single helicity-harmonic dominates the dynamics, only the flux-surface averaged $E_\parallel$ enters Ampere's law.[3]

We next use Eqs. (3), (7)-(9), (15) and (17) to write

$$\tilde{A}_{\parallel} = -\frac{1}{R} A_0 \cos m\beta = -A_{\parallel} F_*(\kappa , \theta_*) \quad , \tag{19}$$

where $A_{\parallel} \equiv (A_0/R)$ is the time-dependent amplitude of the vector potential,

$$F_*(\kappa , \theta_*) = \begin{cases} 1 - 2\kappa^2 \mathrm{sn}^2 [\frac{2K(\kappa)}{\pi} \theta_*] & (\kappa \lesssim 1) \\ \mathrm{cn}^2 [\frac{2K(1/\kappa)}{\pi} \theta_*] - \mathrm{sn}^2 [\frac{2K(1/\kappa)}{\pi} \theta_*] , & (\kappa > 1) \end{cases} \quad , \tag{20}$$

and cn and sn are Jacobian elliptic functions.[7] The term involving $E_{\parallel}$ is linearized, taking $\partial f/\partial\varepsilon \ll \partial f_0/\partial\varepsilon$; this approximation is self-consistent in the sense that the remaining nonlinearity allows the island width to saturate, or evolve to the Rutherford regime of slow growth. The collision operator is approximated by a particle-conserving Krook model

$$C(f_T , f_T) \approx -\nu f + \nu \delta n f_0 \quad ,$$

where ν is the collision frequency, and $\delta n = \int d^3\underset{\sim}{v} f$ is the density perturbation. Experience with linear theory[8] indicates that this model will give at least qualitatively correct results. We also restrict ourselves to a single helicity in f . Then, since $\cos(m\beta)$ can be expressed as a function of κ and θ_* , we take $(\partial f/\partial\zeta) = 0$. Combining these simplifications, the kinetic equation becomes

$$\frac{\partial f}{\partial t} + \omega_b \frac{\partial f}{\partial \theta_*} + \nu f = \frac{ev_\parallel}{cT_e}\frac{\partial A_\parallel}{\partial t} f_0 F_*(\kappa, \theta_*) + \nu f_0 \delta n \quad , \tag{21}$$

where T_e is the electron temperature. We have also introduced $\omega_b \equiv [v_\parallel / Rq_*(\kappa)]$, which is the frequency with which particles orbit the magnetic islands. The term $\omega_b(\partial f/\partial \theta_*)$ in Eq. (21) thus accounts for the tearing mode modification of the electron orbit.

To complete the formulation of our problem, Ampère's law is written in island flux variables. Adopting the "constant ψ" approximation[6], Ampère's law is

$$\Delta' A_\parallel(t) = \frac{4\pi ne}{c}\int_{-\infty}^{+\infty} dx \int_0^{2\pi} d\beta \, \frac{\cos m\beta}{\pi} \int d^3\underset{\sim}{v} v_\parallel f \quad , \tag{22}$$

where x is the radial variable, and $\Delta' \equiv [(\partial A_\parallel/\partial x)|_{-L}^{+L}/A_\parallel]$ gives the discontinuity in $(\partial A_\parallel/\partial x)$ on a scale L, large compared to the tearing layer (but small compared to the minor radius). This approximation thus requires $W \ll L$. The quantity Δ' is given by MHD calculations which have been done for both slab[6] and cylindrical[9] geometry. The β integral in Eq. (22) picks out the m^{th} component of the total current. Transforming from (x, β) to (κ, θ_*), the Jacobian is computed from Eqs. (5), (8) - (9), (15), and (17), and

$$\int_{-\infty}^{+\infty} dx \int_0^{2\pi} \frac{d\beta \cos m\beta}{\pi} \quad \rightarrow$$

$$\frac{4}{\pi}\left(\frac{A_\| L_s}{B_0}\right)^{1/2} \left(\int_0^1 \kappa d\kappa \int_0^{2\pi} d\theta_* \left[\frac{2K(\kappa)}{\pi}\right] \left\{1 - 2\kappa^2 \mathrm{sn}^2 \left[\frac{2K(\kappa)}{\pi}\,\theta_*\right]\right\}\right.$$

