Transformations of mode numbers of kinetic Alfvén waves in toroidal plasmas

M. H. Tyshchenko and Yu. V. Yakovenko

Institute for Nuclear Research, Kyiv, Ukraine

5th IAEA Technical Meeeting on Theory of Plasma Instabilities,
5–7 September 2011, Austin, TX, USA

Outline

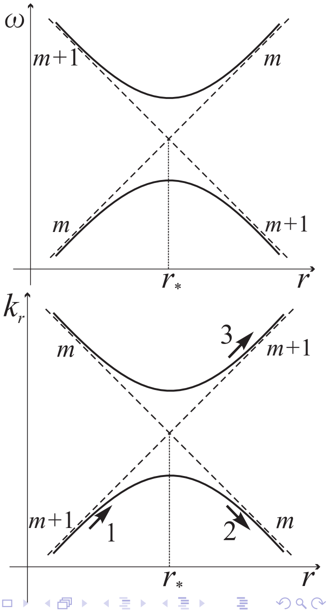
1. Introduction: the idea of the effect
2. Basic equations
3. Calculation of the transformation coefficient
4. Discussion and conclusions

Outline

Aim of the work

- Kinetic Alfvén waves (KAW) often appear in tokamaks and stellarators as a result of continuum and radiative damping of Alfvén instabilities.
- Here we demonstrate that toroidicity (as well as ellipticity and other kind of deviations of the magnetic configuration from the cylindrical geometry) can result in a transformaton of a propagating KAW into another KAW branch, which differs by its mode numbers from the initial wave.
- Our principal aim is to find the coefficient of transformation (the part of the energy which is transformed).

- Coupling is known to result in the formation of a frequency gap in the Alfvén continuum (avoided crossing phenomenon); here dotted lines show the continuum in cylindrical geometry labelled by poloidal mode numbers.
- The same should happen with the branches of the KAW dispersion $k_r = k_r(r)$.
- In the geometrical optics approximation, the wave (1) passes by the gap and gets transformed (2).
- When the wave frequency is above the continuum gap, tunnelling through the gap may save some wave energy from transformation (3).
- When the transformation is efficient?



Outline

Derivation of KAW equations

- Following [Rosenbluth, Rutherford, PRL(1975)], we proceed from Ampère's law:

$$\nabla \cdot (\boldsymbol{j}^{\text{MHD}} + \boldsymbol{j}^{\text{nonideal}}) = 0,$$

- Here the current due to non-ideal effects (FLR and electron dynamics) is

$$4\pi \nabla \cdot \boldsymbol{j}^{\text{nonideal}} = \frac{3}{4}\frac{i\omega c^2}{v_A^2}\rho_i^2 \nabla_\perp^4 \Phi + \frac{i}{4\pi\omega}\nabla_\parallel \nabla^2 E_\parallel,$$

$$E_\parallel = -\rho_s^2 (1 - i\delta_c)\nabla_\parallel \nabla_\perp^2 \Phi.$$

- The MHD current is

$$\boldsymbol{j}^{\text{MHD}} = \boldsymbol{b}\frac{ic^2}{4\pi\omega B}\nabla \cdot \left(B^2 \nabla_\perp \frac{\nabla_\parallel \Phi}{B} \right) + \frac{i\omega c^2}{4\pi v_A^2}\nabla_\perp \Phi,$$

Derivation of KAW equations in toroidal geometry, cont.

