Nonlinear Theory and Simulation of Energetic Particle-induced Geodesic Acoustic Mode

Guoyong Fu

Princeton Plasma Physics Laboratory

In collaboration with

R. Nazikian, R.V. Budny, M. Gorelenkova, M. Van Zeeland

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Outline

• Introduction
• Nonlinear Simulation of EGAM in DIII-D
• Nonlinear Theory of EGAM: generation of second harmonic
Energetic Particle-driven GAM (EGAM)

- Energetic particle-driven GAM-like modes have been observed in tokamaks and stellarators (JET, DIII-D, LHD etc).

- The first observation came from JET where an $n=0$ mode was excited by the fast ICRF tail ions. The mode was interpreted as a MHD GAM eigenmode with frequency above the maximum of GAM continuum. (C. J. Boswell et al., Phys. Lett. A 358, 154 (2006) H. L. Berk et al., Nucl. Fusion 46, S888 (2006).)

- Recent DIII-D results showed count-injected beam ions can excite a $n=0$ GAM-like global mode. The mode frequency is well inside the GAM continuum. (R. Nazikian et al., Phys. Rev. Lett. 101,185001 (2008).)

- Analytic theory showed existence of EGAM with global radial mode structure determined by energetic particle’s finite orbit width effects. (G.Y. Fu, Phys. Rev. Lett. 101,185002 (2008))
Instability Mechanism of EGAM

• The GAM is usually stable due to $n=0$ (i.e., no universal drive due to radial gradient). Thus it is typically driven nonlinearly by micro-turbulence.
• However, energetic particles can provide instability drive via velocity space gradient for inverted distribution function (inverse Landau damping for $dF/dE > 0$). In this way, the energetic particle-driven GAM is similar to the bump-on-tail instability.
• $dF/dE > 0$ is possible for NBI beam ions and ICRF tail ions due to bump-on-tail distribution.
• The wave particle resonance ($\omega_{GAM} \sim \omega_{bh}$) is possible for large value of safety factor ($v_h \sim qv_t$)
Mode Frequency Well Below ideal GAM frequency

• $n=0$ GAM continuum
  \[ \omega \approx \frac{2C_s}{R} \]
  - ideal GAM can only exist above the continuum
  - no NOVA solution

• Mode frequencies well below peak in the continuum
  - not the ideal GAM

• Mode structure is global, not the local kinetic GAM

\[ \delta B/B \sim 10^{-5}, \; n=0 \; \text{at wall} \]

Energetic particle effects induce two new branches of eigenmode (EGAM)

Hybrid Simulation Model for EGAM

- n=0 electrostatic perturbation (Er only)
- fluid model for thermal plasma;
- drift-kinetic model for energetic particles.
Nonlinear Hybrid Equations for EGAM

\[ E = -\nabla \Phi = E_r \nabla r, \quad v = \frac{E \times B}{B^2} \]

**Radial electric field response**

\[ \frac{\partial}{\partial t} < \frac{\rho |\nabla r|^2}{B^2} > = E_r = -< G(r, \theta)(2P_{th} + P_{\parallel h} + P_{\perp h}) > \]

**Geodesic curvature!**

\[ G(r, \theta) = -\frac{B_\phi R}{J B^3} \frac{\partial B}{\partial \theta} \]

**Density response**

\[ \frac{\partial}{\partial t} \rho + v \cdot \nabla \rho = -\nabla \cdot v \rho \]

**Thermal pressure response**

\[ \frac{\partial}{\partial t} P_{th} + v \cdot \nabla P_{th} = -\gamma \nabla \cdot v P_{th} \]

**Energetic particle response**

\[ P_{\parallel h} + P_{\perp h} = \int d^3v (mv^2_{\parallel} + \frac{1}{2}mv^2_{\perp})f_h \]
Parameters and Profiles

\[ R/a=3, \quad q_{\text{min}}=4.0 \]

\[ P=P_0 (1-\psi)^2 \quad \rho=\text{constant} \]

\[
f = \frac{1}{v^3 + v_{\text{crit}}^3} \exp \left[ - \frac{P_\phi}{e\Delta\Psi} - \left( \frac{\Lambda - \Lambda_0}{\Delta\Lambda} \right)^2 \right]
\]

\[ \Lambda_0=0.5, \quad \Delta\Lambda=0.2 \]
Linear evolution of perturbed pressures

δP_{th}

δP_{h}
Nonlinear simulations show bursting behavior
Nonlinear simulation shows frequency chirping of EGAM
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Strong beam-driven GAM was observed in DIII-D (shot #134503)
Nonlinear Simulation Model of Beam-driven
EGAM in DIII-D

