Fluid description of ion dynamics in a toroidally confined plasma

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Fluid equations describing ion dynamics in a toroidally confined plasma at low collision frequency are derived. The principle motivation is to present a framework for incorporating basic neoclassical effects into a fluid theory. The ions are assumed to be magnetized in the sense that relevant scale lengths are much longer than the ion gyroradius, and time scales of interest are assumed long compared to the ion bounce time. These assumptions are consistent with, for example, the evolution of unstable magnetic islands, as well as conventional transport. A special case of the present description is the quasistatic, axisymmetric state with nearly uniform pressure and density on flux surfaces. In that case the equations reproduce the radial ion heat transport predicted by neoclassical transport theory. The essential feature of our derivation is its emphasis on heat flow in the direction of the magnetic field. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1881534]

I. INTRODUCTION

A. Goals and synopsis

This work is intended as a step toward the goal of a fluid description of toroidally confined, magnetized plasma. It derives a set of equations describing the dynamics of *ions* in such a plasma in terms of a few fluid variables.[1] The fluid equations include effects of finite Larmor radius (FLR), as well as key effects of magnetic trapping. In particular, they describe ion thermal transport in the so-called "banana" regime of neoclassical transport theory.[2,3]

As the focus is on a method to capture neoclassical physics, other important effects, such as kinetic and Landau damping type responses, are not addressed. Admittedly, this is a weakness of the theory and is discussed briefly below.

Other collisionality regimes of neoclassical theory are not considered in detail, either. However we point out that a fluid description of the collisional dominated, or Pfirsch–Schlüter regime, is entirely straightforward.

Neoclassical effects were originally derived under strict assumptions, appropriate to an axisymmetrically confined plasma in quasistatic equilibrium. The neoclassical fluid equations take their relatively simple form only because they have been averaged over the flux surfaces. Plasma instability and the evolution of such symmetry breaking phenomena as magnetic islands contradict these assumptions, depending upon effects that the averaged equations cannot show. Thus the more general dynamics require a generalized treatment. Nonetheless, if the nonequilibrium processes are slow compared to the bounce frequency of magnetically trapped particles, key neoclassical processes will continue to have important effects.

The literature contains a number of fluid models that include FLR as well as selected neoclassical effects (see, for example, Ref. 4). The present model is distinguished from its predecessors by its emphasis on heat flow parallel to the magnetic field. At low collisionality such heat flow is rapid, but not instantaneous. Thus the parallel heat flow can play an important role in various confined plasma contexts, such as the evolution and effects of magnetic islands. Furthermore, as we show in Sec. II, it lies at the root of neoclassical anisotropy and radial transport. (In the case of electron dynamics, to be considered in future work, parallel heat flow can be viewed as the fundamental source for the bootstrap current.)

An additional reason for our emphasis on parallel heat flow is that the known ion distribution function in the banana regime predicts such a flow. Indeed, to lowest order in the collisionality parameter, parallel heat flow is predicted to the exclusion of other fluid effects such as density perturbation or pressure anisotropy. We demonstrate these features of the distribution in Sec. III, where its most important moments are computed.

The transport approach to studying parallel heat is to express the flux $q_\parallel$ in terms of lower moment driving forces. Results exist for arbitrary collisionality in homogeneous[5,6] and inhomogeneous fields.[7,8] The method presented here, however, instead promotes the parallel heat flux to a dynamical quantity which must be determined through the evolution of $\partial q_\parallel / \partial t$. While this increases the number of moment equations, the parallel heat is endowed with a more robust means of variation. Although the expected banana-regime result is returned in the appropriate limit, in general, $q_\parallel$ is determined only by the full dynamics of the system of moment equations.

In Sec. IV we combine well-known FLR fluid equations with neoclassical effects to derive a closed set of equations for the ion fluid. These equations comprise a model of confined ion dynamics which, while neither exact nor complete, displays the effects of finite Larmor radius and finite orbit widths in a manner consistent with the underlying physics.

B. Drift ordering

We assume that the ion gyroradius is smaller than any relevant scale length; thus FLR effects are retained perturbatively. We denote the gyroradius ordering parameter by

$$\delta \equiv \frac{\rho}{L_\perp},$$

where ρ is the ion thermal gyroradius and $L_\perp$ is a scale length perpendicular to the magnetic field. In many cases $L_\perp$ will be smaller than the perpendicular scale length of the equilibrium (plasma radius). Parallel scale lengths are assumed to be longer:

$$L_\| \sim \frac{L_\perp}{\delta}, \tag{1}$$

as pertains to the most important perturbations in toroidally confined plasma.[9]

The plasma flow velocity $\boldsymbol{V}$ and heat flow $\boldsymbol{q}$ perpendicular to the magnetic field are impeded by the field:

$$\frac{V_\perp}{v_t} \sim \delta,$$

$$\frac{q_\perp}{p v_t} \sim \delta,$$

where $v_t = \sqrt{2T/m}$ is the thermal speed and p the pressure. Notice that we do not yet constrain the parallel flows: while gyration constrains the perpendicular dynamics, there is no *a priori* analog for the parallel direction.

We assume that the collision frequency is smaller than the transit frequency. It is convenient to formalize this restriction using the gyroradius parameter:

$$\frac{\nu}{\omega_t} \sim \delta. \tag{2}$$

Here ν measures the ion-ion collision frequency and

$$\omega_t = \frac{v_t}{R}$$

is the transit frequency, with R the major toroidal radius.

Finally we assume that evolution occurs on the drift time scale:

$$\omega \sim \omega_* \sim \delta\omega_t, \tag{3}$$

where ω is a frequency scale of the dynamics and ω_* is the drift frequency.

Notice in particular that Eqs. (1) and (3) imply

$$\omega \sim k_\| v_t,$$

where $k_\|$ is the parallel wave number of some plasma disturbance. At these phase velocities one has both an appreciable number of particles and significant slope of the distribution function, and so therefore would expect kinetic resonances (e.g., Landau damping effects) to be of importance.

The omission of Landau damping is a serious flaw in a model for plasma behavior at $\omega \sim k v_t$. Methods have been proposed for incorporating such kinetic effects into fluid models.[10] However, since the focus of this paper is on a particular form of moment closure and we do not yet have a consistent model for kinetic resonances in the regimes of interest, we make no attempt to incorporate Landau damping into our model and only present a framework for capturing *neoclassical* effects. Although the resulting system will necessarily be incomplete, such a modular approach to the problem is by no means without merit.

The above ordering assumptions summarize what is sometimes called the "drift ordering." To compare it to the "transport ordering," adopted in neoclassical theory, we recall that the latter assumes that time variation is second order,

$$\omega \sim \delta^2 \omega_t,$$

and that it considers only spatial variation associated with fluid equilibrium. As a result, the parallel scale lengths of plasma densities and temperatures turn out to be longer than in the drift ordering. For example, in the banana regime one finds

$$\boldsymbol{\nabla} T \sim \left(\frac{\nu}{\omega_t}\right)\delta T \quad \text{(equilibrium variation)}.$$

The present system allows much stronger variation.

Such ordering differences are important. However, we should emphasize the key difference between neoclassical transport theory and the system presented here: the neoclassical equations are flux-surface averages, unable to describe phenomena in which poloidal or toroidal variation plays a role. Moreover, the averages of neoclassical theory are essential to the closure and not trivially undone. (For example, the averages eliminate contributions from parts of the distribution function that the theory does not determine.) The present closure provides a fully three-dimensional, if approximate, description by departing from neoclassical methods. Most importantly our theory includes additional moments, describing the evolution of heat flow.

The assumed drift ordering seriously limits the range of phenomena that can be described. Much less serious is a conventional large aspect ratio assumption,

$$a \ll R. \tag{4}$$

This simplification is helpful for computing numerical coefficients, but its qualitative effect is minor. It turns out to be consistent with Eq. (1) provided

$$\left(\frac{a}{R}\right)^2 \sim \delta.$$

The small collisionality assumption is consistent with the banana regime of neoclassical theory; indeed only banana-regime physics is incorporated. In this regard we recall that the bounce frequency ω_b of magnetically trapped particles is comparable to, if somewhat smaller than, ω_t.

C. Underlying physics

We close this section by describing the basis of our model, which uses an analogy between bounce motion and gyromotion. A more explicit version of this discussion can be found in Sec. IV.

