Infernal Alfvén eigenmodes in low-shear tokamaks

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The present work is devoted to the new class of the Alfvén modes in tokamaks, the so called infernal Alfvén eigenmodes (IAE). These modes exist in plasmas with broad low-shear central core, when the safety factor in this region, q_0, is slightly above an integer or low-order rational. The IAE can appear in cascade with dominant harmonics satisfying $0 < q_0 - m/n \sim \epsilon$, where $m(n)$ is the poloidal (toroidal) mode number and ϵ is the global inverse aspect ratio. The IAE eigenfrequency lies below the Alfvén continuum for the dominant harmonic, but the coupling with the upper toroidal sideband is crucial for mode localization and its continuum damping, which, together with the ion Landau damping, determines the threshold for IAE excitation by energetic ions. In the case of $q_0 \simeq 1$ (i.e. for the "hybrid" discharges) the strong ion Landau damping limits IAE excitation only to plasmas with high pressure, slightly below ideal MHD stability limit, which results in very low mode frequency. In the case $q_0 \geq 2$ the modes can be excited also at low plasma pressure, in which case the frequency spectrum for different modes in the cascade is almost degenerate in the plasma frame.

I. INTRODUCTION

The so called "hybrid" regime attracted much attention in tokamak research [1]. Such equilibria are often characterized by the flat profile of the safety factor, q, in the wide central core, with $q_0 \simeq 1$ in this region. The main advantage of the hybrid shots is the absence of sawteeth, which are the main triggers of the harmful neoclassical tearing modes. For this reason the hybrid regime has been included as a third operational scenario for ITER [2]. Recently the $n = 1$ mode with extremely low frequency (few kilohertz in the plasma frame) has been observed in hybrid shots with record plasma pressures and perpendicular neutral beam injection (NBI) on the JT-60 Upgrade tokamak [3]. At highest NBI powers, these modes not only deteriorate confinement of energetic ions, but also destroy the H-mode pedestal [3]. At maximum power the $n = 1$ appears in cascade with $m = n > 1$ modes.

On the other hand, Alfvén cascade (AC) is an ubiquitous phenomenon in tokamak discharges with reversed magnetic shear in the central core [4]. In particular, the so called "grand" cascades are routinely used for diagnostics of the safety factor profile and internal transport barrier location in such plasmas [5]. Existing theory of the ACs [6, 7] suggests that real eigenmodes, which can be excited by energetic ions (i.e., those propagating in the cocurrent direction, $n > 0$), exist only when $m > nq_{min}$, and therfore must exhibit upward frequency sweeping as current penetrates to the central core. Here $m(n)$ is the poloidal (toroidal) mode number and q_{min} is the safety factor minimum around which the modes are localized. This result is broadly consistent with observations. There is, however, a relatively small database of discharges with downward sweeping ACs (hereafter DACs). One possible explanation is that coupling to the kinetic Alfvén waves allows formation of the potential well for eigenmode in the case $q_{min} > m/n$ [8]. However, detailed analyses of the database with DACs performed in Ref.[9] have not revealed any anomaly in the thermal ion Larmor radius in comparison with discharges exhibiting an upward sweeping cascades. What actually distinguish this database are the very flat q-profiles in the broad central region, which are similar to the JT-60U case mentioned above. Below it is shown that both the low-frequency modes observed in JT-60U and DACs observed in JET are manifestations of the new class of the Alfvén eigenmodes - Infernal Alfvén Eigenmodes (IAE), so called in the name of the ideal MHD instabilities the corresponding equilibria are prone to [10**?**].

The paper is organized as follows. The low-frequency IAEs in hybrids close to MHD stability limit are described in Sec.II, and their excitation threshold by energetic ions is estimated in Sec.III. The higher frequency DACs in plasmas with $q_0 \geq 2$ are described in Sec.IV. Section V includes discussion and summary.

II. LOW-FREQUENCY INFERNAL ALFVÉN EIGENMODES IN HYBRIDS

We assume that $(\omega_0/\omega_A)^2 \ll \gamma_s\beta$ and $q_0 - 1 \sim \epsilon$, where ω_0 is the mode frequency, $\omega_A = V_A/R$ with V_A as the Alfvén speed and R as the major radius of the torus, $\epsilon = a/R$ with a the minor radius, γ_s is the adiabatic index, and β is the plasma pressure normalized to the magnetic field pressure. Then the eigenmode equations in the central core take the form [12, 13]

$$\frac{d}{dr}\left\{\epsilon^{-2}\left[(\iota-1)^2-3\left(\frac{\omega_0}{\omega_A}\right)^2\right]r^3\frac{d\xi_1}{dr}\right\}-4\left(\frac{r}{4}\beta_p'+\beta_p\right)^2 r^3\xi_1=\left(\frac{r}{4}\beta_p'+\beta_p\right)\frac{d}{dr}\left(r^3\hat{\xi}_2\right), \tag{1}$$

