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Remarks on the discrete Alfvén wave spectrum induced by the Hall current

Atsushi Ito and Akira Hirose
Plasma Physics Laboratory, University of Saskatchewan, 116 Science Place, Saskatoon, Saskatchewan S7N 5E2, Canada

Swadesh M. Mahajan
Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712

Shuichi Ohsaki
Graduate School of Frontier Sciences, The University of Tokyo, 5-1-5 Kashiwanoha, Chiba 277-8561, Japan

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It is shown that the discrete Alfvén wave induced by the Hall current [S. Ohsaki and S. M. Mahajan, Phys. Plasmas **11**, 898 (2004)] is equivalent to the kinetic Alfvén wave (KAW). The KAW is, thus, accessible in a fluid description. It is further shown that the dispersion relation for the Hall magnetohydrodynamic waves can be reproduced from kinetic theory only if the ion temperature is negligible compared with the electron temperature. © *2004 American Institute of Physics*.
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I. INTRODUCTION

In inhomogeneous magnetohydrodynamics (MHD), nonideal effects may remove the singularity in the Alfvén continuous spectrum[1] by inducing higher order derivative in the mode equation. For example, inclusion of higher order ion finite Larmor radius terms $\lambda_i = (k_\perp \rho_i)^2$ transforms the Alfvén continuous spectrum into a discrete spectrum known as the kinetic Alfvén wave (KAW).[2,3] However, a later study[4] showed that the discrete mode disappears if expansion of the function $e^{-\lambda_i} I_n(\lambda_i)$ is avoided. Here $I_n(\lambda_i)$ is the modified Bessel function. Mahajan[6] found a discrete Alfvén mode in a more accurate analysis based on full Maxwell's equations.[5] (Earlier studies were done in two-field approximation with scalar potential ϕ and parallel vector potential $A_\parallel$ in which magnetosonic perturbation was ignored.)

Recently, it has been shown[7] that the mode equation based on Hall MHD (HMHD) also induces a discrete Alfvén mode in addition to the mode found earlier by Mahajan.[6] In the present work we wish to clarify the relation between the discrete spectrum induced by the Hall current effect and the KAW. It will be shown that the HMHD mode equation can be recovered exactly from the kinetic dispersion relation provided full Maxwell's equations are employed. Two-potential approximation based on ϕ, the scalar potential, and $A_\parallel$, the vector potential parallel to the magnetic field, often used in low β plasmas is unable to recover the HMHD equation in which magnetosonic perturbation plays a key role. In the limit of negligible ion temperature $(T_i \ll T_e)$ and adiabatic electrons, $\omega \ll k_\parallel v_{Te}$, the HMHD equation is exact. The discrete mode found in Ref. 7 and attributed to the coupling between the ion acoustic perturbation and Hall current is nothing but the exact fluid reincarnation of the KAW. In this paper, we correct the erroneous estimate of the mode width found in Ref. 7.

In Sec. II, we review the formulation of the Alfvén wave equation including the Hall current effect, present numerical results based on the HMHD equations, and discuss the estimation of the width of the eigenfunction. The dispersion relation for the HMHD waves is derived from kinetic theory. In Sec. IV, relation between HMHD and kinetic theory is summarized.

II. DISCRETE SPECTRUM

We begin with a review of the formulation of the eigenmode equations derived in Ref. 7 without normalization. We consider a slab geometry in Cartesian coordinates x, y, and z, and a static equilibrium with an inhomogeneous magnetic field that is a function of only x. The thickness of x in the slab geometry is the system size L_0. The ambient magnetic field is assumed to be

$$\mathbf{B}_0(x) = B_0 \frac{\mathbf{e}_z + f(x)\mathbf{e}_y}{\sqrt{1+f^2(x)}}, \tag{1}$$

which satisfies the force-free condition

$$\nabla \times \mathbf{B}_0 = h\mathbf{B}_0, \quad h(x) = f'(x)/[1+f^2(x)], \tag{2}$$

and introduces the following orthogonal unit vectors:

$$\mathbf{e}_x, \quad \mathbf{e}_\parallel = \mathbf{B}_0/B_0, \quad \mathbf{e}_\perp = \frac{\mathbf{e}_y - f\mathbf{e}_z}{\sqrt{1+f^2}}. \tag{3}$$