$$\left. + \int_1^\infty d\kappa \int_0^{2\pi} d\theta_* \left[\frac{2K(1/\kappa)}{\pi}\right] \left\{\mathrm{cn}^2 \left[\frac{2K(1/\kappa)}{\pi}\,\theta_*\right] - \mathrm{sn}^2 \left[\frac{2K(1/\kappa)}{\pi}\,\theta_*\right]\right\} \right) \tag{23}$$

results. Equations (21) - (23) form a closed system in island flux variables for the time evolution of $A_\|$. Noting that $d\tilde{\alpha} \approx B_0 r dr$, where r is the minor radial coordinate, W is in turn related to $A_\|$ by

$$W = 2\left(\frac{A_\| L_s}{B_0}\right)^{1/2} , \tag{24}$$

provided $W << r_s$, where $q(r_s) = q_0$.

III. COLLISIONLESS SOLUTION

A formal solution of the collisionless kinetic equation [Eq. (21) for $\nu = 0$], is

$$f(\kappa, \theta_*, v, v_\|, t) =$$

$$\left(\frac{e v_\|}{T_e c}\right) f_0 \int_{-\infty}^{t} dt' \frac{\partial A_\|(t')}{\partial t'} F_*(\kappa, \hat{\theta}_*) , \tag{25}$$

where the characteristic is

$$\hat{\theta}_*(t' \; ; \kappa \; , \; \theta_* \; , \; v_\parallel \; , \; t) \;=\; \int_t^{t'} dt''\omega_b(t'') + \theta_* \tag{26}$$

and $f \to 0$ as $t \to -\infty$. Substituting Eq. (25) in Eq. (22) and using Eq. (23), six integrals in κ , θ_* , and $v_\parallel$, for $\kappa < 1$ and $\kappa > 1$, result. To compute these integrals analytically, $F_*(\kappa \, , \, \theta_*)$ is evaluated in the asymptotic limits

$$F_*(\kappa \, , \, \theta_*) \;\approx\; \begin{cases} \left(1 - \kappa^2\right) + \kappa^2 \cos\left(2\theta_*\right) & (\kappa < 1) \\ \\ \cos 2\theta_* & (\kappa > 1) \end{cases} \;. \tag{27}$$

Similarly, we take $(2/\pi)K(\kappa) \approx 1$ for $\kappa < 1$ and $(2/\pi)K(1/\kappa) \approx 1$ for $\kappa > 1$. These approximations alter the contributions to the current from particles in a small layer near the separatrix. Multiple helicity effects are expected to destroy flux surfaces inside this layer, which therefore would require special treatment. However, the separatrix layer is unlikely to affect the simple disturbances which are under consideration here. The integrals for $\kappa < 1$ are then elementary. For $\kappa > 1$, an integral of the form

$$\frac{1}{I}\int_I^\infty dy\left(1 - 2y^2\right) \exp\left(-y^2\right)$$

must be evaluated. Here

$$I \;\equiv\; \frac{1}{2}\int_t^{t'} dt'' \;\; \mathscr{H}(t'') \quad , \tag{28}$$

and $\mathcal{W}$ is a dimensionless island width

$$\mathcal{W} \equiv \frac{W}{\Delta_k} , \qquad (29)$$

where Δ_k is the collisionless current channel width[10] $\Delta_k \equiv \left(\Delta'/2k_0^2\pi^{1/2}\right)$, $k_0 \equiv \omega_{pe}/c$, and ω_{pe} is the plasma frequency. The integral over y is singular at $I = 0$ and so is evaluated by a limiting procedure, replacing $\int_I^\infty dy$ by

$$\left(\lim_{R\to\infty} \int_0^{RI} dy - \int_0^I dy \right) ;$$

the integral can then be evaluated straightforwardly using

$$\lim_{a\to 0} (\pi a)^{-1/2} \exp\left(\frac{-I^2}{a}\right) = \delta(I) \quad .$$