$$\frac{d}{dr}\left(r^3\frac{d\hat{\xi}_2}{dr}\right)-3r\hat{\xi}_2=-4r^3\frac{d}{dr}\left[\left(\frac{r}{4}\beta_p'+\beta_p\right)\xi_1\right], \tag{2}$$

where radius r is normalized to the plasma minor radius $a, \iota = 1/q, \beta_p = 8\pi(\langle p\rangle - p)/B_p^2$, with $\langle\ldots\rangle = (2/r^2)\int_0^r(\ldots)\hat{r}d\hat{r}, p$ is the plasma pressure, B_p is the poloidal magnetic field, prime denotes radial derivative, $\xi_2 \equiv \epsilon\hat{\xi}_2, \xi_{1(2)}$ is the amplitude of the $m = 1(2)$ poloidal harmonic of the plsma radial displacement, and it has been assumed that $V_A(r) = const.$ The general solution of Eq.(2), which is regular on the magnetic axis, is given by

$$\hat{\xi}_2=r^{-3}\int_0^r\hat{r}^4\beta_p\left(\hat{r}\right)\frac{d\xi_1}{d\hat{r}}d\hat{r}+\left[C-\beta_p(r)\xi_1(r)\right]r, \tag{3}$$

where C is an integration constant. Putting Eq.(3) into Eq.(1) and integrating, we find

$$\frac{d\xi_1}{dr}=\frac{\epsilon^2 Cr\beta_p}{(\iota-1)^2-3(\omega_0/\omega_A)^2}. \tag{4}$$

The dispersion relation can be obtained by matching the solution in the inner (shear-free) region to the solution in the outer (sheared) region. This procedure is accurate provided the transition between these regions is sufficiently abrupt [13]. Then in the outer region $|\iota-1|\sim 1$ and $\xi_1\sim\epsilon^2$, as follows from Eq.(4). Therefore, in the outer region, we can neglect the toroidal coupling and write the Euler-Lagrange equation for the $m = 2$ harmonic in the form

$$\frac{d}{dr}\left[\left(\iota-\frac{1}{2}\right)^2 r^3\frac{d\hat{\xi}_2}{dr}\right]-3\left(\iota-\frac{1}{2}\right)^2 r\hat{\xi}_2=0. \tag{5}$$

Equation (5) has the following asymptotic solution in the shear-free region:

$$\hat{\xi}_2\propto\frac{r}{r_2}+\sigma\left(\frac{r}{r_2}\right)^{-3}, \tag{6}$$

where $q(r_2) = 2$ and the constant σ can be determined by integrating Eq.(5) through the outer region. Matching Eq.(6) with the asymptotic form of Eq.(3) in the outer region, we obtain the dispersion relation as follows:

$$\sigma=\left(\frac{r_2}{a}\right)^2\int_0^a\frac{[\epsilon\beta_p(r)]^2}{(\iota-1)^2-3(\omega_0/\omega_A)^2}\left(\frac{r}{r_2}\right)^5 d\left(\frac{r}{r_2}\right). \tag{7}$$

Note that, for $3(\omega_0/\omega_A)^2 < (\iota_0-1)^2$ and $q_0 > 1$, the integral on the right-hand side (RHS) of Eq.(7) converges, which justifies interpretation of the novel mode as a global Alfvén eigenmode (GAE) [14]. But in contrast to the conventional GAE, the frequency of the present IAE lies well below the minimum of the Alfvén continuum, and the mode can exist in plasmas with $V_A(r) = const.$

The constant σ can be calculated analytically for the following model profile of the rotational transform:

$$\iota=\frac{1}{2}+\left(\iota_0-\frac{1}{2}\right)\left[1-\left(\frac{r}{r_2}\right)^{2\lambda}\right], \tag{8}$$

where abruptness of the transition requires $\lambda\gg 1$. Then the general solution of Eq.(5) is given by

$$\xi_2=\frac{r}{r_2}\left\{C_1F\left[1+\frac{3}{2\lambda},\frac{1}{2\lambda};1+\frac{2}{\lambda};\left(\frac{r}{r_2}\right)^{2\lambda}\right]+\left(\frac{r}{r_2}\right)^{-4}C_2F\left[1-\frac{2}{\lambda},-\frac{3}{2\lambda};1-\frac{2}{\lambda};\left(\frac{r}{r_2}\right)^{2\lambda}\right]\right\}, \tag{9}$$

where F(a, b; c; z) is the hypergeometric function. For $r\to r_2-0$, Eq.(9) yields [15]