The perturbations are assumed to take the form of $\tilde{q} = q(x)\exp[i(k_y y + k_z z - \omega t)]$. The wave numbers according to the coordinate system (3) are

$$k_\parallel = \frac{k_z + f k_y}{\sqrt{1+f^2}}, \quad k_\perp = \frac{k_y - f k_z}{\sqrt{1+f^2}}. \tag{4}$$

In HMHD, relevant equations are the equation of motion and Faraday's induction law for the perturbed velocity and magnetic field, $\mathbf{v}$ and $\mathbf{b}$,

$$-i\omega\mathbf{v} = \frac{1}{\mu_0 nM}[(\nabla\times\mathbf{b})\times\mathbf{B}_0 + (\nabla\times\mathbf{B}_0)\times\mathbf{b}] + \frac{iV_s^2}{\omega}\nabla(\nabla\cdot\mathbf{v}), \quad (5)$$

$$-i\omega\mathbf{b} = \nabla\times(\mathbf{v}\times\mathbf{B}_0) - \frac{1}{\mu_0 ne}\nabla\times[(\nabla\times\mathbf{b})\times\mathbf{B}_0 + (\nabla\times\mathbf{B}_0)\times\mathbf{b}] = B_0\nabla\times[\mathbf{v}\times\mathbf{e}_\parallel + i(\omega/\omega_{ci})\mathbf{v}] \equiv B_0\nabla\times\mathbf{g}, \quad (6)$$

where V_s is the speed of sound, $V_s=\sqrt{\gamma P/\rho}$, and $\omega_{ci}=eB_0/M$ is the ion cyclotron frequency. If V_s is small compared to the Alfvén velocity $V_A=B_0/(\mu_0 nM)^{1/2}$, that is, in low β plasmas, the eigenmode equation can be approximated as

$$\frac{d}{dx}F\frac{d}{dx}g_\perp - k_\perp^2 F g_\perp - S(S-k_\parallel h)g_\perp \simeq -\frac{k_\parallel^2 V_s^2\alpha_1\omega^2}{\omega_{ci}^2(\omega^2-k_\parallel^2V_s^2)}\left(\frac{d^2}{dx^2}-k_\perp^2\right)^2 g_\perp \simeq -k_\parallel^2\rho_s^2\left(\frac{d^2}{dx^2}-k_\perp^2\right)^2 g_\perp, \quad (7)$$

where $F=\alpha_1\omega^2/V_A^2-k_\parallel^2$, $S=(\omega/\omega_{ci})\alpha_1\omega^2/V_A^2$, $\alpha_1=(1-\omega^2/\omega_{ci}^2)^{-1}$, and $\rho_s=V_s/\omega_{ci}$ is the ion Larmor radius defined in terms of the speed of sound. The singular perturbation appears as a finite ρ_s effect. Omitting the term representing electron Landau damping, the equation for the KAW, Eq. (15) of Ref. 6, is essentially equivalent to Eq. (7). The Hall currents are included in more rigorous kinetic or two-fluid approximation and the agreement is not surprising.

We now proceed to numerical analysis of the mode equation. The equations solved in this analysis are obtained from Eqs. (5) and (6) as follows:[7]

$$\left(\Omega^2-k_\parallel^2+\frac{d^2}{dx^2}\right)g_\perp = i\left[\frac{d}{dx}(k_\perp g_x)+\epsilon\omega\Omega^2 g_x\right] + \frac{d}{dx}\left[\left(\frac{\omega}{\epsilon k_\parallel}-h\right)g_\parallel\right] - k_\parallel k_\perp g_\parallel, \quad (8)$$

$$(\Omega^2-k_\parallel^2-k_\perp^2)g_x = i\left(k_\perp\frac{d}{dx}-\epsilon\omega\Omega^2\right)g_\perp + i\left(k_\parallel\frac{d}{dx}-\frac{k_\perp\omega}{\epsilon k_\parallel}\right)g_\parallel, \quad (9)$$

and

$$g_\parallel = \frac{\epsilon^2V_s^2\Omega^2k_\parallel}{\omega^2-k_\parallel^2V_s^2}\left[i\left(\frac{d}{dx}+\frac{k_\perp}{\epsilon\omega}\right)g_x - \left(\frac{1}{\epsilon\omega}\frac{d}{dx}+k_\perp\right)g_\perp\right], \quad (10)$$

where $\Omega^2=\omega^2/(1-\epsilon^2\omega^2)$. We have normalized the magnetic fields by B_0, the velocities by the Alfvén velocity V_A, the pressures by B_0^2/μ_0, and the scale length by L_0. The coefficient $\epsilon=(c/\omega_{pi})/L_0$ is a measure of the ion skin depth, where $\omega_{pi}=(ne^2/\epsilon_0 M)^{1/2}$ is the ion plasma frequency. We compare