• Assume $n=0$ electric static perturbation ($E_r$ only);
• Use hybrid model — fluid model for thermal species and
drift-kinetic model for beam ions
• Realistic beam distribution from NUBEAM code including
both Co and Counter beams and all energy components.

\[
\left< \frac{\rho |\nabla r|^2}{B^2} \right> \frac{\partial}{\partial t} E_r = - \left< G(r, \theta)(2P_c + P_{\parallel h} + P_{\perp h}) \right> \\
\frac{\partial}{\partial t} P_c = 2\gamma G(r, \theta)E_r P_c \\
G(r, \theta) = -\frac{B_{\varphi} R}{J B^2} \frac{\partial B}{\partial \theta} \\
P_{\parallel h} + P_{\perp h} = \int m(\mathbf{v}_{\parallel}^2 + \frac{1}{2} \mathbf{v}_{\perp}^2) f \, d\mathbf{v}
\]
Beam Ion Distribution from TRANSP/NUBEAM (r/a~0.3)

R. Budny
Parameters and Profiles
(DIII-D shot # 134503 at t=260msec)

- $R=1.64m$, $a=0.62m$
- $B=2.06T$, $n_e=1.4\times10^{13}\text{cm}^{-3}$, $T_e=0.6\text{kev}$
- $q_{\text{min}}\sim5.0$
- $\beta_{\text{th}}=0.15\%$, $\beta_{\text{beam}}=0.16\%$
Evolution of density fluctuation
From $t \sim 260 \text{msec}$
At $Z=3.6 \text{cm}$.

$f \sim 16 \text{ kHz}$

$\delta n_e \sim 3\%$

outward propagation!
Density fluctuation evolution at R=1.88m
$E_r$ and $\delta n_e$ (2D mode structure)

Up-down asymmetric density fluctuation!
Density fluctuation amplitude and phase
Summary: Simulation of beam-driven EGAM in DIII-D

• Carried out hybrid simulations with realistic beam distribution function (full f PIC method).

• Nonlinear simulations yield mode frequency, mode amplitude and outward radial propagation consistent with experimental measurement of density fluctuation from BES.
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This Work

• Consider nonlinear generation of the second harmonic in density fluctuation due to beam-driven GAM;

• Motivated by recent DIII-D data which show that there is a large $2^{\text{nd}}$ harmonic in the measured density fluctuation associated with the beam-driven GAM (R. Nazikian, 2009).

\(2\omega_{\text{GAM}}\), \(\omega_{\text{GAM}}\), and DC
Kinetic/MHD Hybrid Equations

\[ \rho \left( \frac{\partial}{\partial t} \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla P_c - \nabla \cdot \mathbf{P}_h + \mathbf{J} \times \mathbf{B} \]

\[ \frac{\partial}{\partial t} \rho + \mathbf{v} \cdot \nabla \rho = -\nabla \cdot \mathbf{v} \rho \]

\[ \frac{\partial}{\partial t} P_c + \mathbf{v} \cdot \nabla P_c = -\gamma \nabla \cdot \mathbf{v} P_c \]

\[ \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \]
\[ E_r \text{ Equation} \]

\[
\rho \left( \frac{\partial}{\partial t} \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla P_c - \nabla \cdot \mathbf{P}_h + \mathbf{J} \times \mathbf{B}
\]

\[
< \frac{\rho |\nabla r|^2}{B^2} > \frac{\partial}{\partial t} E_r + < \frac{\rho}{B^2} (\nabla r \times \mathbf{B}) \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) > = - < \frac{\nabla r \times \mathbf{B}}{B^2} \cdot (\nabla P_c + \nabla \cdot \mathbf{P}_h) >
\]

\[
< \frac{\rho |\nabla r|^2}{B^2} > \frac{\partial}{\partial t} E_r + < \frac{\rho}{B^2} (\nabla r \times \mathbf{B}) \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) > = - < G(r, \theta)(2P_c + P_\parallel + P_\perp) >
\]

\[
G(r, \theta) = -\frac{B_\Phi R}{JB^3} \frac{\partial B}{\partial \theta} \approx -\frac{1}{BR} \sin(\theta)
\]
Linear Fluid Equations

\[
< \frac{\rho | \nabla r |^2}{B^2} > \frac{\partial}{\partial t} E_{r1} = -2 < G(r, \theta) P_{c1} >
\]

\[
\rho \frac{\partial}{\partial t} v_{||1} = -b \cdot \nabla P_{c1}
\]