$$\xi_2 = C_1 \frac{\Gamma\left(1+\frac{2}{\lambda}\right)}{\Gamma\left(1+\frac{3}{2\lambda}\right)\Gamma\left(\frac{1}{2\lambda}\right)} \left\{2\psi(1) - \psi\left(1+\frac{3}{2\lambda}\right) - \psi\left(\frac{1}{2\lambda}\right) - \ln\left[1-\left(\frac{r}{r_2}\right)^{2\lambda}\right]\right\} +$$

$$C_2 \frac{\Gamma\left(1-\frac{2}{\lambda}\right)}{\Gamma\left(1-\frac{1}{2\lambda}\right)\Gamma\left(-\frac{3}{2\lambda}\right)} \left\{2\psi(1) - \psi\left(1-\frac{1}{2\lambda}\right) - \psi\left(-\frac{3}{2\lambda}\right) - \ln\left[1-\left(\frac{r}{r_2}\right)^{2\lambda}\right]\right\}, \tag{10}$$

where $\Gamma(z)$ is the gamma function and $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$. Regularity of Eq.(10) at $r = r_2$ yields

$$\sigma = \frac{C_2}{C_1} = -\frac{\Gamma\left(1+\frac{2}{\lambda}\right)\Gamma\left(1-\frac{1}{2\lambda}\right)\Gamma\left(-\frac{3}{2\lambda}\right)}{\Gamma\left(1-\frac{2}{\lambda}\right)\Gamma\left(1+\frac{3}{2\lambda}\right)\Gamma\left(\frac{1}{2\lambda}\right)} = \frac{1}{3}\frac{\Gamma\left(1+\frac{2}{\lambda}\right)\Gamma\left(1-\frac{1}{2\lambda}\right)\Gamma\left(1-\frac{3}{2\lambda}\right)}{\Gamma\left(1-\frac{2}{\lambda}\right)\Gamma\left(1+\frac{1}{2\lambda}\right)\Gamma\left(1+\frac{3}{2\lambda}\right)}. \tag{11}$$

For the safety factor profile given by Eq.(8) and pressure profile given by $p(r) = p_0[1-(r/a)^{2\nu}]$, so that $\beta_p \propto r^{2\nu-2}$, one can obtain from Eq.(7) in the limit $(\omega_0/\omega_A)^2 \ll (\iota_0-1)^2$ [compare with Eq.(52) of Ref.[13] for the growth rate of the infernal mode]

$$\omega_0 \simeq \omega_A \sqrt{A\left[(\iota_0-1)^2 - B(\epsilon\beta_{p1})^2\right]}, \tag{12}$$

where

$$A = [3(1-\zeta/3)(1-\zeta/2)]^{-1}, \quad B = \frac{1}{2\sigma(2\nu+1)}\left(\frac{r_1}{a}\right)^2\left(\frac{r_1}{r_2}\right)^4 \frac{\pi\zeta(1-\zeta)}{\sin(\pi\zeta)},$$

$$\zeta = \frac{2\nu+1}{\lambda}, \quad r_1 = r_2\left(\frac{1-\iota_0}{\iota_0-0.5}\right)^{1/2\lambda}, \quad \beta_{p1} \equiv \beta_p(r_1),$$

and σ is given by Eq.(11). Note that Eq.(12) is valid only close to the ideal MHD stability limit.

III. IAE EXCITATION BY TRAPPED ENERGETIC IONS

The growth and damping rates of the IAE can be calculated perturbatively, using the lowest order eigenfunction and eigenvalue given by Eqs.(4) and (12), respectively. The sum of the fluid and kinetic parts of the fast ion energy is given by [16]

$$\delta W_\alpha = \frac{m_\alpha \omega_{c\alpha}}{2}\omega_0 \int d^3r r\left|\xi_r\right|^2 \int d\Gamma \frac{\omega_{d\alpha}}{\omega_0 - \overline{\omega}_{d\alpha}}\frac{\partial F_\alpha}{\partial r}, \tag{13}$$

where $\omega_{c\alpha}(\omega_{d\alpha})$ is the fast ion gyrofrequency (magnetic drift frequency), overline denotes bounce average, $d\Gamma$ is the velocity space volume element, F_α is the equilibrium distribution of the fast ions, and ξ_r is the radial component of the plasma displacement.