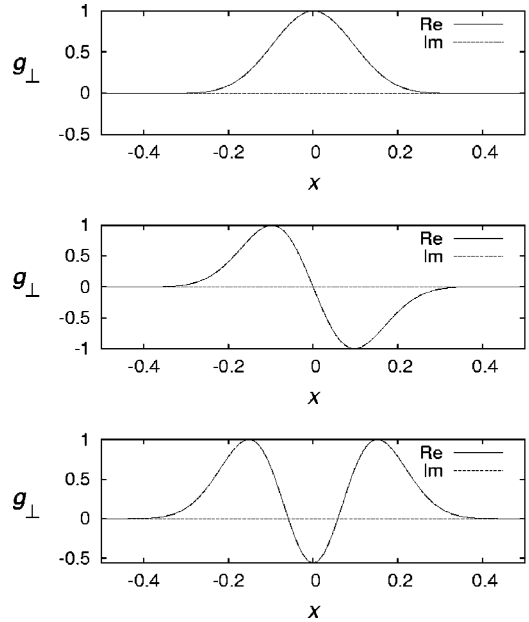

FIG. 1. First three eigenfunctions of $g_\perp$ for $k_z=15$. The other parameters are determined automatically: $k_y=-15+\sqrt{2}$, $\Omega_0=1$, $\epsilon=V_s=k_\perp^{-1}=4.95\times10^{-2}$, and $\lambda=0.315$.

numerical results with approximations for localized modes. The assumptions are as follows: the region of x in the interval $(-0.5,0.5)$, the function $f=\cos(\lambda x)$, $k_z>0$ and $k_y=-k_z+\sqrt{2}$ that give $\Omega_0^2\equiv k_\parallel^2(0)=1$ and $k_\perp^2\equiv k_\perp^2(0)\gg\Omega_0^2$, $\lambda=(2/k_\perp|\Omega_0|)^{1/2}$ so that $k_\parallel^2\simeq 1+x^2$ for small $|x|$, and $\epsilon\sim V_s\sim k_\perp^{-1}\ll 1$. For the localized modes around $x=0$, Eq. (7) is approximately Fourier-transformed to the Schrödinger-type equation,[6,7]

$$\frac{d^2}{d\zeta^2}\psi + [\mu - V(\zeta)]\psi = 0, \quad (11)$$

$$V(\zeta) = \frac{1}{(1+\zeta^2)^2} - \frac{\alpha}{1+\zeta^2} + k_\perp^4\beta(1+\zeta^2), \quad (12)$$

where $\mu=k_\perp^2(\Omega^2-\Omega_0^2)$, $\alpha=\epsilon^2\omega_0^2\Omega_0^4$, $\beta=\epsilon^2\delta^2\Omega_0^4$, $\delta^2=V_s^2/(\omega_0^2-k_\parallel^2V_s^2)\simeq V_s^2/\omega_0^2$, $\omega_0^2=\Omega_0^2/(1+\epsilon^2\Omega_0^2)$, and

$$\psi = \sqrt{(1+\zeta^2)/2\pi}\int_{-\infty}^{\infty} g_\perp \exp[-i(k_\perp x)\zeta]d(k_\perp x). \quad (13)$$

