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Non-Invariance of Both Magnetic Fluxes within Flux Tubes and Global Helicities in an Ideal Plasma and Numerical Demonstrations of a Generalized Self-Organization Theory

KONDOH Yoshiomi , SERIZAWA Shunsuke, TAKAHASHI Toshiki and VAN DAM James W.[1]

Dept. of Electronic Engineering, Gunma University, Kiryu Gunma 376-8515, Japan

[1]Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712, USA.

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Abstract

It is analytically proved that the magnetic flux within a flux tube, the generalized vorticity flux within a vortex flux tube, the magnetic helicity, and the self-helicity are not conserved in the ideally conducting, fully ionized plasma confined by perfectly conducting walls. Self-organization theories based on helicities is clarified to lose their theoretical and physical basis. Numerical demonstrations are presented to show the usefulness of a generalized theory of self-organization based on minimizing the rate of change of global auto-correlations for multiple dynamical field quantities.

Keywords:

self-organization, minimum change rate of global auto-correlation, open or closed dissipative nonlinear dynamical system, numerical demonstration

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1. Introduction

Ever since J. B. Taylor published his theory [1] with a model to derive the Taylor state $\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$ and to explain the reversed field pinch configuration [2], magnetic helicity has been believed to play an important role as a global invariant in the process of self-organization and relaxation for magnetized plasmas. However, the Taylor state had been previously derived by another theory [3] with a completely different physical model to find the minimum dissipative state of magnetic energy. This theory using uniform resistivity is shown to be included in the general self-organization theories [4,5,6] to find decaying self-similar states with niminum change rate of global auto-correlations for multiple physical quantities. The general theories further lead to wider relaxed states such as the non-Taylor states with non-uniform resistivity, as was shown by simulations that agree with analytical results in [7]. Those facts indicate that the introduction of magnetic helicity itself is not physically required in the relaxation process. The general theories can lead to correct decaying self-similar states for other dynamical systems, as was analytically and numerically demonstrated for dissipative solitons described by a viscid Korteweg-de Vries equation [8] and a two-dimensional (2-D) incompressible viscous fluid in a friction-free box [9]. Several authors [10-12] introduced a two-fluid plasma model, having the same theoretical structure with Taylor's theory. Note that Taylor's theory and new ones are never based on either a variational principle (e.g., as in classical mechanics [13]) or an energy principle (e.g., to describe perturbations in an ideal magnetohydrodynamic (MHD) plasma [14]), either of which leads to dynamical equations for the time evolution of the system of interest.

In this paper, we present analytical proofs in Section II which clarify self-organization theories based on helicities lose their theoretical and physical basis. In Section III we present breafly a novel general theory of self-organization [6] and show three applications of the general theory.

2. Non-invariance of magnetic fluxes, vorticity fluxes, and helicities in ideal plasmas

Physically and theoretically inevitable processes to guarantee usefulness of self-organization theories based on topological quantities [1,10-12] are to definitely clear following two issues: 1) Reality of the implicit assumption that both lines of the magnetic field and the generalized vortex are frozen in ideally conducting, fully ionized plasmas. 2) Rigorous invariance of both the magnetic helicity $K \equiv \int_{V_p} \boldsymbol{A} \cdot \boldsymbol{B}\, \mathrm{d}V$ [1] and the self-helicity $K_\alpha \equiv \int_{V_p} \boldsymbol{P}_\alpha \cdot \boldsymbol{\Omega}\, \mathrm{d}V$ [11] in those plasmas confined by perfectly conducting walls, where $\boldsymbol{P}_\alpha \equiv m_\alpha \boldsymbol{u}_\alpha + q_\alpha \boldsymbol{A}$ is the canonical momentum and $\boldsymbol{\Omega}_\alpha \equiv \nabla \times \boldsymbol{P}_\alpha$ is the generalized vorticity.