To calculate IAE continuum damping, we need expression for the toroidal coupling operator, C^+, near the $q = 2$ surface, where the mode frequency crosses the $m = 2$ cylindrical Alfvén continuum. In geometry with shifted circular magnetic surfaces one can obtain

$$C^+ \left.\xi_1\right|_{q=2} \simeq -\frac{1}{2}\left[r\iota'\Delta' + \frac{1}{4}\left(r\Delta'' + 3\Delta' - \frac{r}{R}\right)\right] r^2\frac{d\xi_1}{dr} = \frac{R}{2}\frac{d\beta}{dr}r^2\frac{d\xi_1}{dr}, \tag{14}$$

where Δ is the Shafranov shift and in the last expression we used the relation [16]

$$r\Delta'' = \frac{r}{R} - (3-2s)\Delta' + \alpha_p, \tag{15}$$

with $s = rq'/q$ and $\alpha_p = -q^2R\beta'$. Using Eq.(14), we can rewrite the Euler-Lagrange equation for the $m = 2$ harmonic in the form

$$\frac{d}{dr}\left\{\left[\left(\iota - \frac{1}{2}\right)^2 - \left(\frac{3\omega_0}{2\omega_A}\right)^2\right] r^3\frac{d\xi_2}{dr}\right\} = \frac{R}{2}\frac{d}{dr}\left(\frac{d\beta}{dr}r^2\xi_1\right) - \frac{R}{2}\xi_1\frac{d}{dr}\left(\frac{d\beta}{dr}r^2\right), \tag{16}$$

where on the left-hand side we retained only the term with second derivative and we assumed the $(\omega_0/\omega_A)^2 \ll \gamma_s\beta(r_2)$. Note that only the first term on the RHS of Eq.(16) contributes to the continuum damping.

Treating the fast ion drive and continuum damping perturbatively, we obtain the following expression for the corresponding shift of the eigenvalue, $\delta\omega$:

$$\frac{6\omega_0\delta\omega}{\omega_A^2}\int_0^a r^3\left(\frac{d\xi_1}{dr}\right)^2 dr = -\frac{I\left(\kappa_0^2\right)}{3}\frac{R^{3/2}}{r_1}i\pi\frac{\omega_0}{\overline{\omega}_{dm}}\int_0^a r^{3/2}\frac{d\langle\beta_\alpha\rangle}{dr}\xi_1^2 dr - \frac{R}{2}\left(\frac{d\beta}{dr}r^2\xi_1\right)_{r=r_2}\sum_{i=1,2}\int_{r_{Ai}-0}^{r_{Ai}+0}\frac{d\xi_2}{dr}dr. \tag{17}$$

In calculating the fast ion drive [the first term on the RHS of Eq.(17)], we have chosen F_α in the form of a slowing down energy, ε, distribution with a δ-function in the pitch angle $\Lambda = \mu B_0/\varepsilon$, and we retained only the imaginaty part of the fast ion response, which is associated with precessional resonance $\omega_0 = \overline{\omega}_{d\alpha}$. Furthermore, β_α is the fast ion beta, angular brackets denote the flux surface average, $I(\kappa_0^2) = 2E(\kappa_0^2)/K(\kappa_0^2) - 1$, E and K are the complete elliptic integrals, $\kappa_0^2 = \kappa^2(\Lambda_0)$ with $\kappa^2(\Lambda) \equiv (R/2r)(1/\Lambda - 1 + r/R), \overline{\omega}_{dm} = (\varepsilon_\alpha/m_\alpha\omega_{c\alpha}r_1 R)I(\kappa_0^2)$ with ε_α the injection energy, r_{Ai} are the radii of the Alfvén resonance given by $\iota(r_{Ai}) = 0.5 \pm 1.5\omega_0/\omega_A$, and $\xi_1(\omega_0)$ is given by Eq.(4) [Eq.(12)]. Using Eq.(16), we obtain for the IAE excitation threshold

$$-\frac{I\left(\kappa_0^2\right)}{3}\frac{R^{3/2}}{r_1}\frac{\omega_0}{\overline{\omega}_{dm}}\int_0^a r^{3/2}\frac{d\langle\beta_\alpha\rangle}{dr}\xi_1^2 dr = \frac{R^2}{6}r_2\left(\frac{d\beta}{dr}\xi_1\right)^2_{r=r_2}\frac{\omega_A}{\omega_0\left|\iota'\right|_{r=r_2}}. \tag{18}$$

Using Eq.(4) with boundary condition $\xi_1(a) = 0$, one can obtain from Eq.(18) for the model fast ion pressure profile $\langle\beta_\alpha\rangle(r) = \beta_{\alpha 0}[1-(r/a)^2]^2$,

$$\beta_{\alpha 0}^{th} \simeq \frac{7\nu^2(4\nu+7)(8\nu+7)}{2^8\lambda(\nu-2\lambda)^2}\beta_0^2\frac{\epsilon^{-1/2}}{I\left(\kappa_0^2\right)}\frac{\omega_A\overline{\omega}_{dm}}{\omega_0^2}\left(\iota_0-\frac{1}{2}\right)^{(\nu/\lambda)-5}(1-\iota_0)^{4-(2\nu+1)/\lambda}\left[1-\left(\iota_0-\frac{1}{2}\right)^{2-(\nu/\lambda)}\right]^2, \tag{19}$$

where we have taken into account that $\iota(a) = 0$.