\[
\frac{\partial}{\partial t} P_{c1} = -\gamma (B \cdot \nabla (\frac{v_{||1}}{B}) - 2G(r, \theta)E_{r1})P_c
\]

\[
\frac{\partial}{\partial t} P_{c1} = 2\gamma P_c (L^{-1}G(r, \theta))E_{r1}
\]

\[
L = 1 + \frac{\gamma P_0}{\rho \omega^2} B \cdot \nabla \frac{1}{B^2} B \cdot \nabla
\]
GAM dispersion relation

\[ \frac{\rho |\nabla r|^2}{B^2} \frac{\partial}{\partial t} E_{r_1} = -2 \langle G(r, \theta)P_{c1} \rangle \]

\[ \frac{\partial}{\partial t} P_{c_1} = 2\gamma P_c (L^{-1}G(r, \theta))E_{r_1} \]

\[ \omega^2 = \omega_{GAM}^2 \equiv \frac{4\gamma P}{\rho} \frac{\langle GL^{-1}G \rangle}{\langle |\nabla r|^2 / B^2 \rangle} \approx \frac{2\gamma P}{\rho R^2} (1 + \frac{1}{2q^2}) \]
Nonlinear Fluid Theory: second order
radial electric field is zero

\[
\left< \frac{\rho |\nabla r|^2}{B^2} \right> \frac{\partial}{\partial t} E_{r2} = \left< \frac{\rho_1 |\nabla r|^2}{B^2} \right> \frac{\partial}{\partial t} E_{r1} - \left< \frac{\rho}{B^2} (\nabla r \times \mathbf{B}) \cdot (\mathbf{v}_1 \cdot \nabla \mathbf{v}_1) \right> - 2 \left< G(r, \theta) P_{c2} \right>
\]

\[
\frac{\partial}{\partial t} P_{c2} = -\mathbf{v}_1 \cdot \nabla P_{c1} - \gamma \nabla \cdot \mathbf{v}_1 P_{c1} - \gamma \nabla \cdot \mathbf{v}_2 P_{c}
\]

\[
P_{c2} = P_{21} + P_{22} E_{r2}
\]

\( P_{21} \) is an even function of \( \theta \) \rightarrow \( E_{r2} = 0 \)

This result is the same as the gyrokinetic result of H.S. Zhang et al. 2009
Nucl. Fusion 49, 125009
Nonlinear Fluid Theory: second order density perturbation

\[ \frac{\partial}{\partial t} \rho_2 = -v_1 \cdot \nabla \rho_1 - \nabla \cdot v_2 \rho_0 \approx -E_{r_1} \frac{\nabla r \times B}{B^2} \cdot \nabla \rho_1 \]

\[ \rho_1 = 2G \int E_{r_1}(r, \tau) d\tau \approx -\frac{2}{BR} \sin(\theta) \int_0^t E_{r_1}(r, \tau) d\tau \]

\[ \frac{\rho_2}{\rho_0} = -\frac{r}{R} \cos(\theta) \left( \frac{1}{rB} \int_0^t E_{r_1}(r, \tau) d\tau \right)^2 \]

\[ \approx -\frac{r}{2R} \cos(\theta) \left( \frac{1}{rB \omega} \hat{E}_{r_1}(r, t) \right)^2 (1 - \cos(2\omega t)) \]

\[ E_{r_1}(r, t) = \hat{E}_{r_1}(r, t) \cos(\omega t) \]
Properties of the second order density perturbation

\[
\frac{\rho_2}{\rho_0} \approx -\frac{r}{2R} \cos(\theta) \left( \frac{1}{rB\omega} \hat{E}_{r1}(r,t) \right)^2 (1 - \cos(2\omega t))
\]

- Zero frequency and second harmonic have equal amplitude;
- Second order perturbation breaks up-down asymmetry;
- Second order density perturbation is negative on low field side;
- Near mid-plane, \( \rho_2 \) is comparable to \( \rho_1 \) since \( \rho_1 \sim \sin \theta \)
Simulation of density fluctuation including fluid nonlinearity shows clear negative perturbation
Kinetic/MHD hybrid equations

\[ < \frac{\rho |\nabla r|^2}{B^2} > \frac{\partial}{\partial t} E_r + < \frac{\rho}{B^2} (\nabla r \times B) \cdot (v \cdot \nabla v) >= - < G(2P_c + P_{||} + P_{\perp}) > \]

\[ P_{||} + P_{\perp} = \int m(v_{||}^2 + \frac{1}{2} v_{\perp}^2) f \, d\mathbf{v} \]