Corresponding author's e-mail: kondohy@el.gunma-u.ac.jp

We start with Taylor's theory. The generalized Ohm's law of $(m_e m_i/Ze^2\rho_m)(\partial \boldsymbol{j} / \partial t) = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} - \eta\boldsymbol{j} - (m_i - Zm_e/Ze^2\rho_m)\, \boldsymbol{j} \times \boldsymbol{B} + (1/Ze\rho_m)\,(m_i\nabla p_e - Zm_e\nabla p_i)$ for a fully ionized MHD plasma is known to express correctly experimental plasmas better than the simplified Ohm's law of $\eta\boldsymbol{j} = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}$. Applying Faraday's law and the generalized Ohm's law with $\eta = 0$, we obtain the time derivative of the magnetic flux Φ linking any closed contour C fixed and moving with the ideal plasmas and that of K, respectively, as

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}t} &= \oint_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} + \oint_C \boldsymbol{B} \cdot (\boldsymbol{u} \times \mathrm{d}\boldsymbol{s}) \\ &= \oint_C \left[-\frac{m_i - m_e}{Zep_m}\, \boldsymbol{j} \times \boldsymbol{B} - \frac{m_e - m_i}{Ze^2 p_m} \frac{\partial \boldsymbol{j}}{\partial t} \right. \\ &\quad \left. + \frac{m_i}{Zep_m} (\nabla p_e - \frac{Zm_e}{m_i} \nabla p_i) \right] \cdot \mathrm{d}\boldsymbol{s} \neq 0, \end{aligned} \tag{1}$$

$$\begin{aligned}\frac{\partial K}{\partial t} &= 2 \int_{V_p} \left\{ \left[\frac{m_e - m_i}{Ze^2 p_m} \frac{\partial \boldsymbol{j}}{\partial t} \right. \right. \\ &\quad \left. \left. - \frac{m_i}{Zep_m} (\nabla p_e - \frac{Zm_e}{m_i} \nabla p_i) \right] \cdot \boldsymbol{B} \right\} \mathrm{d}V \neq 0. \end{aligned} \tag{2}$$

Next, we deal with the theories in [10-12]. Using the equations of motion for the two fluid model [10-12], and without using any Ohm's law (e.g., two Ohm's lows shown above), we straightforwardly get the time derivative of the vorticity flux Φ_g linking any closed contour C fixed and moving with local velocities $\boldsymbol{u}_\alpha$ of the ideal fluid α (=e or i for electrons or ions, respectively) without friction force and that of K_α, respectively, as

$$\frac{\mathrm{d}\Phi_g}{\mathrm{d}t} = -\oint_C \left(\frac{\nabla p_\alpha + \nabla \cdot \mathbf{\Pi}_\alpha}{n_\alpha} \right) \cdot \mathrm{d}\boldsymbol{s} \neq 0, \tag{3}$$

$$\begin{aligned}\frac{\partial K_\alpha}{\partial t} &= 2\int_{V_p} \left\{ \left[m_\alpha \boldsymbol{u}_\alpha \times \omega_\alpha - \nabla \left(\frac{m_\alpha u_\alpha^2}{2} + q_\alpha \phi_E \right) \right. \right. \\ &\quad \left. \left. - \frac{\nabla p_\alpha + \nabla \cdot \mathbf{\Pi}_\alpha}{n_\alpha} + q_\alpha \boldsymbol{u}_\alpha \times \boldsymbol{B} \right] \cdot (m_\alpha \omega_\alpha + q_\alpha \boldsymbol{B}) \right\} \mathrm{d}V \\ &\quad + \oint_{S_p} \left\{ \left[m_\alpha \boldsymbol{u}_\alpha \times \omega_\alpha - \nabla \left(\frac{m_\alpha u_\alpha^2}{2} + q_\alpha \phi_E \right) + q_\alpha \boldsymbol{u}_\alpha \times \boldsymbol{B} \right. \right. \\ &\quad \left. \left. - \frac{\nabla p_\alpha + \nabla \cdot \mathbf{\Pi}_\alpha}{n_\alpha} \right] \times (m_\alpha \boldsymbol{u}_\alpha + q_\alpha \boldsymbol{A}) \right\} \cdot \mathrm{d}\boldsymbol{S} \neq 0. \end{aligned} \tag{4}$$

We find from Eqs. (1) and (3) that even if any magnetic and any vortex flux tubes can be defined at each instance due to $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot \Omega_\alpha = 0$, their identities cannot be maintained in the ideal plasmas with changeful flow due to terms of Hall effect, pressure gradients and others. These results exactly indicate that the topological feature of field and vortex lines is always changing even in the ideally conducting plasmas during relaxation process. Directly connecting with these facts, Eqs. (2) and (4) clearly show that K and K_α cannot be invariant even in the ideally conducting, fully ionized plasmas.

