An Exact Nonlinear Hall-MHD Waves

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A fully nonlinear three-dimensional propagating wave solution of Hall MHD is constructed. In the appropriate limits, the solution reproduces the known nonlinear Alfvénic state and the circularly polarized Whistler states.

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Walén's classic solution of the Nonlinear Alfvén wave [1] has served as an essential reference point for all studies of Alfvénic turbulence in incompressible magnetohydrodynamics (MHD). The essence of this arbitrary amplitude wave lies in the relation $\boldsymbol{b} = \pm \boldsymbol{v}$, the requirement that the velocity and the magnetic field perturbations are either parallel or anti-parallel and have the same magnitude [the magnetic (velocity) field is normalized to the uniform ambient filed B_0 (the Alfvén speed $V_A = B_0/\sqrt{4\pi\rho}$, where ρ is the uniform mass density].

When $\boldsymbol{b}$ and $\boldsymbol{v}$ are so related, the nonlinear terms in the time dependent MHD vanish; it is this effective linearization that yields the waves

$$\boldsymbol{b} = pm\boldsymbol{v}, \tag{1}$$

$$\boldsymbol{b} = \widehat{\boldsymbol{b}} \exp\left(i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp \pm i(k_s\, s + k_s t)\right) \tag{2}$$

with an effective frequency $\omega = -(+)k_s$ propagating in a direction antiparallel (parallel) to the ambient field $\boldsymbol{B}_0 = \widehat{e}_s$. Notice that $k_s = \boldsymbol{k}\cdot\widehat{e}_s$ is the projection of the wave vector along the direction of the field line, and $\perp$ is perpendicular to $\hat{e}_s$. In Eq. (2), time and space variables are, respectively, measured in units of the ion gyroperiod $\omega_c^{-1} = mc/qB_0$, and the ion skin depth $\lambda_i = c/\omega_{pi}$, where $\omega_{pi} = (4\pi q^2 n/m_i)^{1/2}$ is the ion plasma frequency.

The purely Alfvénic state, characterized by (1) has been an object of much investigation [2]. It is believed that the eventual state (reached only asymptotically) of MHD turbulence will consist of wave trains given by (1)-(2). This is particularly true if $|\boldsymbol{v}| \ll V_A$. For $|\boldsymbol{v}|\ V_A$, the Alfvénic description will be useful only after a time long enough to allow the disentanglement of the oppositely propagating waves [3].

The importance of the nonlinear Alfvénic state for MHD prompts one to speculate if a similar kind of an exact solution exists for Hall MHD (HMHD), a system which encompasses MHD, but can sustain a much richer spectrum of plasma states not accessible to MHD [4]. In this letter we demonstrate that HMHD, indeed, admits an exact nonlinear wave solution which in the long wave length limit ($k\lambda_i = k \ll 1$) reduces to the nonlinear shear or the compression wave of MHD while for $k \gg 1$, the shear branch goes over to the ion cyclotron mode while the compressional branch goes over to the whistler wave. The general solution spans the entire k range for which the HMHD is valid. Naturally the MHD relation between the velocity and the magnetic field perturbations is fundamentally transformed; it becomes a function of k.

In the Alfvénic units defined above, the following dimensionless equations

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times [(\boldsymbol{V} - \boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}] \tag{3}$$

$$\frac{\partial (\boldsymbol{B} + \nabla \times \boldsymbol{V})}{\partial t} = \nabla \times [\boldsymbol{V} \times (\boldsymbol{B} + \nabla \times \boldsymbol{V})] \tag{4}$$

constitute Hall MHD. Notice that in (4), obtained by taking the curl of the ion force balance equation, the pressure gradient term $\nabla P/n$ has disappeared because it has been assumed to be a perfect gradient (by invoking an equation of state $P = P(n)$, for example); the pressure has not been neglected.