Guiding-center theory studies phenomena whose typical frequencies ω are slow compared to the gyrofrequency Ω:

$$\omega \ll \Omega. \tag{5}$$

Slow variation of the distribution requires that it depends on gyrophase only through the guiding-center position $\boldsymbol{X}=\boldsymbol{x}-\boldsymbol{\rho}$, where $\boldsymbol{x}$ is the spatial coordinate and $\boldsymbol{\rho}$ is the vector gyroradius:

$$\boldsymbol{\rho} = \frac{\boldsymbol{b}\times \boldsymbol{v}_\perp}{\Omega}, \tag{6}$$

with $\boldsymbol{b}\equiv \boldsymbol{B}/B$ a unit vector along the magnetic field $\boldsymbol{B}$. In other words the guiding-center distribution has the form

$$f(\boldsymbol{x},\boldsymbol{v}) = F(\boldsymbol{X},v_\parallel,v_\perp),$$

where $v_\parallel=\boldsymbol{b}\cdot\boldsymbol{v}$ and $v_\perp=|\boldsymbol{b}\times(\boldsymbol{v}\times\boldsymbol{b})|$. The function F is presumed to vary on a scale length L that is much longer than the gyroradius, whence

$$f(\boldsymbol{x},\boldsymbol{v}) \approx F(\boldsymbol{x},v_\parallel,v_\perp) - \boldsymbol{\rho}\cdot\nabla F(\boldsymbol{x},v_\parallel,v_\perp),$$

where the first term, independent of gyrophase, approximates the gyrophase average of f, denoted by $\bar{f}$. This function is determined by the drift-kinetic equation, although to lowest order it can be taken to be a Maxwellian distribution, denoted by f_M:

$$\bar{f} = f_M + O\left(\frac{\rho}{L}\right). \tag{7}$$

The main point here is that the gyrophase-dependent part of f, denoted by f_g, is given by

$$f_g \approx -\boldsymbol{\rho}\cdot\nabla\bar{f} \approx -\boldsymbol{\rho}\cdot\nabla f_M, \tag{8}$$

or, equivalently,

$$f_g \approx f_M\left[\frac{m}{T}\boldsymbol{V}_\perp\cdot\boldsymbol{v} + \left(\frac{mv^2}{2T}-\frac{5}{2}\right)\boldsymbol{b}\times\nabla \ln T\cdot\boldsymbol{v}\right], \tag{9}$$

with

$$\boldsymbol{V}_\perp \equiv \frac{\boldsymbol{b}}{mn\Omega}\times(\nabla p - en\boldsymbol{E}). \tag{10}$$

Here n and T are the density and temperature appearing in the Maxwellian and $p=nT$. The function f_g is central to magnetized plasma dynamics; it reproduces, for example, the familiar electric and diamagnetic drifts, along with such effects as gyroheat flow and gyroviscosity. All FLR fluid models use Eq. (8), at least implicitly, and the present model is not an exception. But our distribution includes an additional term, the banana analog to Eq. (8).

The new term results from a stronger version, Eq. (5):

$$\omega \ll \omega_b, \tag{11}$$

where ω_b is the trapped-particle bounce frequency. Similarly it assumes radial scale lengths longer than the banana width. Then the argument leading to Eq. (8), with gyromotion replaced by bounce motion, implies a term in the distribution of the form

$$f_b \approx -\Delta_b \frac{df_M}{dr}, \tag{12}$$

where Δ_b is the banana orbit width and the variable r, constant on flux surfaces, identifies the position of the banana center. Indeed the banana-regime distribution,[11] which we review in Sec. III, has this form.

Despite the close analogy between Eqs. (8) and (12), the two functions differ in important respects. For example, the gyrocorrection is independent of parallel velocity, while the banana correction is odd in $v_\parallel$. The most important difference is related to localization: while the gyroradius can be assumed small compared to any relevant scale length, the bounce orbit is narrow only in radial extent; on any given flux surface, it traverses a macroscopic interval. We shall see that this circumstance affects both the magnitude and form of its influence on fluid equations. Here we point out that the present model can be characterized as an attempt to incorporate both types of FLR corrections, Eqs. (8) and (12), into a physically plausible fluid description.

We now review the phenomenon of ion heat flow in a neoclassical context. While the following discussions concern transport regimes more restrictive than the full dynamics we are considering, they are the foundation on which our extension is based.

II. NEOCLASSICAL HEAT FLOW

A. Spatial coordinates and notation

A special case of the present description is the quasistatic state in which the pressure and density are nearly uniform on flux surfaces. In that case our equations reduce to simple forms that contain the physics of radial ion heat transport predicted by neoclassical transport theory. The purpose of this section is to show how the known transport results are reproduced.

Because of the small mass ratio, and because collisional momentum conservation makes ion particle transport proceed on the scale of the electron mass, the dominant ion neoclassical process is ion heat conduction. Hence the burden of this section is to show how ion radial heat flow is captured in the model.

We denote the neoclassically predicted radial ion heat flow by $\langle q_b^r\rangle$, where the subscript b refers to the banana regime and the angle brackets refer to a conventional flux-surface average [see Eq. (34)]. It is convenient to introduce here some additional notation that will be used throughout.

We use toroidal coordinates (r,θ,ζ) where the radius r labels magnetic surfaces, and θ and ζ are the poloidal and toroidal angles, respectively. The *equilibrium* magnetic field in an *axisymmetric* toroidal system has the form[12]

$$\boldsymbol{B} = I(r)\nabla\zeta + \nabla\zeta\times\nabla\chi(r), \tag{13}$$

where χ measures the poloidal flux. The major radius is denoted by R; its maximum value is R_0. Note that $|\nabla\zeta|=1/R$, so that the toroidal magnetic field has magnitude

$$B_T = \frac{I}{R}.$$

The total (equilibrium) field magnitude is expressed as

$$B(r,\theta) = \frac{B_0}{h(r,\theta)}, \tag{14}$$

where B_0 is a constant. We will often use a large-aspect ratio approximation in which

$$\epsilon \equiv \frac{r}{R_0}$$

is assumed small. Then $B \approx B_T$ whence

$$h \approx 1 + \epsilon \cos\theta. \tag{15}$$

A convenient measure of the ion gyrofrequency is

$$\Omega_0 \equiv \frac{eB_0}{mc},$$

where $e > 0$ is the magnitude of the electronic charge and m is the ion mass. The frequency that appears most commonly in neoclassical theory is the poloidal gyrofrequency, which is conveniently defined by

$$\Omega_p \equiv \left(\frac{\chi'}{I}\right)\Omega_0. \tag{16}$$

Here the prime denotes a radial derivative.

B. Parallel and radial heat flows

The simplest moment description of heat flow involves the energy flow vector

$$\boldsymbol{Q} = \int d^3v mw\boldsymbol{v}f, \tag{17}$$

where

$$w \equiv \frac{v^2}{2}$$

measure the kinetic energy and f is the distribution function. The heat flow $\boldsymbol{q}$ differs in that each factor of $\boldsymbol{v}$ is replaced by $\boldsymbol{v}-\boldsymbol{V}$, where $\boldsymbol{V}$ is the mean plasma flow velocity. For a nearly isotropic plasma, the two flows are related approximately by

$$\boldsymbol{q} = \boldsymbol{Q} - \tfrac{5}{2}p\boldsymbol{V}.$$

Indeed, the *radial* components of the two flows, q^r and Q^r, do not significantly differ, because the radial particle flow V^r is negligible on the scale of the ion mass. The point is that ambipolar radial diffusion, enforced by collisional momentum conservation, makes V^r proportional to the squared electron gyroradius. On the other hand, the $p\boldsymbol{V}$ term is important with regard to the *parallel* flows.

Now we recall the exact $mv^2\boldsymbol{v}$ moment of the Boltzmann equation (see, for example, Ref. 9):

$$\frac{\partial \boldsymbol{Q}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{R} - \frac{3}{2}\frac{e}{m}p\boldsymbol{E} - \frac{1}{2}enV^2\boldsymbol{E} - \frac{e}{m}\boldsymbol{E}\cdot\boldsymbol{P} - \frac{e}{mc}\boldsymbol{Q}\times\boldsymbol{B} = \boldsymbol{G}, \tag{18}$$

where $\boldsymbol{E}$ is the electric field,

$$\boldsymbol{R} \equiv \int d^3v mw\boldsymbol{v}\boldsymbol{v}f, \tag{19}$$

and

$$\boldsymbol{G} \equiv \int d^3v mw\boldsymbol{v}C(f). \tag{20}$$

Here C represents the collision operator. A conventional argument[9] takes advantage of the large gyrofrequency in the last term on the right-hand side of Eq. (18) to extract the lowest-order (nondissipative) heat flow,

$$\boldsymbol{q}_\perp = \frac{5}{2}\frac{pT}{m\Omega}\boldsymbol{b}\times\boldsymbol{\nabla}\ln T \tag{21}$$

as well as the classical collisional energy flow

$$\boldsymbol{Q}_c = -\frac{\boldsymbol{b}}{\Omega}\times\boldsymbol{G}. \tag{22}$$

Neoclassical theory extends this argument to produce a corresponding expression for $\langle q^r_{NC}\rangle$: one multiplies Eq. (18) with $R^2\boldsymbol{\nabla}\zeta$ to isolate its covariant ζ component and performs a flux surface average. The result is[2,3,9]

$$\langle q^r\rangle = -\frac{c}{e\chi'}\langle mG_\zeta\rangle.$$

This expression includes both classical and neoclassical flow; in view of Eq. (22), the neoclassical component is

$$\langle q^r_{NC}\rangle = -\frac{c}{e\chi'}\langle mG_{\|\zeta}\rangle,$$

where, for any vector $\boldsymbol{A}$,

$$A_{\|\zeta} = \left(\frac{B_\zeta}{B}\right)A_\| = \left(\frac{I}{B}\right)A_\|$$

since $I(\chi) \equiv B_\zeta$. Hence we have

$$\langle q^r_{NC}\rangle = -\frac{mcI}{e\chi'}\left\langle\frac{G_\|}{B}\right\rangle. \tag{23}$$

Next we consider the collisional moment

$$G_\| = \int d^3v mwv_\| C(f).$$

Because of the rotational invariance of C, it is clear that $G_\|$ depends only on the first Legendre component of f—the part of f that gives parallel particle and parallel heat flow. Since only ion-ion collisions matter, a parallel velocity displacement of f will not contribute, suggesting that

$$G_\| = -\bar{\nu}q_\|. \tag{24}$$

The proportionality factor $\bar{\nu}$ is of the same order as the ion-ion collision frequency, but since numerical accuracy is not of primary importance, we do not attempt an explicit calculation. Instead, we use $\bar{\nu}$ to ensure that simple well-known transport results are reproduced (see the end of Sec. IV). Notice that no factors of the inverse aspect ratio occur in the relation—only the collision frequency and a numerical coefficient of order unity.