As a particular example, we consider the following set of parameters: $\epsilon = 1/3, \beta_0 = 0.1, \iota_0 = 0.9, \nu = 1[\beta_p(r) = const], \lambda = 3, \overline{\omega}_{dm} = \omega_0$, and $\kappa_0^2 = 0$ (deeply trapped particles). From Eq.(12) it follows that $\omega_0/\omega_A \simeq 3.2\times 10^{-2}$ and Eq.(19) yields $\beta_{\alpha 0}^{th} \simeq 2.8\times 10^{-3}$, a fairly low value.

IV. ALFVÉN CASCADES WITH DOWNWARD FREQUENCY SWEEPING

As in previous sections, we consider a plasma characterized by extended low-shear central core with $0 < q_0 - m/n \sim \epsilon$, which is separated from the wall by finite region with large shear. In contrast to hybrid case, we assume $q_0 \geq 2$. This means that we must take into account the compressional gap in the Alfvén continuum and the average magnetic well. Then, in the shear-free core, the Euler-Lagrange equations for the toroidally coupled m and $m\pm 1$ harmonics of the plasma displacement take the form

$$(L_m + T_m)\xi_m - \left(\frac{\epsilon}{mn}\right)^2\left[\frac{1}{2}\left(r\beta_p' + 4\beta_p\right)^2 + \left(1-\frac{n^2}{m^2}\right)\left(r\beta_p' + 4\beta_p\right)\right]r^3\xi_m =$$
$$\sum\frac{\epsilon^2}{2nm^2(1\pm m)}r^{1\mp m}\left(r\beta_p' + 4\beta_p\right)\frac{d}{dr}\left(r^{2\pm m}\xi_{m\pm 1}\right), \tag{20}$$

$$\frac{d}{dr}\left(r^3\frac{d\xi_{m\pm 1}}{dr}\right) - [(m\pm 1)^2 - 1]r\xi_{m\pm 1} = -\frac{1\pm m}{2n}r^{2\pm m}\frac{d}{dr}\left[\left(r\beta_p' + 4\beta_p\right)r^{1\mp m}\xi_m\right], \tag{21}$$

where

$$L_m\xi_m = \frac{d}{dr}\left[\left(\frac{1}{nq_0}-\frac{1}{m}\right)^2 r^3\frac{d\xi_m}{dr}\right] - (m^2-1)\left(\frac{1}{nq_0}-\frac{1}{m}\right)^2 r\xi_m, \tag{22}$$

$\sum$ denotes summation over sidebands, and normalizations are the same as in previous sections. In the limit $2q_0^2 \gg 1$ (i.e., taking into account the geodesic compression but neglecting the coupling with sound waves) the inertial operator T_m takes the form [17]

$$T_m\xi_m = \frac{d}{dr}\left(\frac{\omega_G^2-\omega_m^2}{m^2\omega_A^2}r^3\frac{d\xi_m}{dr}\right) - (m^2-1)\frac{\omega_G^2-\omega_m^2}{m^2\omega_A^2}r\xi_m, \tag{23}$$

where ω_G is the geodesic frequency [18, 19] and ω_m is the mode frequency.

Now we again use the property that ξ_m becomes negligible ($\sim \epsilon^2$) in the finite-shear outer region and assume that transition between shear-free and finite-shear regions is sufficiently abrupt. Taking for simplicity $q_0(r) \simeq const, \omega_G(r) \simeq const, \omega_A(r) \simeq cinst, \beta_p(r) \simeq const$ (the latter condition corresponds to the parabolic pressure profile) and imposing the boundary condition $\xi_m(r_1) = 0$ with r_1 the transition radius, one can obtain from Eqs.(20)-(23)

$$n\xi_{m\pm1} = -2\beta_p(1 \pm m)r^{-(2\pm m)} \int_0^r \hat{r}^{2\pm m}\xi_m d\hat{r} + e_\pm r^{m\pm1-1}, \tag{24}$$

$$\xi_m = \frac{C}{Bx}\left[\frac{I_m\left(\sqrt{\frac{B}{A}}x\right)}{I_m\left(\sqrt{\frac{B}{A}}\right)} - x^m\right], \tag{25}$$

$$A = \left(\frac{1}{nq_0} - \frac{1}{m}\right)^2 + \frac{\omega_G^2 - \omega_m^2}{m^2\omega_A^2}, \tag{26}$$

$$B = \left(\frac{2\epsilon}{mn}\right)^2 \left(\frac{r_1}{a}\right)^2 \left(1 - \frac{n^2}{m^2}\right)\beta_p, \tag{27}$$

$$C = \left(\frac{r_1}{a}\right)^{m+1} \left(\frac{2\epsilon}{mn}\right)^2 e_+\beta_p, \tag{28}$$

where $x = r/r_1, I_m(z)$ is the modified Bessel function, and $e_\pm$ is the integration constant. Note that the homogeneous part of Eq.(24) for the lower sideband do not contribute to Eqs.(20) and (25).