To look for wave like solutions' we split the fields into their ambient and the fluctuating parts (there is no ambient flow),

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}; \qquad \boldsymbol{V} = \boldsymbol{v} \tag{5}$$

and substitute in (3)-(4),

$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times [(\boldsymbol{v} - \nabla \times \boldsymbol{b}) \times \boldsymbol{B}_0 + (\boldsymbol{v} - \nabla \times \boldsymbol{b}) \times \boldsymbol{b}] \tag{6}$$

$$\frac{\partial}{\partial t}(\boldsymbol{b} + \nabla \times \boldsymbol{v}) = \nabla \times [\boldsymbol{v} \times (\nabla \times \boldsymbol{v} + \boldsymbol{b}) + \boldsymbol{v} \times \boldsymbol{B}_0]\,. \tag{7}$$

The horribly nonlinear problem represented by (6)-(7) is converted to a set of linear problems (the time honored method for solving nonlinear equations) by imposing the following conditions reminiscent of the double Beltrami conditions of [4]:

$$\boldsymbol{v} - \boldsymbol{\nabla} \times \boldsymbol{b} = \alpha \boldsymbol{b} \tag{8}$$

$$\boldsymbol{b} + \boldsymbol{\nabla} \times \boldsymbol{v} = \beta \boldsymbol{v} \tag{9}$$

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where α and β are like the separation constants. With the nonlinearities so taken care of, we are left with the remaining time dependent linear equations

$$\frac{\partial \boldsymbol{b}}{\partial t} = \alpha \nabla \times [\boldsymbol{b} \times \boldsymbol{B}_0] \qquad (10)$$

$$\frac{\partial}{\partial t}(\boldsymbol{b} + \nabla \times \boldsymbol{v}) = (1/\beta) \nabla \times [\boldsymbol{v} \times \boldsymbol{B}_0]\,. \qquad (11)$$

Apparently we have traded a close nonlinear system (6 equations for six variables) for an overdetermined linear system (8)-(**??**) with 12 equations in six variables. Acceptable solutions, therefore, will be possible only under some particular conditions that will remove the over determination. To seek them, we first notice that (10) and (11) admit

$$\boldsymbol{b} \;=\; \boldsymbol{b}_k \exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} + i\alpha B_0(\widehat{e}_s \cdot \boldsymbol{k})t\right) \qquad (12)$$

$$\boldsymbol{v} \;=\; \boldsymbol{v}_k \exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} + i\frac{B_0}{\beta}(\widehat{e}_s \cdot \boldsymbol{k})t\right). \qquad (13)$$

If the exponential solutions (12) and (13) are to satisfy the linear equations (8) and (9), we must require $\beta = 1/\alpha$. In addition, substituting (12) and (13) into (**??**) and (**??**) leads to

$$\boldsymbol{v}_k - i\boldsymbol{k} \times \boldsymbol{b}_k = \alpha \boldsymbol{b}_k, \qquad (14)$$

$$\boldsymbol{b}_k + i\boldsymbol{k} \times \boldsymbol{v}_k = \frac{1}{\alpha}\boldsymbol{v}_k \qquad (15)$$

which, after simple manipulation, yield

$$\boldsymbol{v}_k - \alpha \boldsymbol{b}_k = i\alpha \boldsymbol{k} \times \boldsymbol{v}_k$$
$$\boldsymbol{v}_k - \alpha \boldsymbol{b}_k = i(\boldsymbol{k} \times \boldsymbol{b}_k).$$

Two consequences immediately follow:

$$\boldsymbol{b}_k = \alpha \boldsymbol{v}_k \qquad (16)$$

relating $\boldsymbol{b}_k$ and $\boldsymbol{v}_k$, and

$$\boldsymbol{k} \times \boldsymbol{v}_k = -i\frac{1-\alpha^2}{\alpha}\boldsymbol{v}_k = -i\lambda \boldsymbol{v}_k. \qquad (17)$$

The first of these establishes the HMHD equivalent of the Alfvénic condition for MHD and the second,the Fourier transform of a Beltrami equation ($\nabla \times \boldsymbol{G} = \lambda \boldsymbol{G}$) has to be solved to complete the story; the solvability constraint will end up relating α with k giving the "dispersion relation" $\alpha = \alpha(k)$.