Combining Eqs. (23) and (24), and recalling the definition Eq. (16), we conclude that

$$\langle q^r_{NC}\rangle = \frac{\bar{\nu}}{\Omega_P}\langle q_\parallel h\rangle. \tag{25}$$

The neoclassical heat flow across a flux surface is proportional to the parallel heat flow on the surface. Notice that this statement pertains for any collision frequency; that is, it pertains in all the neoclassical regimes of collisionality. Note also that to compute the neoclassical flow, the parallel heat flow is needed only to zeroth order in the collision frequency.

We conclude that the inclusion of ion neoclassical dissipation in any fluid model depends upon accurately incorporating ion *parallel* heat flow in the model. In order to understand the origin of parallel heat flow and to compute it, we now turn our attention to the neoclassical distribution function.

III. NEOCLASSICAL DISTRIBUTION FUNCTION

A. Banana kinetic equation

Consider the full drift-kinetic equation,

$$\frac{\partial \bar{f}}{\partial t} + u\nabla_\parallel \bar{f} + \boldsymbol{V}_d\cdot\nabla\bar{f} + \frac{dU}{dt}\frac{\partial \bar{f}}{\partial U} + \frac{d\mu}{dt}\frac{\partial \bar{f}}{\partial \mu} - \mathcal{C}(\bar{f}) = 0. \tag{26}$$

An expansion in gyroparameter, $\bar{f}=f_0+f_1$, using the drift ordering $\partial/\partial t \sim \boldsymbol{V}_d \sim f_1 \sim \delta$ results in the two lowest-order equations

$$u\nabla_\parallel f_0 - \mathcal{C}(f_0) = 0, \tag{27}$$

and

$$\frac{\partial f_0}{\partial t} + u\nabla_\parallel f_1 + \boldsymbol{V}_d\cdot\nabla f_0 + \frac{dU}{dt}\frac{\partial f_0}{\partial U} - \mathcal{C}(f_1) = 0, \tag{28}$$

where collisions have not yet been ordered.

In a standard transport ordering, time variation is completely neglected compared to collisions, even if collisions are small, as in Eq. (2). In this case, Eq. (27) can be used as it stands and provides well-known solutions for f_0: Maxwellians constant over flux surfaces. Clearly the physical processes of parallel streaming and collisions determine the solutions, but the Maxwellian result does not depend on collision frequency and therefore is obtained in any collisionality regime.[9]

Equations (2) and (3) neglect neither collisions nor time variation with respect to each other, but prevent us from deriving such simple conclusions concerning f_0, since keeping $\partial f_0/\partial t \sim \mathcal{C}(f_0)$ would invalidate the arguments for flux surface Maxwellians. Physically, it is evident that there must not be any rapid time-dependent phenomena (no $\partial/\partial t \gtrsim \nu$) and that a collision time must elapse before the plasma will undergo any substantial relaxation to a Maxwellian. Therefore, since we order $\partial/\partial t \sim \nu$ our model cannot capture lowest-order Maxwellianization. However, our fluid model is not intended to apply to far-from-equilibrium conditions, but instead to the relatively slow evolution of a plasma in an already established near-equilibrium state. Such conditions are observed to develop in tokamak plasmas. Even under the onset of turbulence or mild instabilities the plasma can be usefully described as Maxwellian with small, but crucial, corrections (such as gyration and bounce motion). Thus, we do not attempt to rigorously derive the lowest-order distribution function, but instead use the common *ansatz* that f_0 is Maxwellian. Since plasma experiments often confirm and fluid theories generally rely on such assumptions, we believe it to be a reasonable element to include in our theory.

Equation (27) is then satisfied identically, and separating the f_1 and $f_0=f_M$ terms in the first-order equation, Eq. (28) gives

$$u\nabla_\parallel f_1 - \mathcal{C}(f_1) = -\frac{\partial f_M}{\partial t} - \frac{dU}{dt}\frac{\partial f_M}{\partial U} - \boldsymbol{V}_d\cdot\nabla f_M. \tag{29}$$

If the right-hand side were to vanish, this equation would reduce to Eq. (27) and give the same Maxwellian solutions. Thus, the derivatives of f_M can be considered the source terms which drive the solutions for f_1 away from Maxwellians. The terms with $\partial f_M/\partial t$ and dU/dt, however, allow for the weak temporal variation of the lowest-order flux quantities to upset the aforementioned streaming/collisional relaxation. The presence of these terms is generic in the sense that they enter independently of other aspects of the dynamics such as gyromotion, bounce motion, etc. This is not the case for the $\boldsymbol{V}_d\cdot\nabla f_M$ term, which enters the kinetic equation explicitly in this form due to the motion of the guiding centers in a strong magnetic field: particle gyration is the source. Because of the nonvanishing gyroradius, particles are not strictly restricted to flux surfaces, but traverse regions with differing density, temperature, etc. This sampling prevents f_1 from undergoing simple streaming/collisional relaxation on flux surfaces.

As mentioned, transport analysis of Eq. (29) in the banana regime neglects the time derivatives, resulting in a kinetic equation

$$u\nabla_\parallel f_b - \mathcal{C}(f_b) = -\boldsymbol{V}_d\cdot\nabla f_M \tag{30}$$

predicting familiar banana motion with the widths of the banana orbits originating from the $\boldsymbol{V}_d\cdot\nabla f_M$ term. Informally, this reduction occurs as a result of time-scale separation: the bulk motion of the plasma is neglected on the scales of gyration and bounce motion. Since we seek a description that focuses on gyration and bounce motion, this effect is of primary importance.

To explicitly extract this banana-regime response from the complete first-order distribution function we write

$$f_1 = f_b + f_r,$$

where f_b is governed by Eq. (30) and the remainder f_r evidently solves

$$u\nabla_{\|}f_r - \mathcal{C}(f_r) = -\frac{\partial f_M}{\partial t} - \frac{dU}{dt}\frac{\partial f_M}{\partial U}.$$

The familiar solutions of Eq. (30) are presented in the following sections, but we do not attempt to solve the above kinetic equation for f_r. Instead, as it represents more general dynamics than gyration and bouncing, f_r is subsumed into the parts of the distribution function that must be represented by the evolution of the fluid moments. By this separation, the lowest order bounce dynamics appear directly and distinctly through f_b, which is only part of the complete distribution (discussed more fully in Sec. IV.)

Another note regarding time variation is worth mentioning here. In the literature there are numerous investigations of time dependence in a neoclassical context. One approach is to refrain from placing any ordering restrictions on the time derivative [unlike our use of Eq. (3)] and to examine the time dependence of the solutions to the resulting differential equations. Such studies of the relaxation of poloidal flow have resulted in a variety of time scales characterizing $\partial/\partial t$.[13–15] Results usually fall somewhere in the interval

$$\frac{\nu}{\epsilon} \gtrsim \frac{\partial}{\partial t} \gtrsim \nu.$$

Under our adopted frequency ordering,

$$\frac{\nu}{\epsilon} \gg \frac{\partial}{\partial t} \sim \omega_* \sim \nu \sim \delta\omega_t. \tag{31}$$

The condition of large aspect ratio, $\epsilon \ll 1$, allows a large difference between ν and ν/ϵ and the ordering of Eq. (31) allows a major simplification in our treatment of the banana kinetic equation. In particular,

$$\frac{\partial f_b}{\partial t} \sim \omega_* f_b$$

is small compared to

$$\mathcal{C}(f_b) \sim \frac{\nu}{\epsilon} f_b$$

and $\partial f_b/\partial t$ does not enter the equation for the banana distribution, Eq. (30). Notice that while the proper frequency scale for $\mathcal{C}(f_b)$ is ν/ϵ, other collisions are ordered by

$$\mathcal{C}(f_g) \sim \nu f_g,$$

$$\mathcal{C}(f_{\|}) \sim \nu f_{\|}.$$

Thus the particular sensitivity of banana orbits to collisions makes time dependence of secondary importance for determining f_b. The same is not true for other parts of the distribution function, however.

In summary, Eq. (31) prevents our model from capturing effects that proceed as fast as ν/ϵ, but greatly simplifies the treatment of neoclassical time dependence and still allows for time variation at the drift frequency ω_*, the scale for a number of interesting phenomena such as ion temperature gradient modes and slow evolution of unstable magnetic islands.