Since we have analytically proved that the two issues of 1) and 2) pointed above cannot be cleared, we can conclude that the theories in [1,10-12] are not theoretically available for real experimental and space plasmas due to their physically incorrect basis. Note, moreover, that those theories simply use variational calculus—not a variational principle or an energy principle—to find solutions with minimum energy from a set of solutions with the same helicity. We should remind that the value of a topological quantity can be calculated after the configuration of the magnetic field or vortex lines is determined, but the value itself cannot inversely determine the configuration, which is passively formed not by topological restrictions but through physical process itself. We comment that helicity injection experiments do not drive plasma current; this is always driven by energy injection because of external supply for decayed magnetic energy (e.g., externally applied DC voltage, radiofrequency waves, neutral beam injection, plasmoid injection, etc.).

Since, according to the discussion with the proofs shown above, the theories based on helicities are not physically available for experimental and space plasmas, a different theory for self-organization is needed, which is applicable to general dissipative dynamical systems.

3. A general theory of self-organization and application

After we present briefly a general theory extended from [6] and originated from [4,5] for how to find self-organized states in arbitrary dissipative nonlinear dynamical systems, we will show three applications of the general theory, in order to demonstrate predictions by the theory to be correct. Consider a set of N dynamical variables $\boldsymbol{q} \equiv \boldsymbol{q}\,[\xi^k] \equiv (q_1[\xi^k],\ldots,q_N[\xi^k])$, with M-dimensional independent variables $[\xi^k](k = 1, 2,\ldots,M)$. Using generalized symbolic dynamical operators, we may write the general nonlinear set of N simultaneous equations for an open or a closed dynamical system as

$$\partial q_i\,[\xi^k] / \partial\xi^j = D_i^j[\boldsymbol{q}], \tag{5}$$

where $D_i^j[\boldsymbol{q}]$ $(i = 1, 2,\ldots,N)$ represents dynamical operators which include both nonlinear and dissipative terms for the change of a dynamical variable q_i. Here, the dynamical system of interest always has fluctuations of the dynamical variables $q_i[\xi^k]$ along the axis of the variable ξ^j, one of which is expressed as τ_{ci} giving the ordering of the relaxation time scale. Since the self-organized states must have the most unchangeable configurations along ξ^j during the evolution of the dynamical system, we are able to judge and identify the self-organized states as those states for which the rate of change of global auto-correlations for multiple dynamical field quantities is minimized, that are exactly written by

$$\min \left| \frac{\int q_i\,[\xi^j]q_i[\xi^j + (\Delta\xi^j/\ \tau_{ci})]\ |\ J_{k\neq j}\ |\ \prod_{k\neq j} \mathrm{d}\xi^k}{\int (q_i[\ \xi^k])^2 |\ J_{k\neq j}\ |\ \prod_{k\neq j} \mathrm{d}\xi^k} - 1 \right| \quad (6)$$

Expanding Eq. (6), and using Eq. (5), we obtain two equations to judge and identify the self-organized states, as shown in [4,5]. Using the variational calculus for the two equations, we obtain the Euler-Lagrange equations and the solutions of the decaying self-similar, self-organized state with the slowest decay within the time scale τ_{ci}, denoted by a superscript #, as follows;

$$D_i^j\,[\boldsymbol{U}] + \tau_{ci}\Lambda_{im}\boldsymbol{U}_{im}\,[\xi^k_{k\neq j}] = 0. \quad (7)$$

$$q_i^{j\#}[\xi^k] = \boldsymbol{U}_{i1}[\xi^k] = \exp(-\tau_{ci}\Lambda_{i1}\,\xi^j)\boldsymbol{U}_{i1}\,[\xi^k_{k\neq j}]. \quad (8)$$

where $U_{i1}[\xi^k_{k\neq j}]$ is the lowest eigensolutions with the smallest eigenvalue Λ_{i1} for the given boundary conditions.