The solutions of (17 are well-known and we could just quote them. But for completeness, we recapitulate a few steps in the process. Suppressing the indices for a simplified notation, we derive from (**??**): 1) dotting with $\boldsymbol{v}$ yields $\boldsymbol{v}\cdot\boldsymbol{v} = \boldsymbol{v}_r^2 - \boldsymbol{v}_i^2 + 2i\boldsymbol{v}_r\cdot\boldsymbol{v}_i = 0$ implying $v_i = \pm v_r$, 2) and dotting with $\boldsymbol{k}$ gives $\boldsymbol{k}\cdot\boldsymbol{v} = 0 \Rightarrow \boldsymbol{k}\cdot\boldsymbol{v}_r = 0 = \boldsymbol{k}\cdot\boldsymbol{v}_i$. Clearly the suffix r(i) denotes the real(imaginary) part. Crossing (**??**) with $\boldsymbol{k}$ and using $\boldsymbol{k}\cdot\boldsymbol{v} = 0$, we obtain the dispersion relation (remembering that λ is a function of α)

$$\lambda = \pm k. \qquad (18)$$

Keeping track of the $\pm$ may be notationally complicated. Since the physics is the same, we will investigate the option, $v_i = v_r$ and $\lambda = k$. For this choice, it is straightforward to show that $\widehat{\boldsymbol{v}}_r, \widehat{\boldsymbol{v}}_i$, and $\widehat{\boldsymbol{k}}$ form a right-handed orthogonal triad of unit vectors.

Let us first choose $\widehat{\boldsymbol{v}}_r, \widehat{\boldsymbol{v}}_i$, and $\widehat{\boldsymbol{k}}$ to be $\widehat{e}_x, \widehat{e}_y$, and $\widehat{e}_z$ respectively. This choice dictates the following expressions for the velocity and the magnetic field $[\boldsymbol{k} = k\widehat{e}_z], B$, a constant amplitude;

$$\boldsymbol{b} \;=\; \alpha \boldsymbol{v}, \qquad (19)$$

$$\boldsymbol{v} \;=\; B[\widehat{e}_x + i\widehat{e}_y] \exp\left(ikz + i\alpha k(\widehat{e}_z \cdot \widehat{e}_s)B_0 t\right) \qquad (20)$$

with α determined by

$$k = \lambda \;=\; \frac{1-\alpha^2}{\alpha}, \qquad (21)$$

$$\alpha_\pm \;=\; \left[-\frac{k}{2} \pm \left(\frac{k^2}{4} + 1\right)^{1/2}\right]. \qquad (22)$$

From (20) and (22), we extract the effective frequency of the circularly polarized wave $[B_0 = 1]$,

$$\omega_\pm = k\left[-\frac{k}{2} \pm \left(\frac{k^2}{4} + 1\right)^{1/2}\right](\widehat{e}_z \cdot \widehat{e}_s), \qquad (23)$$

a result which is valid over a wide range of k from $k \ll 1$ MHD end to the $k \gg 1$ Hall dominated regime. The k dependence of the separation constant (we could equally legitimately think of $k = k(\ alpha)$, and label the solutions by the separation constant α), implying a k dependent relationship between $\boldsymbol{b}$ and $\boldsymbol{v}$ is one of the defining and distinguishing characteristic of the new broadband fully nonlinear wave.Just to make contact with the familiar, let us examine the two extreme limits of the general result. For $k \ll 1$,

$$\alpha \to \pm 1, \omega \to \mp k(\widehat{e}_z \cdot \widehat{e}_s, \qquad (24)$$

reproducing the$k$independent MHD Alfvénic relationship for both the co- and the counter propagating waves. In the $k \gg 1$ regime, however,

$$\alpha_+ \to 1/k, \alpha_- \to -k, \qquad (25)$$

with

$$\omega_+ \to -\widehat{e}_z \cdot \widehat{e}_s, \omega_- \to \widehat{e}_z \cdot \widehat{e}_s)k^2. \qquad (26)$$