Next we turn to solutions of Eq. (30).

B. Velocity coordinates

Convenient velocity variables are the gyrophase angle γ and the approximate invariants $w=v^2/2$ and λ, which is the pitch-angle variable:

$$\lambda \equiv \frac{B_0 v_\perp^2}{B v^2} = h\frac{v_\perp^2}{v^2}.$$

Thus we have

$$\boldsymbol{v} = \boldsymbol{b} v_{\|}(\lambda,w) + v_\perp(\lambda,w)(\boldsymbol{e}_1 \cos\gamma - \boldsymbol{e}_2 \sin\gamma), \tag{32}$$

where $\boldsymbol{b}=\boldsymbol{B}/B$ and $(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{b})$ forms an orthogonal triplet of unit vectors.

The sign of the parallel velocity is denoted by σ, and we abbreviate

$$u \equiv v_{\|}.$$

Then we have

$$u = \sigma\sqrt{2w}\sqrt{1-\frac{\lambda}{h}}.$$

The maximum value of λ is evidently

$$\lambda_{\max} = h.$$

The smallest λ that allows $u=0$ on a given flux surface marks the boundary between the trapped and passing regions of velocity space. Denoting this value by λ_c we have

$\lambda < \lambda_c$, untrapped or passing region;

$\lambda \geq \lambda_c$, trapped region.

In the large-aspect ratio case, $\lambda_c = 1-\epsilon$.

C. Banana distribution

We next introduce the function $u^*(r,\theta,\lambda,w)$ such that

$$hu^*(r,\theta,\lambda,w) \equiv \sigma\sqrt{2w}L(r,\theta,\lambda),$$

where

$$L(\theta,\lambda) \equiv h\sqrt{1-\frac{\lambda}{h}} - \frac{\Theta(\lambda_c-\lambda)}{2}\int_\lambda^{\lambda_c} \frac{d\lambda'}{\left\langle \sqrt{1-\frac{\lambda'}{h}} \right\rangle}. \tag{33}$$

Here Θ denotes a step function and the angular brackets denote a flux-surface average, which has the form

$$\langle \mathcal{F} \rangle \equiv \oint d\theta\, \mathcal{F}h^2 \Big/ \oint d\theta\, h^2. \tag{34}$$

It is important to notice that the function L is localized in λ, being roughly concentrated in the trapped particle region. This follows from the observation that, as $\lambda \to 0$, the flux-surface average in Eq. (33) has no effect; then the λ integral in L can be performed and the two terms in Eq. (33) can be seen to cancel:

$$\lim_{\lambda\to 0} L(\lambda) = 0.$$

On the other hand the localization is weak: L has a tail, ordered by $\sqrt{\epsilon}$, that extends into the passing region.

It follows that the function u^* is a sort of truncated version of the parallel velocity u. Both u and u^* have the same energy and σ dependence, and they precisely coincide in the trapped region (where the step function vanishes). They differ only because u^*, being localized to the trapped region, decays away for small λ, while u increases as $\lambda \to 0$.

Finally it is convenient to introduce here the traditional driving forces

$$A_1 = \frac{\partial \ln p}{\partial r} + \frac{e}{T}\frac{\partial \ln \Phi}{\partial r}, \tag{35}$$

$$A_2 = \frac{\partial \ln T}{\partial r}, \tag{36}$$

the poloidal gyrofrequency

$$\Omega_p(r) = \left(\frac{eB_0}{m}\right)\left(\frac{\chi'}{I}\right),$$

and the neoclassical flow parameter

$$\alpha \approx \tfrac{4}{3}.$$

Through first order in the (poloidal) gyroradius and zeroth order in the collision frequency, the banana-regime neoclassical ion distribution is

$$\frac{f_b}{f_M} = -\frac{hu}{\Omega_P}\left[A_1 + \left(\alpha - \frac{5}{2}\right)A_2\right] - \frac{hu^*}{\Omega_P}\left(\frac{mw}{T} - \alpha\right)A_2. \tag{37}$$

This expression is the explicit version of Eq. (12), the function discussed in Sec. I. Notice that its first term corresponds to a Maxwellian, displaced in parallel velocity; we denote this contribution by

$$f_d \equiv -f_M \frac{hu}{\Omega_P}\left[A_1 + \left(\alpha - \frac{5}{2}\right)A_2\right].$$

The remaining term in f_b is, as we have noted, weakly localized to the trapped region; we denote it by

$$f_l \equiv -f_M \frac{hu^*}{\Omega_P}\left(\frac{mw}{T} - \alpha\right)A_2. \tag{38}$$

Thus we express Eq. (37) as

$$f_b = f_d + f_l. \tag{39}$$

The parallel between Eqs. (9) and (39), discussed in Sec. I, is clear.

D. Collisions and the flow parameter

We treat collisions in the conventional[11] manner, omitting ion scattering by electrons, which is small in the mass ratio, and exploiting the localized property of the banana distribution. Thus the ion-ion collision operator C is approximated by

$$C(f) \approx C_0(f_l), \tag{40}$$

where C_0 describes scattering in pitch angle. It is given by

$$C_0(f) \equiv \nu(w)\mathcal{L}f, \tag{41}$$

where ν is the energy-dependent ion-ion collision frequency,

$$\nu(w) \equiv \frac{3\sqrt{2\pi}}{\tau_i}\eta^{-5}[\eta\,\mathrm{erf}(\eta) + (2\eta^2 - 1)\mathrm{erf}(\eta)] \tag{42}$$

with $\eta = v/v_t$, and $\mathcal{L}$ is the pitch-angle scattering operator,

$$\mathcal{L} \equiv 2h\sqrt{1 - \frac{\lambda}{h}}\frac{\partial}{\partial\lambda}\lambda\sqrt{1 - \frac{\lambda}{h}}\frac{\partial}{\partial\lambda}. \tag{43}$$

The parameter τ_i is the conventional ion-ion collision time.[16]

Since it describes like-species collisions, the operator C_0 must conserve momentum: the parallel "friction force,"

$$F_\parallel \equiv \int d^3v\, mv_\parallel C(f) \tag{44}$$

must vanish. This requirement,

$$\int d^3v\, mv_\parallel C_0(f_l) = 0,$$

determines the parameter α appearing in f_l. One finds that

$$\alpha = [2 - \sqrt{2}\ln(1 + \sqrt{2})]^{-1} \approx 1.327\,05. \tag{45}$$

E. Galilean transformation

The form of the distribution in a frame moving with the local fluid velocity $\boldsymbol{V}$ is especially useful. We denote this function by $g(\boldsymbol{x},\boldsymbol{s},t)$, where $\boldsymbol{s} \equiv \boldsymbol{v} - \boldsymbol{V}$, and write

$$g(\boldsymbol{x},\boldsymbol{s},t) = f[\boldsymbol{x},\boldsymbol{s} + \boldsymbol{V}(\boldsymbol{x},t),t]. \tag{46}$$

When the fluid velocity reflects FLR physics, V is small compared to v_t, and expansion of Eq. (46) gives

$$g_0(\boldsymbol{x},\boldsymbol{s},t) = f_M(\boldsymbol{x},\boldsymbol{s},t)$$

and

$$g_1(\boldsymbol{x},\boldsymbol{s},t) = f_1(\boldsymbol{x},\boldsymbol{s},t) + \boldsymbol{V}\cdot\frac{\partial f_M}{\partial \boldsymbol{s}}$$

or

$$g_1(\boldsymbol{x},\boldsymbol{v},t) = f_1 - \frac{m}{T}\boldsymbol{V}\cdot\boldsymbol{v}f_M. \tag{47}$$

From here on we suppress the subscript 1.

The gyrophase-dependent part of g is easily found from Eq. (9):

$$\tilde{g} = f_M\left(\frac{mw}{T} - \frac{5}{2}\right)(\boldsymbol{b}\times\boldsymbol{\nabla} r)\cdot\boldsymbol{v}A_2. \tag{48}$$

The gyrophase-averaged part of g, distinguished by an overbar, is more complicated. It is obvious that

$$\bar{g} = \bar{f} - \frac{m}{T}V_\parallel u f_M$$

whence

$$g_b = f_b - \frac{m}{T} V_{\parallel b} u f_M, \tag{49}$$

where $V_{\parallel b}$ is the parallel flow contained in f_b and the overbar is suppressed. One finds that

$$V_{\parallel b} = -\frac{hT}{m\Omega_p}\left[A_1 + \left(\alpha - \frac{5}{2}\right)\left(\frac{1-\sqrt{2\varepsilon}}{h^2}\right)A_2\right] \tag{50}$$

and therefore

$$g_b = -f_M \frac{hu}{\Omega_p}\left(\alpha - \frac{5}{2}\right)\left(\frac{h^2 - 1 + \sqrt{2\varepsilon}}{h^2}\right)A_2 + f_l$$

or, after using Eq. (38),

$$g_b = f_M \frac{A_2}{\Omega_p}\left[\left(\frac{5}{2} - \alpha\right)\left(\frac{\sqrt{2\varepsilon} + h^2 - 1}{h^2}\right)hu - \left(\frac{mw}{T} - \alpha\right)hu^*\right]. \tag{51}$$

F. Moments of the distribution

We consider moments measured in the moving frame: moments of the function g, rather than of f. Since there is no particle flow associated with g, by definition, the relevant low-order moments are the heat flow [compare Eq. (17)]

$$\boldsymbol{q} = \int d^3v m w \boldsymbol{v} g(\boldsymbol{x},\boldsymbol{v},t), \tag{52}$$

the pressure tensor

$$\boldsymbol{p} \equiv \int d^3v m \boldsymbol{v}\boldsymbol{v} g(\boldsymbol{x},\boldsymbol{v},t), \tag{53}$$

and the energy-weighted pressure [compare Eq. (19)]

$$\boldsymbol{r} \equiv \int d^3v m w \boldsymbol{v}\boldsymbol{v} g(\boldsymbol{x},\boldsymbol{v},t). \tag{54}$$

The Maxwellian contributions to these moments, distinguished with a subscript M, are easily computed: $\boldsymbol{q}_M=0$, $pM=nT$, and

$$\boldsymbol{r}_M = \frac{5}{2}\frac{nT^2}{m}\boldsymbol{I},$$

where $\boldsymbol{I}$ is the unit tensor.