From further algebra and elementary integrations, we obtain

$$\mathcal{W} = 2\dot{\mathcal{W}} + \frac{\mathcal{W}^2}{3\pi^{1/2}} - \frac{1}{3\pi^{1/2}} \int_{-\infty}^{\tau} d\tau' \left[\frac{\partial \mathcal{W}^2(\tau')}{\partial \tau'} \right] \left(\frac{5}{2} + I^2\right) \exp\left(-I^2\right) \quad . \qquad (30)$$

Here, τ is the dimensionless time coordinate, $\tau \equiv \gamma_k t$, where $\gamma_k = \left(mv_e\Delta_k/rL_s\right)$ is the linear collisionless growth rate[10], and the dot denotes differentiation with respect to τ. Equation (30) describes both the linear (through the first term on the right-hand side) and the nonlinear (through the remaining terms) island evolution.

To solve Eq. (30), the integral term is approximately evaluated. For large enough τ, I increases rapidly to large values as $|\tau' - \tau|$ increases. We then expand the integrand about the maximum of $\exp(-I^2)$ at $\tau' = \tau$, where $I = 0$. Defining $\tilde{\tau} \equiv (\tau' - \tau)$ and $\tilde{\tau}' \equiv (\tau'' - \tau)$,

$$I = \frac{1}{2}\int_0^{\tilde{\tau}} d\tilde{\tau}' \, \mathscr{W}(\tau + \tilde{\tau}') \approx \frac{\mathscr{W}(\tau)}{2}\tilde{\tau} + \frac{1}{2} \, 2\dot{\mathscr{W}}(\tau)(\tilde{\tau})^2 + \ldots \, , \qquad (31)$$

and substituting for I to lowest order,

$$\int_{-\infty}^{\tau} d\tau' \, \frac{\partial \mathscr{W}^2(\tau')}{\partial \tau'} \left(\frac{5}{2} + I^2 \right) \exp(-I^2) \approx$$

$$\frac{\partial \mathscr{W}^2}{\partial \tau} \int_0^{\infty} d\tau' \left[\frac{5}{2} + \frac{\mathscr{W}^2(\tau)}{4}(\tau')^2\right] \exp\left[\frac{-\mathscr{W}^2(\tau)(\tau')^2}{4}\right]$$

$$= -2\dot{\mathscr{W}} + 0\left[\left(\frac{\dot{\mathscr{W}}}{\mathscr{W}}\right)^2\right] \, . \qquad (32)$$

Equation (30) then yields $\mathscr{W} \approx 3\pi^{1/2}$, $\dot{\mathscr{W}} \approx 0$, so that the corrections to Eq. (32) are self-consistently small, and the integral in Eq. (30) is itself small. In the limit $\tau \to -\infty$, the τ' range of integration goes to zero in Eq. (30), and the integral term may again be neglected, yielding purely linear growth, $\mathscr{W} \propto \exp(\tau/2)$. Since the integral term is small in both the large and small τ limits, Eq. (30) may thus be approximated by

$$\mathscr{W} = 2\dot{\mathscr{W}} + \frac{\mathscr{W}^2}{3\pi^{1/2}} \, , \qquad (33a)$$

whose solution is

$$\mathscr{W} = \frac{C_L \exp(\tau/2)}{1 + (C_L/3\pi^{1/2})\exp(\tau/2)} \quad . \qquad (33b)$$

This simple result exhibits linear growth for $C_L \exp(\tau/2) \ll 3\pi^{1/2}$ and nonlinear saturation $\mathscr{W} = 3\pi^{1/2}$ for the reverse inequality. The arbitrary constant, C_L, corresponds to the indeterminancy of the mode amplitude in the linear regime. Saturation occurs because of the enhanced Lenz's law effect which occurs when the island restricts the poloidal motion of the electrons.[2] The analytic result $\mathscr{W} = 3\pi^{1/2}$ is in good agreement with previous, partly numerical, computations[1-2] which gave $\mathscr{W} \approx (5/2)\pi^{1/2}$.