In the outer region both inertia and toroidal coupling can be neglected to the lowest order, leaving equation for the upper sideband in the form,

$$\frac{d}{dr}\left[\left(\frac{1}{nq} - \frac{1}{m+1}\right) r^3 \frac{d\xi_{m+1}}{dr}\right] - [(m+1)^2 - 1]\left(\frac{1}{nq} - \frac{1}{m+1}\right)^2 r\xi_{m+1} = 0. \tag{29}$$

Note that, in the vicinity of the rational surface $nq(r_{m+1}) = m + 1$, a pair of the Alfvén resonances arises when in Eq.(29) the inertia is taken into account to the next order. With toroidal coupling due to residual $\xi_m \sim \epsilon^2$, these resonances lead to the continuum damping of the mode [20], which together with the ion Landau damping determine a threshold for the mode excitation by energetic ions. These higher-order effects are beyond the scope of the present work.

The asymptotic form of the solution of Eq.(29) in the shear-free core is given by

$$\xi_{m+1} \propto \left(\frac{r}{r_{m+1}}\right)^m + \sigma_m \left(\frac{r}{r_{m+1}}\right)^{-(2+m)}, \tag{30}$$

where the constant σ_m is determined by integrating Eq.(29) through the outer region. The dispersion relation for the eigenfrequency ω_m is obtained by matching Eq.(30) to the asymptotic form of Eq.(24) [with $\xi_m(r > r_1) \simeq 0$] in the outer region. The result is given by

$$\sigma_m = \beta_p \left(\frac{r_1}{r_{m+1}}\right)^{2(m+1)} \frac{m^2}{m^2 - n^2}\left[1 - 2(m+1)\sqrt{\frac{A}{B}}\frac{I_{m+1}\left(\sqrt{\frac{B}{A}}\right)}{I_m\left(\sqrt{\frac{B}{A}}\right)}\right]. \tag{31}$$

Taking $B \ll A$ in Eq.(31) leads to the solution, which is similar in form to Eq.(12),

$$\omega_m^2 = \left(\frac{m}{nq_0} - 1\right)^2 \omega_A^2 + \omega_G^2 - \left(\frac{\epsilon\beta_p}{n}\right)^2 \frac{\omega_A^2}{(m+1)(m+2)\sigma_m}\left(\frac{r_1}{r_{m+1}}\right)^{2(m+1)} \left(\frac{r_1}{a}\right)^2. \tag{32}$$

From Eq.(32) it follows that (i) as in the hybrid case, the mode frequency lies below the Alfvén continuum and (ii) the spectrum of DAC is almost degenerate in the plasma frame, except for high pressure plasmas and for lowest mode numbers in the cascade.

The constant σ_m can be calculated for the following model profile of the rotational transform:

$$\iota = \frac{n_0}{m_0+1} + \left(\iota_0 - \frac{n_0}{m_0+1}\right)\left[1-\left(\frac{r}{r_0}\right)^{2\lambda}\right], \tag{33}$$

where again $\lambda \gg 1$ to ensure abruptness of the transition between shear-free and finite-shear regions. With Eq.(33) the solution of Eq.(29) takes the familiar form

$$\xi_{m+1} = C_1\left(\frac{r}{r_0}\right)^m F\left[a_1, b_1; c_1; \frac{\iota_0 - n_0/(m_0+1)}{\iota_0 - n/(m+1)}\left(\frac{r}{r_0}\right)^{2\lambda}\right] +$$

$$C_2\left(\frac{r}{r_0}\right)^{-(2+m)} F\left[a_2, b_2; c_2; \frac{\iota_0 - n_0/(m_0+1)}{\iota_0 - n/(m+1)}\left(\frac{r}{r_0}\right)^{2\lambda}\right], \tag{34}$$

where

$$a_{1,2} = 1 \pm \frac{m+1 \pm \sqrt{(2\lambda+1)^2 + m(m+2)}}{2\lambda}, \tag{35}$$

$$b_{1,2} = 1 \pm \frac{m+1 \mp \sqrt{(2\lambda+1)^2 + m(m+2)}}{2\lambda}, \tag{36}$$

$$c_{1,2} = 1 \pm \frac{m+1}{\lambda}. \tag{37}$$

The condition that Eq.(34) is bounded at the rational surface $nq = m+1$ again yields

$$\sigma_m = \frac{C_2}{C_1} = -\left[\frac{\iota_0 - n_0/(m_0+1)}{\iota_0 - n/(m+1)}\right]^{\frac{m+1}{\lambda}} \frac{\Gamma(c_1)\Gamma(a_2)\Gamma(b_2)}{\Gamma(c_2)\Gamma(a_1)\Gamma(b_1)}. \tag{38}$$