The contributions coming from gyration, that is from the function $\tilde{g}$, are also easily found. Since $\tilde{g}$ is odd in $\boldsymbol{v}$, it cannot contribute to $\boldsymbol{p}$ or $\boldsymbol{r}$; its contribution to heat flow is given by Eq. (21).

Consider next the contribution from the lowest-order banana function g_b. Being odd in u, g_b cannot contribute to ion density or temperature, or to any component of the ion pressure tensor. Its significant contribution is to the parallel heat flow,

$$q_{\parallel b} \equiv \int d^3v m w u g_b.$$

The evaluation of this integral is very similar to that for $V_{\parallel b}$; one finds that

$$q_{\parallel b} = -\frac{5nT^2 A_2}{m\Omega_p}\left(\frac{\sqrt{2\varepsilon} + h^2 - 1}{h}\right). \tag{55}$$

The heat flows resulting from gyration and from banana motion are generic, in the sense that they occur whatever dynamical process may be underway, as long as the orderings, Eqs. (5) and (11), are satisfied. They are not the complete flows, however, but only partial contributions. Additional heat flow may occur as part of dynamics; such heat is taken into account by the closed set of moments derived in Sec. IV.

G. Collisional correction

Our fluid closure also uses the collisional correction to the banana distribution. Here we revert to the laboratory frame distribution f_b, which again solves the kinetic equation

$$u\nabla_\parallel f_b - C(f_b) = -v_D \frac{\partial f_M}{\partial r}, \tag{56}$$

where v_D is the radial guiding-center drift, to lowest order in the small collision frequency. Denoting the next-order correction, proportional to ν, by f^1, we have

$$u\nabla_\parallel f^1 = C(f_b). \tag{57}$$

Without explicitly solving Eq. (57), we note that its solution will be even in u (that is, independent of the discrete variable σ), because f_b is odd. It follows that any neoclassical contributions to the tensors $\boldsymbol{p}$ or $\boldsymbol{r}$ come from f^1.

However, Eq. (57) allows a much stronger statement: that the neoclassical contribution to the parallel divergence of the stress, $\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{p}_b$, vanishes. (Here the subscript b indicates the contribution from f_b only.) To verify this statement, first recall that

$$\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{p} = \nabla_\parallel p_\parallel - (p_\parallel - p_\perp)\nabla_\parallel \ln B \tag{58}$$

for any stress computed from a gyrophase-averaged distribution. Now the identity

$$u\nabla_\parallel f = \nabla_\parallel(uf) + (mu)^{-1} f\mu\nabla_\parallel B,$$

where $\mu = m\lambda w/B_0$ is the magnetic moment, implies

$$\int d^3v m u^2 \nabla_\parallel f = \int d^3v m u \nabla_\parallel(uf) + \int d^3v f\mu\nabla_\parallel B.$$

Recalling the definition of $p_\perp$,

$$p_\perp = \int d^3v f\mu B,$$

as well as the velocity-space Jacobian,[9]

$$\int d^3v = \frac{2\pi}{h}\sum_\sigma \int_0^h d\lambda \int_0^\infty \frac{w\,dw}{|u|}, \tag{59}$$

we see that

$$\int d^3vmu^2\nabla_\| f = \frac{1}{h}\nabla_\|(hp_\|) + p_\perp \nabla_\| \ln B$$

or, in view of Eq. (58),

$$\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{p}_b = \int d^3vmu^2\nabla_\| f_b. \tag{60}$$

(Terms corresponding to the parallel divergence of convective inertia $mnV_\|^2$ are suppressed in this argument because they contribute only in higher order.) Now consider the mu moment of the kinetic equation, Eq. (56):

$$\int d^3vmu^2\nabla_\| f_b = \int d^3vmu[C(f_b) - \boldsymbol{v}_D\cdot\nabla f_M].$$

Noticing that the right-hand side vanishes because of collisional momentum conservation and antisymmetry in u, we conclude that

$$\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{p}_b = 0 \tag{61}$$

as anticipated. It is significant that no flux-surface average enters this relation.

Physical arguments can also show that ion anisotropy can be neglected due to ambipolarity requirements. Roughly speaking, a nonvanishing ion pressure anisotropy leads to radial ion flux that the electrons cannot match. To see how this comes about, consider the expressions for a gyrotropic tensor and its divergence,

$$\boldsymbol{p} = p\boldsymbol{I} + \Delta p\left(\boldsymbol{bb} - \tfrac{1}{3}\boldsymbol{I}\right),$$

$$\nabla\cdot\boldsymbol{p} = \nabla p - \tfrac{1}{3}(\nabla_\perp - 2\nabla_\|)\Delta p + B\Delta p\nabla_\|\left(\frac{\boldsymbol{b}}{B}\right),$$

where $p\equiv \mathrm{Tr}(\boldsymbol{p})/3$ is the trace of the pressure tensor and $\Delta p \equiv p_\| - p_\perp$ the pressure anisotropy. This divergence enhances the perpendicular flux

$$n\boldsymbol{V}_p = \frac{1}{m\Omega}\boldsymbol{b}\times\nabla\cdot\boldsymbol{p}.$$

Clearly, the isotropic pressure leads to the familiar diamagnetic drift while the remaining terms arise from the presence of anisotropy.

Consider now those parts of the distribution function which can contribute to anisotropy. Symmetry annihilates terms from all parts of $\bar{f}$ except f_b, which is small by a factor of the ion gyroradius.[17] Similarly, in the expression for the electron particle flux, the only possible sources for anisotropy will scale as the electron gyroradius. The ratio of the two gyroradii goes as

$$\frac{\rho_i}{\rho_e} = \sqrt{\frac{m_iT_i}{m_eT_e}}.$$

Thus, the electron flux will be too small by a factor of the square root of the mass ratio. To prevent the unphysical appearance of a large charge separation due to this flux imbalance, the ions must attain a state of negligible anisotropy. Thus, the result

$$\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{p}_b = 0.$$

Following the discussions in Ref. 13 and above, we say that the nonambipolar flux due to time dependence has relaxed on the faster scale ν/ϵ leaving ambipolar flow and isotropic pressure for the ions on the drift time scale $\partial/\partial t \sim \delta\omega_t$. Electron pressure anisotropy, however, is expected to be an important feature of the electron dynamics to be treated in future work.

Thus ion bounce motion does not contribute to pressure anisotropy. However the energy-weighted collisional moment $\boldsymbol{G}$ of Eq. (20) is not small in the mass ratio, so that bounce motion does contribute to the parallel divergence of the tensor $\boldsymbol{r}$. Indeed, applying to $\boldsymbol{r}$ precisely the same argument as was used above for $\boldsymbol{p}$, we find that the banana kinetic equation implies

$$\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{r}_b = \int d^3vmwuC_0(f_l).$$

After substituting from Eqs. (41) and (38) we find that

$$\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{r}_b = \left(\frac{9}{4} - \alpha\right)\frac{\sqrt{2\epsilon} + h^2 - 1}{h}\frac{pT}{m\Omega_p\tau_i}A_2. \tag{62}$$

IV. FLUID CLOSURE

A. Dynamical variables

We describe the ion fluid in terms of four dynamical variables: the ion density, pressure, and the parallel flow of particles and heat:

$$n, p, V_\|, q_\|. \tag{63}$$

The corresponding perpendicular flows are

$$n\boldsymbol{V}_\perp = \frac{1}{m\Omega}\boldsymbol{b}\times(\nabla p - en\boldsymbol{E}),$$

$$\boldsymbol{q}_\perp = \frac{5}{2}\frac{p}{m\Omega}\boldsymbol{b}\times\nabla T, \tag{64}$$

where $T = p/n$.

