

# Simulation and adiabatic models for spontaneous frequency sweeping of energetic particle-driven Alfvén eigenmodes

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TAE mode is characterized as being a standing wave that is excited within a frequency gap that is surrounded by the MHD continuum [1]. In the presence of dissipation due to the background plasma, and at low enough collisionality, the TAE wave is likely to chirp rapidly in frequency. The initial evolution of the wave follows the now standard formation of a hole/clump pair, which applies to the most simplified model of the electrostatic bump-on-tail instability [2]. However, once the frequency shift becomes comparable to the frequency shift from the continuum, the stand dynamics of a phase space structure changes considerably. Here we will exhibit a new paradigm for describing phase space chirping emanating wave excitation that chirps into the surrounding continuum.

We derive a dynamic TAE model, based on the linear tip theory of Rosenbluth, Berk and Van Dam (RBV) [3], which is valid for large inverse aspect ratio. The RBV theory leads to a Volterra integral equation in the time domain. Then with enough extrinsic dissipation so that the system is close to marginal stability, it is possible for the excited wave to chirp into the continuum. A single couplet for internal TAE modes is driven by the energetic particles due to effective currents of a single set of action-angle variables  $(p, \theta)$  which produces an instability driving with the same structure as bump-on-tail problem. The final wave equation takes the form,

$$(\Delta_m - i)A(t) = \int_0^t (J_0(t - \tau) - iJ_1(t - \tau))e^{-\gamma_d(t-\tau)}A(\tau)d\tau - \frac{i\eta}{\pi} \int_{-\infty}^{\infty} f(p, \theta, t)e^{-i\theta} dpd\theta, \quad (1)$$

where  $\gamma_d$  mocks up of damping rate from background dissipation. In our simulation, the full kinetic Vlasov equation is used to describe the dynamics of particles for a collisionless plasma in a distribution function characterized by a single pair of action-angle pair  $(p, \theta)$  variables,

$$\frac{\partial f}{\partial t} + p \frac{\partial f}{\partial \theta} - \Re\{A(t)e^{i\theta}\} \frac{\partial f}{\partial p} = 0. \quad (2)$$

An improved algorithm is developed from the original Petviashvili work [4]. In the original numerical scheme of Petviashvili, the Vlasov equation in the Fourier transform space  $(s, n)$  of the action-angle variables  $(p, \theta)$  along the characteristic line  $s - n\Delta t = \text{constant}$ , where  $\Delta t$  is the time step, is integrated. Because the integral depends on the updated distribution function, Picard iterative method is used to calculate the updated distribution function. The iterations come at the cost of computational time and complications for code parallelization. Our improved Vlasov solver avoids the iterative process and converts Fourierized Vlasov equation along each individual characteristic line into a tri-diagonal matrix equation. The resulting code is ready for

parallelization on a set of characteristic lines. In this method the computational time is reduced by an order magnitude in a single quad-core machine.

A tracer of phase space structure is also designed in our numerical scheme. The tracer allows a shift of the computation frame from a lab frame to a wave frame. In this computational frame the chirping signal we are interested in propagates at a very small frequency. Then a moderately large time step can be taken and the targeted phase space structure can evolve and remain accurate even with a fixed  $\Delta t$  according to Nyquist–Shannon sampling theorem [5]. The tracer informs the program at what velocity the calculation frame ought to be in. We have developed a method that is an improvement from that of using a short time window Fourier transformation to determine the wave frame. This latter method sacrifices accuracy due to Heisenberg uncertainty principle associated with the time window. Our tracer method is developed to instantaneously determine the velocity and acceleration by an ensemble averaging on phase space structure. Therefore, the resonance condition  $\omega - kv = 0$  determines the frequency shifting of calculation frame directly from the velocity of the targeted phase space structure. (see Figure 1)