IV. SOLUTION WITH FINITE COLLISIONALITY

Again, we begin with a formal expression for f ,

$$f = \int_{-\infty}^{\tau} d\tau' \exp[-\frac{\nu}{\gamma_k}(\tau - \tau')] f_0$$

$$[\frac{ev_\parallel}{T_e c}\frac{\partial A_\parallel(\tau')}{\partial \tau'} F_*(\kappa , \hat{\theta}_*) + \frac{\nu}{\gamma_k}\delta n(\hat{\theta}_* , \kappa , \tau')] \quad . \qquad (34)$$

Taking the zeroeth velocity moment, $\int d^3\underset{\sim}{v}$, of Eq. (34) yields an integral equation for δn ,

$$\delta n(\theta_* , \kappa , \tau) = \int_{-\infty}^{+\infty} dv_\parallel \int_{-\infty}^{\tau} d\tau' \exp[-\frac{\nu}{\gamma_k}(\tau - \tau')] f_0(v_\parallel)$$

$$\times \left[\left(\frac{ev_\parallel}{T_e c}\right) \frac{\partial A_\parallel}{\partial \tau'} F_*(\kappa , \hat{\theta}_*) + \frac{\nu}{\gamma_k} \delta n(\hat{\theta}_* , \kappa , \tau') \right] , \quad (35)$$

where $f_0(v_\parallel) \equiv \int d^2 v_\perp f_0$. Equations (34) and (22), together with Eq. (35), determine $A_\parallel$ and δn .

Equation (35) may be solved by changing variables from $v_\parallel$ to $\hat{\theta}_* = I(v_\parallel / v_e) + \theta_*$ for fixed θ_* , where $v_e \equiv (2T_e/m_e)^{1/2}$. This puts the right-hand side of Eq. (35) in convolution form, so that taking the Fourier transform,

$$\delta n_\ell(\kappa , \tau) \equiv \int_{-\infty}^{+\infty} d\theta_* \exp(-i\ell\theta_*)\, \delta n(\theta_* , \kappa , \tau) =$$

$$\int_0^\infty ds \exp\left(\frac{-\nu}{\gamma_k} s\right) \exp\left(- \frac{\ell^2 I^2}{4}\right) \left[\left(\frac{ev_e}{T_e c}\right) \frac{\partial A(\tau')}{\partial \tau'}\Big|_{\tau'=\tau-s} \left(\frac{\ell}{2i}\right) I F_\ell(\kappa) \right.$$

$$\left. + \left(\frac{\nu}{\gamma_k}\right) \delta n_\ell(\kappa , \tau-s) \right] , \quad (36)$$

where

$$F_\ell(\kappa) = 2\pi(1 - \kappa^2)\delta(\ell) + \kappa^2\pi[\delta(2 + \ell) + \delta(2 - \ell)]$$

for $\kappa < 1$, and

$$F_\ell(\kappa) = \pi[\delta(2 + \ell) + \delta(2 - \ell)]$$

for $\kappa > 1$, and $s \equiv -\tilde{\tau}$. Proceeding as in the large τ limit of the collisionless problem, the integrand in Eq. (36) is expanded about $s = 0$. Then, using Eq. (31), we get

$$\delta n_\ell(\kappa\,,\,\tau) \approx F_\ell(\kappa)\left(\frac{ev_e}{T_e c}\right)\left(\frac{i\ell}{4}\right)\mathcal{W}(\tau)\dot{A}(\tau)I_1$$

$$+\left(\frac{\nu}{\gamma_k}\right) I_2 \delta n_\ell(\kappa\,,\,\tau) - F_\ell(\kappa)\left(\frac{iev_e}{T_e c}\right)\frac{\ell}{4}\left(\dot{\mathcal{W}}\dot{A} + \mathcal{W}\ddot{A}\right)$$

$$\times\left\{\frac{8}{(\ell g\mathcal{W})^2}\left(\frac{\gamma_k}{\nu}\right) - I_1\left[\frac{8\nu}{(\ell g\mathcal{W})^2\gamma_k} + \frac{\gamma_k}{\nu}\right]\right\} - \left(\frac{\nu}{\gamma_k}\right)\delta\dot{n}_\ell I_1 \quad . \tag{37}$$