B. Truncation

To appreciate how the present closure of moment equations works, it is simplest to consider the form of the distribution function that is assumed. We assume that, through first order in δ, the ion distribution has the form

$$f(\boldsymbol{x},\boldsymbol{v},t) = f_M + f_\| + f_g + f_b. \tag{65}$$

Here the first term is a Maxwellian whose density and temperature vary arbitrarily, except as constrained by Eqs. (1) and (3), while the second term, also independent of gyrophase, represents parallel heat and mass flow; it is similarly constrained and discussed in more detail below. The two remaining terms represent the effects of faster processes: gyration (time scale Ω) enters through $f_g = -\boldsymbol{\rho}\cdot\nabla\bar{f}$ and bounce motion (time scale ω_b) enters through f_b, given by Eq. (37).

Notice that only the Maxwellian is zeroth order in δ, making its implicit appearance in the last two terms consistent. Hence all terms in Eq. (65) are known except the second. If one had a rigorous and general expression for $f_\parallel$ in the drift ordering, then the dynamics of a toroidally confined plasma would be solved (through first δ order). Of course in that case one would have little use for moment equations.

Because a general solution to the drift-kinetic equation is neither known nor easily constructed, we resort to representing $f_\parallel$ and its evolution through moment equations. Clearly this is possible only if $f_\parallel$ can be specified in terms of the four dynamical variables:

$$f_\parallel(\boldsymbol{x},\boldsymbol{v},t) = \bar{F}(n,p,V_\parallel,q_\parallel;w,\lambda). \tag{66}$$

We point out that, while we expect $\bar{F}$ to contribute to both parallel flows, the variables $V_\parallel$ and $q_\parallel$ refer to the *total* flows, including the neoclassical contributions.

Since the density and temperature appearing in f_M are exact moments, $f_\parallel$ can contribute to neither integral, implying that it will be an odd function of parallel velocity.

Equation (65) distinguishes two kinds of perturbations to the Maxwellian: the kinetic response $f_\parallel$ to whatever particular process may be underway; and the generic perturbations, f_g and f_b, which are always present in the drift ordering, regardless of the process. In other words, our truncation is based on the fact that, for any evolution slow compared to Ω and ω_b, the ions will gyrate and the trapped ions will bounce—and those generic motions will affect the evolution.

The $f_\parallel$ term in Eq. (65) describes parallel flows of particles and heat beyond those required by equilibrium neoclassical theory, thus permitting the description of a broad domain of fluid behavior. In particular, it allows more rapid parallel flow. We expect the parallel flows to be larger than $O(\delta)$ but still smaller than the thermal velocity. This is indicated through the introduction of a small parameter relevant to the parallel quantities:

$$V_\parallel \sim \delta_\parallel v_t,$$

$$q_\parallel \sim \delta_\parallel p v_t.$$

Thus, our perturbative analysis will involve independent expansions in powers of the parameters δ and $\delta_\parallel$. At the same time, the form of Eq. (66) prevents additional dynamical variables from entering the system.

The present fluid closure depends on Eq. (66) and one additional simplification. In the equations for the evolution of $V_\parallel$ and $q_\parallel$, which necessarily involve second-order terms, one needs the non-Maxwellian contributions to $\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{p}$ and $\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{r}$. We express these quantities in terms of our basic variables by calculating and incorporating the contributions from gyration (such as gyroviscosity) and banana motion, as in Eq. (62). However we do not attempt to include other possible contributions to these terms, such as those from a second-order version of the function $\bar{F}$. In other words we use the forms

$$\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{p} = \boldsymbol{b}\cdot\nabla p + \boldsymbol{b}\cdot\nabla\cdot\boldsymbol{p}_g, \tag{67}$$

$$\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{r} = \boldsymbol{b}\cdot\nabla\left(\frac{5}{2}\frac{pT}{m}\right) + \boldsymbol{b}\cdot\nabla\cdot\boldsymbol{r}_g + \left(\frac{9}{4}-\alpha\right)\frac{\sqrt{2\epsilon}+h^2-1}{h}\frac{pT}{m\Omega_p\tau_i}A_2. \tag{68}$$

Here we have recalled Eqs. (61) and (62), the odd $v_\parallel$ symmetry of $f_\parallel$, and introduce the subscript g to indicate contributions from gyromotion, calculated from $f_g = -\boldsymbol{\rho}\cdot\nabla\bar{f}$.

C. Kinetic ansatz and orderings

We now characterize the fluid closure explicitly and completely. Following the above discussions we have for the distribution function

$$f = f_M + f_\parallel - \boldsymbol{\rho}\cdot\nabla f_M - \boldsymbol{\rho}\cdot\nabla f_\parallel + f_b, \tag{69}$$

where the lowest-order term is Maxwellian,

$$f_M = n\left(\frac{m}{2\pi T}\right)^{3/2}\exp\left(-\frac{mv^2}{2T}\right) \sim O(1),$$

implying $-\boldsymbol{\rho}\cdot\nabla f_M \sim O(\delta)$. For the banana distribution $f_b \sim O(\delta)$ we use Eq. (37).

To complete the description, we allow dynamical parallel flows above and beyond standard neoclassical theory using

$$\frac{f_\parallel}{f_M} = \frac{m}{T}(V_\parallel - V_{\parallel b})v_\parallel + \frac{2m}{5pT}(q_\parallel - q_{\parallel b})v_\parallel\left(\frac{mv^2}{2T}-\frac{5}{2}\right). \tag{70}$$

This form introduces the parallel flows into the parametrization of f, and ensures the identities $nV_\parallel \equiv \int d^3v v_\parallel f$ and $q_\parallel \equiv \int d^3v m w v_\parallel f$ when integrating the full distribution function; the banana terms $V_{\parallel b}$ and $q_{\parallel b}$ enter $f_\parallel$ to cancel the contributions from f_b so that the full fluxes $V_\parallel$ and $q_\parallel$ are returned when integrating. Notice that the smallness of the gyroradius implies that the parallel banana terms will enter $-\boldsymbol{\rho}\cdot\nabla f_\parallel$ only to second order, $O(\delta^2)$, which we neglect, leaving the contribution from the full parallel flows to be

$$-\boldsymbol{\rho}\cdot\nabla f_\parallel \sim O(\delta_\parallel\delta).$$

Thus our distribution function is accurate to first order, excluding $O(\delta^2)$ and $O(\delta_\parallel^2)$.

As before, we consider moments in the moving frame. In terms of the moving-frame distribution function $g(\boldsymbol{x},\boldsymbol{v},t)$, the kinetic equation takes the form

$$\frac{\partial g}{\partial t} + \nabla\cdot[(\boldsymbol{V}+\boldsymbol{v})g] + \frac{\partial}{\partial\boldsymbol{v}}\cdot\left[\left(\frac{1}{mn}\nabla\cdot\boldsymbol{p} + \frac{e}{m}\boldsymbol{v}\times\boldsymbol{B} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{V}\right)g\right] = C. \tag{71}$$

Integration of this expression over phase-space velocity results in the moment equation hierarchy derived below. Specification of the distribution function, Eq. (69), allows truncation of the hierarchy, approximating higher unknown moments in terms of the dynamical variables, Eq. (63), parametrizing Eq. (69).

D. Density and velocity evolution

The zeroth moment of Eq. (71) reproduces particle conservation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{V}) = 0, \tag{72}$$

and a trivial identity follows from the use of the exact equation for momentum evolution

$$mn\frac{d\boldsymbol{V}}{dt} + \nabla \cdot \boldsymbol{p} - en(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}) = 0 \tag{73}$$

to derive Eq. (71). Here we use the familiar notation $d/dt \equiv \partial/\partial t + \boldsymbol{V}\cdot\nabla$ for the full-time derivative and neglect the friction force on account of collisional momentum conservation.

The continuity equation, Eq. (72), is already expressed in terms of the dynamical variables, Eq. (63), and needs no modification. Momentum evolution, Eq. (73), however, depends on the tensor $\boldsymbol{p}$ which must be related to the scalar pressure p and the other variables of Eq. (63).

E. Pressure tensor

Under the adopted orderings, $d\boldsymbol{V}_\perp/dt \sim \delta^2$ and $d\boldsymbol{V}_\parallel/dt \sim \delta_\parallel \delta$. Therefore all other terms in Eq. (73) must be kept to at least the same accuracy. Since the distribution function used does not contain terms of $O(\delta^2)$ it is not possible to evaluate $\boldsymbol{p}$ to the necessary accuracy by direct integration of Eq. (53). The method by which the fluid equations can be manipulated to arrive at an $O(\delta^2)$ expression is well known[9] and the results are familiar:

$$\boldsymbol{p} = p\boldsymbol{I} + \boldsymbol{\Pi} \tag{74}$$

where $\boldsymbol{I}$ is the diagonal identity tensor and the symmetric gyroviscosity tensor $\boldsymbol{\Pi}$ captures the higher-order contributions to $\boldsymbol{p}$ induced by particle gyration:

$$\boldsymbol{\Pi} = \boldsymbol{\Pi}_\perp + \boldsymbol{b}\boldsymbol{\Pi}_\parallel + \boldsymbol{\Pi}_\parallel b,$$

$$\boldsymbol{\Pi}_\perp = \frac{1}{2\Omega}\left[p(\boldsymbol{b}\times\nabla)\boldsymbol{V}_\perp + p\nabla_\perp(\boldsymbol{b}\times\boldsymbol{V}) + \frac{2}{5}(\boldsymbol{b}\times\nabla)\boldsymbol{q}_\perp + \frac{2}{5}\nabla_\perp(\boldsymbol{b}\times\boldsymbol{q})\right] \sim O(\delta^2), \tag{75}$$

$$\boldsymbol{\Pi}_\parallel = \frac{1}{\Omega}\left[p(\boldsymbol{b}\times\nabla)V_\parallel + \frac{2}{5}(\boldsymbol{b}\times\nabla)q_\parallel\right] \sim O(\delta_\parallel\delta).$$

Here we keep only the necessary accuracy.