At first, we apply this general theory to the following Korteweg-deVries equation with a viscosity ν term with periodic boundary conditions

$$\partial u/\partial t = -6u(\partial u/\partial x) - \partial^3 u/\partial x^3 + \upsilon\partial^2 u/\partial x^2. \quad (9)$$

Using Eq. (8), we obtain the analytical solution for the self-organizied state, which has the minimum rate of change and appears after interchange between nonlinear and dissipative terms:

$$u^{\#}(t\ , x) = \exp(-\Lambda_1 t)\exp[\pm i(\Lambda_1/\upsilon)^{1/2}x]. \quad (10)$$

This analytical solution agrees very well with simulations reported elsewhere [8].

Next, we apply the general theory to a 2-D incompressible viscous fluid equation with periodic boundary conditions in the x, y plane (normalized to unit length) in the following vorticity ω form with $\omega\,(x, y) = \nabla \times \boldsymbol{u}$;

$$\partial\omega/\partial t = -(\boldsymbol{u}\cdot\nabla)\omega + \upsilon\nabla^2\omega \quad (11)$$

Similarly to the first application shown above, we obtain the analytical solution for the self-similar, slowest decay phase with smallest eigenvalue Λ_1 as

$$\omega\#(t,x,y) = \exp(-\frac{4\pi^2}{R}t)[\cos 2\pi x + \cos 2\pi y]\boldsymbol{k}. \quad (12)$$

Here, the kinematic viscosity υ is the reciprocal of the Reynolds number in dimensionless units for unit length and unit initial rms velocity, i.e., $\upsilon = R^{-1}$, and the eigenvalue $\Lambda_1 = 4\pi^2 / R$ of the lowest mode {(1, 0) + (0, 1)} is used. Equation (11) is numerically solved with the use of two relations $\boldsymbol{u} = \nabla\psi \times \boldsymbol{k}$ and $\nabla^2\psi = -\omega$, where the stream function $\psi = \psi\,(x, y, t)$ and other field variables are independent of z. When we combine $\nabla^2\psi = -\omega$ with the lowest eigenmode solutions for of Eq. (12), we can derive the relation between ω and ψ of the self-similar, self-organized state as follows;

$$\omega = 4\pi^2\psi. \quad (13)$$

We numerically solved Eq. (11) by the Kernel Optimum Nearly-analytical Discretization algorithm with high numerical accuracy [16], with the use of the Jacobi scheme to solve $\nabla^2\psi = -\omega$. We also numerically calculated the correlation coefficient $C(f, g)$ between the analytical solution and simulation data, where $C(f, g)$ is defined for two functions f and as g $C(f, g) = \{\overline{(f-\overline{f})(g-\overline{g})}\} / \{(\overline{(f-\overline{f})^2}\cdot\overline{(g-\overline{g})^2})^{1/2}\}$. In order to find the self-organized state with minimum rate of change of the global autocorrelations, simulations must be performed with rather long effective computation times, i. e., more than ten times longer than the simulation time indicated in Ref. [15]. Figure 1 shows the time evolution of the spectral components of vorticity. Horizontal scale shows the square of spectral eigenvalues $\Lambda_k = \pi^2\ (l_k^2 + m_k^2)$ for eigenmodes $(l_k,\ m_k)$. Vertical scale is normalized by the maximum absolute value of either the positive or the negative spectral components C_{wk} in each figure, where the positive spectra are shown by bold bars, and the negative ones by shaded ones attached to the right hand side of the bold bars. Even though the initial flow at $t = 0$ does not contain the lowest eigenmodes of {(1, 0) + (0, 1) }, the time evolution of spectra clarifies the greater dissipation of the higher spectral components