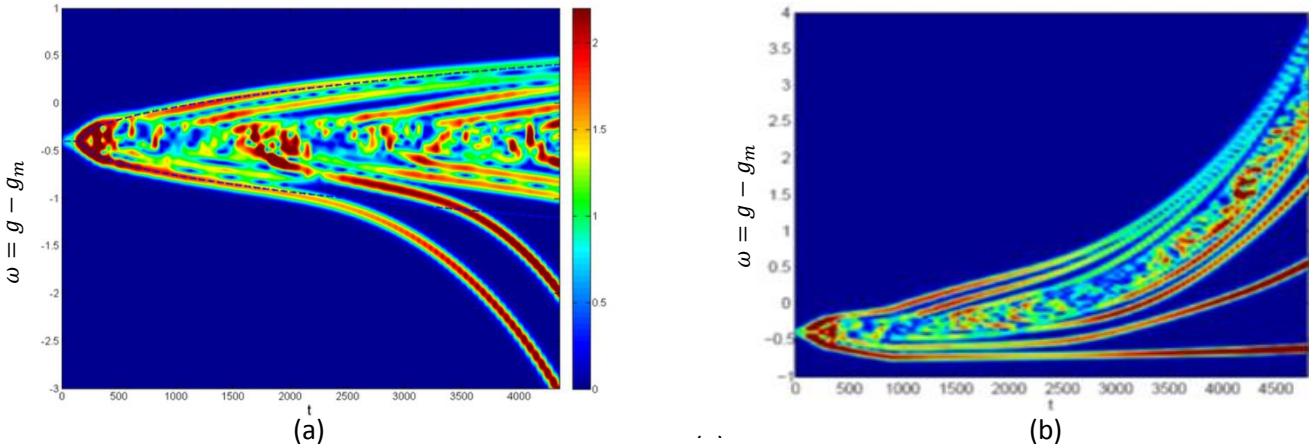


Figure 1. (a) Frequency response in a lab frame. (b) Frequency response in a calculation frame that tracks frequency of clump structure to improve the computational accuracy.

The resonant structure in phase space is commonly analyzed by using action-angle variables. With an appropriate canonical transformation the Hamiltonian for the system becomes,

$$H = \frac{p^2}{2} - \omega_b^2(\cos\theta + \alpha\theta). \quad (3)$$

This Hamiltonian, as a function of wave amplitude  $\omega_b^2$  and chirping rate  $\alpha \equiv \omega_b^{-2}d\omega/dt$ , leads to a skewed separatrix that characterizes the output at various times. A normalized Hamiltonian can be used to simplify analysis to one parameter  $\alpha$ ,

$$h = \frac{\tilde{p}^2}{2} - \cos\theta + \alpha\theta, \quad (4)$$

where  $\tilde{p} = P/\omega_b$ ,  $h = H/\omega_b^2$ . In the adiabatic approximation,  $f(J)$  remains only a function of the action,

$$J = \oint \tilde{p} d\theta = \sqrt{2}\omega_b \oint \sqrt{e + \cos\theta - \alpha\theta} d\theta. \quad (5)$$

Then TAE wave equation can be written in terms of a new pair of action-angle variable  $(J, \phi)$ ,