When these expressions are inserted into the momentum equation, Eq. (73), one obtains a useful simplification, the so-called "gyroviscous cancellation." The parallel component results in the evolution of $V_\parallel$,

$$mn\left(\frac{\partial V_\parallel}{\partial t} + \boldsymbol{V}_E\cdot\nabla V_\parallel\right) + \nabla_\parallel p - enE_\parallel = 0, \tag{76}$$

where $\mathbf{V}_E$ is the $\mathbf{E}\times\mathbf{B}$ drift.

The scalar pressure p is itself also a dynamical variable and has its own evolution equation, given by the trace of the $m\boldsymbol{vv}$ moment of Eq. (71),

$$\frac{3}{2}\frac{dp}{dt} + \nabla\cdot\boldsymbol{q} + \frac{3}{2}p\nabla\cdot\boldsymbol{V} + \boldsymbol{p}:\nabla\boldsymbol{V} = 0. \tag{77}$$

Collisional energy exchange does not appear due to its smallness in the mass ratio. With Eq. (74) for $\boldsymbol{p}$ this becomes, to first order in δ and $\delta_\parallel$,

$$\frac{3}{2}\frac{dp}{dt} + \nabla\cdot\boldsymbol{q} + \frac{5}{2}p\nabla\cdot\boldsymbol{V} = 0. \tag{78}$$

Thus the equations, Eqs. (72), (76), and (78), provide the evolution equations for $\{n, p, V_\parallel\}$. The final dynamical variable to be addressed is the parallel heat flow $q_\parallel$.

F. Heat evolution

Multiplication of Eq. (71) by mwu followed by integration over velocity space results in the exact equation for the evolution of $q_\parallel$ (neglecting the friction force, as before),

$$\frac{dq_\parallel}{dt} + \boldsymbol{b}\cdot\nabla\cdot\boldsymbol{r} + (\boldsymbol{m}:\nabla\boldsymbol{V})_\parallel + \boldsymbol{V}\cdot\nabla q_\parallel + \boldsymbol{q}\cdot\nabla V_\parallel - \frac{3}{2}\frac{p}{mn}\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{p} - \frac{1}{mn}(\boldsymbol{p}\cdot\nabla\cdot\boldsymbol{p})_\parallel = g_\parallel \tag{79}$$

with the moving-frame collision integral given by

$$g_\parallel \equiv \int d^3v\, mwuC(g).$$

This collisional moment should not be confused with parts of the rest-frame distribution function, also denoted by the symbol g. Comparison of the definitions of $g_\parallel$ and $G_\parallel$, neglecting collisional friction and energy exchange, results in the relation

$$g_\parallel = G_\parallel - \boldsymbol{V}\cdot\int d^3v\, mu\boldsymbol{v}C(f).$$

For Eq. (79) to be useful, the two moments $\boldsymbol{r}$ and

$$\boldsymbol{m} \equiv \int d^3v\, m\boldsymbol{vvv}g \tag{80}$$

must be expressed in terms of the set of dynamical variables, Eq. (63). As in the momentum evolution equation, the time derivative here is of order $dq\|/dt \sim \delta_\parallel\delta$, making it necessary to keep $\delta_\parallel\delta$ accuracy for all other terms. Since the velocity is first order, we see that it is only necessary to compute $\boldsymbol{m}$ to first order to have $\boldsymbol{m}:\nabla\boldsymbol{V}$ accurate to second order. The distribution function, Eq. (69), is accurate to first order, so $\boldsymbol{m}$ can be calculated directly from Eq. (80).

A related ordering argument applies to the distinction between $g_\parallel$ and $G_\parallel$. Terms in the expression

$$\boldsymbol{V}\cdot\int d^3v\, mu\boldsymbol{v}C(f)$$

vanish by either odd symmetries in the velocity integrations or higher order smallness (since $\nu/\omega_t \sim \delta$), so that

$$g_\parallel \approx G_\parallel = -\bar{\nu}q_\parallel.$$

Applying the drift ordering to Eq. (79), using Eqs. (68) and (69) for the pressure tensor, and using the distribution, Eq. (69), to evaluate $\boldsymbol{m}$ then gives the following for the evolution of parallel heat flow:

$$\frac{dq_\parallel}{dt} + \boldsymbol{b}\cdot\nabla\cdot\boldsymbol{r} + \frac{7}{5}(q_\parallel\nabla\cdot\boldsymbol{V}_\perp + \boldsymbol{q}_\perp\cdot\nabla V_\parallel) - \frac{5}{2}\frac{p}{mn}(\nabla_\parallel p + \nabla\cdot\boldsymbol{\Pi}_\parallel) - \frac{1}{mn}\boldsymbol{\Pi}_\parallel\cdot\nabla p = -\bar{\nu}q_\parallel. \quad (81)$$

1. Gyromotion in heat evolution

In this section the calculation of $\boldsymbol{r}$ is presented. While it is possible to use the moments of the moving-frame kinetic equation, Eq. (71) directly, the approach presented here first evaluates the fixed-frame moment $\boldsymbol{R}$ defined in Eq. (19). Transformation between $\boldsymbol{R}$ and $\boldsymbol{r}$ follows from comparing Eqs. (19) and (54). Thus, the exact expression

$$\boldsymbol{R} = \tfrac{1}{2}V^2(mn\boldsymbol{VV}+\boldsymbol{p}) + \tfrac{1}{2}\boldsymbol{VV}\ \mathrm{Tr}(\boldsymbol{p}) + (\boldsymbol{p}\cdot\boldsymbol{V})\boldsymbol{V} + \boldsymbol{V}(\boldsymbol{p}\cdot\boldsymbol{V}) + \boldsymbol{m}\cdot\boldsymbol{V} + \boldsymbol{qV} + \boldsymbol{Vq} + \boldsymbol{r}. \quad (82)$$

Neglecting the unnecessary terms of second order, this rearranges to

$$\boldsymbol{r} = \boldsymbol{R} - \tfrac{7}{2}p(\boldsymbol{V}_\perp\boldsymbol{V}_\parallel + \boldsymbol{V}_\parallel\boldsymbol{V}_\perp) - \boldsymbol{m}\cdot\boldsymbol{V} - (\boldsymbol{q}_\parallel\boldsymbol{V}_\perp + \boldsymbol{V}_\perp\boldsymbol{q}_\parallel + \boldsymbol{q}_\perp\boldsymbol{V}_\parallel + \boldsymbol{V}_\parallel\boldsymbol{q}_\perp), \quad (83)$$

where $\boldsymbol{m}$ need only be evaluated to first order because of its multiplication with $\boldsymbol{V}$. Neoclassical effects are included elsewhere using f_b, so here we directly integrate

$$\boldsymbol{R}_g = \int d^3umw\boldsymbol{vv}(-\rho\cdot\nabla)f_\parallel.$$

Only the parallel component is needed:

$$\boldsymbol{b}\cdot R_g = \frac{7T}{2m}\left(\frac{p}{\Omega}\boldsymbol{b}\times\nabla V_\parallel + \frac{4}{5\Omega}\boldsymbol{b}\times\nabla q_\parallel\right) + \frac{7}{2}pV_\parallel\boldsymbol{V}_\perp + \frac{14}{5}q_\parallel\boldsymbol{V}_\mathbf{E} + \frac{7\boldsymbol{q}_\perp}{2p}\left(\frac{26}{25}q_\parallel + \frac{2}{5}pV_\parallel\right). \quad (84)$$

With this result we use Eq. (83) to calculate the divergence

$$\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{r}_g = -\frac{7}{5}p(\boldsymbol{V}_\perp\cdot\nabla)V_\parallel - \frac{7}{5}(\boldsymbol{q}_\perp\cdot\nabla)V_\parallel - \frac{63}{25p}(\boldsymbol{q}_\perp\cdot\nabla)q_\parallel + \frac{7}{5}(\boldsymbol{V}_\perp - 2\boldsymbol{V}_\mathrm{d})\cdot\nabla q_\parallel + \frac{7}{5}q_\parallel\nabla\cdot\boldsymbol{V}_\perp + \frac{28}{25}\frac{q_\parallel}{p^2}(\boldsymbol{q}_\perp\cdot\nabla)p. \quad (85)$$

Inserting the expressions, Eq. (75) for $\boldsymbol{\Pi}_\parallel$ and Eq. (85) for $\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{r}_g$ into the heat equation, Eq. (81), gives the final expression for the time evolution of $q_\parallel$

$$\frac{dq_\parallel}{dt} + \boldsymbol{b}\cdot\nabla\cdot\boldsymbol{r}_b + \frac{5p}{2m}\nabla_\parallel T + \frac{14}{5}q_\parallel\nabla\cdot\boldsymbol{V}_\perp - \frac{7}{2}p\boldsymbol{V}_\mathbf{E}\cdot\nabla V_\parallel + \left(\frac{7}{5}\boldsymbol{V}_\mathbf{E} - \boldsymbol{V}_\mathbf{d} - \frac{63}{25p}\mathbf{q}_\perp\right)\cdot\nabla q_\parallel + \frac{14}{5m\Omega}\frac{q_\parallel}{p}\boldsymbol{b}\cdot\nabla\boldsymbol{T}\times\nabla p = -\bar{\nu}q_\parallel, \quad (86)$$

with the banana contribution $\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{r}_b$ given by Eq. (62). To complete the description, we next examine the significance of $\bar{\nu}$.