Equation (11) are obtained ignoring the contribution from boundaries. We solve Eq. (11) numerically using a shooting code. Equations (8)–(10) are solved with the numerical method used in Refs. 8 and 9. If we discretize the equations and combine them with the boundary conditions, we obtain a matrix equation for the discretized variables. The eigenvalues are then obtained by finding the roots of the determinant of this matrix. We choose here the boundary conditions of rigid and conducting walls, i.e., $v_x=b_x=0$ at $x=\pm0.5$, which correspond to

$$-g_\perp + i\omega\epsilon g_x = 0, \quad (14)$$

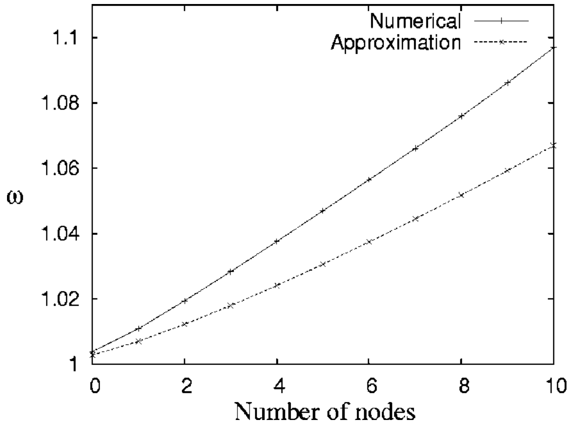

FIG. 2. Frequency as a function of the number of nodes of the eigenfunction for $k_z=15$. "Numerical" represents solutions of Eqs. (8)–(10), and "Approximation" represents solutions of Eq. (11).

$$k_\perp g_\parallel - k_\parallel g_\perp = 0. \tag{15}$$

Figure 1 shows the first three modes of the eigenfunction $g_\perp$ for $k_z=15$. The eigenfunctions are well localized around $x=0$, where $k_\parallel^2$ is minimum. Figure 2 shows the discrete eigenvalues obtained from Eqs. (8)–(10) and those obtained from Eq. (11). Discrete modes appear above the minimum of $k_\parallel^2$. Since the HMHD equations do not include the Landau damping term, all of the discrete eigenvalues are real. For higher modes, the difference between the two results is larger because the eigenfunction becomes broader (see Fig. 1) and may be affected by the boundaries. Since $k_\perp^4\beta\sim 1$ in the quadratic term of the potential (12), the width of the eigenfunction in ζ space is estimated as $\Delta_\zeta\sim 1$; thus the normalized width of $g_\perp$ is $\overline{\Delta}_x\sim(k_\perp\Delta_\zeta)^{-1}\sim\epsilon$. In fact, the lowest mode of $g_\perp$ in Fig. 1 is well fitted by a Gaussian curve of form $\exp(-x^2/2\overline{\Delta}_x'^2)$ with $\overline{\Delta}_x'\simeq 1.89\epsilon$. These results are in agreement with those obtained previously using kinetic theory in Refs. 6 and 10.

The width of the eigenfunction may also be estimated directly from Eq. (7). In the neighborhood of a singular point x_0, where $F(x_0)=0$, Eq. (7) is approximated as

$$F'(x_0)\Delta_x\frac{d^2}{dx^2}g_\perp + F'(x_0)\frac{d}{dx}g_\perp - k_\perp^2F'(x_0)\Delta_x g_\perp - S^2g_\perp \simeq -k_\parallel^2\rho_s^2\left(\frac{d^2}{dx^2}-k_\perp^2\right)^2 g_\perp, \tag{16}$$

where $F'=-(d/dx)(k_\parallel^2)$ and $\Delta_x=x-x_0$. In the absence of the fourth-order derivative, the eigenfunction contains a logarithmic function $\ln|x-x_0|$ (Ref. 1) and the derivative at $x=x_0+\Delta_x$ is thus estimated as

$$\frac{1}{g_\perp}\frac{d}{dx}g_\perp\sim\Delta_x^{-1}. \tag{17}$$

As $\Delta_x\to 0$, the fourth-order term increases and becomes comparable to the second-order term when Δ_x satisfies the relation

$$\frac{F'(x_0)}{\Delta_x}\sim\frac{k_\parallel^2\rho_s^2}{\Delta_x^4}. \tag{18}$$

In this scale of Δ_x, the fourth-order term modifies the singular eigenfunction to the regular one. This Δ_x gives the width of the eigenfunction. Since $k_\parallel^2\simeq k_\parallel^2(0)[1+(x/L_0)^2]$ and the singularity is located near the edge of the Alfvén continuum, i.e., $|x_0|=|\Delta_x|$, in the present analysis, $F'(x_0)$ is written as

$$F'(x_0)\sim 2k_\parallel^2\Delta_x/L_0^2. \tag{19}$$

Thus, we obtain

$$\Delta_x\sim(\rho_sL_0)^{1/2}=\left(\frac{V_s}{V_A}\frac{c}{\omega_{pi}}L_0\right)^{1/2}\sim\epsilon L_0. \tag{20}$$