It is easy to recognize that, in analogy with the linear theory, the (+) wave is the shear-cyclotron branch, while the (−) represents the compressional-whistler mode. the frequency of the (+) wave approaches some fraction of the ion gyro frequency (normalizing frequency) — it is only when $\boldsymbol{k}$ and $\boldsymbol{B}_0$ are parallel ($\widehat{e}_z \cdot \widehat{e}_s = \pm 1$) that the wave reaches the cyclotron frequency asymptotically. In this limit the magnitudes of the velocity and magnetic fields can vastly differ (they still remain parallel). The respective relationships are

$$\boldsymbol{v} \to k\boldsymbol{b}. \qquad (27)$$

for the (+) branch, and

$$\boldsymbol{b} \to k\boldsymbol{v}. \qquad (28)$$

for the (−) branch; the compressional-whistler mode has an abundance of turbulent magnetic energy over the turbulent kinetic energy, while in the shear-cyclotron mode, the kinetic energy dominates. The nonlinearly correct relationship of (27) strongly strengthens the results of [5] where exactly this relationship was invoked to predict the solar granulation spectrum.The triad $\widehat{\boldsymbol{v}}_r = \widehat{e}_x, \widehat{\boldsymbol{v}}_i = \widehat{e}_y$, and $\widehat{\boldsymbol{k}} = \widehat{e}_z$ could be replaced by two other independent cyclic combinations. Since (17) is linear, the solutions can be added. The most general three-dimensional solution, therefore, may be written as

$$\begin{aligned} \boldsymbol{v} &= B(\widehat{e}_x + i\widehat{e}_y)\exp\left(ikz + i\alpha k(\widehat{e}_z \cdot \widehat{e}_s)B_0 t\right) \\ &+ C(\widehat{e}_y + i\widehat{e}_z)\left(ikx + i\alpha k(\widehat{e}_x \cdot \widehat{e}_s)B_0 t\right) \\ &+ A(\widehat{e}_z + i\widehat{e}_x)\left(iky + i\alpha k(\widehat{e}_y \cdot \widehat{e}_s)B_0 t\right). \end{aligned} \qquad (29)$$

It is straightforward to verify that (29) exactly solves the original system (10, (11), (8), and (9) with $\beta = 1/\alpha$,

$$\frac{\partial \boldsymbol{b}}{\partial t} = \alpha\nabla \times [\boldsymbol{b} \times \boldsymbol{B}_0] \qquad (30)$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = \alpha\nabla \times (\boldsymbol{v} \times \boldsymbol{B}_0), \qquad (31)$$

$$\boldsymbol{b} = \alpha\boldsymbol{v} \qquad (32)$$

and $(\lambda = \left(1 - \alpha^2)/\alpha\right))$

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \lambda\boldsymbol{v}. \qquad (33)$$

Since the defining equations consist of only real variables, either $\mathrm{Im}\,\boldsymbol{v}$ or $\mathrm{Re}\,\boldsymbol{v}$ could be a solution. Let us write down the imaginary part,

$$\begin{aligned} \boldsymbol{v} &= \widehat{e}_x[A\cos(ky + \alpha k\widehat{e}_y \cdot \widehat{e}_s t) \\ &\quad + B\sin(kz + \alpha k\widehat{e}_z \cdot \widehat{e}_s t)] \\ &+ \widehat{e}_y[B\cos(kz + \alpha k\widehat{e}_z \cdot \widehat{e}_s t) \\ &\quad + C\sin(kx + \alpha k\widehat{e}_x \cdot \widehat{e}_s t)] \\ &+ \widehat{e}_z[C\cos(kx + \alpha k\widehat{e}_x \cdot \widehat{e}_s t) \\ &\quad + A\sin(ky + \alpha k\widehat{e}_y \cdot \widehat{e}_s t)]. \end{aligned} \qquad (34)$$

This is the time-dependent generalization of the famous ABC solution of $\nabla \times \boldsymbol{G} = k\boldsymbol{G}$. This is no surprise because the original system (10, 11, 8, and 9) can be cast as a single Beltrami equation with a new ∇ defined in terms of the "coordinates" $X_i = x_i + \alpha k\widehat{e}_(x_i) \cdot \widehat{e}_s$. Exact time dependent three-dimensional solutions to interacting field theories are quite rare. To the best of our knowledge, this is the only time dependent, three-dimensional exact and fully nonlinear wave solution to a physical system of great interest and complication as Hall MHD. The importance of knowing an exact nonlinear solution can never be overestimated. It is hoped and expected to provide a reference solution for a variety of enquiries relating to plasma turbulence , in particular, in elucidating the role of short wavelength perturbations which cannot be treated within the framework of MHD.

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