- Assuming that the wave consists of only two Fourier harmonics, we arrive at equations for the scalar potentials of two coupled harmonics of the wave, Φ_1 and Φ_2 (see details in [Fesenyuk *et al*, Plasma Phys. Control. Fusion (2004)]).

$$\widehat{\tau}\left(\frac{d^2}{dx^2}-1\right)^2\Phi_1+\left[\frac{d}{dx}(\hat{\Delta}+x)\frac{d}{dx}-(\hat{\Delta}+x)\right]\Phi_1+\left(\hat{\varepsilon}\frac{d^2}{dx^2}+\hat{A}\frac{d}{dx}-\tilde{\varepsilon}\right)\Phi_2=0,$$
$$\widehat{\tau}\left(\frac{d^2}{dx^2}-1\right)^2\Phi_2+\left[\frac{d}{dx}(\hat{\Delta}-x)\frac{d}{dx}-(\hat{\Delta}-x)\right]\Phi_2+\left(\hat{\varepsilon}\frac{d^2}{dx^2}-\hat{A}\frac{d}{dx}-\tilde{\varepsilon}\right)\Phi_1=0.$$

- Here x is the dimensionless radius; $\hat{\Delta}$, dimensionless frequency; $\tilde{\varepsilon}$, $\hat{\varepsilon}$ and $\hat{A}$ represent coupling of the wave harmonics due to toroidicity; $\widehat{\tau}$, non-ideal kinetic effects.
- Actually, only the coupling via $\hat{\varepsilon}$ (the angular modulation of $|\nabla\psi|^2$ and B), which is responsible for the continuum gap, will be important.

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Transformation

To reduce the order of equations, we use the Fourier transformation

$$\psi_{1,2}(p) = \int_{-\infty}^{+\infty} \Phi_{1,2}(p) \exp(-ipx) dx$$

Introducing the new variables $\chi_1 = (\psi_1 + \psi_2)$, $\chi_2 = i(\psi_1 - \psi_2)$ and $z = \widehat{\tau}^{\frac{1}{2}} p$, we obtain the following set of first-order equations:

$$\widehat{\tau}^{\frac{1}{2}} \frac{d}{dz} \chi_1 = -\alpha_1 \chi_2 + \beta_1 \chi_1,$$

$$\widehat{\tau}^{\frac{1}{2}} \frac{d}{dz} \chi_2 = \alpha_2 \chi_1 + \beta_2 \chi_2,$$

where $\alpha_1 = (H + M)$, $\alpha_2 = (H - M)$, $\beta_1 = -(1 - \hat{A}) \widehat{\tau}^{\frac{1}{2}} z/(z^2 + \widehat{\tau})$, $\beta_2 = -(1 + \hat{A}) \widehat{\tau}^{\frac{1}{2}} z/(z^2 + \widehat{\tau})$, $H = z^2 + \hat{\tau} - \hat{\Delta}$, $M = (\hat{\epsilon} z^2 + \hat{\tau} \tilde{\epsilon})/(z^2 + \hat{\tau})$.

Continuation

Excluding one of the dependent variables, we obtain:

$$\widehat{\tau}\frac{d}{dz}\left[f_1\dot{\phi}_1\right] = \alpha_2\phi_1 \exp\left(\int dz(\beta_1 - \beta_2)\right), \quad (1)$$

$$\widehat{\tau}\frac{d}{dz}\left[f_2\dot{\phi}_2\right] = -\alpha_1\phi_2 \exp\left(\int dz(\beta_2 - \beta_1)\right). \quad (2)$$

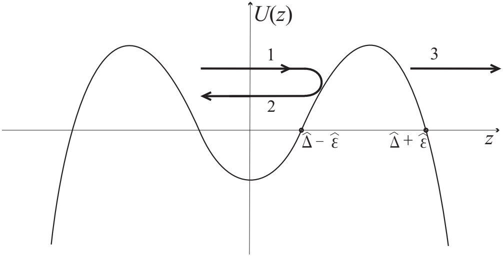
where $\phi_{1,2} = \chi_{1,2}e^{-\int dz\,\beta_{1,2}}$, $f_1 = (1/\alpha_1)e^{\int dz(\beta_1-\beta_2)}$ and $f_2 = (1/\alpha_2)e^{\int dz(\beta_2-\beta_1))}$. Equation (1) and (2) are singular at the points $\alpha_1 = 0$ $(z = \sqrt{\hat{\Delta} - \hat{\varepsilon}})$ and $\alpha_2 = 0$ $(z = \sqrt{\hat{\Delta} + \hat{\varepsilon}})$, respectively.