The width of the eigenfunction observed in Ref. 7, $\tilde{\Delta}_x\sim(V_s/k_\parallel\omega)^{1/2}$, is given by the relation

$$S^2\sim\frac{k_\parallel^2\rho_s}{\tilde{\Delta}_x^4}. \tag{21}$$

The fourth-order term is too small to modify the logarithmic eigenfunction in the scale of $\tilde{\Delta}_x$ since

$$\frac{k_\parallel^2\rho_s^2/\tilde{\Delta}_x^4}{F'(x_0)/\tilde{\Delta}_x}\sim\frac{k_\parallel^2L_0^2}{2}\frac{\omega^2}{\omega_{ci}^2}\sim k_\parallel^2L_0^2\epsilon^2\sim\epsilon^2. \tag{22}$$

III. KINETIC DISPERSION RELATION

In this section, we derive the dispersion relation for the MHD waves from the kinetic equations. The wave equation for a homogeneous plasma is given by[11]

$$\mathbf{n}\times(\mathbf{n}\times\mathbf{E})+\boldsymbol{\epsilon}\cdot\mathbf{E}=0, \tag{23}$$

where $\mathbf{n}=\mathbf{k}c/\omega$. The magnetic field is in the z direction and the wave number vector is $\mathbf{k}=k_x\mathbf{e}_x+k_\parallel\mathbf{e}_z$. Assuming $v_{Ti}\ll\omega/|k_\parallel|\ll v_{Te}$, $T_i\ll T_e$ and $\omega\ll\omega_{ci}$, the dielectric tensor $\boldsymbol{\epsilon}$ (Refs. 5 and 11) can be approximated as follows:

$$\epsilon_{xx}\simeq(c^2/V_A^2)\alpha_1, \tag{24}$$

$$\epsilon_{xy}=-\epsilon_{yx}\simeq -i(c^2/V_A^2)\alpha_1(\omega/\omega_{ci}), \tag{25}$$

$$\epsilon_{xz}=\epsilon_{zx}\simeq 0, \tag{26}$$

$$\epsilon_{yy}\simeq(c^2/V_A^2)\alpha_1, \tag{27}$$

$$\epsilon_{yz}=-\epsilon_{zy}\simeq i\frac{c^2}{V_A^2}\frac{k_x}{k_\parallel}\frac{\omega_{ci}}{\omega}, \tag{28}$$

$$\epsilon_{zz}\simeq\left(\frac{\omega_{pi}}{c_sk_\parallel}\right)^2\left(1-\frac{k_\parallel^2c_s^2}{\omega^2}\right), \tag{29}$$

where $c_s=(T_e/M)^{1/2}$ is the ion acoustic speed. The dispersion relation can be found from the determinant of Eq. (23),

$$\begin{vmatrix} \alpha_1\omega^2 - k_\parallel^2 V_A^2 & -i(\omega/\omega_{ci})\alpha_1\omega^2 & k_x k_\parallel V_A^2 \\ i(\omega/\omega_{ci})\alpha_1\omega^2 & \alpha_1\omega^2 - k^2 V_A^2 & i(k_x/k_\parallel)(\omega_{ci}/\omega)\omega^2 \\ k_x k_\parallel (c/\omega_{pi})^2 (k_\parallel c_s)^2 & -i(k_x/k_\parallel)(\omega/\omega_{ci})(k_\parallel c_s)^2 & \omega^2 - (k_\parallel c_s)^2 - (k_\parallel c_s)^2 k_x^2 (c/\omega_{pi})^2 \end{vmatrix} = 0, \tag{30}$$

which yields

$$(\omega^2 - k_\parallel^2 V_A^2)[\omega^4 - k^2(V_A^2 + c_s^2)\omega^2 + k^2 k_\parallel^2 c_s^2 V_A^2] - k^2 V_A^2 k_\parallel^2 V_A^2 \omega^2(\omega^2 - k^2 c_s^2)/\omega_{ci}^2 = 0, \tag{31}$$