Here, I_1 and I_2 are tabulated integrals[11] which can be written as

$$I_1 = \frac{8}{(\ell g\mathcal{W})^2}\left\{1 - \left(\frac{2\pi^{1/2}\nu}{\gamma_k \ell g\mathcal{W}}\right)\exp\left[\left(\frac{2\nu}{\gamma_k \ell g\mathcal{W}}\right)^2\right]\operatorname{erfc}\left(\frac{2\nu}{\gamma_k \ell g\mathcal{W}}\right)\right\}$$

and

$$I_2 = \left(\frac{2\pi^{1/2}}{\ell g\mathcal{W}}\right)\exp\left[\left(\frac{2\nu}{\gamma_k \ell g\mathcal{W}}\right)^2\right]\operatorname{erfc}\left(\frac{2\nu}{\gamma_k \ell g\mathcal{W}}\right) \quad ,$$

and the function

$$g(\kappa) \equiv \begin{cases} 1\,, & \kappa \leq 1 \\ \kappa\,, & \kappa > 1 \end{cases}$$

has been introduced for convenience.

We now order $(\nu/\gamma_k \mathscr{W}) \ll 1$; this allows further analytic progress, and is consistent with realistic collision frequency, relaxing as $\mathscr{W}$ increases with τ. We also note that the ratio of higher to lower order terms in Eq. (37) must be self-consistently small for the validity of the expansion about $s = 0$; this requires

$$\left(\frac{\dot{\mathscr{W}}}{\mathscr{W}}\right)^2 , \qquad \left(\frac{\ddot{\mathscr{W}}}{\mathscr{W}}\right) \ll \dot{\mathscr{W}} \tag{38a}$$

and

$$\delta\dot{n}_\ell \ll \delta n_\ell \left(\frac{I_2}{I_1}\right) . \tag{38b}$$

Here, $(2\pi^{1/2}/\ell g) \sim 1$ has been assumed for typical ℓ. After expanding I_1 and I_2 in $(\nu/\gamma_k \mathscr{W})$, some algebra yields

$$\delta n_\ell(\kappa , \tau) = \left(\frac{e v_e}{T_e c}\right) \left(\frac{2i}{g^2 \ell}\right) \frac{\dot{A}(\tau)}{\mathscr{W}(\tau)} F_\ell . \tag{39}$$

With this result, the condition described in Eq. (38b) reduces to $(\ddot{\mathscr{W}}/\mathscr{W}) \ll \dot{\mathscr{W}}$, one of the conditions in Eq. (38a). To invert δn_ℓ , we exclude the contribution from $\ell = 0$. This is required, since

$$\delta n_{\ell=0}(\kappa , \tau) \propto \int_0^{2\pi} \frac{d\theta_*}{2\pi} \delta n(\theta_* , \kappa , \tau) ,$$

which is directly computed to be zero from Eq. (34). We then obtain the simple result

$$\delta n(\kappa\ ,\ \theta_*\ ,\ \tau) \;=\; \left(\frac{-\kappa^2}{g^3}\right)\left(\frac{ev_e}{T_e c}\right)\frac{\dot{A}(\tau)}{\mathscr{W}(\tau)}\sin(2\theta_*)\ . \tag{40}$$

Combining Eqs. (22), (23), (34), and (40), the equation for $\mathscr{W}$ becomes

$$\mathscr{W} = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\tau} d\tau'\ \exp\left[-\left(\frac{\nu}{\gamma_k}\right)(\tau-\tau')\right]\frac{\partial\mathscr{W}^2(\tau')}{\partial\tau'}$$

$$\left\{\frac{1}{3}+\pi^{1/2}\delta(I)-\exp(-I^2)\left(\frac{5}{6}+\frac{I^2}{3}\right)\right.$$

$$\left.-\left[\frac{\nu}{\gamma_k\mathscr{W}(\tau')}\right]\left[\frac{I}{3}\exp(-I^2)+\pi^{1/2}\ \mathrm{erfc}(I)\right]\right\}\ .$$