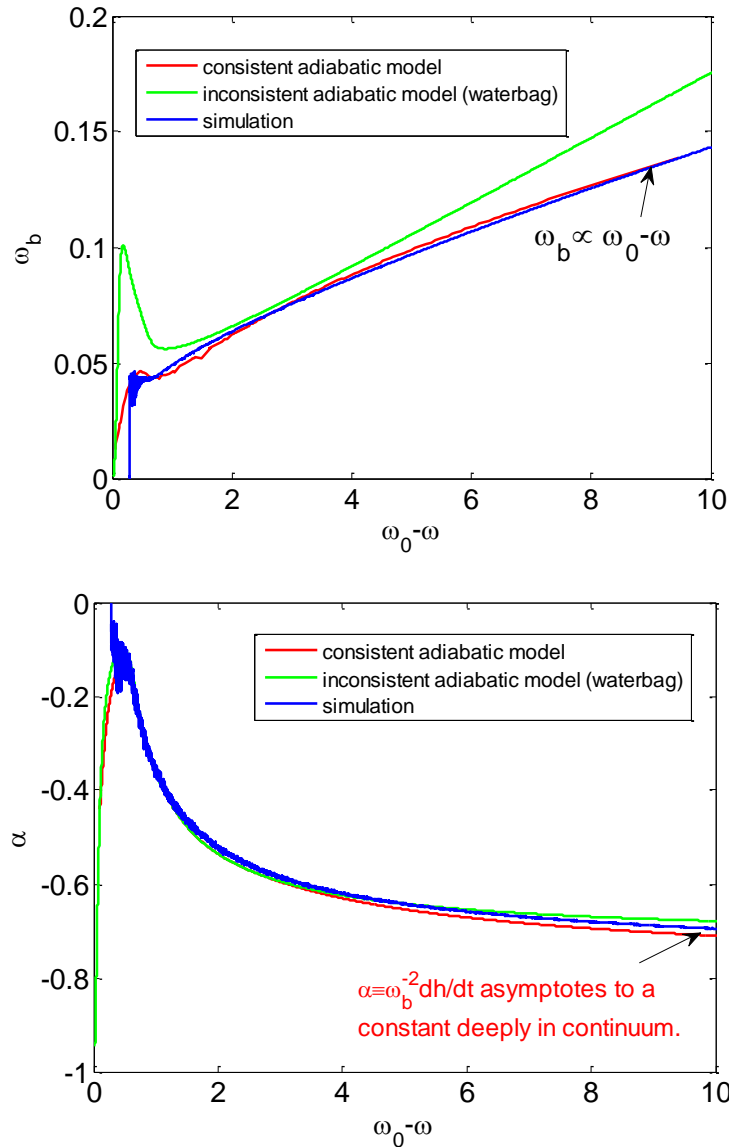
$$\Re \left[ \Delta_m - \left( \frac{1+\tilde{\omega}}{1-\tilde{\omega}} \right)^{\frac{1}{2}} \right] \omega_b = \frac{\eta}{\pi} \oint f(J) \cos\theta(\phi) d\phi dJ,$$

$$\Im \left[ \Delta_m - \left( \frac{1+\tilde{\omega}}{1-\tilde{\omega}} \right)^{\frac{1}{2}} \right] \omega_b = -\alpha \frac{\eta}{\pi} \oint f(J) d\phi dJ,$$

where  $\tilde{\omega} = \omega + i\gamma_d$ . For the adiabatic evolution of energetic particles, the distribution function is evolved in a relatively simple way. If  $J$  at the separatrix,  $J_{sep}$  increases in time, new phase space regions are entrained in the trapping region,

$$f(J_{sep}(t)) = F_0 \left( \omega(t) \pm \frac{\Delta\omega}{2} \right). \quad (6)$$

Here  $F_0$  is the unperturbed distribution function. '+' or '-' is applied to hole or clump, respectively and  $\Delta\omega$  is the momentum extension of the structure, which is the consequence of a wake formed by the leaked particles behind the moving separatrix. If  $J_{sep}$  decreases, the range of  $J$  in the trapping region decreases. These rules allow much faster predictions of the wave evolution than that from the direct Vlasov simulation. However, these new rules are only certain to be correct if the response remains adiabatic.



- Figure 2 The consistent adiabatic model gives much better agreement with the Vlasov simulation results than the waterbag model (inconsistent adiabatic model) for the clump, indeed very accurate predictions. Physical parameters are set as  $\Delta_m = 0.654, \eta = 0.00248, \gamma_L = 0.1, \gamma_d = 0.08$ .

The comparison between Vlasov simulation and adiabatic model for the clump is shown in Figure 2. An inconsistent adiabatic model simplifies the dependence of action  $J$  on the distribution function  $f(J)$  by using a waterbag energetic particle distribution during the entire evolution. However, the consistent adiabatic model takes on a waterbag energetic distribution only when the phase space structure is born near the resonance region and recalculates the action at the separatrix at each time step to update the energetic particle distribution  $f(J)$ . The agreement between Vlasov simulation and consistent adiabatic model is impressively good. We also find  $\alpha$  approaches the asymptotic value  $\alpha_\infty = -0.7165$  when the clump chirps deeply into

the continuum in our case. Then the adiabatic theory gives an asymptotic explosive response (see Figure 3),

$$\omega_b(t) = \frac{\omega_{b0} t_\infty}{t_\infty - t}, (t_\infty = 6488, \omega_{b0} = 0.0322)$$

Vlasov simulation, as far as it can go, gives compatible results. Perhaps this is a feature of an experimental TAE avalanche by M. Podesta, et. al [6,7].