Over-barrier scattering problem

Proceeding to the variables $u_{1,2} = \sqrt{f_{1,2}}\phi_{1,2}$, we get the following two equations.

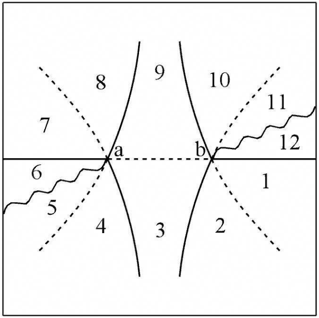
$$\hat{\tau}\frac{d^2 u_1}{dz^2} + [\alpha_1\alpha_2 + O(\hat{\tau})]u_1 = 0, \qquad \hat{\tau}\frac{d^2 u_2}{dz^2} + [\alpha_1\alpha_2 + O(\hat{\tau})]u_2 = 0.$$

They look identical, but they have different dependent variables and different domains of applicability.



The potential barrier ($U(z) = \alpha_1\alpha_2 > 0$) corresponds to the gap in the wave dispersion, and the reflection coefficient corresponds to the coefficient of the wave transformation.

- To evaluate tunnelung through the barrier, we use the WKB approach.
- We take the WKB solution that includes no in-coming wave at the right side.
- We draw a Stokes diagram and use the Heading rules [Heading, 1962] to continue the solution from domain 1 to domain 6.



- Finally, our equations conserve flux $(\chi_1\chi_2^* - \chi_1^*\chi_2)$, which enables us to improve the accuracy.
- This procedure is well-known, the only technical difficulty is that the coefficients possess second-order poles $\Rightarrow$ we need to switch from u_1 to u_2 in domain 3.

Transformation efficiency

- The probabilities of reflection and transmission are

$$|r|^2 = \frac{1}{1+e^{-2W}}, \qquad |t|^2 = \frac{e^{-2W}}{1+e^{-2W}}$$

with $W = \frac{1}{2}\int_a^b dz(\beta_1+\beta_2) + \widehat{\tau}^{-1/2}\int_a^b dz(\alpha_1\alpha_2)^{1/2}$.

- W characterizes the efficiency of the transformation: the transformation predominates when $W > 1/2$; the wave tunnels through the gap without transformation when $W < 1/2$.
- In terms of physical quantities,

$$W = \frac{\pi}{4\sqrt{7}} \frac{\epsilon_{TAE}^2 k_{\parallel}^2 R r v_A}{\hat{s}\iota\rho_i\sqrt{\Delta\omega^2}\, m\sqrt{\delta_0}}$$

- The transformation is stronger for low-m modes (note that $1/m$ is not the small parameter of our WKB).
- For example, in NSTX it occurs for $m\hat{s} < 3$ in NSTX and $m\hat{s} < 8$ in ITER.

Outline

Practical consequences

- The transformation weakens the Landau damping of the KAW on electrons since the wave is directed to the branch with larger $k_{\|}$.
- In particular, in the work [Kolesnichenko et al., Phys. Rev. Lett. (2005)] the assumption that such transformation takes place was used to explain why the wave affects a wide plasma region; now we substantiated this assumption.
- The transformation may affect the instability Fourier spectrum observed by Mirnov diagnostics if KAW radiated by the instability reaches the plasma boundary.

Conclusions

- A deviation from the cylindrical symmetry (in particular, toroidicity) can result in transformation of the mode numbers of the kinetic Alfvén waves. The cause of the transformation is the appearance of gaps in the dispersion curves.
- The transformation studied here occurs for the KAWs with frequencies above the corresponding continuum gap (the TAE-gap for toroidicity).
- The transformation is strong for low-m modes in, e.g., NSTX and ITER.
- The transformation may modify the spectrum of the mode numbers observed externally by Mirnov diagnostics and reduce the wave absorption.