Taking for the transition radius $\iota(r_1) = 2\iota_0 - n/m$, we obtain from Eqs.(31) and (38) an explicit dispersion relation,

$$-\frac{\Gamma(c_1)\Gamma(a_2)\Gamma(b_2)}{\Gamma(c_2)\Gamma(a_1)\Gamma(b_1)} = \left(\frac{n/m - \iota_0}{\iota_0 - n/(m+1)}\right)^{\frac{m+1}{\lambda}} \beta_p \frac{m^2}{m^2-n^2}\left[1 - 2(m+1)\sqrt{\frac{A}{B}}\frac{I_{m+1}(\sqrt{B/A})}{I_m(\sqrt{B/A})}\right] \tag{39}$$

Shown in Fig.1 are the solutions of Eq.(39) for the particular case $m/n = 2, q_0 = 2+\delta q, \beta_p = 0.5, \epsilon = 1/3, (\omega_G/\omega_A)^2 = 0.016\beta_p, \lambda = 10, n = 1 \div 4$. Shown is a spectrum in the lab frame for plasma rotation frequency $\omega_{rot}/\omega_A = 0.025$. The spectrum exhibits downward sweeping as current penetrates to the central core ($\delta q \to +0$), and is almost degenerate in the plasma frame. Both these features are consistent with the spectrogram of DAC shown in Fig.3 of Ref.[9].

V. DISCUSSION AND SUMMARY

The following properties of the IAE in hybrids described in Sec.II are consistent with recent observations in JT-60U tokamak [3]: (i) the mode is certainly driven by trapped energetic ions and its frequency correlates with precession frequency of the injected ions; (ii) the mode structure shown in Fig.14 of Ref.[3] is peaked on axis and decreases abruptly at the boarder between shear-free and finite shear regions [see Fig.6(c) of Ref.[3] for the q-profile], consistent with Eq.(4); (iii) the cascade at the highest NBI power shown in Fig.11 of Ref.[3] is consistent with excitation of IAEs with $m = n > 1$, which, similar to DAC considered in Sec.IV, have the almost degenerate spectrum in the plasma frame [compare with Eq.(32)]

$$\omega_n = \omega_A\sqrt{\frac{1}{3}\left[\left(\frac{1}{q_0}-1\right)^2 - K_m\left(\frac{\epsilon}{n}\beta_p\right)^2\right]}, \tag{40}$$

where K_m decreases with m. The reason why higher modes in the cascade are excited only at highest NBI power is clear: the pair of Alfvén resonances around the $q = (n+1)/n$ surface is located much closer to the shear-free core,

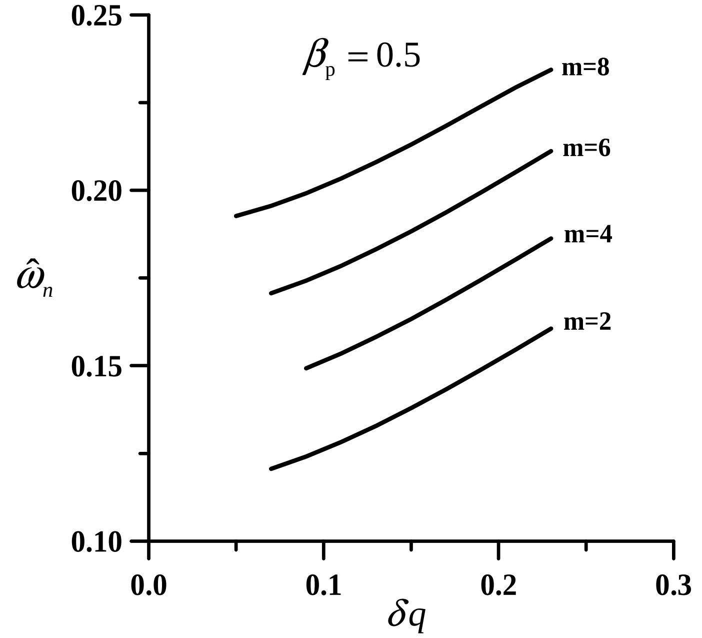

Figure 1: Spectrum of IAEs with $n = 1 \div 4$ in the lab frame for $m/n = 2, q_0 = 2 + \delta q, \beta_p = 0.5, \epsilon = 1/3, (\omega_G/\omega_A)^2 = 0.016\beta_p, \omega_{rot}/\omega_A = 0.025$, and $\lambda = 10$. Normalization is $\hat{\omega}_n = \omega_n/\omega_A$ with $\omega_n = \omega_m + n\omega_{rot}$.