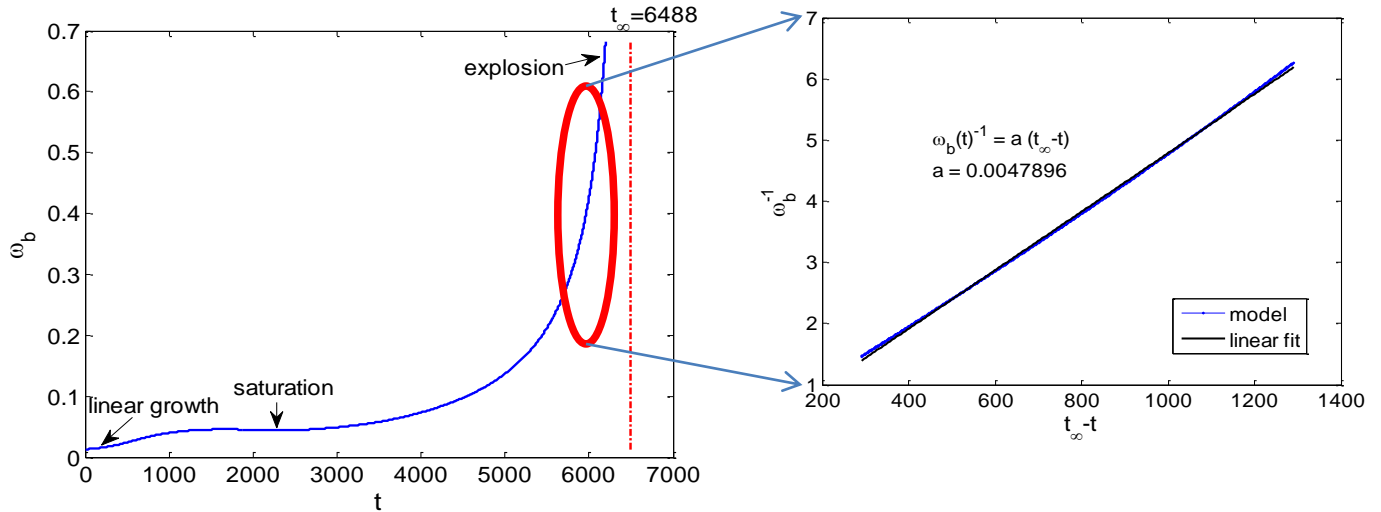


Figure 3 The characteristic solution of wave amplitude, which is a linear growth initially, saturates afterwards and ends with an explosive solution.

The adiabatic model also predicts that up-chirping TAE mode survives through the upper tip boundary while up-frequency chirping prediction from the Vlasov simulation never reaches continuum boundary shown in Figure 4. This discrepancy is likely due to a breakdown of adiabaticity when the chaotic region around the separatrix permeates too much into the trapped particle structure.

In conclusion, we have reported on our recent progress on a TAE chirping model. The Vlasov simulation algorithm has been greatly improved in the accuracy and efficiency. A sophisticated adiabatic model is also developed and gives an impressive agreement with the results from simulation. It is possible that this simulation captures essential features to explain avalanches phenomena that have been reported experimentally.