where $k^2 = k_x^2 + k_\parallel^2$. This equation is identical to the dispersion relation for HMHD waves,[7,12] provided the sound speed $V_s = \sqrt{\gamma P/\rho}$ in HMHD is identified as the ion acoustic speed $c_s = \sqrt{T_e/M}$, that is, when $T_i \ll T_e$. In other words, HMHD is valid (in the sense that it can be fully recovered from the kinetic dispersion relation) only if T_i is ignorable.[9] The consistency between HMHD and kinetic theory when $T_i = 0$ can be seen more clearly through the recovery of two important components b_z and j_z found in HMHD (Ref. 7) from kinetic theory; see the Appendix.

The coupling term

$$k^2 V_A^2 k_\parallel^2 V_A^2 \omega^2(\omega^2 - k^2 c_s^2)/\omega_{ci}^2 = \left(\frac{ck}{\omega_{pi}}\right)^2 k_\parallel^2 V_A^2 \omega^2(\omega^2 - k^2 c_s^2)$$

originates from the product of the diagonal components,

$$\left(\omega^2 - \frac{1}{\alpha_1}k_\parallel^2 V_A^2\right)\left(\omega^2 - \frac{1}{\alpha_1}k^2 V_A^2\right)\{\omega^2 - (k_\parallel c_s)^2[1 + (ck_x/\omega_{pi})^2]\}$$
$$= \left[\omega^2\left(1 + \frac{c^2 k_\parallel^2}{\omega_{pi}^2}\right) - k_\parallel^2 V_A^2\right]\left[\omega^2\left(1 + \frac{c^2 k^2}{\omega_{pi}^2}\right) - k^2 V_A^2\right] \times \{\omega^2 - (k_\parallel c_s)^2[1 + (ck_x/\omega_{pi})^2]\},$$

and entirely disappears in two-field approximation based on ϕ and $A_\parallel$ alone, namely, if the magnetosonic perturbation is ignored.

The last term contains fourth-order derivative,

$$\omega^2 k^2 V_A^2 k_\parallel^2 V_A^2 k^2 \rho_s^2 \propto k^4,$$

and is responsible for the appearance of localized eigenfunctions of discrete Alfvén mode. Here $\rho_s = c_s/\omega_{ci}$ is the ion acoustic Larmor radius. In this form, the coupling between the shear Alfvén mode $k_\parallel V_A$ and the magnetosonic mode kV_A through the ion acoustic Larmor radius is explicitly seen. When $k_\parallel^2 \ll k^2$ and $c_s^2 \ll V_A^2$, the dispersion relation for the shear Alfvén mode is obtained,[12]

$$\omega^2 \simeq \frac{k_\parallel^2 V_A^2 + k_\parallel^2 c_s^2 (ck/\omega_{pi})^2}{1 + (ck_\parallel/\omega_{pi})^2} = \frac{k_\parallel^2 V_A^2}{1 + (k_\parallel V_A/\omega_{ci})^2}(1 + k^2 \rho_s^2). \tag{32}$$

It is noted that ρ_s appears isotropically $k^2\rho_s^2$ in contrast to the form based on two-field approximation,

$$\omega^2 \simeq \frac{k_\parallel^2 V_A^2}{1 + (k_\parallel V_A/\omega_{ci})^2}(1 + k_\perp^2 \rho_s^2).$$

IV. SUMMARY

In this paper, we have shown that the mode equation for the Alfvén wave in Hall MHD can be recovered exactly from the kinetic dispersion relation in the limit of $T_i \ll T_e$ and $\omega/|k_\parallel| \ll v_{Te}$, i.e., cold ions and adiabatic electrons. The ability of the simpler one-fluid theory (Hall MHD) in capturing the essential features of the wave spectrum (removal of the Alfvénic singularity, for instance), which are normally considered to be associated with kinetic effects, is important and bodes well for the future of HMHD theory and codes alike.