where the dominant $m = n$ harmonic is localized, so that modes with $n > 1$ suffer from higher continuum damping. The abrupt disappearence of the higher modes after replacement of one of the NBI units slanted in the co-current direction to that slanted counter to the plasma current can be explained by reduction of the fast ion content in the shear-free core for the same total NBI power. The fact that IAEs in hybrids have been observed only at record plasma pressures can be explained by the huge ion Landau damping at low pressures. Indeed, for $q_0 \simeq 1$ the ordering $(\omega_0/\omega_A)^2 \sim (q_0 - 1)^2 \sim \beta$ is equivalent to $\omega_0 \sim \omega_{ti}$ with ω_{ti} the ion transit frequency.

On the other hand, IAEs in equilibria with $q_0 \geq 2$ can be responsible for the Alfvén cascades with downward frequency sweeping, which have been observed exclusively in discharges with almost flat q-profiles in the central core [9]. Few words are in order to compare IAE with "quasimodes" proposed as an alternative interpretation of the DAC in Ref.[9]. The radiative damping of the quasimode, γ_{rd}, is given by [cf. Eqs.(90) and (92) of Ref.[9] and note the different definition of ω_A given by Eq.(71)],

$$\frac{\gamma_{rd}}{\omega_A} \simeq \sqrt{\frac{1}{8m}\frac{\omega_{c\alpha}}{\omega_A}\frac{n_\alpha}{n_i}\frac{r}{L_\alpha}\frac{r}{m}\sqrt{\frac{q''}{q}}}\left(\frac{nq}{m} - 1\right)^{-1/2}\left[\left(\frac{m - nq}{q}\right)^2 + \left(\frac{\omega_G}{\omega_A}\right)^2\right]^{1/4}, \tag{41}$$

where $\omega_A = V_A/R_0, n_{i(\alpha)}$ is the background (fast) ion density, and L_α is the fast ion density gradient scale length. For $\omega_{c\alpha}/\omega_A = 10^2, n_\alpha/n_i = 10^{-3}, r/L_\alpha = 3, q - 2 = 0.2, m = 2/n = 1, (\omega_G/\omega_A)^2 = 0.016, q''r^2/q = 0.4$ (for the latter parameter see Table I of Ref.[9] for the JET shot No. 66 550 shown in Fig.3), Eq.(41) yields $\gamma_{rd}/\omega_A \simeq 5 \times 10^{-2}$. The continuum damping of the IAE due to toroidal coupling with the upper sideband, γ_{cd}, can be estimated similar to Sec.III,

$$\frac{\gamma_{cd}}{\omega_A} \sim \epsilon_{m+1}^2\left[\frac{\xi_m(r_{m+1})}{\xi_m(0)}\right]^2\left(\frac{\omega_A}{\omega_0}\right)^2\left|r_{m+1}\iota'_{m+1}\right|^{-1} \lesssim \epsilon_{m+1}^2(\delta q)^4\left(\frac{\omega_A}{\omega_G}\right)^2, \tag{42}$$

where subscript $m + 1$ denotes the value at the surface $nq = m + 1$ and in the last expression we have taken into account that, since $q - m/n = \delta q \ll 1$ in the shear-free core and ~ 1 in the outer region, then $\xi_m(r_{m+1})/\xi_m(0) \lesssim (\delta q)^2$ due to continuity of ξ_m and $\xi_m(a) = 0$. For $\epsilon_{m+1} = \delta q = 0.2, (\omega_G/\omega_A)^2 = 0.016$ (i.e., the same parameters as in the estimate of γ_{rd}) Eq.(42) yields $\gamma_{cd}/\omega_A \lesssim 4 \times 10^{-3}$, more than an order of magnitude lower than the radiative damping of quasimode given above.

In summary, it is shown that both the low-frequency cascades in the high-pressure hybrids observed in JT-60U and relatively rare, but challenging for existing theory Alfvén cascades with downward frequency sweeping observed in JET can be interpreted as a new class of modes, the so called infernal Alfvén eigenmodes. Such modes reside in equilibria with flat q-profiles in the central core, when the value of the safety factor in this region is slightly above the integer or low-order rational. The corresponding cascades exhibit almost degenerate spectrum in the plasma frame.

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