Again, expanding terms involving $\exp(-I^2)$ about $\tau = \tau'$ leads to

$$\mathscr{W} = -2\dot{\mathscr{W}}\left\{\exp\left[\left(\frac{\nu}{\gamma_k\mathscr{W}}\right)^2\right]\mathrm{erfc}\left(\frac{\nu}{\gamma_k\mathscr{W}}\right)-1\right\}-\frac{4}{3}\dot{\mathscr{W}}\left(\frac{\nu}{\gamma_k\mathscr{W}}\right)^2$$

$$\times\exp\left[\left(\frac{\nu}{\gamma_k\mathscr{W}}\right)^2\right]\mathrm{erfc}\left(\frac{\nu}{\gamma_k\mathscr{W}}\right)+\frac{4}{3\pi^{1/2}}\left(\frac{\nu}{\gamma_k\mathscr{W}}\right)\dot{\mathscr{W}}+\frac{1}{3\pi^{1/2}}$$

$$\times\int_{-\infty}^{\tau} d\tau'\ \exp\left[-\frac{\nu}{\gamma_k}(\tau-\tau')\right]\frac{\partial\mathscr{W}^2(\tau')}{\partial\tau'}\left[1-\frac{3\pi^{1/2}}{\mathscr{W}(\tau')}\frac{\nu}{\gamma_k}\mathrm{erfc}(I)\right]\ . \tag{41}$$

By inspection, Eq. (41) reproduces the correct $\tau \to \infty$ limit, $\mathscr{W} = 3\pi^{1/2}$, of the collisionless limit.

We again expand to lowest order in $(\nu/\gamma_k) \ll 1$. We also differentiate Eq. (41) with respect to τ , so that the erfc(I) integrand leads to a factor $\exp(-I^2)$, which we treat as before. This leads to

$$\dot{\mathscr{W}} = -\frac{\nu}{\gamma_k}\mathscr{W} + \left(\frac{2}{3\pi^{1/2}}\right)\mathscr{W}\dot{\mathscr{W}} + \left(\frac{\nu}{\gamma_k \mathscr{W}}\right)\left(\frac{16}{3\pi^{1/2}}\right)\left[\ddot{\mathscr{W}} - \mathscr{W}\left(\frac{\dot{\mathscr{W}}}{\mathscr{W}}\right)^2\right] .$$

The last two terms on the right-hand side of this equation are smaller than the first two by factors $8(\nu/\gamma_k\mathscr{W})(\ddot{\mathscr{W}}/\dot{\mathscr{W}})$ and $8(\nu/\gamma_k\mathscr{W})[(\dot{\mathscr{W}}/\mathscr{W})^2/\dot{\mathscr{W}}]$ respectively. Consistent with the basic expansion [Eq. (38)], and taking $8(\nu/\gamma_k\mathscr{W}) \lesssim 1$, we thus obtain our final result

$$\dot{\mathscr{W}} = -\frac{\nu}{\gamma_k}\mathscr{W} + \frac{2}{3\pi^{1/2}}\mathscr{W}\dot{\mathscr{W}} . \tag{42}$$

Comparing this result to a previous result obtained by taking the θ_*-averaged equation[1], the analytic coefficient $(2/3\pi^{1/2})$ is very close to the numerical result of about $(0.8/\pi^{1/2})$. The ordering $(\nu/\gamma_k\mathscr{W}) \ll 1$ is also very similar to the ordering $[(\nu/\gamma_k)^{2/3}/\mathscr{W}] \ll 1$ adopted in Ref. 1, although our derivation is not further restricted to the "semi-collisional" ordering $1 \ll (\nu/\gamma_k) \ll (\rho_i/\Delta_k)^{3/2}$.