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APPENDIX: ALTERNATIVE DERIVATION OF THE KINETIC DISPERSION RELATION

Here we demonstrate an alternative derivation of the dispersion relation (31) from kinetic theory. The components b_z and j_z (perturbed magnetic field and current parallel to the unperturbed magnetic field) can be found from the following equations:

$$-i\omega\mathbf{b} = -i\mathbf{k} \times \mathbf{E}, \tag{A1}$$

$$\mu_0\mathbf{j} = i\mathbf{k} \times \mathbf{b}, \tag{A2}$$

$$\mathbf{E} \simeq \frac{i}{\epsilon_0\omega}\boldsymbol{\epsilon}^{-1} \cdot \mathbf{j}, \tag{A3}$$

$$\mathbf{k} \cdot \mathbf{b} = k_\perp b_x + k_\parallel b_z = 0, \tag{A4}$$

where

$$\boldsymbol{\epsilon}^{-1} = \frac{1}{|\boldsymbol{\epsilon}|}\begin{pmatrix} \epsilon_{yy}\epsilon_{zz} - \epsilon_{yz}\epsilon_{zy} & -\epsilon_{xy}\epsilon_{zz} & \epsilon_{xy}\epsilon_{yz} \\ -\epsilon_{yx}\epsilon_{zz} & \epsilon_{xx}\epsilon_{zz} & -\epsilon_{xx}\epsilon_{yz} \\ \epsilon_{yx}\epsilon_{zy} & -\epsilon_{xx}\epsilon_{zy} & \epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}\epsilon_{yx} \end{pmatrix}, \tag{A5}$$

$$|\boldsymbol{\epsilon}| = \epsilon_{xx}\epsilon_{yy}\epsilon_{zz} - \epsilon_{xx}\epsilon_{yz}\epsilon_{zy} - \epsilon_{xy}\epsilon_{yx}\epsilon_{zz}.$$

Substituting Eq. (A3) into Eq. (A1) yields

$$\epsilon_0\omega^2\mathbf{b} = i\mathbf{k} \times (\boldsymbol{\epsilon}^{-1} \cdot \mathbf{j}). \tag{A6}$$

From Eqs. (A2) and (A4), we obtain

$$j_x = -\frac{k_\parallel}{k_\perp} j_z, \quad j_y = -i\frac{k^2}{\mu_0 k_\perp} b_z, \quad b_y = -i\frac{\mu_0}{k_\perp} j_z.$$

The y component of Eq. (A6) gives

$$\begin{aligned}\frac{\omega^2}{c^2}|\boldsymbol{\epsilon}| j_z = &- k_\parallel\{k_\parallel(\epsilon_{yz}\epsilon_{zy} - \epsilon_{yy}\epsilon_{zz}) + k_\perp \epsilon_{yx}\epsilon_{zy}\} j_z \\ &- k_\perp\{k_\parallel \epsilon_{xy}\epsilon_{yz} + k_\perp(\epsilon_{xy}\epsilon_{yx} - \epsilon_{xx}\epsilon_{yy})\} j_z \\ &- i\mu_0^{-1}(k_\parallel \epsilon_{xy}\epsilon_{zz} - k_\perp \epsilon_{xx}\epsilon_{zy}) k^2 b_z, \end{aligned} \tag{A7}$$

which yields

$$\mu_0 j_z = -\frac{c^2}{\omega_{pi}^2} \frac{k^2 k_\parallel \omega_{ci}\omega}{\omega^2 - k_\parallel^2 V_A^2} b_z. \tag{A8}$$

The z component of Eq. (A6) gives

$$\frac{\omega^2}{c^2}|\boldsymbol{\epsilon}| b_z = i\mu_0(k_\parallel \epsilon_{yx}\epsilon_{zz} - k_\perp \epsilon_{xx}\epsilon_{yz}) j_z + k^2 \epsilon_{xx}\epsilon_{zz} b_z,$$

which yields

$$-\omega b_z = -\frac{k^2 V_A^2(\omega^2 - k_\parallel^2 c_s^2)}{\omega(\omega^2 - k^2 c_s^2)} b_z + \frac{c^2}{\omega_{pi}^2} k_\parallel \omega_{ci}\mu_0 j_z. \tag{A9}$$

Equations (A8) and (A9) are equivalent to Eqs. (11) and (8) of Ref. 7 derived from the HMHD and yield the dispersion relation for the HMHD waves (31). The importance of the magnetosonic mode b_z and its coupling to the parallel current j_z associated with the shear Alfvén mode may be better appreciated in this derivation.

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