\[ \frac{\partial f}{\partial t} + (v_{||} b + v_d + \frac{E \times B}{B^2}) \cdot \nabla f + \frac{dE}{dt} \frac{\partial f}{\partial E} = 0 \]

\[ \frac{dE}{dt} = (mv_{||}^2 + \frac{1}{2} mv_{\perp}^2) G(r, \theta) E_r \]
Nonlinear expansion of distribution function

\[
< \frac{\rho \left| \nabla r \right|^2}{B^2} > \frac{\partial}{\partial t} E_{r2} = -< G(r, \theta)(2P_{c2} + P_{\parallel h2} + P_{\perp h2}) >
\]

\[
f = f_0 + f_1 + f_2 + ....
\]

\[
\frac{df_1}{dt} = \frac{\partial f_1}{\partial t} + (v_{\parallel}b + v_{d}) \cdot \nabla f_1 = -\left( m v_{\parallel}^2 + \frac{1}{2} m v_{\perp}^2 \right) G(r, \theta) E_{r1} \frac{\partial f_0}{\partial E}
\]

\[
\frac{df_2}{dt} = -\frac{E_1 \times B}{B^2} \cdot \nabla f_1 - \left( m v_{\parallel}^2 + \frac{1}{2} m v_{\perp}^2 \right) G(r, \theta) \left( E_{r1} \frac{\partial f_1}{\partial E} + E_{r2} \frac{\partial f_0}{\partial E} \right)
\]
Energetic particles can generate a second harmonic in radial electric field

\[ f_{1,\text{non-res}} = \frac{mv_{\|}^2 + \frac{1}{2}mv_{\perp}^2}{2BR} \hat{E}_r \frac{\partial f_0}{\partial E} \left( \frac{\cos(\omega t - \theta)}{\omega - \omega_b} - \frac{\cos(\omega t + \theta)}{\omega + \omega_b} \right) \]

\[ f_2 = \frac{3(mv_{\|}^2 + \frac{1}{2}mv_{\perp}^2)}{2B^2rR} \hat{E}^2 \frac{\partial f_0}{\partial E} \left( \frac{\omega \omega_b}{(4\omega^2 - \omega_b^2)(\omega^2 - \omega_b^2)} \right) \sin \theta \cos 2\omega t + ... \]

\[ E_{r,2} = -\frac{\omega_h^2}{4\omega^2 - \omega_{EGAM}^2} \frac{\hat{E}^2}{\omega Br} \sin(2\omega t) \]

Energetic particle effects can generate second harmonic in \( E_r \)

\[ \omega_h^2 = -\frac{3}{2\rho R^2} \int \left( \frac{\omega^3 \omega_b}{(4\omega^2 - \omega_b^2)(\omega^2 - \omega_b^2)} \right) (mv_{\|}^2 + \frac{1}{2}mv_{\perp}^2)^2 \frac{\partial f_0}{\partial E} d^3v \]
Second order density perturbation due to energetic particles

\[
\frac{\rho_{2,h}}{\rho_0} = -\frac{\omega_h^2}{4\omega^2 - \omega_{EGAM}^2} \frac{\hat{E}^2}{\omega^2 B^2 R r} \sin(\theta) \cos(2\omega t)
\]

\[
\frac{\rho_{2,h}}{\rho_{2,f}} = -\frac{\omega_h^2}{4\omega^2 - \omega_{EGAM}^2} \frac{\sin(\theta)}{\cos(\theta)} \sim \frac{P_{||h}}{P_c} + \frac{P_{\perp h}}{P_c} \frac{\sin(\theta)}{\cos(\theta)}
\]

Spatial structure of energetic particle-induced second harmonic is the same as the fundamental harmonic!
Simulations without fluid nonlinearity shows evidence of second harmonic due to energetic effects.

\[ \delta n_e \]

\[ f \text{ (kHz)} \]

\[ R=1.74\text{m} \]
\[ Z=0.18\text{m} \]
Summary: Nonlinear theory of generation of second harmonic

- **Fluid Model:**
  - GAM self-interaction cannot generate a second harmonic in the radial electric field;
  - A second harmonic of density fluctuation is generated by convective nonlinearity. A DC component is also present.
  - The density perturbation is negative near the mid-plane for strong instability.
  - These results are consistent with DIII-D experiments.

- **Energetic Particle Effects:**
  - Energetic particle effects can generate a second harmonic in radial electric field;
  - The energetic particle-induced second harmonic of the density perturbation scales as \( \sin \theta \). The EP contribution is small near the mid-plane.