Integrating Eq. (42), and requiring $\mathscr{W} = 3\pi^{1/2}$ for $\nu = 0$,

$$\mathscr{W} - \frac{3\pi^{1/2}}{2}\frac{\nu}{\gamma_k}\tau - 3\pi^{1/2} = \frac{3\pi^{1/2}}{2}\left(\ln\mathscr{W} - \ln 3\pi^{1/2}\right) . \tag{43}$$

This equation may be solved iteratively, taking $\mathscr{W} \gtrsim 3\pi^{1/2} \gg \ln(\mathscr{W})$, which yields

$$\mathscr{W} = 3\pi^{1/2} + \frac{3\pi^{1/2}}{2} \ln\left(\frac{\nu}{2\gamma_k}\tau + 1\right) + \frac{3\pi^{1/2}}{2}\left(\frac{\nu}{\gamma_k}\right)\tau \quad . \qquad (44)$$

This result smoothly joins collisionless saturation to Rutherford type island growth. Thus, when $(\nu\tau/2\gamma_k) \gg 1$, $\mathscr{W} \approx (3\pi^{1/2}/2)(\nu/\gamma_k)\tau$, which is fairly close to the result of Rutherford's MHD, partly numerical computation[3], $\mathscr{W} \approx \pi^{1/2}(\nu/\gamma_k)\tau$.

V. DISCUSSION

We have developed and applied island flux coordinates to a simplified nonlinear tearing mode island evolution problem. The behavior arising from island modification of electron orbits has been computed analytically. The collisionless solution evolves from linear growth to nonlinear saturation. The asymptotic behavior with finite collisionality varies with ν from collisionless saturation to Rutherford type growth.

Regarding the self-consistency of our approximations, the time asymptotic approximations, Eq. (38), clearly improve as τ and $\mathscr{W}$ increase for values greater than one. The condition $(\nu/\gamma_k\mathscr{W}) \ll 1$ allows for realistic collision frequencies ($\nu \sim 10^4 \text{sec.}^{-1}$) when $\mathscr{W}$ is a few times the collisionless saturation width for $m = 2$, and for smaller widths in inverse proportion to higher m's . In order to neglect electrostatic potential contributions, our calculation assumes that W is smaller than an ion gyro-radius; observed islands are often much larger.[12] Note that the collision-dominated (fluid) theory need not be restricted to thin islands, if one assumes that the parallel current, $J_\parallel$, is constant on flux surfaces[3]:

$$J_\parallel = J_\parallel(\psi) \ . \tag{45}$$

Equation (45) allows a formulation in terms of θ_*-averaged quantities, and the electrostatic potential becomes irrelevant for any $\mathscr{W}$. However, Eq. (45) can hold only close to saturation; it is strongly contradicted in linear regimes and difficult to justify from kinetic theory. Thus any general description of the nonlinear behaviour of wide islands would have to take electrostatic effects into account, presumably by means of a two-species generalization of the kinetic theory outlined here.

Despite such differences, it is not surprising that our collisional results (Eqs. (42) and (44)) agree with these of the fluid theory.[3] Since the limit in which we obtain our collisional solution is approximately $\nu \ll (v_e/v_\parallel q)\omega_b(v_\parallel)$, we must have $\nu \ll \omega_b$ unless $v_\parallel \ll (v_e/q)$, so that the θ_*-averaged response is adequate for most particles. While the response of slow particles $(v_\parallel \ll v_e/q)$ differs from the θ_*-averaged response, their contribution to the current, and consequent effect on the island growth, turn out to be negligible in our approximation. Thus, if Eq. (45) is assumed a priori, and Eqs. (22)-(23) are combined with the θ_* average of Eqs. (34) and (40), Eq. (42) is trivially obtained. Since, furthermore, (ν/γ_k) is typically much greater than one, our final result, Eq. (44), is dominated by Rutherford-type island growth.

Use of island flux variables may simplify the study of more complicated problems. Such problems include the generation and evolution of multiple helicities, the inclusion of guiding-center drifts, and use of a more realistic collision operator. Formally, these effects are all included in Eq. (18). With regard to the collision operator, we note that use of the particle-conserving Krook model is in agreement with previous work, where a more realistic pitch-angle scattering operator was taken.[1]

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