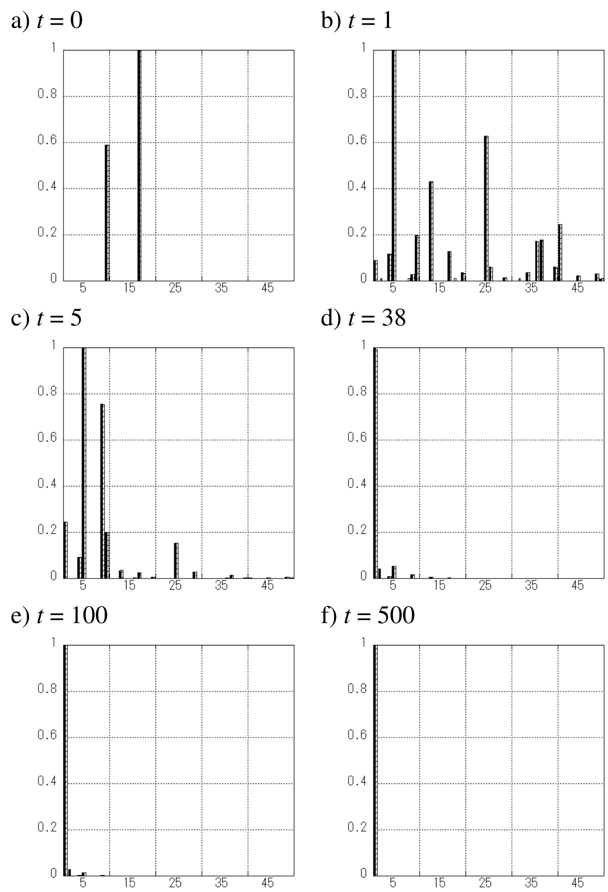

Fig. 1 Time evolution of spectral componets of vorticity during self-organization for the simulation data with R = 14,000.

and the rapid spectrum accumulation at the lowest eigenmodes, where the amplitude of (1,0) mode is equal to that of (0,1) mode at t = 500. Figure 2 shows the time evolution of the correlation coefficient of the simulation data with the relation $\omega = 4\pi^2\psi$ and with $\omega = c\sinh(\beta\psi)$ which is not the self-similar solution of Eq. (11) but only a solution of $(\boldsymbol{u} \cdot \nabla)\omega = 0$ derived by the maximum entropy theory in Ref. [15]. The bold line is the coefficient for $\omega = 4\pi^2\psi$, and the chain-dotted line is for $\omega = c\sinh(\beta\psi)$. It is seen from this figure that the value of the correlation coefficient for $\omega = 4\pi^2\psi$ almost completely becomes 1.0 after around t =280, but that for $\omega = c\sinh(\beta\psi)$ can never become 1.0, as was analitically predicted by the present general theory. From the spectral analysis for the simulation data such as Fig. 1, we can find a sufficient physical picture of self-organization, i.e., a) simultaneous normal and inverse cascading by the nonlinear term, b) the faster spectral decay of higher eigenmodes to result in the rapid spectral accumulation to the lowest eigenmode by the dissipative term. Details about these simulations will be reported elsewhere.

Finally, we apply the general theory to a fully ionized, compressible, resistive, viscid MHD fusion plasma expressed by three nonlinear power balance equations for kinetic flow energy, magnetic field energy, and internal thermal energy. Using Eq. (7), we obtain the following three Euler-Lagrange equations as the set of dynamical equations including time derivative terms:

$$
\begin{aligned}
& v\nabla\times\nabla\times\boldsymbol{u} + \nabla\times(\nabla v\times\boldsymbol{u}) - v\nabla(\nabla\cdot\boldsymbol{u}) \\
& -\frac{1}{3}\nabla(v\nabla\cdot\boldsymbol{u}) - \nabla(\nabla v\cdot\boldsymbol{u}) + \boldsymbol{u}\nabla^2 v + \frac{\rho_m\boldsymbol{u}}{2}\nabla\cdot\boldsymbol{u} \\
& +\nabla\left(\frac{\rho_m\boldsymbol{u}\cdot\boldsymbol{u}}{2}\right) - \left[\frac{(m_i - Zm_e)\rho_e}{Ze\rho_m} + 1\right](\boldsymbol{J}\times\boldsymbol{B} \\
& -\nabla p_e - \nabla p_i) - \frac{\rho_e}{Ze\rho_m}(Zm_e\nabla p_e + m_i\nabla p_i) \\
& -\eta\rho_e\boldsymbol{j} - \frac{m_i m_e\rho_e}{Ze^2\rho_m}\frac{\partial\boldsymbol{j}}{\partial t} = \lambda_u\frac{\rho_m\boldsymbol{u}}{2},
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
& \nabla\times\Big[\frac{\eta}{\mu_0}\nabla\times\boldsymbol{B} - \boldsymbol{u}\times\boldsymbol{B} - \frac{(m_i - Zm_e)}{Ze\rho_m}\Big(\frac{\boldsymbol{B}}{\mu_0}\times\nabla\times\boldsymbol{B} \\
& +\nabla p_e + \nabla p_i\Big) - \frac{1}{Ze\rho_m}(Zm_e\nabla p_e - m_i\nabla p_i) \\
& +\frac{m_i m_e}{Z\mu_0 e^2\rho_m}\nabla\times\frac{\partial\boldsymbol{B}}{\partial t}\Big]\Big\} = \lambda_B\boldsymbol{B},
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
& \frac{\nabla\cdot(p\boldsymbol{u})}{\gamma - 1} + p(\nabla\cdot\boldsymbol{u}) - \boldsymbol{j}\cdot\Big[\eta\boldsymbol{j} \\
& +\frac{m_i}{Ze\rho_m}\Big(\nabla p_e + \frac{Zm_e}{m_i}\nabla p_i\Big) - \frac{m_i m_e\rho_e}{Ze^2\rho_m}\frac{\partial\boldsymbol{j}}{\partial t}\Big] \\
& -\nabla\cdot(k_e\nabla T_e + k_i\nabla T_i) = \lambda_p\Big(\frac{p}{\gamma - 1}\Big).
\end{aligned}
\tag{16}
$$

Equations (14), (15), and (16) indicate that the relaxed states gradually change and eventually become unstable due to the time derivative terms, and the relaxation process repeats itself. This is consistent with observations from fusion plasmas and with simulation results. From Eq. (15), we obtain $\nabla\times\nabla\times\boldsymbol{B} = (\mu_0\,\eta)\lambda_B\boldsymbol{B}$ which include the Taylor state $\nabla\times\boldsymbol{B} = \lambda\boldsymbol{B}$ in the limiting case of uniform resistivity, $\boldsymbol{u}$ = 0, and quasi-steady zero-pressure plasma. The relaxed states by this limiting case are the same with the states with minimum dissipation of magnetic energy derived in [3].

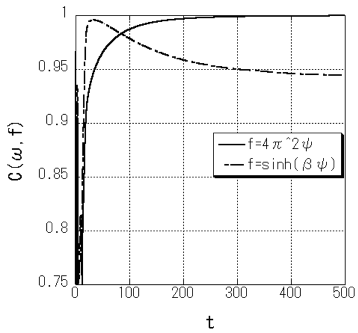

Fig. 2 Time evolutions of the correlation coefficients of the simulation data with $\omega = 4\pi^2\psi$ and with $\omega = c\sinh$. $(\beta\psi)R$ = 14,000.

4. Summary

In Section I, showing both the physical model of the theory in [3] and that of the theories in [4,5,6], we clarified that the introduction of the magnetic helicity itself is not physically required to derive the Taylor state in the relaxation process.

We analytically proved in Section II that even in the ideally conducting, fully ionized plasmas with changeful flow, the identities of the magnetic and the general vortex lines cannot be maintained, and both the magnetic and the self helicities used in [1,10-12] cannot be invariant [cf. Eqs. (1), (3), (2) and (4)]. From these analytical results, we conclude that the theories in [1,10-12] are not theoretically available for real experimental and space plasmas due to their physically incorrect basis.

In Section III, we presented briefly a new general theory of self-organization and three typical applications with simulation data in order to demonstrate correct predictions by the new theory. The Taylor state was shown to be analytically included in the third application. We also presented a sufficient physical picture of self-organization from the spectral analysis for simulation data. The most remarkable feature of the general theory is that it can be applicable not only to magnetized plasmas but also to any dissipative nonlinear dynamical systems, written by Eq. (5), giving suitable self-organized states as the Euler-Lagrange equations.

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