G. Transport reduction

The dynamics thus derived are more general than conventional treatments of transport and therefore describe a greater range of physical phenomena. While it is precisely this feature that is of primary importance, it is still useful to reproduce well-known transport results under the appropriate approximations. In this section, we consider the form of the ion dynamics under the transport ordering and specify the parameter $\bar{\nu}$ to ensure a simple and reasonable result.

As discussed in Sec. I, the essence of the transport ordering lies in the restriction to slow variation,

$$\frac{\partial}{\partial t} \sim \delta^2\omega_t,$$

slow plasma flows,

$$V_\perp \sim V_\parallel \sim \delta v_t,$$

$$q_\perp \sim q_\parallel \sim \delta p v_t,$$

and the neglect of $O(\delta^2)$ terms. Under these stricter constraints, Eq. (86) becomes

$$\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{r}_b + \frac{5p}{2m}\nabla_\parallel T = -\bar{\nu}q_\parallel.$$

This expression now gives $q_\parallel$ directly, rather than through a differential equation. In addition to the transport ordering, neoclassical theory makes use of flux-surface averages, which would approximately annihilate $p\nabla_\parallel T$, leaving

$$-\bar{\nu}q_\parallel = \boldsymbol{b}\cdot\nabla\cdot\boldsymbol{r}_b = \left(\frac{9}{4}-\alpha\right)\frac{\sqrt{2\epsilon}+h^2-1}{h}\frac{pT}{m\Omega_p\tau_i}A_2 = -\left(\frac{9}{4}-\alpha\right)\frac{q_{\parallel b}}{5\tau_i}$$

and showing that the judicious choice of

$$\bar{\nu} = \left(\frac{9}{4}-\alpha\right)\frac{1}{5\tau_i} \approx \frac{0.183}{\tau_i} \quad (87)$$

ensures our system returns the banana parallel heat flow, Eq. (55), in the neoclassical regime.

Restoring the parallel temperature gradient, the complete system, Eqs. (72), (76), (78), and (86), *under the transport ordering* then becomes

$$\nabla \cdot (n\boldsymbol{V}) = 0,$$

$$\nabla_{\|} p - enE_{\|} = 0,$$

$$\nabla \cdot \boldsymbol{q} + \frac{5}{2} p \, \nabla \cdot \boldsymbol{V} = 0,$$

$$\bar{\nu}(q_{\|} - q_{\|b}) + \frac{5p}{2m} \nabla_{\|} T = 0. \quad (88)$$

We explicitly note again that the parallel heat flow is equal to the banana prediction only in a flux-surface averaged sense.

Although these equations are considerably simpler than the full system, they are only appropriate under conditions too restrictive for our purposes. We return to our more general expressions below.

V. SUMMARY

For clarity, we summarize the full ion system. The evolution of the dynamical variables n, p, $V_{\|}$, $q_{\|}$ is given by

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{V}) = 0,$$

$$mn\left(\frac{\partial V_{\|}}{\partial t} + \boldsymbol{V}_E \cdot \nabla V_{\|}\right) + \nabla_{\|} p - enE_{\|} = 0,$$

$$\frac{3}{2}\frac{dp}{dt} + \nabla \cdot \boldsymbol{q} + \frac{5}{2} p \, \nabla \cdot \boldsymbol{V} = 0,$$

$$\frac{dq_{\|}}{dt} + \bar{\nu}(q_{\|} - q_{\|b}) + \frac{5p}{2m}\nabla_{\|} T + \frac{14}{5} q_{\|} \nabla \cdot \boldsymbol{V}_{\perp} - \frac{7}{2} p \boldsymbol{V}_{\boldsymbol{E}} \cdot \nabla V_{\|} + \left(\frac{7}{5}\mathbf{V}_{\mathbf{E}} - \mathbf{V}_{\mathbf{d}} - \frac{63}{25p}\boldsymbol{q}_{\perp}\right) \cdot \nabla q_{\|} + \frac{14}{5m\Omega}\frac{q_{\|}}{p} \boldsymbol{b} \cdot \nabla T \, \nabla p = 0.$$

To first order, the perpendicular flows are

$$\boldsymbol{V}_{\perp} = \boldsymbol{V}_E + \boldsymbol{V}_d = \frac{e}{m\Omega}\boldsymbol{E} \times \boldsymbol{b} + \frac{1}{mn\Omega}\boldsymbol{b} \times \nabla \boldsymbol{p}$$

$$\boldsymbol{q}_{\perp} = \frac{5}{2}\frac{pT}{m\Omega}\boldsymbol{b} \times \nabla \ln T.$$

Thus, the description of the ion dynamics is completed, in the sense that expressions for the evolution of all the ion variables have been derived. Bounce motion, gyromotion, and parallel heat flow have been included in a systematic manner to expand the range of physical phenomena addressed. The above discussion of transport shows that the theory has advantages over neoclassical theory in that it allows larger flows and time derivatives, and allows quantities that are eliminated by flux-surface averaging.

Of course, Maxwell's equations are needed for the evolution of the electromagnetic fields which couple the evolution of the electrons to the ions. Complete closure involves knowledge of the electron dynamics, an area of future work.

[1] Equations describing electron behavior are necessary for a complete closure with Maxwell's equations. Several more interesting complications and physical effects, such as bootstrap current and conductivity reduction, are expected to appear fully when the electron problem is addressed, a subject of future work.

[2] F. L. Hinton and R. D. Hazeltine, "Theory of plasma transport in toroidal confinement systems," Rev. Mod. Phys. **48**, 239 (1976).

[3] P. Helander and D. J. Sigmar, *Collisional Transport in Magnetized Plasmas* (Cambridge University Press, Cambridge, 2002).

[4] J. D. Callen, W. X. Qu, K. D. Siebert, B. A. Carreras, K. C. Shaing, and D. A. Spong, "Neoclassical MHD equations, instabilities and transport in tokamaks," Plasma Phys. Controlled Nucl. Fusion Res. **2**, 157 (1986).

[5] R. D. Hazeltine, "Transport theory in the collisionless limit," Phys. Plasmas **5**, 3282 (1998).

[6] E. D. Held, J. D. Callen, C. C. Hegna, and C. R. Sovinec, "Conductive electron heat flow along magnetic field lines," Phys. Plasmas **8**, 1171 (2001).

[7] R. D. Hazeltine, "Collisionless transport parallel to the magnetic field in a toroidal plasma," Phys. Plasmas **6**, 550 (1999).

[8] E. D. Held, J. D. Callen, and C. C. Hegna, "Conductive electron heat flow along an inhomogeneous magnetic field," Phys. Plasmas **10**, 3933 (2003).

[9] R. D. Hazeltine and J. D. Meiss, *Plasma Confinement* (Addison-Wesley, Reading, Massachusetts, 1992).

[10] G. W. Hammett, W. Dorland, and F. W. Perkins, "Fluid models of phase mixing, Landau damping, and nonlinear gyrokinetic dynamics," Phys. Fluids B **4**, 2052 (1992).

[11] M. N. Rosenbluth, R. D. Hazeltine, and F. L. Hinton, "Plasma transport in toroidal confinement systems," Phys. Fluids **15**, 116 (1972).

[12] Note that this expression is only valid in the special case of axisymmetric equilibrium. We use this form only to discuss canonical neoclassical results, and not for our general analysis.

[13] S. P. Hirshman, "The ambipolarity paradox in toroidal diffusion, revisited," Nucl. Fusion **18**, 917 (1978).

[14] C. T. Hsu, K. C. Shaing, and R. Gormley, "Time dependent parallel viscosity and relaxation rate of poloidal rotation in the banana regime," Phys. Plasmas **1**, 132 (1994).

[15] R. C. Morris, M. G. Haines, and R. J. Hastie, "The neoclassical theory of poloidal flow damping in a tokamak," Phys. Plasmas **3**, 4513 (1996).

[16] S. I. Braginskii, "Transport processes in a plasma," in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1.

[17] See Sec. IV for a full discussion of the constituents of $\bar{f}$. Contributions to the stress tensor from f_g are second order [see Eq. (75)]: the resulting third-order flux is negligibly small.