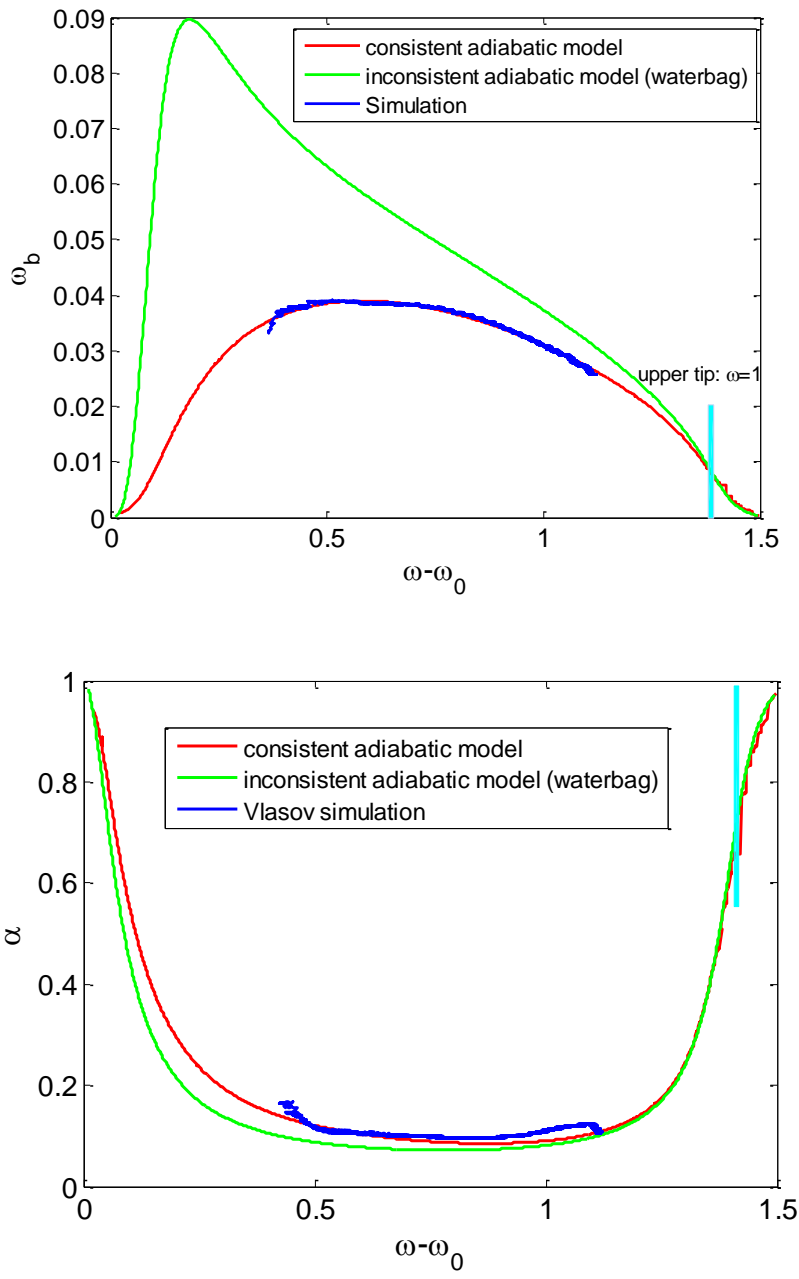


Figure 4 Comparison between Vlasov simulation and adiabatic models for the hole structure.

***Reference :***

- [1] Cheng CZ, Chance MS. Phys Fluids 1986, 29:3695-701
- [2] Berk HL, Breizman BN, Petviashvili NV. Phys Lett A 1997, 234:213-8
- [3] Rosenbluth MN, Berk HL, Van Dam JW, Lindberg DM. Phys. Fluids B 1992, 4:2189-202
- [4] Petviashvili NV. Coherent structures in nonlinear plasma dynamics, Ph.D dissertation, University of Texas at Austin, Department of Physics, 1998.
- [5] H. Nyquist, Trans. AIEE 1928, 617–44. Reprint as classic paper in : Proc. IEEE, Vol. 90, No. 2, Feb 2002.
- [6] Podesta M, Heidbrink WW, Liu D, Ruskov E, Bell RE, Darrow DS, et al. Phys.Plasma 2009, 16:056104
- [7] Fredrickson ED, Crocker NA, Bell RE, Darrow DS, Gorelenkov NN, Kramer GJ, et al. Phys. Plasma 2009, 16:122505